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# A Fuzzy Imperfect Production Inventory Model Based on Fuzzy Differential and Fuzzy Integral Method

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**Abstract:** In the inventory theory, to treat the uncertainty, the fuzzy set concept is used in order to provide a feasible approach to deal with the uncertainty problem. In this research work, a fuzzy economic production quantity model with interactive fuzzy demands is proposed. In a production process, in the beginning, the system is assumed to be in a controlled state in which only perfect items are manufactured. Later, the manufacturing production process shifts to be an out-of-control-state system; producing both perfect and imperfect items simultaneously, this is considered as a fuzzy state. The defective production rate is also taken into account as a fuzzy state. Here, the selection process of produced items is realized during the production period. With the aim of studying the practical feasibility of the fuzzy economic production inventory model along with a sensitivity analysis of some parameters, different numerical examples are illustrated.

**Keywords:** fuzzy economic production quantity; fuzzy imperfect production process; fuzzy integral method; fuzzy demand; fuzzy programming technique



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## 1. Introduction

It is well known that the fuzzy set concept is applied into the inventory models to treat the uncertainty. The fuzzy set theory was introduced by (Zadeh 1965) with the aim of providing a feasible approach to deal with the fuzzy uncertainty problem. In the literature, the fuzzy set theory, also known as uncertain sets, has attracted attention for treating uncertainty in a variety of circumstances. For example, fuzzy inventory costs in the economic order quantity model are used in (Park 1987; Priyan and Uthayakumar 2016). Obtaining the economic production quantity when the quantity of demand is uncertain is analyzed in (Chang 1999). To treat the inventory problem considering all the parameters and variables being fuzzy numbers, a fuzzy economic production model is established by (Chen and Hsieh 2000). Different types of production inventory models for fuzzy environments are proposed by studies such as (Dey et al. 2005; Hsieh 2002; Lee and Yao 1998; Lin and Yao 2000; Manna et al. 2014, 2017a). Furthermore, other, different research works solve uncertainty issues using fuzzy set theory, such as (Das et al. 2015; Soni and Joshi 2015). Bera and Jana (2017) developed an imperfect production inventory model for multi-items under bi-fuzzy environments. (Dey 2019) introduced an imperfect production inventory problem under a fuzzy random environment. Recently, (Maiti 2021) incorporated the demand-dependent production rate into an inventory model with imperfect production process under a cloudy fuzzy environment.

Traditional economic production quantity (EPQ) models assume that in manufacturing systems, all items are made of perfect quality. However, in the real world, due to many factors, such as the operator's skills, machine components, raw materials, and/or other aspects, the generation of defective items is inevitable. An imperfect production process is considered if all industries produce a certain percent of imperfect quality items (see, for instance, (Salameh and Jaber 2000) and (Yoo et al. 2009)). An inventory model for imperfect quality items is presented by (Salameh and Jaber 2000), assuming that defective items are vended as a single batch at the end of the total inspection process. Moreover, the probability of switching from an "in-control" state to an "out-of-control" state is analyzed by (Porteus 1986). An EPQ model dealing with imperfect quality, assuming that at some random point in time the process might shift from an "in-control" to an "out-of-control" state, and a fixed percentage of defective items is generated, is proposed by (Rosenblatt and Lee 1986). Furthermore, an economic order quantity (EOQ) inventory model with demand-dependent unit production cost and imperfect production processes is examined by (Cheng 1991). Different types of imperfect production inventory models are presented by several studies, such as (Chiu et al. 2009; Chung and Hou 2003; Jaber 2006; Lee 2005; Lin et al. 2003; Lo et al. 2007; Pasandideh et al. 2015). Based on these imperfect production inventory models, Cárdenas-Barrón (2000) and Goyal and Cárdenas-Barrón (2002) formulated an economic production quantity model for items with imperfect quality. Furthermore, the determining of the replenishment lot size and shipment policy for an EPQ inventory model with delivery and rework were investigated by Cárdenas-Barrón et al. (2015). An EPQ inventory model with a random defective rate, rework process, and backorders for a single-stage production system is addressed by (Sarkar et al. 2014). An EPQ model with promotional demand in a random planning horizon with a population-varying genetic algorithm approach is analyzed in (Manna et al. 2016). Furthermore, an imperfect production-inventory model with a production-rate-dependent defective rate and advertisement-dependent demand is given in (Manna et al. 2017b). (Shah and Vaghela 2018) developed an imperfect production-inventory model with time-dependent demand under inflation and reliability. (Khara et al. 2020) derived a sustainable manufacturing and remanufacturing model with quality-level-dependent development cost and rework policy of imperfect items. (Shaikh et al. 2020) considered price-dependent demand with inflation and reliability in an imperfect production model for deteriorating items under partial trade credit policy. Recently, (Mallick et al. 2021) formulated a bi-level supply chain model for perishable goods with complete backlogged shortages under fuzzy lead time. Based on the shown literature, some research works discussed fuzzy costs and fuzzy demand, among others, but up to now, no one has included a fuzzy detective rate and fuzzy time on imperfect production. Therefore, to study the fuzzy economic production quantity (FEPQ) model, considering the above literature with imperfect unrepaired products is very important in a vague environment.

Recently, it has been noted that good-quality products are increasingly demanded by most consumers. Defective products delivered to consumers increase the replacement cost and repair services cost, causing huge damage to the company's credibility. Therefore, companies must establish a production procedure to adjust the inspection systems to assure the customer's quality. To assist a manager to perform operational control and quality assurance, it is important to have understanding of production, inspection, inventory, and maintenance and their relationships to prevent imperfect systems. In production processes, the process starts in a controlled state and could change to an "out-of-control" state, appearing as defective or non-conforming items. Through the inspection, as a part of the process, the defective items need to be detected to ensure that consumers will not receive the non-conforming items. The state of the product quality and production system is determined during the inspection process. The quality and inspection costs need to be balanced and it depends on the frequency of the inspections to be performed. Therefore, the inspections need to be scheduled. In this way, (Wang and Sheu 2001) generalized the model of (Porteus 1986) by introducing a product inspection policy. In addition, (Wang 2005) extended the (Kim and Hong 1999) work, taking into account a product inspection policy at the end of the production run, instead of full inspections during a production run. Bera and Jana (2017) incorporated the fuzzy screening cost into an imperfect production-inventory model for separating the produced imperfect items. Taleizadeh et al. (2019) addressed quality screening and reworking policies in a single machine multi-products production-inventory model. They also considered rework policy of produced defective items. Recently, Banu et al. (2021) considered the screening cost in a manufacturer–retailer supply chain model under type-2 fuzzy environment. They supposed that the demand rate is dependent on both credit period and stock. The main features of the proposed research work and some related models are presented in Table 1.

**Table 1.** Major characteristics of imperfect production inventory models on selected articles.

| Author(s) | Imperfect Production | Fuzzy Defective Rate | Fuzzy Time ($\tilde{\tau}$) Which Shifts "In-Control" to "Out-Control" | Fuzzy Demand Rate | Formulation Using FDE | Imprecise Environment |
|---|---|---|---|---|---|---|
| Park (1987) | | | | √ | | √ |
| Cheng (1991) | √ | | | | | |
| Lee and Yao (1998) | √ | | | √ | | √ |
| Chang (1999) | √ | | | | | √ |
| Chen and Hsieh (2000) | √ | | | | | √ |
| Lin and Yao (2000) | √ | | | | | √ |
| Salameh and Jaber (2000) | √ | | | | | |
| Kao and Hsu (2002) | √ | | | √ | | √ |
| Wang (2005) | √ | | | | | √ |
| Dutta et al. (2005) | √ | | | √ | | √ |
| Shao and Ji (2006) | √ | | | | | √ |
| Taleizadeh et al. (2009) | | | | √ | | √ |
| Bera and Jana (2017) | √ | | | | | √ |
| Shaikh et al. (2018) | | | | | | √ |
| Dey (2019) | √ | | | | | √ |
| Moghdani et al. (2020) | | | | √ | | √ |
| Banu et al. (2021) | √ | | | | | √ |
| Maiti (2021) | √ | | | | | √ |
| Present research work | √ | √ | √ | √ | √ | √ |

The sales and manufacturing environments are applied to classical inventory models focused on a constant demand rate. In some circumstances, the consumption rate may be influenced by the stock levels for certain consumer products. Due to the lack of historical data, it is not easy to estimate the probability distribution of market demand. Thus, it is necessary to use linguistic terms, such as "the market demand is about $d_M$, but not less than $d_L$ and not larger than $d_R$", in order to describe the fuzzy market demand. That means that, based on experience, the demand quantity can be specified approximately. Applying fuzzy theory, some research papers deal with this case, such as (Petrovic et al. 1996), who proposed a newsboy-type problem with discrete fuzzy demand. A single-period inventory problem in an imprecise and uncertain mixed environment is studied by (Dutta et al. 2005). They included the demand as a fuzzy random variable. Moreover, the multi-product newsboy problem with fuzzy demands under budget constraint is investigated by (Shao and Ji 2006). On the one hand, a single-period inventory model with fuzzy demand is established by (Kao and Hsu 2002). On the other hand, (Li et al. 2002) proposed, in a fuzzy environment, two single-period inventory models, assuming that in one of them, the demand is stochastic and the holding and shortage costs are fuzzy. The other one considers that the costs are deterministic but the demand is fuzzy. The production-inventory problem for fuzzy demand quantity is discussed by (Lee and Yao 1998). An uncertain EOQ inventory model for joint replenishment policy with incremental discount scheme and fuzzy rough demand is explored by (Taleizadeh et al. 2009). (Moghdani et al. 2020) considered demand as triangular fuzzy numbers in a production-quantity model for multi items. Recently, (Manna et al. 2021) examined a deteriorating two-warehouse inventory problem with time-dependent demand under all-unit discount and partial backlogged shortages via a metaheuristic algorithm.

In this research work, an imperfect production-inventory model is proposed with fuzzy defective and demand rates in which the production starts with a constant production rate up to a variable time in a controlled state, manufacturing perfect items. During production run time, after a certain time that follows a fuzzy number, the manufacturing process may shift to an "out-of-control" state, in which a percent of fabricated items are defective. The defective items are vended at a single lot at a reduced cost after that the production ends. Then, two profit functions are formulated and optimized through some numerical illustration.

The main contributions of this research work are highlighted as follows:

- A fuzzy production-inventory model in which the manufacturing system shifts from "in-control" state to "out-of-control" state at any time is developed.
- The defective and demand rates are taken in triangular, trapezoidal, parabolic, and general fuzzy numbers.
- The fuzzy production-inventory model is formulated mathematically by fuzzy differential equations and fuzzy integration approach.
- The corresponding optimization problem is fuzzy interval-valued which is maximized by fuzzy programming technique (FPT).
- To check the validity of the fuzzy production-inventory model, five numerical examples are presented and solved.
- The statistic test (Fisher's $t$-test) is performed for comparing between the optimum profit of triangular fuzzy demand and defective rates with trapezoidal fuzzy demand and defective rates.

The remainder of this paper is organized as follows: Section 2 describes notation and assumptions. Section 3 formulates the fuzzy production-inventory model. Section 4 provides a solution procedure. Section 5 solves some numerical examples and provides a sensitivity analysis. Finally, Section 6 gives conclusions and some future research directions.

## 2. Notation and Assumptions

The fuzzy production-inventory model is formulated using the notation (given at end of this article) and the following assumptions.

*Assumptions*

The following assumptions support the model here proposed:

(i) The fuzzy production-inventory model is developed only for a single item manufacturer which fabricates the items at a rate of $P$.

(ii) In the production process, in the beginning, the system is assumed to be in a controlled state in which only perfect items are fabricated. Later, the manufacturing process shifts to be an out-of-control state system, producing both perfect and imperfect items simultaneously; this is considered as a fuzzy state. Therefore, the manufacturing system may be in either "in-control" state or "out-of-control" state. Thus, it is assumed that the manufacturing system starts as a controlled state up to time $\widetilde{\tau}$, which is considered as a fuzzy state; after that, the production system goes to "out-of-control" state.

(iii) Usually, the rate of defectiveness is not a constant, due to the production rate, machine components, raw materials, operator's skills, or many other factors. Therefore, the production system may vary. In this way, the defectiveness quantity is uncertain. This fuzzy production-inventory model considers that the defective rate $(\widetilde{\beta})$ is fuzzy.

(iv) The fuzzy production-inventory model considers the cycle period $(T)$ as a decision variable.

(v) The production rate $(P)$ is constant and known.

(vi) The demand for the retailer changes due to various factors. It is imprecise and vague according to the business policy. In consequence, it is considered as a fuzzy demand from the retailer to the manufacturer. According to Zimmermann (1996), in fuzzy set theory, there are several standard fuzzy numbers. Here, the triangular, trapezoidal, parabolic, and general fuzzy numbers are considered for illustration purposes.

(vii) All produced items in "out-of-control" state (considered imperfect) are all vended together at a reduced price at the end of the production period.

## 3. Mathematical Formulation of the Fuzzy Production-Inventory Model

To formulate the fuzzy production-inventory model, the fuzzy concepts of (Mizumoto and Tanaka 1976; Román-Flores et al. 2001; Seikkala 1987; Zadeh 1965; Zimmermann 1996), the differential equation solutions (Chalco-Cano and Roman-Flores 2009), the fuzzy integration concepts (Wu 2000), and fuzzy programming techniques (Zimmermann 1996) are applied. The production starts at time $t = 0$ at the rate of $P$. Initially, up to time $\widetilde{\tau}$, the system produces perfect items. Then, during $[\widetilde{\tau}, t_1]$, it fabricates both good and defective items. At the time $t_1$, the production stops. After that, from the customers' stock, the demand is fulfilled up to time $T$. According to the assumptions, $\widetilde{\tau}$, $\widetilde{\beta}$, and $\widetilde{D}$ are taken as fuzzy numbers. Due to the existence of fuzzy parameters, at any time $t$, the inventory level is also fuzzy. Since there exist productions of perfect items and imperfect items, two separate inventory levels are considered.

### 3.1. Inventory Level for the Items with Perfect Quality

In Figure 1, the initial stock of perfect items is taken as zero, then the production begins at the rate of $P$. Good-quality items are fabricated by the system during $[0, \widetilde{\tau}]$, and both good and defective items are manufactured during $[\widetilde{\tau}, t_1]$, $\widetilde{\tau} \in (0, t_1)$. All the good items produced during $[0, t_1]$ are used to meet the perfect item demand up to time $T$. At the time $t_1$, the production of the cycle stops.

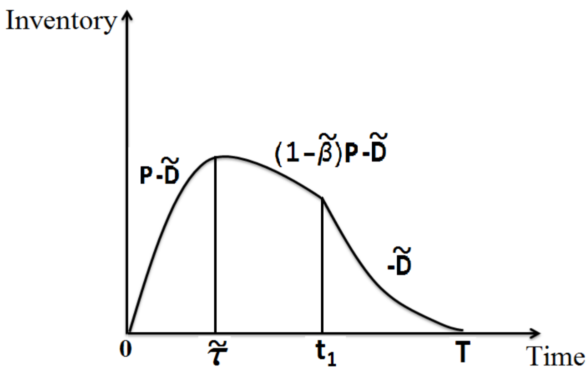


**Figure 1.** Pictorial representation of the manufacturer's inventory of a perfect-quality item.

Under such consideration, the inventory level of perfect items $\widetilde{q}_1(t)$ is modeled with the following differential equations at the time $t$:

$$\frac{d\widetilde{q}_1(t)}{dt} = \begin{cases} P - \widetilde{D}, & 0 \le t \le \widetilde{\tau} \\ (P - \widetilde{D}) - \widetilde{\beta}P, & \widetilde{\tau} \le t \le t_1 \\ -\widetilde{D}, & t_1 \le t \le T \end{cases}, \tag{1}$$

subject to $\widetilde{q}_1[0] = 0$, $\widetilde{q}_1[T] = 0$.

To solve the fuzzy differential Equation (1) first, according to (Chalco-Cano and Roman-Flores 2009), the solution $q_1(t)$ in the crisp differential environment is determined as follows:

$$\frac{dq_1(t)}{dt} = \begin{cases} P - D, & 0 \le t \le \tau \\ (P - D) - \beta P, & \tau \le t \le t_1 \\ -D, & t_1 \le t \le T \end{cases}, \tag{2}$$

subject to, $q_1[0] = 0$, $q_1[T] = 0$.

The differential equations solutions are

$$q_1(t) = \begin{cases} (P-D)t, & 0 \le t \le \tau \\ (P-D)t + \beta P\tau - \beta Pt, & \tau \le t \le t_1 \\ D(T-t), & t_1 \le t \le T \end{cases} \tag{3}$$

The manufacturer's production rate ($P$) and demand rate ($D$), and also the production time ($t_1$) and business time ($T$), must satisfy the following condition:

$$(1-\beta)Pt_1 + \beta P\tau = DT \tag{4}$$

From the continuity condition of $q_1(t)$ at $t = t_1$, the following relations are computed:

$$(P-D)t_1 + \beta P\tau - \beta Pt_1 = -D(t_1 - T)$$
i.e., $(1-\beta)Pt_1 + \beta P\tau = DT$

As $q\ (t)$ is continuous for $t \ge 0$, the unique fuzzy solution of (1), according to (Chalco-Cano and Roman-Flores 2009), is given by

$$\widetilde{q}_1(t) = \begin{cases} (P-\widetilde{D})t, & 0 \le t \le \widetilde{\tau} \\ (P-\widetilde{D})t + \widetilde{\beta} P\widetilde{\tau} - \widetilde{\beta} Pt, & \widetilde{\tau} \le t \le t_1 \\ \widetilde{D}(T-t), & t_1 \le t \le T \end{cases} \tag{5}$$

with the condition $(1-\widetilde{\beta})Pt_1 + \widetilde{\beta}P\widetilde{\tau} = \widetilde{D}T$.

$$\widetilde{q}_1(t)[\alpha] = [q_1^L(\alpha, t), q_1^R(\alpha, t)], \tag{6}$$

where

$$q_1^L(\alpha, t) = \begin{cases} (P-D_\alpha^R)t, & 0 \le t \le \tau_\alpha^L \\ (P-D_\alpha^R)t + P\beta_\alpha^L\tau_\alpha^L - P\beta_\alpha^R t, & \tau_\alpha^R \le t \le t_1 \\ D_\alpha^L(T-t), & t_1 \le t \le T \end{cases}$$

and

$$q_1^R(\alpha, t) = \begin{cases} (P-D_\alpha^L)t, & 0 \le t \le \tau_\alpha^R \\ (P-D_\alpha^L)t + P\beta_\alpha^R\tau_\alpha^R - P\beta_\alpha^L t, & \tau_\alpha^L \le t \le t_1 \\ D_\alpha^R(T-t), & t_1 \le t \le T \end{cases}$$

In a fuzzy environment, the manufacturer's production time period ($t_1$) and business time period ($T$) must satisfy the condition either (i) $\left\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\right\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$ or (ii) $\left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$.

From the $q_1^L(\alpha, t)$ and $q_1^R(\alpha, t)$ at $t = t_1$, continuity conditions, the following relations are calculated, respectively: (i) $(P - D_\alpha^R)t_1 + P\beta_\alpha^L\tau_\alpha^L - P\beta_\alpha^R t_1 = D_\alpha^L(T - t_1)$ i.e., $\left\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\right\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$ or, (ii) $(P - D_\alpha^L)t_1 + P\beta_\alpha^R\tau_\alpha^R - P\beta_\alpha^L t_1 = D_\alpha^R(T - t_1)$ i.e., $\left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$.

Now, it is seen that there exist variabilities of $t_1$ and $T$ for crisp value of $\beta$, $\tau$, and $D$, but in fuzzy environment, two relations are obtained. If $t_1$ and $T$ satisfy both these two relations simultaneously, then there will be loss of variability of $t_1$ and $T$. Therefore, to maintain variabilities of $t_1$ and $T$, they must satisfy either $\left\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\right\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$ or, $\left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$.

### 3.2. Inventory Level at Time t for the Imperfect Quality Items

At the end of the selection process, as a single lot, the imperfect quality items are vended. At time $t$, the inventory level $\widetilde{q}_2(t)$ satisfies the differential equation shown below.

$$\frac{d\widetilde{q}_2(t)}{dt} = \widetilde{\beta}P, \;\; \widetilde{\tau} \leq t \leq t_1 \tag{7}$$

subject to, $\widetilde{q}_2[\widetilde{\tau}] = 0$.

Now, (Chalco-Cano and Roman-Flores 2009) recommended, first, to find out the solution $q_2(t)$ of the crisp differential equation

$$\frac{dq_2(t)}{dt} = \beta P, \;\; \tau \leq t \leq t_1, \tag{8}$$

subject to $q_2[\tau] = 0$.

The solution is provided below.

$$q_2(t) = \beta P(t - \tau), \;\; \tau \leq t \leq t_1. \tag{9}$$

For each $t \geq 0$, the $q(t)$ is continuous, then, based on (Chalco-Cano and Roman-Flores 2009), the unique fuzzy solution of Equation (7) is given by

$$\widetilde{q}_2(t) = \widetilde{\beta}P(t - \widetilde{\tau}), \;\; \widetilde{\tau} \leq t \leq t_1 \tag{10}$$

Therefore, $\alpha$-cut of Equation (10) is provided by

$$\widetilde{q}_2(t)[\alpha] = [q_2^L(\alpha, t), q_2^R(\alpha, t)], \tag{11}$$

where

$$q_2^L(\alpha, t) = P\beta_\alpha^L(t - \tau_\alpha^R), \;\; \tau_\alpha^R \leq t \leq t_1, \tag{12}$$

$$q_2^R(\alpha, t) = P\beta_\alpha^R(t - \tau_\alpha^L), \;\; \tau_\alpha^L \leq t \leq t_1. \tag{13}$$

### 3.3. The Profit Function of the Fuzzy Production-Inventory System

During the cycle $(0, T)$, in the manufacturing system, the total production cost $(PC)$ is computed by

$$PC = c_p \int_0^{t_1} P\,dt = c_p P t_1$$

During the cycle $(0, T)$, in the production system, the total screening cost $(SC)$ is calculated by

$$SC = c_{sr} \int_0^{t_1} P\,dt = c_{sr} P t_1$$

During the cycle $(0, T)$, in the fabrication system, the total setup cost is equal to $A_m$. Now, from Wu (2000), $\alpha$-cut of the total holding cost $(HC)$ during the cycle $(0, T)$ in the production system is determined by

$$\begin{aligned} \widetilde{HC}[\alpha] &= \left(h_m \int_0^T \widetilde{q}_1(t)\,dt + h'_m \int_{\widetilde{\tau}}^{t_1} \widetilde{q}_2(t)\,dt\right)[\alpha] \\ &= \left[h_m \int_0^T q_1^L(t,\alpha)\,dt + h'_m \int_{\tau_\alpha^R}^{t_1} q_2^L(t,\alpha)\,dt,\; h_m \int_0^T q_1^L(t,\alpha)\,dt + h'_m \int_{\tau_\alpha^L}^{t_1} q_2^R(t,\alpha)\,dt\right] \\ &= \left[HC_\alpha^L, HC_\alpha^R\right], \end{aligned}$$

where

$$\begin{aligned}
HC^L_\alpha &= h_m \int_0^T q_1^L(t,\alpha)\,dt + h'_m \int_{\tau^R_\alpha}^{t_1} q_2^L(t,\alpha)\,dt \\
&= h_m\Big[\int_0^{\tau^L_\alpha} q_1^L(t,\alpha)\,dt + \int_{\tau^R_\alpha}^{t_1} q_1^L(t,\alpha)\,dt + \int_{t_1}^{T} q_1^L(t,\alpha)\,dt\Big] + h'_m \int_{\tau^R_\alpha}^{t_1} q_2^L(t,\alpha)\,dt \\
&= \frac{h_m}{2}\Big[(P-D^R_\alpha)\{t_1^2+(\tau^L_\alpha)^2-(\tau^R_\alpha)^2\} + 2P\beta^L_\alpha\tau^L_\alpha(t_1-\tau^R_\alpha) - P\beta^R_\alpha\{t_1^2-(\tau^R_\alpha)^2\} \\
&+ D^L_\alpha(T-t_1)^2\Big] + \frac{h'_m}{2}P(t_1-\tau^R_\alpha)^2, and \\
HC^R_\alpha &= h_m \int_0^T q_1^R(t,\alpha)\,dt + h'_m \int_{\tau^L_\alpha}^{t_1} q_2^R(t,\alpha)\,dt \\
&= h_m\Big[\int_0^{\tau^R_\alpha} q_1^R(t,\alpha)\,dt + \int_{\tau^L_\alpha}^{t_1} q_1^R(t,\alpha)\,dt + \int_{t_1}^{T} q_1^R(t,\alpha)\,dt\Big] + h'_m \int_{\tau^L_\alpha}^{t_1} q_2^R(t,\alpha)\,dt \\
&= \frac{h_m}{2}\Big[(P-D^L_\alpha)\{t_1^2+(\tau^R_\alpha)^2-(\tau^L_\alpha)^2\} + 2P\beta^R_\alpha\tau^R_\alpha(t_1-\tau^L_\alpha) - P\beta^L_\alpha\{t_1^2-(\tau^L_\alpha)^2\} \\
&+ D^R_\alpha(T-t_1)^2\Big] + \frac{h'_m}{2}P(t_1-\tau^L_\alpha)^2.
\end{aligned}$$

The $\alpha$-cut of total revenue $(\widetilde{SR})$ during the cycle $(0,T)$ in the manufacturing system is obtained with

$$\begin{aligned}
\widetilde{SR}[\alpha] &= \Big(s\int_0^T \widetilde{D}\,dt + s'\widetilde{\beta}P(t_1-\widetilde{\tau})\Big)[\alpha] \\
&= \Big[s\int_0^T D^L_\alpha\,dt + s'\beta^L_\alpha P(t_1-\tau^R_\alpha),\ s\int_0^T D^R_\alpha\,dt + s'\beta^R_\alpha P(t_1-\tau^L_\alpha)\Big] \\
&= \Big[sD^L_\alpha T + s'\beta^L_\alpha P(t_1-\tau^R_\alpha),\ sD^R_\alpha T + s'\beta^R_\alpha P(t_1-\tau^L_\alpha)\Big] \\
&= \Big[SR^L_\alpha,\ SR^R_\alpha\Big],
\end{aligned}$$

where

$$\begin{aligned}
SR^L_\alpha &= s\int_0^T D^L_\alpha\,dt + s'P\beta^L_\alpha(t_1-\tau^R_\alpha) = sD^L_\alpha T + s'P\beta^L_\alpha(t_1-\tau^R_\alpha), \\
SR^R_\alpha &= s\int_0^T D^R_\alpha\,dt + s'P\beta^R_\alpha(t_1-\tau^L_\alpha) = sD^R_\alpha T + s'P\beta^R_\alpha(t_1-\tau^L_\alpha).
\end{aligned}$$

The $\alpha$-cut of the total profit $(\widetilde{TP})$ during the cycle $(0,T)$, in the fabrication system is found with

$$\widetilde{TP}[\alpha] = [TP^L_\alpha(t_1,T),\ TP^R_\alpha(t_1,T)],$$

where

$$\begin{aligned}
TP^L_\alpha(t_1,T) &= \frac{1}{T}\Big[SR^L_\alpha - HC^R_\alpha - PC - SC - A_m\Big], \\
TP^R_\alpha(t_1,T) &= \frac{1}{T}\Big[SR^R_\alpha - HC^L_\alpha - PC - SC - A_m\Big]
\end{aligned}$$

Finally, from the (Zimmermann 1996) method, the optimization problem becomes

$$\begin{aligned}
&\text{Max } TP_\alpha^L(t_1, T) = \frac{1}{T}\left[SR_\alpha^L - HC_\alpha^R - PC - SC - A_m\right],\\
&\text{Max } TP_\alpha^R(t_1, T) = \frac{1}{T}\left[SR_\alpha^R - HC_\alpha^L - PC - SC - A_m\right]\\
&\text{such that: } 0 < \tau_\alpha^R < t_1,\ \ 0 < t_1 < T,\\
&\text{and } \left\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\right\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T\\
&\text{or } \left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T.
\end{aligned}$$

## 4. Solution Procedure

The following steps are needed to calculate the production run time ($t_1$), business period ($T$), and profit optimal values in the fuzzy production-inventory model.

Step 1. Input the crisp and fuzzy suitable parameters values of $TP_\alpha^L(t_1, T)$ and $TP_\alpha^R(t_1, T)$.

Step 2. Compute the left $\alpha$-cut ($\tau_\alpha^L$) and right $\alpha$-cut ($\tau_\alpha^R$) of fuzzy parameter $\widetilde{\tau}$ as follows:

(i) If $\widetilde{\tau}$ be TFN such as $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2)$ then $\tau_\alpha^L = (\tau_0 - \Delta_1) + \alpha\Delta_1$ and $\tau_\alpha^R = (\tau_0 + \Delta_2) - \alpha\Delta_2$.

(ii) If $\widetilde{\tau}$ be TrFN such as $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4)$ then $\tau_\alpha^L = (\tau_0 - \Delta_1) + \alpha(\Delta_1 - \Delta_2)$ and $\tau_\alpha^R = (\tau_0 + \Delta_4) - \alpha(\Delta_4 - \Delta_3)$.

(iii) If $\widetilde{\tau}$ be PFN such as $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2)$ respectively then $\tau_\alpha^L = \tau_0 - \sqrt{\alpha}\Delta_1$ and $\tau_\alpha^R = \tau_0 + \sqrt{\alpha}\Delta_2$.

(iv) If $\widetilde{\tau}$ be GFN such that $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4)$ then $\tau_\alpha^L = (\tau_0 - \Delta_1) + \sqrt{\alpha}(\Delta_1 - \Delta_2)$ and $\tau_\alpha^R = (\tau_0 + \Delta_4) - \sqrt{\alpha}(\Delta_4 - \Delta_3)$.

Similarly, calculate left and right $\alpha$-cuts of other two fuzzy parameters $\widetilde{\beta}$ and $\widetilde{D}$ for TFN, TrFN, PFN, and GFN.

Step 3. Maximize the profit $TP_\alpha^L(t_1, T)$ and obtain the best possible values of $t_1$, $T$, $TP_\alpha^R(t_1, T)$, and $TP_\alpha^{L*}(t_1, T)$ for different values of $\alpha$ using the LINGO software.

Step 4. Maximize the profit $TP_\alpha^R(t_1, T)$ and determine the best possible values of $t_1$, $T$, $TP_\alpha^L(t_1, T)$, and $TP_\alpha^{R*}(t_1, T)$ for different values of $\alpha$ using the LINGO software.

Step 5. Maximize both profits $TP_\alpha^L(t_1, T)$ and $TP_\alpha^R$ by fuzzy programming technique (FPT) and find the best possible values of $t_1$, $T$ and profit for different values of $\alpha$ using the LINGO software as follows:

Step 6. From the results of step 3 and step 4, the following pay-off matrix is constructed:

$$\begin{pmatrix} TP_\alpha^{L*}(t_1, T) & TP_\alpha^R(t_1, T) \\ TP_\alpha^L(t_1, T) & TP_\alpha^{R*}(t_1, T) \end{pmatrix}$$

Step 7. From this pay-off matrix, two values, $U_j$ and $L_j$, are defined as the upper and lower bounds of the $j$-th objective for each $j$ = 1,2, respectively. Here, the higher acceptable level of achievement is $L_j$, and the aspired level of achievement for maximization is $U_j$; these are computed as follows:

$$\begin{aligned}
U_1 = \max\{TP_\alpha^{L*}(t_1, T)\},\ U_2 = \max\{TP_\alpha^R(t_1, T)\}\\
L_1 = \min\{TP_\alpha^L(t_1, T)\},\ L_2 = \min\{TP_\alpha^{R*}(t_1, T)\}
\end{aligned}$$

Step 8. Then, the membership functions $\mu_1(TP_\alpha^L(t_1, T))$ and $\mu_2(TP_\alpha^R(t_1, T))$ corresponding to the objective functions of $TP_\alpha^L(t_1, T)$ and $TP_\alpha^R(t_1, T)$ are built linearly as follows:

$$\mu_1(TP_\alpha^L(t_1, T)) = \begin{cases} 0, & \text{if } TP_\alpha^L(t_1, T) \le L_1 \\ \frac{TP_\alpha^L(t_1, T) - L_1}{U_1 - L_1}, & \text{if } L_1 \le TP_\alpha^L(t_1, T) \le U_1 \\ 1, & \text{if } TP_\alpha^L(t_1, T) \ge U_1 \end{cases}$$

$$\mu_2(TP^R_\alpha(t_1,T)) = \begin{cases} 0, & \text{if } TP^R_\alpha(t_1,T) \le L_2 \\ \frac{TP^R_\alpha(t_1,T)-L_2}{U_2-L_2}, & \text{if } L_2 \le TP^R_\alpha(t_1,T) \le U_2 \\ 1, & \text{if } TP^R_\alpha(t_1,T) \ge U_2 \end{cases}$$

Step 9. Finally, the multi-objective programming problem, according to the (Zimmermann 1996) method, is reduced to the following single-objective programming problem:

$$\begin{aligned} &\text{Max } \lambda \\ &\text{such that} \\ &\mu_1(TP^L_\alpha(t_1,T)) \ge \lambda, \\ &\mu_2(TP^R_\alpha(t_1,T)) \ge \lambda, \\ &0 < \tau^R_\alpha < t_1,\ \ 0 < t_1 < T,\ \ \lambda \in [0,1], \\ &\text{and } \left\{(1-\beta^R_\alpha)P + D^L_\alpha - D^R_\alpha\right\}t_1 + P\beta^L_\alpha\tau^L_\alpha = D^L_\alpha T \\ &\text{or } \left\{(1-\beta^L_\alpha)P + D^R_\alpha - D^L_\alpha\right\}t_1 + P\beta^R_\alpha\tau^R_\alpha = D^R_\alpha T. \end{aligned}$$

## 5. Numerical Illustrations

To illustrate numerically the fuzzy production-inventory model, five numerical examples are presented and solved. It is necessary to have the following two relations: $\left\{(1-\beta^R_\alpha)P + D^L_\alpha - D^R_\alpha\right\}t_1 + P\beta^L_\alpha\tau^L_\alpha = D^L_\alpha T$ and $\left\{(1-\beta^L_\alpha)P + D^R_\alpha - D^L_\alpha\right\}t_1 + P\beta^R_\alpha\tau^R_\alpha = D^R_\alpha T$, one of which is taken to compute $t_1$ and $T$ in order to optimize the objective function. At first, the relation $\left\{(1-\beta^R_\alpha)P + D^L_\alpha - D^R_\alpha\right\}t_1 + P\beta^L_\alpha\tau^L_\alpha = D^L_\alpha T$ is considered, with which the fuzzy production-inventory model is optimized for the following examples, and then the other relation $\left\{(1-\beta^L_\alpha)P + D^R_\alpha - D^L_\alpha\right\}t_1 + P\beta^R_\alpha\tau^R_\alpha = D^R_\alpha T$, is considered to obtain the optimum solution, but in this latter case, it has been shown that this relation is not acceptable due to some infeasibility of the solution.

**Example 1.** *The following parameter values are used:* $C_p = \$30$, $C_{sr} = \$2$, $A_m = USD5200$, $s = USD59$, $s' = USD35$, $h_m = USD1.00$, $h'_m = USD0.50$, $P = 3380$. *The fuzzy parameters, in this case, are considered as triangular fuzzy numbers (TFNs), and their different values are provided as follows:* $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5)$, $\Delta_1 = 1$, $\Delta_2 = 0.5$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15)$, $\sigma_1 = 0.03$, $\sigma_2 = 0.05$, $\widetilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631)$, $\rho_1 = 59$, $\rho_2 = 72$. *This example is solved using LINGO-12.0. Tables 2 and 3 present the optimum results when* $TP^L_\alpha$, $TP^R_\alpha$ *are maximized separately for different values of* $\alpha$, *and Table 4 shows the optimum results when* $TP^L_\alpha$ *and* $TP^R_\alpha$ *are maximized simultaneously for different values of* $\alpha$.

**Table 2.** Optimum results of Example 1 for maximizing $TP^L_\alpha$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP^{L*}_\alpha$ (Max) | $TP^R_\alpha$ |
|---|---|---|---|---|
| 0.00 | 8.445339 | 9.452128 | 47,447.09 | 69,007.36 |
| 0.25 | 7.884611 | 9.066641 | 52,386.18 | 68,234.01 |
| 0.50 | 7.100508 | 8.402024 | 57,183.69 | 67,540.85 |
| 0.75 | 5.982998 | 7.315060 | 61,904.18 | 66,988.16 |
| 0.99 | 4.345648 | 5.555799 | 66,499.80 | 66,701.16 |

**Table 3.** Optimum results of Example 1 for maximizing $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^L$ | $TP_\alpha^{R*}$ (Max) |
|---|---|---|---|---|
| 0.00 | 10.71312 | 11.93943 | 47,248.61 | 69,111.75 |
| 0.25 | 9.064714 | 10.38858 | 52,327.64 | 68,270.08 |
| 0.50 | 7.526072 | 8.888575 | 57,175.25 | 67,546.87 |
| 0.75 | 5.976583 | 7.307580 | 61,904.18 | 66,988.16 |
| 0.99 | 4.332328 | 5.539977 | 66,499.78 | 66,701.17 |

**Table 4.** Optimum results of Example 1 for maximizing both $TP_\alpha^L$ and $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^{R*}$ (Max) | Average Profit |
|---|---|---|---|---|---|
| 0.00 | 9.513103 | 10.62325 | 47,397.64 | 69,085.74 | 58,241.69 |
| 0.25 | 8.454594 | 9.705132 | 52,371.56 | 68,261.07 | 60,316.32 |
| 0.50 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| 0.75 | 5.982997 | 7.315060 | 61,904.18 | 66,988.16 | 64,446.17 |
| 0.99 | 4.339744 | 5.548787 | 66,499.79 | 66,701.17 | 66,600.48 |

The optimum business time period ($T^*$), optimum production time period ($t_1^*$), and profit interval ($[TP_\alpha^{L*}, TP_\alpha^{R*}]$) are obtained for the given parameter values. As is expected, the optimum profit increases at the left and decreases at the right with the increment of $\alpha$. At $\alpha = 0.99$, the profit values are almost same.

**Example 2.** *The fuzzy parameters, in this case, are given as trapezoidal fuzzy numbers (TrFNs), and their different values are provided as follows:* $\tilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4) = (2, 2.5, 3.5, 5)$, $\Delta_1 = 1.25$, $\Delta_2 = 0.75$, $\Delta_3 = 0.25$, $\Delta_4 = 1.75$, $\tilde{\beta} = (\beta_0 - \sigma_1, \beta_0 - \sigma_2, \beta_0 + \sigma_3, \beta_0 + \sigma_4) = (0.07, 0.10, 0.13, 0.15)$, $\sigma_1 = 0.05$, $\sigma_2 = 0.02$, $\sigma_3 = 0.01$, $\sigma_4 = 0.03$, $\tilde{D} = (d_0 - \rho_1, d_0 - \rho_2, d_0 + \rho_3, d_0 + \rho_4) = (2500, 2549, 2571, 2631)$, $\rho_1 = 65$, $\rho_2 = 16$, $\rho_3 = 6$, $\rho_4 = 66$. *The same parameter values of Example 1 are utilized here. LINGO-12.0 is applied in order to solve this example.*

*Tables 5 and 6 display the optimum results when $TP_\alpha^L$, $TP_\alpha^R$ are maximized separately, and Table 7 represent, the optimum results when $TP_\alpha^L$ and $TP_\alpha^R$ are maximized simultaneously.*

**Table 5.** Optimum results of Example 2 for maximizing $TP_\alpha^L$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^R$ |
|---|---|---|---|---|
| 0.00 | 8.445339 | 9.452128 | 47,447.09 | 69,007.36 |
| 0.25 | 8.211077 | 9.327867 | 50,801.53 | 68,435.14 |
| 0.50 | 7.869975 | 9.079349 | 54,132.44 | 67,864.09 |
| 0.75 | 7.394903 | 8.672050 | 57,450.46 | 67,305.73 |
| 0.99 | 6.772804 | 8.081556 | 60,639.97 | 66,795.06 |

**Table 6.** Optimum results of Example 2 for maximizing $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^L$ | $TP_\alpha^{R*}$ (Max) |
|---|---|---|---|---|
| 0.00 | 10.71312 | 11.93943 | 47,248.61 | 69,111.75 |
| 0.25 | 9.666378 | 10.94183 | 50,716.17 | 68,484.49 |
| 0.50 | 8.696047 | 10.00548 | 54,103.70 | 67,882.37 |
| 0.75 | 7.772620 | 9.100049 | 57,444.15 | 67,310.16 |
| 0.99 | 6.907287 | 8.235475 | 60,639.13 | 66,795.72 |

**Table 7.** Optimum results of Example 2 for maximizing both $TP_\alpha^L$ and $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^{R*}$ (Max) | Average Profit |
|---|---|---|---|---|---|
| 0.00 | 9.513103 | 10.62325 | 47,397.64 | 69,085.74 | 58,241.69 |
| 0.25 | 8.909673 | 10.10263 | 50,780.23 | 68,472.17 | 59,626.20 |
| 0.50 | 8.273025 | 9.531221 | 54,125.26 | 67,877.80 | 61,001.53 |
| 0.75 | 7.581600 | 8.883601 | 57,448.88 | 67,309.05 | 62,378.97 |
| 0.99 | 6.840087 | 8.158563 | 60,639.76 | 66,795.55 | 63,717.66 |

The optimum business time period ($T^*$), optimum production time period ($t_1^*$), and profit interval ($[TP_\alpha^{L*}, TP_\alpha^{R*}]$) are found for the given parameter values. The results for different values of $\alpha$ are presented in Tables 5–7. As expected, the optimum profit increases at the left and decreases at the right with the increase of $\alpha$.

**Example 3.** *The fuzzy parameters, in this case, are considered as parabolic fuzzy numbers (PFNs), and their different values are given as follows:* $\tilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5)$, $\Delta_1 = 1$, $\Delta_2 = 2$, $\tilde{\beta} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15)$, $\sigma_1 = 0.03$, $\sigma_2 = 0.05$, $\tilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631)$, $\rho_1 = 59$, $\rho_2 = 72$. *The same parameter values of Example 1 are also used here. LINGO-12.0 is utilized to resolve this example. Tables 8 and 9 represent the optimum results when* $TP_\alpha^L$, $TP_\alpha^R$ *are maximized separately, and Table 10 represents the optimum results when* $TP_\alpha^L$, $TP_\alpha^R$ *are maximized simultaneously.*

**Table 8.** Optimum results of Example 3 for maximizing $TP_\alpha^L$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^R$ |
|---|---|---|---|---|
| 0.00 | 8.445339 | 9.452128 | 47,447.09 | 69,007.36 |
| 0.25 | 8.168350 | 9.274599 | 50,114.31 | 68,585.50 |
| 0.50 | 7.768098 | 8.975275 | 53,218.01 | 68,108.13 |
| 0.75 | 7.100508 | 8.402024 | 57,183.69 | 67,540.85 |
| 0.99 | 5.055331 | 6.336396 | 64,752.49 | 66,769.40 |

**Table 9.** Optimum results of Example 3 for maximizing $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^L$ | $TP_\alpha^{R*}$ (Max) |
|---|---|---|---|---|
| 0.00 | 10.71312 | 11.93943 | 47,248.61 | 69,111.75 |
| 0.25 | 9.808184 | 11.09378 | 50,005.78 | 68,647.74 |
| 0.50 | 8.796376 | 10.13124 | 53,172.85 | 68,136.67 |
| 0.75 | 7.526072 | 8.888575 | 57,175.25 | 67,546.87 |
| 0.99 | 4.980989 | 6.248691 | 64,752.15 | 66,769.71 |

**Table 10.** Optimum results of Example 3 for maximizing both $TP_\alpha^L$ and $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^{R*}$ (Max) | Average Profit |
|---|---|---|---|---|---|
| 0.00 | 9.513103 | 10.62325 | 47,397.64 | 69,085.74 | 58,241.69 |
| 0.25 | 8.951570 | 10.14348 | 50,087.23 | 68,632.21 | 59,359.72 |
| 0.50 | 8.266637 | 9.535721 | 53,206.74 | 68,129.54 | 60,668.14 |
| 0.75 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| 0.99 | 5.018012 | 6.292368 | 64,752.40 | 66,769.63 | 65,761.02 |

The optimum business time period ($T^*$), optimum production time period ($t_1^*$), and profit interval ($[TP_\alpha^{L*}, TP_\alpha^{R*}]$) are determined for the given parameter values. The results for different values of $\alpha$ are presented in Tables 8–10. As is expected, the optimum profit increases at the left and decreases at the right with the increase of $\alpha$. At $\alpha = 0.99$, the profit values are almost the same.

**Example 4.** *The fuzzy parameters, in this case, are considered as general fuzzy numbers (GFNs), and their different values are given as follows:* $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4) = (2, 2.5, 3.5, 5)$, $\Delta_1 = 1$, $\Delta_2 = 0.25$, $\Delta_3 = 0.75$, $\Delta_4 = 2$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0 - \sigma_2, \beta_0 + \sigma_3, \beta_0 + \sigma_4) = (0.07, 0.10, 0.13, 0.15)$, $\sigma_1 = 0.05$, $\sigma_2 = 0.02$, $\sigma_3 = 0.01$, $\sigma_4 = 0.03$, $\widetilde{D} = (d_0 - \rho_1, d_0 - \rho_2, d_0 + \rho_3, d_0 + \rho_4) = (2500, 2549, 2571, 2631)$, $\rho_1 = 65$, $\rho_2 = 16$, $\rho_3 = 6$, $\rho_4 = 66$. *The same parameter values provided in Example 1 are utilized here. LINGO-12.0 is used to solve this example. Tables 11 and 12 represent the optimum results when* $TP_\alpha^L$, $TP_\alpha^R$ *are maximized separately, and Table 13 represents the optimum results when* $TP_\alpha^L$, $TP_\alpha^R$ *are maximized simultaneously.*

**Table 11.** Optimum results of Example 4 for maximizing $TP_\alpha^L$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^R$ |
|---|---|---|---|---|
| 0.00 | 8.445339 | 9.452128 | 47,447.09 | 69,007.36 |
| 0.25 | 7.869975 | 9.079349 | 54,132.44 | 67,864.09 |
| 0.50 | 7.487523 | 8.755217 | 56,881.46 | 67,400.04 |
| 0.75 | 7.117747 | 8.415065 | 58,990.51 | 67,054.82 |
| 0.99 | 6.757742 | 8.066704 | 60,706.47 | 66,784.82 |

**Table 12.** Optimum results of Example 4 for maximizing $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^L$ | $TP_\alpha^{R*}$ (Max) |
|---|---|---|---|---|
| 0.00 | 10.71312 | 11.93943 | 47,248.61 | 69,111.75 |
| 0.25 | 8.696047 | 10.00548 | 54,103.70 | 67,882.37 |
| 0.50 | 7.928822 | 9.254359 | 56,872.92 | 67,405.93 |
| 0.75 | 7.353021 | 8.682959 | 58,988.00 | 67,056.67 |
| 0.99 | 6.889372 | 8.217389 | 60,705.66 | 66,785.45 |

**Table 13.** Optimum results of Example 4 for maximizing both $TP_\alpha^L$ and $TP_\alpha^R$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_\alpha^{L*}$ (Max) | $TP_\alpha^{R*}$ (Max) | Average Profit |
|---|---|---|---|---|---|
| 0.00 | 9.513103 | 10.62325 | 47,397.64 | 69,085.74 | 58,241.69 |
| 0.25 | 8.273025 | 9.531221 | 54,125.26 | 67,877.80 | 61,001.53 |
| 0.50 | 7.704997 | 9.001196 | 56,879.33 | 67,404.46 | 62,141.89 |
| 0.75 | 7.234143 | 8.547598 | 58,989.88 | 67,056.21 | 63,023.05 |
| 0.99 | 6.823119 | 8.141546 | 60,706.27 | 66,785.29 | 63,745.78 |

The optimum business time period ($T^*$), optimum production time period ($t_1^*$), and profit interval ($[TP_\alpha^{L*}, TP_\alpha^{R*}]$) are obtained for the given parameter values. The results for different values of $\alpha$ are presented in Tables 11–13. As is expected, the optimum profit increases at the left and decreases at the right with the increase of $\alpha$.

### 5.1. When $\left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$

**Example 5.** *To illustrate the model, the next parametric values are used:* $C_p = USD30$, $C_{sr} = USD2$, $A_m = USD5200$, $s = USD55$, $s' = USD35$, $h_m = USD1.0$, $h_m' = USD0.50$, $P = 3380$. *The fuzzy parameters, in this case, are considered as triangular fuzzy numbers (TFNs), and their different values are given as follows:* $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5)$, $\Delta_1 = 1$, $\Delta_2 = 2$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15)$, $\sigma_1 = 0.03$, $\sigma_2 = 0.05$, $\widetilde{D} = (D_0 - \rho_1, D_{0,0} + \rho_2) = (2500, 2559, 2631)$, $\rho_1 = 59$, $\rho_2 = 72$. *LINGO-12.0 is used to solve this example. Tables 14 and 15 represent the optimum results when* $TP_\alpha^L$, $TP_\alpha^R$ *are maximized separately.*

**Table 14.** Optimum results of Example 5 for maximizing $TP_{\alpha}^{L}$.

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_{\alpha}^{L*}$ (Max) | $TP_{\alpha}^{R}$ |
|---|---|---|---|---|
| 0.00 | 1.904205 | 3.333382 | 72,465.08 | 91,202.27 |
| 0.25 | 1.992095 | 3.252411 | 70,387.39 | 84,903.91 |
| 0.50 | 1.834233 | 2.883574 | 68,885.25 | 78,970.41 |
| 0.75 | 1.735770 | 2.604554 | 67,219.03 | 72,486.40 |
| 0.99 | 4.184291 | 5.377229 | 66,696.28 | 66,897.66 |

**Table 15.** Optimum results of Example 5 for maximizing $TP_{\alpha}^{R}$

| $\alpha$ | Production Period ($t_1^*$) | Business Period ($T^*$) | $TP_{\alpha}^{L}$ | $TP_{\alpha}^{R*}$ (Max) |
|---|---|---|---|---|
| 0.00 | 1.904205 | 3.333382 | 72,465.08 | 91,202.27 |
| 0.25 | 1.992095 | 3.252411 | 70,387.39 | 84,903.91 |
| 0.50 | 1.834233 | 2.883574 | 68,885.25 | 78,970.41 |
| 0.75 | 1.468407 | 2.282920 | 67,206.21 | 72,582.51 |
| 0.99 | 4.169558 | 5.359706 | 66,696.27 | 66,897.67 |

Now, from Tables 14 and 15, when $\alpha = 0.00$, it is observed that the optimum profit interval is $[TP_{\alpha}^{L}, TP_{\alpha}^{R}] = [72,465.08, 91,202.27]$. Again, when $\alpha = 0.25$, then the optimum profit interval is $[TP_{\alpha}^{L}, TP_{\alpha}^{R}] = [70,387.39, 84,903.91]$. However, here, $[TP_{0.0}^{L}, TP_{0.0}^{R}] \supseteq [TP_{0.25}^{L}, TP_{0.25}^{R}]$ is not satisfied. This infeasibility is also shown when $\alpha$ is further increasing. Therefore, this relation has no role in giving the optimum solution of the model.

### *5.2. Comparison between the Optimum Average Profit Due to TFN and TrFN by Fisher's t-Test*

In the fuzzy EPQ model, using TFN and TrFN, two optimum average profits have been obtained. It is possible to know the differences between these two values by testing that the null hypothesis $H_0$ of the mean of values of average profit for TFN ($\overline{AP}_{TFN}$) is equal to the mean of values of average profit for TrFN ($\overline{AP}_{TrFN}$) against the alternative hypothesis $H_1$: $\overline{AP}_{TFN} \neq \overline{AP}_{TrFN}$, based on the results presented in Tables 4 and 7. Using t-distribution, this hypothesis can be tested.

The statistic test which follows $t$-distribution with $(n_1 + n_2 - 2)$ degrees of freedom is

$$t = \frac{\overline{AP}_{TFN} - \overline{AP}_{TrFN}}{s\sqrt{(1/n_1) + (1/n_2)}}$$

where

$$s^2 = \frac{n_1 s_{TFN}^2 + n_2 s_{TrFN}^2}{n_1 + n_2 - 2}$$

Here, $n_1 = 5$, $n_2 = 5$, $\overline{AP}_{TFN} = 62,393.63$, $\overline{AP}_{TrFN} = 60,993.21$, $s_{TFN}^2 = 10,866,309.90$, $s_{TrFN}^2 = 4,695,651.613$. Therefore, the degrees of freedom are $(n_1 + n_2 - 2) = 8$ and the value of $t = 0.2840$. Since the evaluated value of $t$ is less than the tabulated value of $t_{0.05}$, then the null hypothesis $H_0$ is accepted with 95% of confidence limit and the conclusion is that there is no significant difference between the mean average profit $(AP)$ for TFN and for TrFN.

### *5.3. Sensitivity Analysis*

Example 1 is considered to study, with respect to key parameters, the sensitivity analysis of the proposed model. The optimum results of the model with the changes in the parameters $\Delta_1$, $\Delta_2$, $\sigma_1$, $\sigma_2$, $\rho_1$, $\rho_2$, $P$, and $s$ are shown in Table 16, taking $\alpha = 0.5$.

**Table 16.** Sensitivity analysis on Example 1 with respect to $\Delta_1$, $\Delta_2$, $\sigma_1$, $\sigma_2$, $\rho_1$, $\rho_2$, $P$ and $S$.

| Parameter | Value | Production Period ($t_1$) | Business Period ($T$) | $TP_\alpha^L$ | $TP_\alpha^R$ | Average Profit |
|---|---|---|---|---|---|---|
| | 0.75 | 7.250299 | 8.587478 | 57,389.12 | 67,422.17 | 62,405.64 |
| $\Delta_1$ | 1.00 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 1.25 | 7.361048 | 8.685704 | 56,976.58 | 67,665.90 | 62,321.24 |
| | 1.75 | 7.198704 | 8.514293 | 57,409.15 | 67,504.98 | 62,457.07 |
| $\Delta_2$ | 2.00 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 2.25 | 7.415203 | 8.761817 | 56,956.63 | 67,584.60 | 62,270.62 |
| | 0.02 | 7.229466 | 8.566165 | 57,624.05 | 67,683.21 | 62,653.63 |
| $\sigma_1$ | 0.03 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 0.04 | 7.375419 | 8.699629 | 56,733.72 | 67,410.27 | 62,071.99 |
| | 0.04 | 7.083671 | 8.430101 | 57,728.50 | 67,646.06 | 62,687.28 |
| $\sigma_2$ | 0.05 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 0.06 | 7.526995 | 8.839341 | 56,620.66 | 67,454.61 | 62,037.63 |
| | 55 | 7.317655 | 8.649237 | 57,295.96 | 67,535.26 | 62,415.61 |
| $\rho_1$ | 59 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 64 | 7.301025 | 8.632592 | 57,038.64 | 67,558.02 | 62,298.33 |
| | 67 | 7.300695 | 8.638115 | 57,256.03 | 67,460.07 | 62,358.05 |
| $\rho_2$ | 72 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 77 | 7.319873 | 8.645592 | 57,106.98 | 67,630.55 | 62,368.77 |
| | 3350 | 7.497850 | 8.775979 | 57,266.24 | 67,613.92 | 62,440.08 |
| $P$ | 3380 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 3420 | 7.079061 | 8.478815 | 57,070.02 | 67,458.47 | 62,264.24 |
| | 54 | 7.310260 | 8.641835 | 44,534.08 | 54,570.36 | 49,552.22 |
| $s$ | 59 | 7.310260 | 8.641835 | 57,181.58 | 67,545.36 | 62,363.47 |
| | 64 | 7.310260 | 8.641835 | 69,829.08 | 80,520.36 | 64,188.79 |

Now, from Table 16, the following features of the proposed model are observed:

(i) When $\Delta_1$ increases, the production run time and business period also increase, but the average profit decreases with increasing of $\Delta_1$.

(ii) When $\Delta_2$ increases, the production run time and business period increase, but the average profit decreases with increasing of $\Delta_2$.

(iii) When $\sigma_1$ increases, the production run time and business period also increase, but the average profit decreases with increasing of $\sigma_1$.

(iv) When $\sigma_2$ increases, the production run time and business period also increase, but the average profit decreases with increasing of $\sigma_2$.

(v) When $\rho_1$ increases, the production run time and business period also decrease, but the average profit increases with increasing of $\rho_1$.

(vi) When $\rho_2$ increases, the production run time and business period also increase, but average profit increases with increasing of $\rho_2$.

(vii) When the production rate $(P)$ increases, the inventory increases, as well as the production run time, and business period reduces, but the average profit increases with the production rate.

(viii) When selling price of perfect item per unit $(s)$ increases, the production run time and business period do not change, but the average profit increases with the increasing of selling price $(s)$.

*5.4. Discussion*

The optimum results of the proposed model are obtained from Tables 4, 7, 10 and 13 when the fuzzy numbers $\widetilde{\tau}$, $\widetilde{\beta}$, and $\widetilde{D}$ are considered as TFN, TrFN, PFN, and GFN in Examples 1–4, respectively. From these tables, it is shown that when $\alpha$ increases, $(\frac{\Delta TP_\alpha^L}{\Delta\alpha})$ also increases but $(\frac{\Delta TP_\alpha^R}{\Delta\alpha})$ decreases. Again, Figure 2 shows that the rate of increase of the left optimum profit $\left(\frac{\Delta TP^L_\alpha}{\Delta\alpha}\right)$ is more than the rate of decrease of the right optimum profit $\left(\frac{\Delta TP^R_\alpha}{\Delta\alpha}\right)$. Hence, the average profit $(AP)$, i.e., $\frac{1}{2}(TP^L_\alpha + TP^R_\alpha)$, is not the same for all $\alpha$. Ultimately, the average profit rises with the increase of $\alpha$.

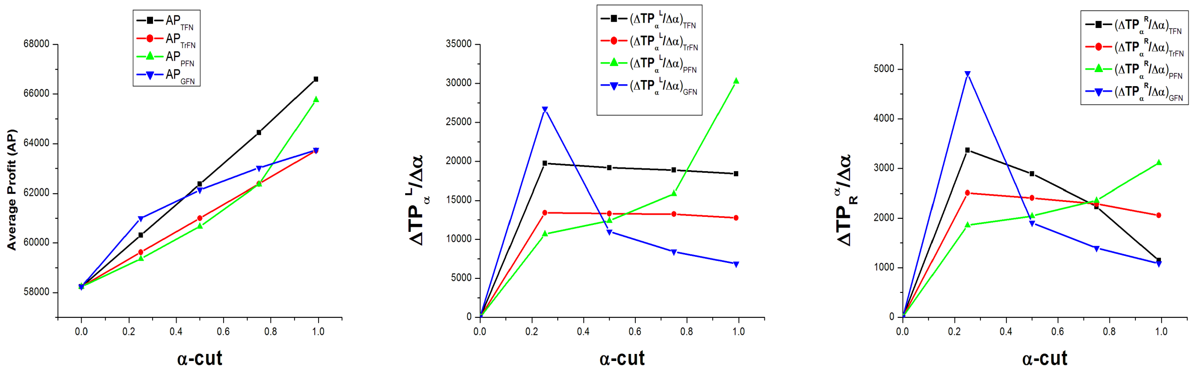

**Figure 2.** Graphical representation of average profit with $\alpha$.

Figure 2 shows that the TFNs give the maximum average profit among others, i.e., the ordering of optimum average profit is $AP_{TrFN} \le AP_{GFN} \le AP_{PFN} \le AP_{TFN}$, where $AP_{FN}$ indicates the optimum average profit for fuzzy number $FN$, though all fuzzy numbers have the same spread.

## 6. Conclusions and Future Research

The development of a fuzzy economic production quantity (FEPQ) model with fuzzy demand for perfect-quality items with inspection of imperfect items is the main contribution of this research work. During the manufacturing process, the system may shift to an "out-of-control" state after a certain time that follows constant/random time. The first development of the fuzzy production-inventory model for determining the time where the manufacturing process may shift to an "out-of-control" state after a certain time that follows a fuzzy number. Then, the process starts to produce defective items during the "out-of-control" state. Moreover, the defective rate is considered a fuzzy number.

A solution procedure is proposed by using a fuzzy differential equation, fuzzy Riemann integration, and fuzzy programming technique (FPT) in order to obtain an optimal decision. The $\alpha$-cut of fuzzy profit is optimized by maximizing the profit in a fuzzy imperfect production. The production-inventory model is optimized for the production run time ($t_1^*$), business period ($T^*$), and profit interval. Here, four fuzzy numbers, TFN, TrFN, PFN, and GNF, have been used to illustrate the production-inventory model for fuzzy parameters. The production-inventory model can be applied in different manufacturing industries, for example, garments industry and electronic goods industry, among others, where the manufacturer produces perfect as well as defective items and the customers' demands are uncertain.

This research work can be extended in several ways. First, this production-inventory model may be extended with machine breakdowns during production run time. Second, the production-inventory model can be extended to incorporate distinctive realistic features, such as production rate, screening cost, inventory holding cost, a quantity discount, and others with uncertain nature regarding the time. Third, different types of optimization techniques can be applied in order to optimize the production-inventory model.

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## Notation

The following abbreviations are used in this manuscript:

| | |
|---|---|
| $q_1(t)$ | Inventory at any on hand time $t$, in crisp environment for perfect quality item |
| $\tilde{q_1}(t)$ | Inventory at any on hand time $t$, in fuzzy environment for perfect quality item |
| $q_2(t)$ | Inventory at any on hand time $t$, in crisp environment for imperfect quality item |
| $\tilde{q_2}(t)$ | Inventory at any on hand time $t$, in fuzzy environment for imperfect quality item |
| $D$ | Perfect quality items demand rate in crisp environment |
| $\widetilde{D}$ | Perfect quality items fuzzy demand rate in fuzzy environment |
| $P$ | Production rate $(P > D)$ |
| $\widetilde{\beta}$ | Imperfect quality items per unit of time with fuzzy percentage and its $\alpha$-cut $[\beta^L_\alpha, \beta^R_\alpha]$ |
| $\widetilde{\tau}$ | Fuzzy time to shifts the production system from 'in-control' state to the 'out-of-control' state with its $\alpha$-cut $[\tau^L_\alpha, \tau^R_\alpha]$ |
| $t_1$ | Production run time duration |
| $c_p$ | Production cost per unit item |
| $c_{sr}$ | Screening cost per unit item |
| $h_c$ | Inventory holding cost per unit for a perfect item in production center per unit of time |
| $h'_c$ | Inventory holding cost per unit for an imperfect item in production center per unit of time |
| $s$ | Perfect item per unit selling price |
| $s'$ | Imperfect item per unit selling price |
| $T$ | Business period length |
| $\sim$ | This mark is placed on the top of a symbol in order to represent fuzzy parameters |

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