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Inertial Optimization Based Two-Step Methods for Solving Equilibrium Problems with Applications in Variational Inequality Problems and Growth Control Equilibrium Models

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Abstract: This manuscript aims to incorporate an inertial scheme with Popov's subgradient extragradient method to solve equilibrium problems that involve two different classes of bifunction. The novelty of our paper is that methods can also be used to solve problems in many fields, such as economics, mathematical finance, image reconstruction, transport, elasticity, networking, and optimization. We have established a weak convergence result based on the assumption of the pseudomonotone property and a certain Lipschitz-type cost bifunctional condition. The stepsize, in this case, depends upon on the Lipschitz-type constants and the extrapolation factor. The bifunction is strongly pseudomonotone in the second method, but stepsize does not depend on the strongly pseudomonotone and Lipschitz-type constants. In contrast, the first convergence result, we set up strong convergence with the use of a variable stepsize sequence, which is decreasing and non-summable. As the application, the variational inequality problems that involve pseudomonotone and strongly pseudomonotone operator are considered. Finally, two well-known Nash–Cournot equilibrium models for the numerical experiment are reviewed to examine our convergence results and show the competitive advantage of our suggested methods.

Keywords: energy production models; optimization problems; control parameters; Lipschitz-type conditions; variational inequality; Nash-Cournot oligopolistic equilibrium model

1. Introduction

An Equilibrium problem (EP) was originally started in the unifying feature by Blum and Oettli [1] in 1994 and provided a detailed investigation of their theoretical properties. This study contributes significantly to the advancement of applied and pure science. This problem is primarily related to Ky Fan Inequity due to his early contributions to this field [2]. It has been established that the equilibrium problem theory has set up an unique approach to investigate an immense range of topics that have appeared in social and physical science. For instance, it might involve physical or mechanical structures, chemical processes [3], the distribution of traffic over computer, and telecommunication networks or public roads [4–7]. In economics, it often refers to production competition [8] or the dynamics of offer and demand [9], exploiting the mathematical model of non-cooperative games and the analogous equilibrium concept by Nash [10,11]. The problem of equilibrium, as a particular case, includes many mathematical problems as a particular case, such as the variational inequality problems (VIP), problems of minimization, the fixed point problems, Nash equilibrium of non-cooperative games, complementarity problems, and saddle point problem (see e.g., [1,12]).

On the other hand, iterative methods are efficient techniques for determining the approximate solution of an equilibrium problem. In that case, two major approaches that are well-known i.e., the proximal point method [13] and auxiliary problem principle [14]. The proximal point method strategy was initially developed by Martinet [15] for the monotone variational inequality problems and later Rockafellar [16] extends this approach for monotone operators. Moudafi [13] proposed the proximal point method for monotone equilibrium problems. Konnov [17] also suggests a different interpretation of the proximal point method with weaker assumptions for equilibrium problems.

In addition, inertial-type methods are additionally significant, depending on the heavy-ball methods of the second-order time dynamic system. Polyak began by considering inertial extrapolation as an acceleration procedure to deal with the problem of smooth convex minimization. Inertial-type algorithms are two-step iterative schemes, and the next iteration is determined by using the previous two iterations and it can be viewed as an accelerating step of the iterative sequence. A large number of methods are the earliest, being set up for solving the problem (EP) in finite and infinite-dimensional spaces, such as the proximal point-like methods [13,18], the extragradient methods [19–23], the subgradient extragradient methods [24–26], the inertia methods [27–32] and others in [33,34].

In this work, our focus is on the proximal point method, in particular projection methods, which are well established and technically easy to implement due to their convenient numerical computation. This manuscript aims to suggest two modifications of the results that appeared in [21,35,36] by applying the inertial scheme that is useful for speeding up the iteration process. The first result includes the two-step inertial Popov's extragradient method for determining a numerical solution to the pseudomonotone equilibrium problems and the weak convergence of the suggested method is achieved based on the standard assumptions. We also propose an alternative inertial-type method, the second variant of the first method. The second method does not need any information regarding the Lipschitz-type and strongly pseudomonotone constants of a bifunction. A practical explanation for the second method is that it uses a diminishing and non-summable sequence of non-negative real numbers, which are useful in achieving the strong convergence.

This manuscript is arranged, as follows: in Section 2, we provide some essential definitions and useful results. Sections 3 and 4 include all of our main methods and corresponding convergence results. Section 5 provides the methods for variational inequality problems. Section 6 sets out the numerical tests to show the numerical efficiency of the proposed methods for the test problems based on the Nash–Cournot equilibrium model compare to other existing methods.

2. Background

Let K be a non-empty, convex, and closed subset of the Hilbert space $\mathbb{E}$. Let $H : K \to \mathbb{E}$ be an operator and $SOL_{VI(H,K)}$ is the solution set of a variational inequality problem relative to the operator H upon the set K. Likewise, $SOL_{EP(f,K)}$ denotes the solution set of an equilibrium problem on the set K and ξ^* is any arbitrary element of the solution set $SOL_{EP(f,K)}$ or $SOL_{VI(H,K)}$.

Definition 1. [1] *Let $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be a bifunction with $f(\widetilde{u}, \widetilde{u}) = 0$, for each $\widetilde{u} \in K$. The equilibrium problem for f upon K is defined, as follows:*

$$\text{Find } \xi^* \in K \text{ such that } f(\xi^*, \widetilde{v}) \geq 0, \ \forall \widetilde{v} \in K.$$

Definition 2. [37] *The metric projection $P_K(\widetilde{u})$ of $\widetilde{u}$ on a closed and convex subset K of $\mathbb{E}$ is determined, as follows:*

$$P_K(\widetilde{u}) = \arg\min\{\|\widetilde{v} - \widetilde{u}\| : \widetilde{v} \in K\}.$$

Next, we take the concept of monotonicity of a bifunction into account (see [1,38] for details).

Definition 3. *Let $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ on K for $\gamma > 0$ is*

(1) *strongly monotone if*
$$f(\widetilde{u}, \widetilde{v}) + f(\widetilde{v}, \widetilde{u}) \leq -\gamma\|\widetilde{u} - \widetilde{v}\|^2, \ \forall \widetilde{u}, \widetilde{v} \in K;$$

(2) *monotone if*
$$f(\widetilde{u}, \widetilde{v}) + f(\widetilde{v}, \widetilde{u}) \leq 0, \ \forall \widetilde{u}, \widetilde{v} \in K;$$

(3) *strongly pseudomonotone if*
$$f(\widetilde{u}, \widetilde{v}) \geq 0 \Longrightarrow f(\widetilde{v}, \widetilde{u}) \leq -\gamma\|\widetilde{u} - \widetilde{v}\|^2, \ \forall \widetilde{u}, \widetilde{v} \in K;$$

(4) *pseudomonotone if*
$$f(\widetilde{u}, \widetilde{v}) \geq 0 \Longrightarrow f(\widetilde{v}, \widetilde{u}) \leq 0, \ \forall \widetilde{u}, \widetilde{v} \in K;$$

(5) *satisfying the Lipschitz-type condition on K if there exist constants $L_1, L_2 > 0$, such that*
$$f(\widetilde{u}, \widetilde{w}) \leq f(\widetilde{u}, \widetilde{v}) + f(\widetilde{v}, \widetilde{w}) + L_1\|\widetilde{u} - \widetilde{v}\|^2 + L_2\|\widetilde{v} - \widetilde{w}\|^2, \ \forall \widetilde{u}, \widetilde{v}, \widetilde{w} \in K,$$
holds.

This section ends with a few essential lemmas that are useful for examining convergence.

Lemma 1. [39] *Assume that K is non-empty, convex, and closed subset of Hilbert space $\mathbb{E}$ and $g : K \to \mathbb{R}$ is a convex, subdifferentiable, and lower semi-continuous function on K. Furthermore, $\widetilde{u} \in K$ is a minimizer of g if and only if $0 \in \partial g(\widetilde{u}) + N_K(\widetilde{u})$ where $\partial g(\widetilde{u})$ and $N_K(\widetilde{u})$ denotes the subdifferential of g at $\widetilde{u}$ and normal cone of K at $\widetilde{u}$, respectively.*

Lemma 2. [40] *Let $\{p_n\}, \{q_n\} \subset [0, +\infty)$ be two sequences and $\sum_{n=1}^{\infty} p_n = \infty$ with $\sum_{n=1}^{\infty} p_n q_n < \infty$, then $\liminf_{n\to\infty} q_n = 0$.*

Lemma 3. [41] *For $\widetilde{u}, \widetilde{v} \in \mathbb{E}$ and $\mu \in \mathbb{R}$ then the following relation is true:*

$$\|\mu\widetilde{u} + (1 - \mu)\widetilde{v}\|^2 = \mu\|\widetilde{u}\|^2 + (1 - \mu)\|\widetilde{v}\|^2 - \mu(1 - \mu)\|\widetilde{u} - \widetilde{v}\|^2.$$

Lemma 4. [42] *Assume that $\widetilde{a}_n$, $\widetilde{b}_n$ and $\widetilde{c}_n$ are sequences in $[0, +\infty)$, such that*

$$\widetilde{a}_{n+1} \leq \widetilde{a}_n + \widetilde{b}_n(\widetilde{a}_n - \widetilde{a}_{n-1}) + \widetilde{c}_n, \ \forall n \geq 1, \ \text{with} \ \sum_{n=1}^{+\infty} \widetilde{c}_n < +\infty,$$

and also with $\widetilde{b} > 0$, such that $0 \leq \widetilde{b}_n \leq \widetilde{b} < 1$ for all $n \in \mathbb{N}$. Subsequently, the following relations are hold.

(i) $\sum_{n=1}^{+\infty} [\widetilde{a}_n - \widetilde{a}_{n-1}]_+ < \infty$, *with* $[s]_+ := \max\{s, 0\}$;

(ii) $\lim_{n\to+\infty}\widetilde{a}_n = a^* \in [0,+\infty)$.

Lemma 5. [43] *Let $\{\widetilde{u}_n\}$ be a sequence in $\mathbb{E}$ and $K \subset \mathbb{E}$ such that the following relations are true:*

(i) *For each $\widetilde{u} \in K$, $\lim_{n\to\infty}\|\widetilde{u}_n - \widetilde{u}\|$ exists;*
(ii) *Every sequentially weak cluster point of $\{\widetilde{u}_n\}$ belongs to K;*

Subsequently, $\{\widetilde{u}_n\}$ weakly converges to a point in K.

A *normal cone of K* at $\widetilde{u} \in K$ is defined as:

$$N_K(\widetilde{u}) = \{\widetilde{z} \in \mathbb{E} : \langle \widetilde{z}, \widetilde{v} - \widetilde{u}\rangle \le 0,\ \forall\, \widetilde{v} \in K\}.$$

Let $g : K \to \mathbb{R}$ be a convex function with *subdifferential of g* at $\widetilde{u} \in K$ is defined as:

$$\partial g(\widetilde{u}) = \{\widetilde{z} \in \mathbb{E} : g(\widetilde{v}) - g(\widetilde{u}) \ge \langle \widetilde{z}, \widetilde{v} - \widetilde{u}\rangle,\ \forall\, \widetilde{v} \in K\}.$$

3. Inertial Popov's Two-Step Subgradient Extragradient Algorithm for Pseudomonotone EP

We present our first method to solve the pseudomonotone equilibrium problems involving the Lipschitz-type condition of a bifunction. It uses an inertial term to boost up the iterative sequence, so we referred it as an "Inertial Popov's Two-step Subgradient Extragradient Algorithm" for a class pseudomonotone equilibrium problems. The detailed algorithm is given below.

Algorithm 1 (Two-step Subgradient Extragradient Algorithm for Pseudomonotone EP)

Initialization: Choose $u_{-1}, u_0, v_0 \in \mathbb{E}$, $0 \le \vartheta_n \le \vartheta < \sqrt{5} - 2$ and $\lambda(\vartheta, L_1, L_2) > 0$. Set

$$u_1 = \arg\min_{y\in K}\{\lambda f(v_0, y) + \frac{1}{2}\|\rho_0 - y\|^2\},$$

$$v_1 = \arg\min_{y\in K}\{\lambda f(v_0, y) + \frac{1}{2}\|\rho_1 - y\|^2\},$$

where $\rho_0 = u_0 + \vartheta_0(u_0 - u_{-1})$ and $\rho_1 = u_1 + \vartheta_1(u_1 - u_0)$.

Iterative steps: Given $u_{n-1}, u_n, v_{n-1}, v_n$ for $n \ge 1$ and construct a half space

$$H_n = \{z \in \mathbb{E} : \langle \rho_n - \lambda\omega_{n-1} - v_n, z - v_n\rangle \le 0\},$$

where $\omega_{n-1} \in \partial f(v_{n-1}, v_n)$.

Step 1: Compute

$$u_{n+1} = \arg\min_{y\in H_n}\{\lambda f(v_n, y) + \frac{1}{2}\|\rho_n - y\|^2\},$$

where $\rho_n = u_n + \vartheta_n(u_n - u_{n-1})$.

Step 2: Compute

$$v_{n+1} = \arg\min_{y\in K}\{\lambda f(v_n, y) + \frac{1}{2}\|\rho_{n+1} - y\|^2\},$$

where $\rho_{n+1} = u_{n+1} + \vartheta_{n+1}(u_{n+1} - u_n)$.

Step 3: If $u_{n+1} = \rho_n$ and $v_n = v_{n-1}$, then STOP. Otherwise, set $n := n + 1$ and go back to **Step 1**.

Assumption 1. *Assume that* $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ *satisfy the following conditions:*

(A1) $f(\widetilde{v}, \widetilde{v}) = 0$ *for all* $\widetilde{v} \in K$ *and* f *is pseudomonotone on* K*;*
(A2) f *satisfy the Lipschitz-type condition on* $\mathbb{E}$ *through two positive constants* L_1 *and* L_2*;*
(A3) $\limsup_{n\to\infty} f(\widetilde{u}_n, \widetilde{v}) \le f(\widetilde{u}^*, \widetilde{v})$ *for all* $\widetilde{v} \in K$ *and* $\{\widetilde{u}_n\} \subset K$ *satisfy* $\widetilde{u}_n \rightharpoonup \widetilde{u}^*$*;*
(A4) $f(\widetilde{u}, .)$ *is convex and subdifferentiable on* $\mathbb{E}$ *for each* $\widetilde{u} \in \mathbb{E}$*.*

Lemma 6. *We have the following crucial inequality that results from the Algorithm 1.*

$$\lambda f(v_n, y) - \lambda f(v_n, u_{n+1}) \ge \langle \rho_n - u_{n+1}, y - u_{n+1} \rangle, \ \forall y \in H_n.$$

Proof. By the value u_{n+1} through Lemma 1, we have

$$0 \in \partial_2 \Big\{ \lambda f(v_n, y) + \frac{1}{2} \|\rho_n - y\|^2 \Big\} (u_{n+1}) + N_{H_n}(u_{n+1}).$$

For $\omega \in \partial f(v_n, u_{n+1})$, there exists $\overline{\omega} \in N_{H_n}(u_{n+1})$, such that

$$\lambda \omega + u_{n+1} - \rho_n + \overline{\omega} = 0.$$

The above implies that

$$\langle \rho_n - u_{n+1}, y - u_{n+1} \rangle = \lambda \langle \omega, y - u_{n+1} \rangle + \langle \overline{\omega}, y - u_{n+1} \rangle, \ \forall y \in H_n.$$

Because $\overline{\omega} \in N_{H_n}(u_{n+1})$ then $\langle \overline{\omega}, y - u_{n+1} \rangle \le 0$, $\forall y \in H_n$. It implies that

$$\lambda \langle \omega, y - u_{n+1} \rangle \ge \langle \rho_n - u_{n+1}, y - u_{n+1} \rangle, \ \forall y \in H_n. \tag{1}$$

Due to $\omega \in \partial f(v_n, u_{n+1})$ and by definition of subdifferentiable, we obtain

$$f(v_n, y) - f(v_n, u_{n+1}) \ge \langle \omega, y - u_{n+1} \rangle, \ \forall y \in \mathbb{E}. \tag{2}$$

From expressions (1) and (2), we have the required result. □

Lemma 7. *We also have the following inequality from Algorithm 1.*

$$\lambda f(v_n, y) - \lambda f(v_n, v_{n+1}) \ge \langle \rho_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \ \forall y \in K.$$

Proof. The proof is the same as that of Lemma 6. □

Lemma 8. *We have the following inequality from Algorithm 1.*

$$\lambda \big\{ f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \big\} \ge \langle \rho_n - v_n, u_{n+1} - v_n \rangle.$$

Proof. Because $u_{n+1} \in H_n$ then the definition of H_n implies that

$$\langle \rho_n - \lambda \omega_{n-1} - v_n, u_{n+1} - v_n \rangle \le 0.$$

The above implies that

$$\lambda \langle \omega_{n-1}, u_{n+1} - v_n \rangle \ge \langle \rho_n - v_n, u_{n+1} - v_n \rangle. \tag{3}$$

From $\omega_{n-1} \in \partial f(v_{n-1}, v_n)$ and due to subdifferential definition, we have

$$f(v_{n-1}, y) - f(v_{n-1}, v_n) \ge \langle \omega_{n-1}, y - v_n \rangle, \ \forall y \in \mathbb{E}.$$

Set $y = u_{n+1}$ in the above expression

$$f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \geq \langle \omega_{n-1}, u_{n+1} - v_n \rangle, \ \forall y \in \mathbb{E}. \tag{4}$$

From expression (3) and (4), we obtain the desired result. □

Now, we are proving the validity of the stopping criterion for Algorithm 1.

Lemma 9. *If $u_{n+1} = \rho_n$ and $v_n = v_{n-1}$ in Algorithm 1, then $v_n \in SOL_{EP(f,K)}$.*

Proof. By substituting $u_{n+1} = \rho_n$ in Lemma 6, we have

$$\lambda f(v_n, y) - \lambda f(v_n, u_{n+1}) \geq 0, \ \forall y \in H_n. \tag{5}$$

Because $u_{n+1} \in H_n$ and $v_n = v_{n-1}$, $u_{n+1} = \rho_n$, then from Lemma 8, we have

$$\lambda f(v_n, u_{n+1}) \geq \|\rho_n - v_n\|^2 \geq 0. \tag{6}$$

The expression (5) and (6) implies that $v_n \in SOL_{EP(f,K)}$. □

Remark 1. *Two more conditions for stopping criterion are $u_{n+1} = v_n = \rho_n$ and $\rho_{n+1} = v_{n+1} = v_n$ for Algorithm 1. The validity of these stopping criterion can be shown easily by Lemma 6 and Lemma 7, respectively.*

Lemma 10. *Let $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfying the Assumption 1. Assume that $SOL_{EP(f,K)}$ is nonempty. Afterwards, for each $\xi^* \in SOL_{EP(f,K)}$, we have*

$$\begin{aligned} &\|u_{n+1} - \xi^*\|^2 \\ &\leq \|\rho_n - \xi^*\|^2 - (1 - 4\lambda L_1)\|\rho_n - v_n\|^2 - (1 - 2\lambda L_2)\|u_{n+1} - v_n\|^2 + 4\lambda L_1 \|\rho_n - v_{n-1}\|^2. \end{aligned} \tag{7}$$

Proof. Substituting $y = \xi^*$ into Lemma 6, we obtain

$$\lambda f(v_n, \xi^*) - \lambda f(v_n, u_{n+1}) \geq \langle \rho_n - u_{n+1}, \xi^* - u_{n+1} \rangle, \ \forall y \in H_n. \tag{8}$$

Since $\xi^* \in SOL_{EP(f,K)}$ then $f(\xi^*, v_n) \geq 0$. Thus, from (A1) the above expression becomes

$$\langle \rho_n - u_{n+1}, u_{n+1} - \xi^* \rangle \geq \lambda f(v_n, u_{n+1}). \tag{9}$$

Because of the Lipschitz-type condition, we have

$$f(v_{n-1}, u_{n+1}) \leq f(v_{n-1}, v_n) + f(v_n, u_{n+1}) + L_1\|v_{n-1} - v_n\|^2 + L_2\|v_n - u_{n+1}\|^2. \tag{10}$$

The expression (9) and (10) implies that

$$\begin{aligned} &\langle \rho_n - u_{n+1}, u_{n+1} - \xi^* \rangle \\ &\geq \lambda \{ f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \} - \lambda L_1 \|v_{n-1} - v_n\|^2 - \lambda L_2 \|v_n - u_{n+1}\|^2. \end{aligned} \tag{11}$$

From expression (11) and Lemma 8, we obtain

$$\begin{aligned} &\langle \rho_n - u_{n+1}, u_{n+1} - \xi^* \rangle \\ &\geq \langle \rho_n - v_n, u_{n+1} - v_n \rangle - \lambda L_1 \|v_{n-1} - v_n\|^2 - \lambda L_2 \|v_n - u_{n+1}\|^2. \end{aligned} \tag{12}$$

We have the following facts:

$$-2\langle \rho_n - u_{n+1}, u_{n+1} - \xi^* \rangle = -\|\rho_n - \xi^*\|^2 + \|u_{n+1} - \rho_n\|^2 + \|u_{n+1} - \xi^*\|^2.$$

$$2\langle \rho_n - v_n, u_{n+1} - v_n \rangle = \|\rho_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|\rho_n - u_{n+1}\|^2.$$

We also have the following inequality

$$\|v_{n-1} - v_n\|^2 \leq \big(\|v_{n-1} - \rho_n\| + \|\rho_n - v_n\|\big)^2 \leq 2\|v_{n-1} - \rho_n\|^2 + 2\|\rho_n - v_n\|^2.$$

From the above two facts and last inequality with (12) provides the required result. □

Now, we are in a position to provide our first convergence result of this work.

Theorem 1. *Assume that $\{u_n\}$, $\{v_n\}$ and $\{\rho_n\}$ sequences in $\mathbb{E}$ generated by Algorithm 1, where the sequence ϑ_n is non-decreasing and λ is a positive real number, such that*

$$0 < \lambda < \frac{\frac{1}{2} - 2\vartheta - \frac{1}{2}\vartheta^2}{L_2(1-\vartheta)^2 + 2L_1(1+\vartheta+\vartheta^2+\vartheta^3)} \quad \textit{and} \quad 0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2.$$

Subsequently, the sequences $\{u_n\}$, $\{v_n\}$ and $\{\rho_n\}$ are converges weakly to an element ξ^* of $SOL_{EP(f,K)}$.

Proof. From Lemma 10, we have

$$\begin{aligned} &\|u_{n+1} - \xi^*\|^2 + 4\lambda L_1\|\rho_{n+1} - v_n\|^2 \\ &\leq \|\rho_n - \xi^*\|^2 - (1 - 4\lambda L_1)\|\rho_n - v_n\|^2 - (1 - 2\lambda L_2)\|u_{n+1} - v_n\|^2 \\ &\quad + 4\lambda L_1\|\rho_n - v_{n-1}\|^2 + 4\lambda L_1\|\rho_{n+1} - v_n\|^2. \end{aligned} \tag{13}$$

By the definition of ρ_n in Algorithm 1, we have

$$\begin{aligned} \|\rho_n - \xi^*\|^2 &= \|(1+\vartheta_n)(u_n - \xi^*) - \vartheta_n(u_{n-1} - \xi^*)\|^2 \\ &= (1+\vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{14}$$

By the definition of ρ_{n+1} in Algorithm 1, we also have

$$\begin{aligned} \|\rho_{n+1} - v_n\|^2 &= \|u_{n+1} + \vartheta_{n+1}(u_{n+1} - u_n) - v_n\|^2 \\ &= \|(1+\vartheta_{n+1})(u_{n+1} - v_n) - \vartheta_{n+1}(u_n - v_n)\|^2 \\ &= (1+\vartheta_{n+1})\|u_{n+1} - v_n\|^2 - \vartheta_{n+1}\|u_n - v_n\|^2 + \vartheta_{n+1}(1+\vartheta_{n+1})\|u_{n+1} - u_n\|^2 \\ &\leq (1+\vartheta_n)\|u_{n+1} - v_n\|^2 + \vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2. \end{aligned} \tag{15}$$

Combining the expression (13)–(15), we obtain

$$\begin{aligned} &\|u_{n+1} - \xi^*\|^2 + 4\lambda L_1\|\rho_{n+1} - v_n\|^2 \\ &\leq (1+\vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 \\ &\quad + 4\lambda L_1\|\rho_n - v_{n-1}\|^2 - (1 - 4\lambda L_1)\|\rho_n - v_n\|^2 - (1 - 2\lambda L_2)\|u_{n+1} - v_n\|^2 \\ &\quad + 4\lambda L_1(1+\vartheta_n)\|u_{n+1} - v_n\|^2 + 4\lambda L_1\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 \end{aligned} \tag{16}$$

$$\begin{aligned} &\leq (1+\vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 \\ &\quad + 4\lambda L_1\|\rho_n - v_{n-1}\|^2 + 4\lambda L_1\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 \\ &\quad - (1 - 4\lambda L_1)\|\rho_n - v_n\|^2 - (1 - 2\lambda L_2 - 4\lambda L_1(1+\vartheta_n))\|u_{n+1} - v_n\|^2 \end{aligned} \tag{17}$$

$$\begin{aligned} &\leq (1+\vartheta_{n+1})\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 \\ &\quad + 4\lambda L_1\|\rho_n - v_{n-1}\|^2 + 4\lambda L_1\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 \\ &\quad - \frac{(1 - 2\lambda L_2 - 4\lambda L_1(1+\vartheta_n))}{2}\Big[2(\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2)\Big]. \end{aligned} \tag{18}$$

By substituting

$$\sigma_n = \frac{1 - 2\lambda L_2 - 4\lambda L_1(1+\vartheta_n)}{2},$$

and due to the inequality $2\|u_{n+1} - v_n\|^2 + 2\|\rho_n - v_n\|^2 \geq \|u_{n+1} - \rho_n\|^2$. From this discussion, the expression (18) turns into following:

$$\Lambda_{n+1} \leq \Lambda_n + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 + 4\lambda L_1 \vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 - \sigma_n\|u_{n+1} - \rho_n\|^2, \tag{19}$$

where $\Lambda_n = \|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + 4\lambda L_1\|\rho_n - v_{n-1}\|^2$. By the value ρ_{n+1}, we have

$$\begin{aligned}
\|u_{n+1} - \rho_n\|^2 &= \|u_{n+1} - u_n - \vartheta_n(u_n - u_{n-1})\|^2 \\
&= \|u_{n+1} - u_n\|^2 + \vartheta_n^2\|u_n - u_{n-1}\|^2 - 2\vartheta_n\langle u_{n+1} - u_n, u_n - u_{n-1}\rangle \qquad (20) \\
&\geq \|u_{n+1} - u_n\|^2 + \vartheta_n^2\|u_n - u_{n-1}\|^2 - 2\vartheta_n\|u_{n+1} - u_n\|\|u_n - u_{n-1}\| \\
&\geq (1-\vartheta_n)\|u_{n+1} - u_n\|^2 + (\vartheta_n^2 - \vartheta_n)\|u_n - u_{n-1}\|^2. \qquad (21)
\end{aligned}$$

Combining the expression (19) and (21) implies that

$$\begin{aligned}
\Lambda_{n+1} &\leq \Lambda_n + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 + 4\lambda L_1\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 \\
&\quad - \sigma_n(1-\vartheta_n)\|u_{n+1} - u_n\|^2 - \sigma_n(\vartheta_n^2 - \vartheta_n)\|u_n - u_{n-1}\|^2 \\
&\leq \Lambda_n + r_n\|u_n - u_{n-1}\|^2 - q_n\|u_{n+1} - u_n\|^2,
\end{aligned} \tag{22}$$

where $r_n := \vartheta_n(1+\vartheta_n) + \sigma_n\vartheta_n(1-\vartheta_n)$ and $q_n := \sigma_n(1-\vartheta_n) - 4\lambda L_1\vartheta_n(1+\vartheta_n)$.

Further, we take $\Gamma_n = \Lambda_n + r_n\|u_n - u_{n-1}\|^2$. It follows from (22) that

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n &= \|u_{n+1} - \xi^*\|^2 - \vartheta_{n+1}\|u_n - \xi^*\|^2 + r_{n+1}\|u_{n+1} - u_n\|^2 + 4\lambda L_1\|\rho_{n+1} - v_n\|^2 \\
&\quad - \|u_n - \xi^*\|^2 + \vartheta_n\|u_{n-1} - \xi^*\|^2 - r_n\|u_n - u_{n-1}\|^2 - 4\lambda L_1\|\rho_n - v_{n-1}\|^2 \\
&= \|u_{n+1} - \xi^*\|^2 - (1+\vartheta_{n+1})\|u_n - \xi^*\|^2 + \vartheta_n\|u_{n-1} - \xi^*\|^2 + 4\lambda L_1\|\rho_{n+1} - v_n\|^2 \\
&\quad - 4\lambda L_1\|\rho_n - v_{n-1}\|^2 - r_n\|u_n - u_{n-1}\|^2 + r_{n+1}\|u_{n+1} - u_n\|^2 \\
&\leq -q_n\|u_{n+1} - u_n\|^2 + r_{n+1}\|u_{n+1} - u_n\|^2 \\
&= -(q_n - r_{n+1})\|u_{n+1} - u_n\|^2.
\end{aligned} \tag{23}$$

Next, we need to compute

$$\begin{aligned}
& q_n - r_{n+1} \\
&= \sigma_n(1-\vartheta_n) - 4\lambda L_1\vartheta_n(1+\vartheta_n) - \vartheta_{n+1}(1+\vartheta_{n+1}) - \sigma_{n+1}\vartheta_{n+1}(1-\vartheta_{n+1}) \\
&\geq \sigma_n(1-\vartheta_n) - 4\lambda L_1\vartheta_n(1+\vartheta_n) - \vartheta_n(1+\vartheta_n) - \sigma_n\vartheta_n(1-\vartheta_n) \\
&\geq \sigma_n(1-\vartheta)^2 - 4\lambda L_1\vartheta(1+\vartheta) - \vartheta(1+\vartheta) \\
&\geq \frac{1 - 2\lambda L_2 - 4\lambda L_1(1+\vartheta)}{2}(1-\vartheta)^2 - 4\lambda L_1\vartheta(1+\vartheta) - \vartheta(1+\vartheta) \\
&= \left(\frac{1}{2} - 2\vartheta - \frac{1}{2}\vartheta^2\right) - \lambda\Big(L_2(1-\vartheta)^2 + 2L_1(1+\vartheta+\vartheta^2+\vartheta^3)\Big) \\
&\geq 0.
\end{aligned} \tag{24}$$

The expression (23) and (24) with some $\delta \geq 0$, implies that

$$\Gamma_{n+1} - \Gamma_n \leq -(q_n - r_{n+1})\|u_{n+1} - u_n\|^2 \leq -\delta\|u_{n+1} - u_n\|^2 \leq 0. \tag{25}$$

The above relation (25) implies that the sequence $\{\Gamma_n\}$ is non-increasing. From Γ_{n+1}, we have

$$\begin{aligned} \Gamma_{n+1} &= \|u_{n+1} - \xi^*\|^2 - \vartheta_{n+1}\|u_n - \xi^*\|^2 + r_{n+1}\|u_{n+1} - u_n\|^2 + 4\lambda L_1\|\rho_{n+1} - v_n\|^2 \\ &\geq -\vartheta_{n+1}\|u_n - \xi^*\|^2. \end{aligned} \tag{26}$$

Additionally, from definition Γ_n, we have

$$\begin{aligned} \|u_n - \xi^*\|^2 &\leq \Gamma_n + \vartheta_n\|u_{n-1} - \xi^*\|^2 \\ &\leq \Gamma_1 + \vartheta\|u_{n-1} - \xi^*\|^2 \\ &\leq \cdots \leq \Gamma_1(\vartheta^{n-1} + \cdots + 1) + \vartheta^n\|u_0 - \xi^*\|^2 \\ &\leq \frac{\Gamma_1}{1-\vartheta} + \vartheta^n\|u_0 - \xi^*\|^2. \end{aligned} \tag{27}$$

Combining the expression (26) and (27), we obtain

$$\begin{aligned} -\Gamma_{n+1} &\leq \vartheta_{n+1}\|u_n - \xi^*\|^2 \\ &\leq \vartheta\|u_n - \xi^*\|^2 \\ &\leq \vartheta\frac{\Gamma_1}{1-\vartheta} + \vartheta^{n+1}\|u_0 - \xi^*\|^2. \end{aligned} \tag{28}$$

It continues to follow from (25) and (28), such that

$$\begin{aligned} \delta\sum_{n=1}^{k}\|u_{n+1} - u_n\|^2 &\leq \Gamma_1 - \Gamma_{k+1} \\ &\leq \Gamma_1 + \vartheta\frac{\Gamma_1}{1-\vartheta} + \vartheta^{n+1}\|u_0 - \xi^*\|^2 \\ &\leq \frac{\Gamma_1}{1-\vartheta} + \|u_0 - \xi^*\|^2, \end{aligned} \tag{29}$$

letting $k \to \infty$ in (29) implies that

$$\sum_{n=1}^{\infty}\|u_{n+1} - u_n\|^2 < +\infty \quad \text{implies} \quad \|u_{n+1} - u_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{30}$$

From the relation (20) and (30), we obtain

$$\|u_{n+1} - \rho_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{31}$$

Next, the expression (28) implies that

$$-\Lambda_{n+1} \leq \vartheta\frac{\Gamma_1}{1-\vartheta} + \vartheta^{n+1}\|u_0 - \xi^*\|^2 + r_{n+1}\|u_{n+1} - u_n\|^2. \tag{32}$$

From the relation (18) we have

$$\begin{aligned} &(1 - 2\lambda L_2 - 4\lambda L_1(1+\vartheta))\left[\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2\right] \\ &\leq \Lambda_n - \Lambda_{n+1} + \vartheta(1+\vartheta)\|u_n - u_{n-1}\|^2 + 4\lambda L_1\vartheta(1+\vartheta)\|u_{n+1} - u_n\|^2. \end{aligned} \tag{33}$$

Set $k \in \mathbb{N}$ and using (33) for $n = 1, 2, \cdots, k$, gives that

$$
\begin{aligned}
&\left(1 - 2L_2\lambda - 4L_1\lambda(1+\vartheta)\right) \sum_{n=1}^{k} \left[\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2\right] \\
&\leq \Lambda_0 - \Lambda_{k+1} + \vartheta(1+\vartheta) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2 + 4\lambda L_1 \vartheta(1+\vartheta) \sum_{n=1}^{k} \|u_{n+1} - u_n\|^2 \\
&\leq \Lambda_0 + \vartheta \frac{\Gamma_1}{1-\vartheta} + \vartheta^{k+1}\|u_0 - \xi^*\|^2 + r_{k+1}\|u_{k+1} - u_k\|^2 \\
&\quad + \vartheta(1+\vartheta) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2 + 4\lambda L_1 \vartheta(1+\vartheta) \sum_{n=1}^{k} \|u_{n+1} - u_n\|^2,
\end{aligned} \tag{34}
$$

letting $k \to \infty$ in (34) implies that

$$
\sum_{n=1}^{\infty} \|u_{n+1} - v_n\|^2 < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|\rho_n - v_n\|^2 < +\infty, \tag{35}
$$

and

$$
\lim_{n\to\infty} \|u_{n+1} - v_n\| = \lim_{n\to\infty} \|\rho_n - v_n\| = 0. \tag{36}
$$

The following relation can easily be derived:

$$
\lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|u_n - \rho_n\| = \lim_{n\to\infty} \|v_{n-1} - v_n\| = 0. \tag{37}
$$

By the definition of ρ_n and using Cauchy inequality, we have

$$
\begin{aligned}
\|\rho_n - v_{n-1}\|^2 &= \|u_n + \vartheta_n(u_n - u_{n-1}) - v_{n-1}\|^2 \\
&= \|(1+\vartheta_n)(u_n - v_{n-1}) - \vartheta_n(u_{n-1} - v_{n-1})\|^2 \\
&= (1+\vartheta_n)\|u_n - v_{n-1}\|^2 - \vartheta_n\|u_{n-1} - v_{n-1}\|^2 + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 \\
&\leq (1+\vartheta)\|u_n - v_{n-1}\|^2 + \vartheta(1+\vartheta)\|u_n - u_{n-1}\|^2.
\end{aligned} \tag{38}
$$

Now, summing up the expression (38) for $n = 1, 2 \cdots, k$, we obtain

$$
\sum_{n=1}^{k} \|\rho_n - v_{n-1}\|^2 \leq (1+\vartheta) \sum_{n=1}^{k} \|u_n - v_{n-1}\|^2 + \vartheta(1+\vartheta) \sum_{n=1}^{k} \|u_n - u_{n-1}\|^2 \tag{39}
$$

The above expression with (30) and (35) implies that

$$
\sum \|\rho_n - v_{n-1}\|^2 < +\infty. \tag{40}
$$

It follows from the relation (16), we obtain

$$
\begin{aligned}
\|u_{n+1} - \xi^*\|^2 \leq (1+\vartheta)\|u_n - \xi^*\|^2 - \vartheta\|u_{n-1} - \xi^*\|^2 + \vartheta(1+\vartheta)\|u_n - u_{n-1}\|^2 \\
+ 4L_1\lambda\|\rho_n - v_{n-1}\|^2,
\end{aligned} \tag{41}
$$

above expression with (30), (40), (37) and Lemma 4 implies that limit of $\|u_n - \xi^*\|$, $\|\rho_n - \xi^*\|$ and $\|v_n - \xi^*\|$ exists for every $\xi^* \in SOL_{EP(f,K)}$, means that the sequences $\{u_n\}$, $\{\rho_n\}$ and $\{v_n\}$ are bounded. Next, we need to show that each weak sequential limit point of the sequence $\{u_n\}$ belongs to $SOL_{EP(f,K)}$. Let z be arbitrary weak cluster point of the sequence $\{u_n\}$, and then there exists a weak convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges to z, this also implies that $\{v_{n_k}\}$ also converge weakly to z. Now our aim to prove that $z \in SOL_{EP(f,K)}$. By Lemma 6, the bifunction Lipschitz-type condition and Lemma 8, we have

$$\begin{aligned}\lambda f(v_{n_k}, y) &\geq \lambda f(v_{n_k}, u_{n_k+1}) + \langle \rho_{n_k} - u_{n_k+1}, y - u_{n_k+1}\rangle \\ &\geq \lambda f(v_{n_k-1}, u_{n_k+1}) - \lambda f(v_{n_k-1}, v_{n_k}) - \lambda L_1 \|v_{n_k-1} - v_{n_k}\|^2 \\ &\quad - \lambda L_2 \|v_{n_k} - u_{n_k+1}\|^2 + \langle \rho_{n_k} - u_{n_k+1}, y - u_{n_k+1}\rangle \\ &\geq \langle \rho_{n_k} - v_{n_k}, u_{n_k+1} - v_{n_k}\rangle - \lambda L_1 \|v_{n_k-1} - v_{n_k}\|^2 \\ &\quad - \lambda L_2 \|v_{n_k} - u_{n_k+1}\|^2 + \langle \rho_{n_k} - u_{n_k+1}, y - u_{n_k+1}\rangle\end{aligned} \tag{42}$$

where y be an any element in H_n. As a result with (31), (36), (37), and due to the boundedness of the sequence $\{u_n\}$ the above inequality tends to zero. By given $\lambda > 0$, the assumption (A3) and $v_{n_k} \rightharpoonup z$, we obtain

$$0 \leq \limsup_{k\to\infty} f(v_{n_k}, y) \leq f(z, y), \ \forall y \in H_n.$$

Due to $z \in K \subset H_n$, we obtain $f(z, y) \geq 0, \forall y \in K$. This implies that z belongs to $SOL_{EP(f,K)}$. Thus Lemma 5, ensures that $\{\rho_n\}$, $\{u_n\}$ and $\{v_n\}$ weakly converges to ξ^* as $n \to \infty$. □

Remark 2. *For $\vartheta_n = \vartheta = 0$ in Algorithm 1 gives the results as in [35,36].*

4. Inertial Popov's Two-Step Subgradient Extragradient Algorithm for Strongly Pseudomonotone EP

The second algorithm is also an inertial algorithm that is able to solve the strongly pseudomonotone equilibrium problem. However, the advantage of this algorithm is that there is no need for prior information regarding the strongly pseudomonotone constant γ and Lipschitz constants L_1, L_2. Let $\{\lambda_n\} \subset (0, +\infty)$ be a non-increasing sequence, so that the following conditions are satisfied:

$$(\text{T1}): \quad \lim_{n\to\infty} \lambda_n = 0 \quad \text{and} \quad (\text{T2}): \quad \sum_{n=1}^{\infty} \lambda_n = +\infty. \tag{43}$$

Assumption 2. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfies the following conditions:*

(B1) *$f(\widetilde{u}, \widetilde{u}) = 0, \forall \widetilde{u} \in K$ and f is strongly pseudomontone on K;*
(B2) *f meet the Lipschitz-type condition on $\mathbb{E}$ with two positive constants L_1 and L_2;*
(B3) *$f(\widetilde{u}, .)$ is sub-differentiable and convex on $\mathbb{E}$ for all $\widetilde{u} \in \mathbb{E}$.*

Lemma 11. *Assume that $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfies the conditions* (B1)–(B3). *Let the solution set $SOL_{EP(f,K)}$ is nonempty. For each $\xi^* \in SOL_{EP(f,K)}$, we have*

$$\begin{aligned}\|u_{n+1} - \xi^*\|^2 \leq \|\rho_n - \xi^*\|^2 - (1 - 4\lambda_n L_1)\|\rho_n - v_n\|^2 - (1 - 2\lambda_n L_2)\|u_{n+1} - v_n\|^2 \\ + 4\lambda_n L_1 \|\rho_n - v_{n-1}\|^2 - 2\gamma\lambda_n \|v_n - \xi^*\|^2.\end{aligned}$$

Now, we are in a position to provide our second convergence result of this work.

Theorem 2. *Assume that $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfies the conditions* (B1)–(B3). *Let $\{u_n\}$, $\{v_n\}$ and $\{\rho_n\}$ are sequences in $\mathbb{E}$ generated by Algorithm 2 and ϑ_n is non-decreasing sequence with $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$. Subsequently, $\{u_n\}$, $\{v_n\}$ and $\{\rho_n\}$ strongly converge to an element ξ^* in $SOL_{EP(f,K)}$.*

Algorithm 2 (Two-step Subgradient Extragradient Algorithm for Strongly Pseudomonotone EP)

Initialization: Choose $u_{-1}, u_0, v_0 \in \mathbb{E}$, $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$ and a sequence $\{\lambda_n\}$ satisfying (43). Set

$$u_1 = \arg\min\{\lambda_0 f(v_0, y) + \frac{1}{2}\|\rho_0 - y\|^2 : y \in K\},$$

$$v_1 = \arg\min\{\lambda_1 f(v_0, y) + \frac{1}{2}\|\rho_1 - y\|^2 : y \in K\},$$

where $\rho_0 = u_0 + \vartheta_0(u_0 - u_{-1})$ and $\rho_1 = u_1 + \vartheta_1(u_1 - u_0)$.

Iterative steps: Assume that u_{n-1}, u_n, v_{n-1} and v_n are known for $n \geq 1$ and

$$H_n = \{z \in \mathbb{E} : \langle \rho_n - \lambda_n \omega_{n-1} - v_n, z - v_n \rangle \leq 0\},$$

where $\omega_{n-1} \in \partial f(v_{n-1}, v_n)$.

Step 1: Compute

$$u_{n+1} = \arg\min\{\lambda_n f(v_n, y) + \frac{1}{2}\|\rho_n - y\|^2 : y \in H_n\},$$

where $\rho_n = u_n + \vartheta_n(u_n - u_{n-1})$.

Step 2: Compute

$$v_{n+1} = \arg\min\{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|\rho_{n+1} - y\|^2 : y \in K\},$$

where $\rho_{n+1} = u_{n+1} + \vartheta_{n+1}(u_{n+1} - u_n)$.

Step 3: If $u_{n+1} = \rho_n$ and $v_n = v_{n-1}$, then STOP. Otherwise set $n := n + 1$ and go to **Step 1**.

Proof. The proof is the identical as the proof of Theorem 1, but there are still few changes. We provide the proof for the readable purpose. By Lemma 11 and adding $4L_1\lambda_n\|\rho_{n+1} - v_n\|^2$ in both sides, we have

$$\begin{aligned}
&\|u_{n+1} - \xi^*\|^2 + 4L_1\lambda_n\|\rho_{n+1} - v_n\|^2 \\
&\leq \|\rho_n - \xi^*\|^2 - (1 - 4L_1\lambda_n)\|\rho_n - v_n\|^2 - (1 - 2L_2\lambda_n)\|u_{n+1} - v_n\|^2 \\
&\quad + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 + 4L_1\lambda_n\|\rho_{n+1} - v_n\|^2.
\end{aligned} \tag{44}$$

By using the definition of ρ_n in Algorithm 2, we have

$$\|\rho_n - \xi^*\|^2 = (1 + \vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + \vartheta_n(1 + \vartheta_n)\|u_n - u_{n-1}\|^2. \tag{45}$$

By using the definition ρ_{n+1} in Algorithm 2, we also have

$$\|\rho_{n+1} - v_n\|^2 \leq (1 + \vartheta_n)\|u_{n+1} - v_n\|^2 + \vartheta_n(1 + \vartheta_n)\|u_{n+1} - u_n\|^2. \tag{46}$$

Combining the expression (44)–(46), we obtain

$$\begin{aligned}
&\|u_{n+1} - \xi^*\|^2 + 4L_1\lambda_{n+1}\|\rho_{n+1} - v_n\|^2 \\
&\leq (1+\vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 \\
&\quad + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2 - (1-4L_1\lambda_n)\|\rho_n - v_n\|^2 - (1-2L_2\lambda_n)\|u_{n+1} - v_n\|^2 \\
&\quad + 4L_1\lambda_n(1+\vartheta_n)\|u_{n+1} - v_n\|^2 + 4L_1\lambda_n\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 \qquad (47) \\
&\leq (1+\vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2 \\
&\quad + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 + 4L_1\lambda_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\
&\quad - \frac{(1-2L_2\lambda_n - 4L_1\lambda_n(1+\vartheta_n))}{2}\Big[2(\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2)\Big]. \qquad (48)
\end{aligned}$$

Next, we let $\varrho_n = \frac{1-2L_2\lambda_n - 4L_1\lambda_n(1+\vartheta_n)}{2}$ and

$$\Phi_n = \|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2.$$

Due to the above substituting the expression (48) turns into the following:

$$\begin{aligned}
\Phi_{n+1} \leq \Phi_n &+ \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 + 4L_1\lambda_n\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 \\
&- \varrho_n\|u_{n+1} - \rho_n\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2,
\end{aligned} \tag{49}$$

By the definition ρ_{n+1}, we have

$$\|u_{n+1} - \rho_n\|^2 \geq (1-\vartheta_n)\|u_{n+1} - u_n\|^2 + (\vartheta_n^2 - \vartheta_n)\|u_n - u_{n-1}\|^2. \tag{50}$$

Combining the expression (49) and (50), we obtain

$$\begin{aligned}
\Phi_{n+1} &\leq \Phi_n + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 + 4L_1\lambda_n\vartheta_n(1+\vartheta_n)\|u_{n+1} - u_n\|^2 \\
&\quad - 2\gamma\lambda_n\|v_n - \xi^*\|^2 - \varrho_n(1-\vartheta_n)\|u_{n+1} - u_n\|^2 - \varrho_n(\vartheta_n^2 - \vartheta_n)\|u_n - u_{n-1}\|^2 \\
&= \Phi_n + R_n\|u_n - u_{n-1}\|^2 - Q_n\|u_{n+1} - u_n\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2,
\end{aligned} \tag{51}$$

where $R_n := \vartheta_n(1+\vartheta_n) + \varrho_n\vartheta_n(1-\vartheta_n)$ and $Q_n := \varrho_n(1-\vartheta_n) - 4L_1\lambda_n\vartheta_n(1+\vartheta_n)$. In addition, we also take $\Psi_n = \Phi_n + R_n\|u_n - u_{n-1}\|^2$. It follows from (51) that

$$\Psi_{n+1} - \Psi_n \leq -(Q_n - R_{n+1})\|u_{n+1} - u_n\|^2 - 2\gamma\lambda_n\|v_n - \xi^*\|^2. \tag{52}$$

Since $\lambda_n \to 0$, then there exists a finite number $n_0 \in \mathbb{N}$ such that

$$0 < \lambda_n < \frac{\frac{1}{2} - 2\vartheta - \frac{1}{2}\vartheta^2}{L_2(1-\vartheta)^2 + 2L_1(1+\vartheta+\vartheta^2+\vartheta^3)}, \; n \geq n_0.$$

Similarly, it follows from (24) and expression (52) implies that

$$\Psi_{n+1} - \Psi_n \leq -\delta\|u_{n+1} - u_n\|^2 \leq 0, \; n \geq n_0. \tag{53}$$

The above implies that the sequence $\{\Psi_n\}$ is non-increasing for $n \geq n_0$. From the value of Ψ_n, we have

$$\begin{aligned} \|u_n - \xi^*\|^2 &\leq \Psi_n + \vartheta_n \|u_{n-1} - \xi^*\|^2 \\ &\leq \Psi_{n_0} + \vartheta \|u_{n-1} - \xi^*\|^2 \\ &\leq \cdots \leq \Psi_{n_0}(\vartheta^{n-n_0} + \cdots + 1) + \vartheta^{n-n_0}\|u_{n_0} - \xi^*\|^2 \\ &\leq \frac{\Psi_{n_0}}{1-\vartheta} + \vartheta^{n-n_0}\|u_{n_0} - \xi^*\|^2. \end{aligned} \tag{54}$$

From the definition of Ψ_{n+1} with the expression (54), we obtain

$$\begin{aligned} -\Psi_{n+1} &\leq \vartheta_{n+1}\|u_n - \xi^*\|^2 \\ &\leq \vartheta \|u_n - \xi^*\|^2 \\ &\leq \vartheta \frac{\Psi_{n_0}}{1-\vartheta} + \vartheta^{n-n_0+1}\|u_{n_0} - \xi^*\|^2 \\ &\leq \vartheta \frac{\Psi_{n_0}}{1-\vartheta} + \|u_{n_0} - \xi^*\|^2. \end{aligned} \tag{55}$$

It is follows from (53) and (55) that

$$\begin{aligned} \delta \sum_{n=n_0}^{k} \|u_{n+1} - u_n\|^2 &\leq \Psi_{n_0} - \Psi_{k+1} \\ &\leq \Psi_{n_0} + \vartheta \frac{\Psi_{n_0}}{1-\vartheta} + \|u_{n_0} - \xi^*\|^2 \\ &\leq \frac{\Psi_{n_0}}{1-\vartheta} + \|u_{n_0} - \xi^*\|^2, \end{aligned} \tag{56}$$

letting $k \to \infty$ in the expression (56), we obtain

$$\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|^2 < +\infty \quad \text{implies that} \quad \|u_{n+1} - u_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{57}$$

From the expression (20) and (57), we obtain

$$\|u_{n+1} - \rho_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{58}$$

The expression (55) implies that

$$-\Phi_{n+1} \leq \vartheta \frac{\Psi_{n_0}}{1-\vartheta} + \|u_{n_0} - \xi^*\|^2 + R_{n+1}\|u_{n+1} - u_n\|^2. \tag{59}$$

It follows from (48) for all $n \geq n_0$, such that

$$\begin{aligned} &\left(1 - 2L_2\lambda_n - 4L_1\lambda_n(1+\vartheta)\right)\left[\|u_{n+1} - v_n\|^2 + \|\rho_n - v_n\|^2\right] \\ &\leq \Phi_n - \Phi_{n+1} + \vartheta(1+\vartheta)\|u_n - u_{n-1}\|^2 + 4L_1\lambda_n\vartheta(1+\vartheta)\|u_{n+1} - u_n\|^2. \end{aligned} \tag{60}$$

Consider the expression (60) for $n_0, n_0+1, \cdots, k$. Summing them up, we obtain

$$\begin{aligned}
&\left(1-2L_2\lambda_n-4L_1\lambda_n(1+\vartheta)\right)\sum_{n=n_0}^{k}\left[\|u_{n+1}-v_n\|^2+\|\rho_n-v_n\|^2\right]\\
&\leq \Phi_{n_0}-\Phi_{k+1}+\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_n-u_{n-1}\|^2+\frac{4L_1}{2L_2+4L_1}\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_{n+1}-u_n\|^2\\
&\leq \Phi_{n_0}+\vartheta\frac{\Phi_{n_0}}{1-\vartheta}+\|u_{n_0}-\xi^*\|^2+R_{k+1}\|u_{k+1}-u_k\|^2\\
&\quad+\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_n-u_{n-1}\|^2+\frac{4L_1}{2L_2+4L_1}\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_{n+1}-u_n\|^2\\
&=\frac{\Phi_{n_0}}{1-\vartheta}+\|u_{n_0}-\xi^*\|^2+R_{k+1}\|u_{k+1}-u_k\|^2\\
&\quad+\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_n-u_{n-1}\|^2+\frac{4L_1}{2L_2+4L_1}\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_{n+1}-u_n\|^2,
\end{aligned} \tag{61}$$

By letting $k \to \infty$ in the expression (61) implies that

$$\sum_n \|u_{n+1}-v_n\|^2 < +\infty \quad \text{and} \quad \sum_n \|\rho_n-v_n\|^2 < +\infty, \tag{62}$$

and

$$\lim_{n\to\infty}\|u_{n+1}-v_n\| = \lim_{n\to\infty}\|\rho_n-v_n\| = 0. \tag{63}$$

We can easily derive the following relationship:

$$\lim_{n\to\infty}\|u_n-v_n\| = \lim_{n\to\infty}\|u_n-\rho_n\| = \lim_{n\to\infty}\|v_{n-1}-v_n\| = 0. \tag{64}$$

By using the value ρ_n, we obtain

$$\|\rho_n-v_{n-1}\|^2 \leq (1+\vartheta)\|u_n-v_{n-1}\|^2+\vartheta(1+\vartheta)\|u_n-u_{n-1}\|^2. \tag{65}$$

Now, summing up equation (65) for $n = n_0, n_0+1\cdots, k$, we obtain

$$\sum_{n=n_0}^{k}\|\rho_n-v_{n-1}\|^2 \leq (1+\vartheta)\sum_{n=n_0}^{k}\|u_n-v_{n-1}\|^2+\vartheta(1+\vartheta)\sum_{n=n_0}^{k}\|u_n-u_{n-1}\|^2 \tag{66}$$

The above expression with (57) and (62) implies that

$$\sum_{n=1}^{\infty}\|\rho_n-v_{n-1}\|^2 < +\infty. \tag{67}$$

Furthermore, the expression (47) gives that

$$\begin{aligned}
&\|u_{n+1}-\xi^*\|^2\\
&\leq (1+\vartheta)\|u_n-\xi^*\|^2-\vartheta\|u_{n-1}-\xi^*\|^2+\vartheta(1+\vartheta)\|u_n-u_{n-1}\|^2+4L_1\lambda_n\|\rho_n-v_{n-1}\|^2.
\end{aligned} \tag{68}$$

The above expression through (57), (67), and Lemma 4 implies that

$$\lim_{n\to\infty}\|u_n-\xi^*\| = l. \tag{69}$$

The expression (64) with (69), we obtain

$$\lim_{n\to\infty} \|\rho_n - \xi^*\| = \lim_{n\to\infty} \|v_n - \xi^*\| = l. \tag{70}$$

Now, we are showing that the sequence $\{u_n\}$ converges strongly to ξ^*. Due to the condition on λ_n for all $n \geq n_0$, we can easily observe the following inequality:

$$0 < \lambda_n < \frac{1}{2L_2 + 4L_1}, \ \forall n \geq n_0.$$

It follows from Lemma 11, such that

$$2\gamma\lambda_n\|v_n - \xi^*\|^2 \leq \|\rho_n - \xi^*\|^2 - \|u_{n+1} - \xi^*\|^2 + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2, \ \forall n \geq n_0. \tag{71}$$

From the expression (45) and (71), we obtain

$$\begin{aligned} 2\gamma\lambda_n\|v_n - \xi^*\|^2 &\leq -\|u_{n+1} - \xi^*\|^2 + (1+\vartheta_n)\|u_n - \xi^*\|^2 - \vartheta_n\|u_{n-1} - \xi^*\|^2 \\ &\quad + \vartheta_n(1+\vartheta_n)\|u_n - u_{n-1}\|^2 + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2 \\ &\leq (\|u_n - \xi^*\|^2 - \|u_{n+1} - \xi^*\|^2) + 2\vartheta\|u_n - u_{n-1}\|^2 \\ &\quad + (\vartheta_n\|u_n - \xi^*\|^2 - \vartheta_{n-1}\|u_{n-1} - \xi^*\|^2) + 4L_1\lambda_n\|\rho_n - v_{n-1}\|^2. \end{aligned} \tag{72}$$

It follows from expression (72) that

$$\begin{aligned} &\sum_{n=n_0}^{k} 2\gamma\lambda_n\|v_n - \xi^*\|^2 \\ &\leq (\|u_{n_0} - \xi^*\|^2 - \|u_{k+1} - \xi^*\|^2) + 2\vartheta\sum_{n=n_0}^{k}\|u_n - u_{n-1}\|^2 \\ &\quad + (\vartheta_k\|u_k - \xi^*\|^2 - \vartheta_{n_0-1}\|u_{n_0-1} - \xi^*\|^2) + \frac{4L_1}{2L_2 + 4L_1}\sum_{n=n_0}^{k}\|\rho_n - v_{n-1}\|^2 \\ &\leq \|u_{n_0} - \xi^*\|^2 + \vartheta\|u_k - \xi^*\|^2 + 2\vartheta\sum_{n=n_0}^{k}\|u_n - u_{n-1}\|^2 + \frac{4L_1}{2L_2 + 4L_1}\sum_{n=n_0}^{k}\|\rho_n - v_{n-1}\|^2 \\ &\leq M, \end{aligned} \tag{73}$$

for $M \geq 0$. It implies that

$$\sum_{n=1}^{\infty} 2\gamma\lambda_n\|v_n - \xi^*\|^2 < +\infty. \tag{74}$$

By the Lemma 2 and (74) implies that

$$\liminf \|v_n - \xi^*\| = 0. \tag{75}$$

Finally, expression (69) and (75) provide that $\lim_{n\to\infty}\|u_n - \xi^*\| = 0$. This completes the proof. □

5. Application to Variational Inequality Problems

For considering Algorithm 1 and Theorem 1, we can able to write the next result for solving variational inequality problems that involve pseudomonotone and Lipschitz continuous operator.

Corollary 1. *Assume that $H : K \to \mathbb{E}$ be a Lipschitz continuous with the constant L and pseudomonotone operator. Let $\{u_n\}$, $\{v_n\}$ and $\{\rho_n\}$ be sequences generated, as follows:*

(i) *Choose* $u_{-1}, u_0, v_0 \in \mathbb{E}$, $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$ *and* $\lambda(\vartheta, L_1, L_2) > 0$. *Compute*

$$\begin{cases} u_1 = P_K(\rho_0 - \lambda H v_0), \text{ where } \rho_0 = u_0 + \vartheta_0(u_0 - u_{-1}), \\ v_1 = P_K(\rho_1 - \lambda H v_0), \text{ where } \rho_1 = u_1 + \vartheta_1(u_1 - u_0). \end{cases}$$

(ii) *Given* u_{n-1}, u_n, v_{n-1}, *and* v_n *for each* $n \geq 1$, *and construct the half-space first as*

$$H_n = \{z \in \mathbb{E} : \langle \rho_n - \lambda H v_{n-1} - v_n, z - v_n \rangle \leq 0\}.$$

(iii) *Evaluate*

$$\begin{cases} u_{n+1} = P_{H_n}(\rho_n - \lambda H v_n), \text{ where } \rho_n = u_n + \vartheta_n(u_n - u_{n-1}), \\ v_{n+1} = P_K(\rho_{n+1} - \lambda H v_n), \text{ where } \rho_{n+1} = u_{n+1} + \vartheta_{n+1}(u_{n+1} - u_n), \end{cases}$$

where $\lambda > 0$, *such that*

$$0 < \lambda < \frac{\frac{1}{2} - 2\vartheta - \frac{1}{2}\vartheta^2}{L_2(1-\vartheta)^2 + 2L_1(1 + \vartheta + \vartheta^2 + \vartheta^3)} \text{ and } 0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2,$$

with $L_1 = L_2 = \frac{L}{2}$. *Subsequently, sequence* $\{u_n\}$, $\{\rho_n\}$ *and* $\{v_n\}$ *converge weakly to* $\xi^* \in SOL_{VI(H,K)}$.

From the consideration on Algorithm 2 and Theorem 2, we state the following result for the class of variational inequality problems involving strongly pseudomonotone and Lipschitz continuous operator.

Corollary 2. *Assume that* $H : K \to \mathbb{E}$ *is a Lipschitz continuous and strongly pseudomonotone operator with the constant* L. *Let* $\{u_n\}$, $\{v_n\}$ *and* $\{\rho_n\}$ *are the sequences generated as follows:*

(i) *Choose* $u_{-1}, u_0, v_0 \in \mathbb{E}$, $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$ *and a sequence* $\{\lambda_n\}$ *satisfying* (43). *Compute*

$$\begin{cases} u_1 = P_K(\rho_0 - \lambda_0 H v_0), \text{ where } \rho_0 = u_0 + \vartheta_0(u_0 - u_{-1}), \\ v_1 = P_K(\rho_1 - \lambda_1 H v_0), \text{ where } \rho_1 = u_1 + \vartheta_1(u_1 - u_0). \end{cases}$$

(ii) *Given* u_{n-1}, u_n, v_{n-1}, *and* v_n *create a half space for each* $n \geq 1$, *such that*

$$H_n = \{z \in \mathbb{E} : \langle \rho_n - \lambda_n H v_{n-1} - v_n, z - v_n \rangle \leq 0\}.$$

(iii) *Compute*

$$\begin{cases} u_{n+1} = P_{H_n}(\rho_n - \lambda_n H v_n), \text{ where } \rho_n = u_n + \vartheta_n(u_n - u_{n-1}), \\ v_{n+1} = P_K(\rho_{n+1} - \lambda_{n+1} H v_n), \text{ where } \rho_{n+1} = u_{n+1} + \vartheta_{n+1}(u_{n+1} - u_n), \end{cases}$$

where $0 \leq \vartheta_n \leq \vartheta < \sqrt{5} - 2$, *with* $L_1 = L_2 = \frac{L}{2}$. *The sequence* $\{u_n\}$, $\{\rho_n\}$ *and* $\{v_n\}$ *converge strongly to* $\xi^* \in SOL_{VI(H,K)}$.

6. Computational Experiment

Some numerical results will be presented in this section to show the performance of our proposed methods. The MATLAB codes run in MATLAB version 9.5 (R2018b) on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70GHz 1.70GHz, RAM 4.00 GB).

6.1. Nash-Cournot Equilibrium Model of Electricity Markets

The Nash–Cournot equilibrium model of electricity markets in [20] is considered in this example. Assume that there are three companies ($i = 1, 2, 3$) generating electricity. These three companies has generating units denoted as $U_1 = \{1\}$, $U_2 = \{2,3\}$ and $U_3 = \{4,5,6\}$, respectively. Let u_j denote the generating power of the each unit for $i = \{1,2,3,4,5,6\}$. Next, we take the electricity price P as $P = 378.4 - 2\sum_{j=1}^{6} u_j$. The cost of generating the j unit is written as:

$$c_j(u_j) := \max\{\overset{\circ}{c}_j(u_j), \overset{\bullet}{c}_j(u_j)\},$$

where $\overset{\circ}{c}_j(u_j) := \frac{\overset{\circ}{\alpha}_j}{2}u_j^2 + \overset{\circ}{\beta}_j u_j + \overset{\circ}{\gamma}_j$ and $\overset{\bullet}{c}_j(u_j) := \overset{\bullet}{\alpha}_j u_j + \frac{\overset{\bullet}{\beta}_j}{\overset{\bullet}{\beta}_j+1}\overset{\bullet}{\gamma}_j^{\frac{-1}{\overset{\bullet}{\beta}_j}}(u_j)^{\frac{(\overset{\bullet}{\beta}_j+1)}{\overset{\bullet}{\beta}_j}}$. Table 1 provides the values of the unknown parameters. Consider that the profit of the firm i is

$$F_i(u) := P\sum_{j\in I_i} u_j - \sum_{j\in I_i} c_j(u_j) = \Big(378.4 - 2\sum_{l=1}^{6} u_l\Big)\sum_{j\in I_i} u_j - \sum_{j\in I_i} c_j(u_j),$$

with $u = (u_1, \cdots, u_6)^T$ corresponding to the constraint set $u \in C := \{u \in \mathbb{R}^6 : u_j^{\min} \le u_j \le u_j^{\max}\}$, with $u_j^{\min}$ and $u_j^{\max}$ values given in Table 2. Consider the equilibrium function f by

$$f(u,v) := \sum_{i=1}^{3}\big(\phi_i(u,u) - \phi_i(u,v)\big),$$

where

$$\phi_i(u,v) := \Big[378.4 - 2\Big(\sum_{j\notin I_i} u_j + \sum_{j\in I_i} v_j\Big)\Big]\sum_{j\in I_i} v_j - \sum_{j\in I_i} c_j(v_j).$$

The Nash–Cournot equilibrium models of electricity markets can be seen as an equilibrium problem in the following way (see [44] for more details):

$$\text{Find } \xi^* \in K \text{ such that } f(\xi^*, y) \ge 0,\ \forall y \in K.$$

During the numerical example in Section 6.1, we take the values $u_{-1} = (10,10,20,17,8,14)^T$, $u_0 = (10,20,30,10,0,1)^T$, $v_0 = (48,48,30,27,18,24)^T$.

Table 1. The values of parameters are used in the cost function.

j	$\overset{\circ}{\alpha}_j$	$\overset{\circ}{\beta}_j$	$\overset{\circ}{\gamma}_j$	$\overset{\bullet}{\alpha}_j$	$\overset{\bullet}{\beta}_j$	$\overset{\bullet}{\gamma}_j$
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

Table 2. The parameter values use for constraint set.

j	1	2	3	4	5	6
u_j^{min}	0	0	0	0	0	0
u_j^{max}	80	80	50	55	30	40

6.1.1. Algorithm 1 Behaviour for Different Values of ϑ_n:

Figure 1 and Table 3 characterize the behaviour of error term $D_n = \|u_{n+1} - u_n\| \le TOL$ regarding Algorithm 1 (`Algo1`) with respect to different values of ϑ_n in terms of the number of iterations and elapsed time, respectively.

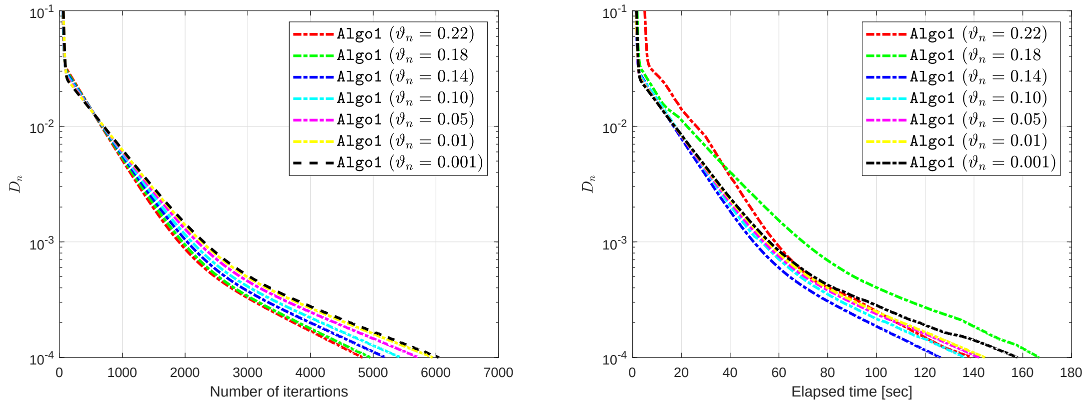

Figure 1. Experiment in Section 6.1.1: Algorithm 1 behaviour for different values of ϑ_n.

Table 3. Experiment in Section 6.1.1: Algorithm 1 performance for varying parameters extrapolation factor ϑ_n.

Algo.name	ϑ_n	λ	ξ^*	Iter.	Time	TOL
`Algo1`	0.22	0.02	$(46.6525, 32.1462, 15.0018, 25.0170, 10.8987, 10.8982)^T$	4824	138.915365	10^{-4}
`Algo1`	0.18	0.02	$(46.6525, 32.1460, 15.0020, 25.0104, 10.9019, 10.9016)^T$	4949	166.620335	10^{-4}
`Algo1`	0.14	0.02	$(46.6525, 32.1460, 15.0020, 25.0035, 10.9050, 10.9053)^T$	5193	127.834772	10^{-4}
`Algo1`	0.10	0.02	$(46.6726, 32.1460, 15.0020, 24.9969, 10.9080, 10.9089)^T$	5432	136.310422	10^{-4}
`Algo1`	0.05	0.02	$(46.6526, 32.1460, 15.0020, 24.9885, 10.9118, 10.9134)^T$	5721	142.108161	10^{-4}
`Algo1`	0.01	0.02	$(46.6526, 32.1460, 15.0020, 24.9818, 10.9149, 10.9170)^T$	5945	144.356535	10^{-4}
`Algo1`	0.001	0.02	$(46.6726, 32.1460, 15.0021, 24.9787, 10.9163, 10.9187)^T$	6043	157.711757	10^{-4}

6.1.2. Algorithm 1 Comparison with Existing Algorithms:

Figure 2 and Table 4 explain the numerical comparison between Algorithm 1 (`EgA`) in [19], Algorithm 1 (`PEgA`) in [21], Algorithm 3.1 (`PSgEgA`) in [35,36] and Algorithm 1(`Algo1`).

Algorithm 1 (`EgA`) in [19]: Choose $u_0 \in \mathbb{E}$ and $0 < \lambda < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$.

$$\begin{cases} v_n = \arg\min\{\lambda f(u_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in K\}, \\ u_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in K\}. \end{cases} \tag{76}$$

Algorithm 1 (`PEgA`) in [21]: Choose $u_0, v_0 \in \mathbb{E}$ and $0 < \lambda < \min \frac{1}{2L_2 + 4L_1}$.

$$\begin{cases} u_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in K\}, \\ v_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_{n+1} - y\|^2 : y \in K\}. \end{cases} \tag{77}$$

Algorithm 3.1 (`PSgEgA`) in [35,36]: Choose $u_0, v_0 \in \mathbb{E}$ and $0 < \lambda < \min \frac{1}{2L_2 + 4L_1}$.

(i)

$$\begin{cases} u_1 = \arg\min\{\lambda f(v_0, y) + \frac{1}{2}\|u_0 - y\|^2 : y \in K\}, \\ v_1 = \arg\min\{\lambda f(v_0, y) + \frac{1}{2}\|u_1 - y\|^2 : y \in K\}. \end{cases}$$

(ii) Given $u_{n-1}, u_n, v_{n-1}, v_n$ for $n \geq 1$ and construct a half space as

$$H_n = \{z \in \mathbb{E} : \langle u_n - \lambda\omega_{n-1} - v_n, z - v_n\rangle \leq 0\}, \text{ where } \omega_{n-1} \in \partial f(v_{n-1}, v_n).$$

(iii)

$$\begin{cases} u_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_n - y\|^2 : y \in H_n\}, \\ v_{n+1} = \arg\min\{\lambda f(v_n, y) + \frac{1}{2}\|u_{n+1} - y\|^2 : y \in K\}. \end{cases} \tag{78}$$

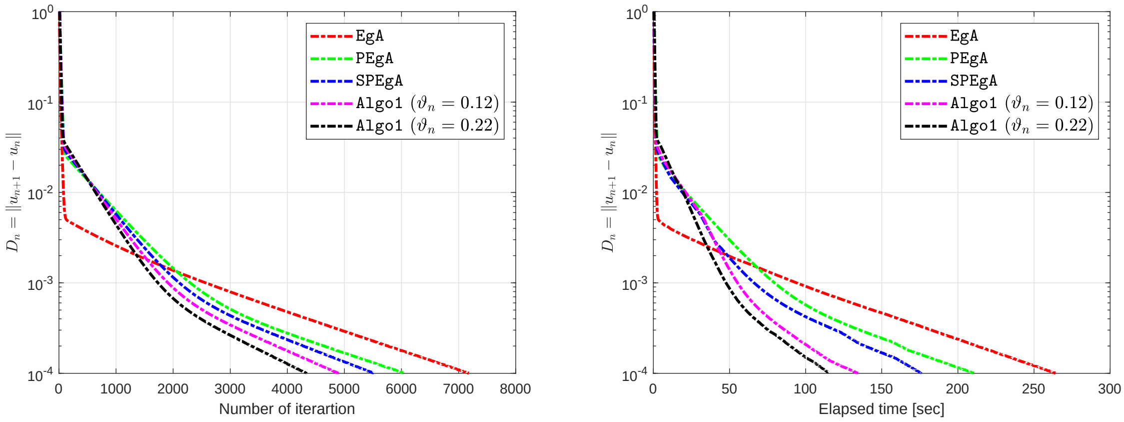

Figure 2. Comparison of Algorithm 1 with Algorithm 1 in [19], Algorithm 1 in [21], and Algorithm 3.1 in [35,36].

Table 4. Experiment in Section 6.1.2: Algorithm 1 comparison with existing algorithms using two different values of ϑ_n.

Algo.name	ϑ_n	λ	ξ^*	Iter.	Time	TOL
EgA	–	0.02	$(46.6526, 32.1469, 15.0012, 24.9783, 10.9154, 10.9200)^T$	7180	264.156236	10^{-4}
PEgA	–	0.02	$(46.6526, 32.1460, 15.0021, 24.9784, 10.8164, 10.9188)^T$	6055	210.681669	10^{-4}
PSgEgA	–	0.02	$(46.6525, 32.1463, 15.0017, 25.0004, 10.9058, 10.9076)^T$	5515	175.840493	10^{-4}
Algo1	0.12	0.02	$(46.6725, 32.1463, 15.0017, 25.0181, 10.8976, 10.8982)^T$	4894	134.245610	10^{-4}
Algo1	0.20	0.02	$(46.6725, 32.1463, 15.0017, 25.0326, 10.8910, 10.8904)^T$	4333	115.599023	10^{-4}

6.1.3. Algorithm 2 Behaviour by Using Different Step-Size Sequences λ_n

Figure 3 and Table 5 describe the numerical results for error term $D_n = \|u_{n+1} - u_n\| \leq TOL$ for Algorithm 2 (`Algo2`).

Table 5. Experiment in Section 6.1.3: Algorithm 2 numerical values by using different step-size sequences λ_n.

Algo.name	ϑ_n	λ	ξ^*	Iter.	Time	TOL
Algo2	0.12	$\frac{1}{n+1}$	$(46.6526, 32.1467, 15.0011, 25.1260, 10.8442, 10.8442)^T$	1254	61.898186	10^{-4}
Algo2	0.12	$\frac{1}{\log(n+1)}$	$(46.6523, 32.1467, 15.0011, 25.1409, 10.8368, 10.8368)^T$	442	29.006584	10^{-4}
Algo2	0.12	$\frac{1}{(n+1)(\log(n+3))}$	$(46.6524, 32.1467, 15.0011, 25.1011, 10.8566, 10.8566)^T$	2311	70.849546	10^{-4}
Algo2	0.12	$\frac{\log(n+3)}{n+1}$	$(46.6523, 32.1467, 15.0011, 25.1371, 10.8387, 10.8387)^T$	662	44.766232	10^{-4}
Algo2	0.12	$\frac{1}{\log(\log(n+20))}$	$(46.6525, 32.1467, 15.0011, 25.1464, 10.8341, 10.8341)^T$	434	31.504484	10^{-4}

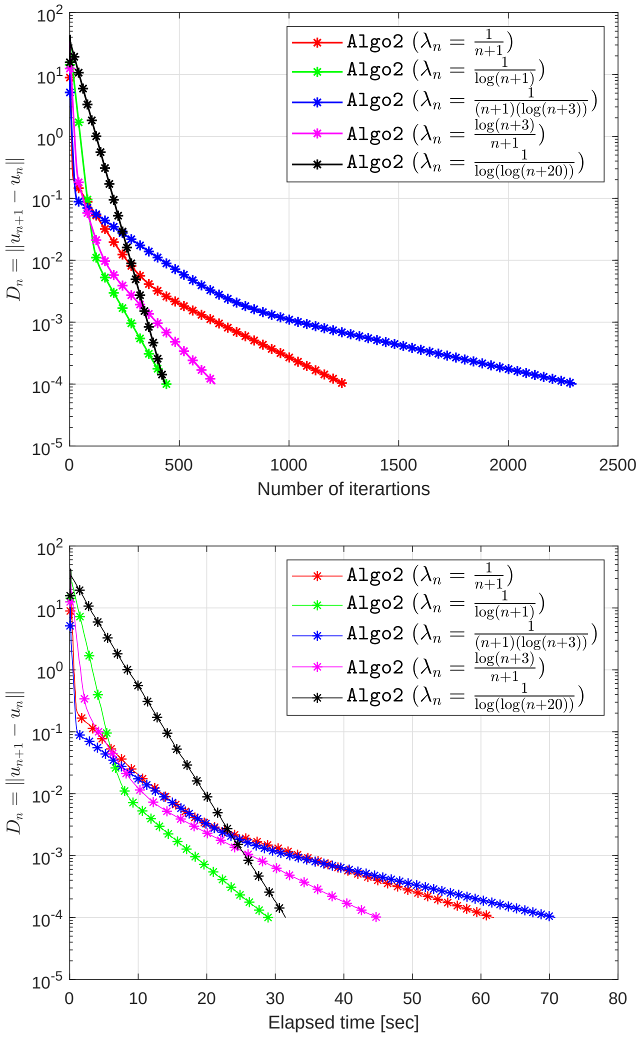

Figure 3. Algorithm 2 behaviour with respect to different step-size sequences λ_n.

6.2. Example 2

Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(u,v) = \tan^{-1}(u)(v-u), \ \forall u, v \in \mathbb{R},$$

where $K = [0,1]$. We can easily see that $f(u,v)$ satisfy all of the conditions (A1)–(A4) with Lipschitz-type constants are $L_1 = L_2 = \frac{1}{2}$ (for more details, see [36]).

6.2.1. Algorithm 1 Performance for Different Values of Extrapolation Factor ϑ_n:

Figure 4 and Table 6 show the numerical results regarding the error term $D_n = \|u_n\|$ of Algorithm 1 using different values of ϑ_n in term of the no.of iterations. For these results, we use values $u_{-1} = \frac{1}{2}$, $u_0 = 1$, $v_0 = 1$ and y-axes depict D_n value, whereas x-axes are depicted as the number of iterations. The input and output values of the parameters are shown in Table 6, which are useful for choosing the best extrapolation factor value.

Table 6. Experiment in Section 6.2.1: Algorithm 1 performance for varying parameters extrapolation factor ϑ_n.

ϑ_n	λ	ξ^*	Iter.	Time	TOL
0.20	0.050	6.5877×10^{-9}	70	0.008866	10^{-8}
0.15	0.050	6.6948×10^{-9}	75	0.010382	10^{-8}
0.10	0.050	6.4466×10^{-9}	80	0.008518	10^{-8}
0.05	0.050	7.5191×10^{-9}	84	0.008378	10^{-8}
0.01	0.050	6.9392×10^{-9}	88	0.008989	10^{-8}

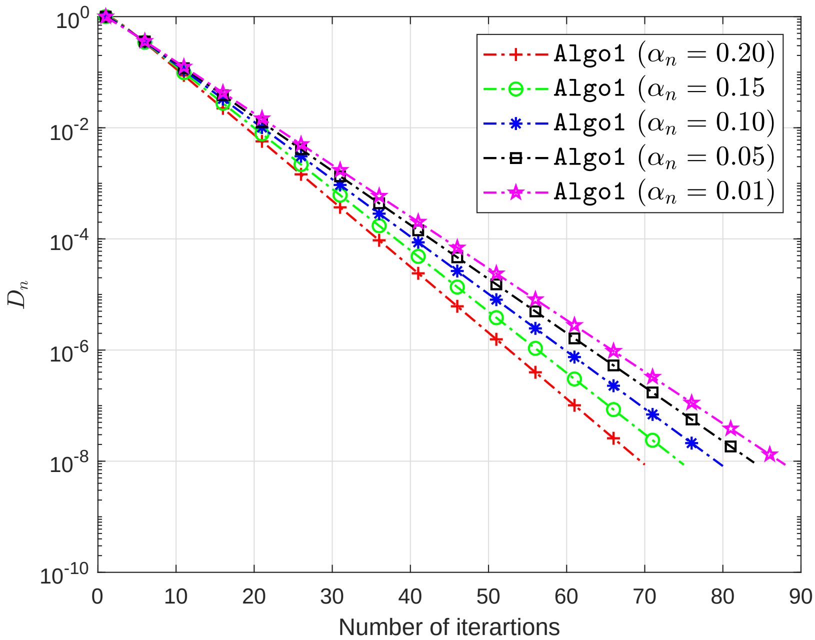

Figure 4. Experiment in Section 6.2.1: Algorithm 1 behaviour regarding different values of ϑ_n.

6.2.2. Algorithm 1 Comparison with Existing Algorithm

Figure 5 and Table 7 illustrate the comparison of our proposed Algorithm 1 (`Algo1`) with the existing Algorithm 3.1 (`PSgEgA`) that appears in the paper of Liu [36]. For these results, the stopping criterion is ($D_n = \|u_n\|$) and y-axes depict D_n value, whereas the x-axes are depicted as the number of iterations. The input and output values for the parameters are written in Table 7.

Table 7. Experiment in Section 6.2.2: Algorithm 1 comparison with Algorithm 3.1 in [35,36].

Algorithm	u_{-1}	u_0	v_0	ϑ_n	λ	ξ^*	Iter.	Time	TOL
`PSgEgA`	—	1	1	—	0.1	7.9278×10^{-11}	110	0.001014	10^{-10}
`Algo1`	0.5	1	1	0.16	0.1	6.5112×10^{-11}	92	0.006082	10^{-10}
`PSgEgA`	—	0.5	0.5	—	0.1	6.9204×10^{-11}	107	0.006580	10^{-10}
`Algo1`	1	0.5	0.5	0.16	0.1	7.1870×10^{-11}	87	0.006919	10^{-10}
`PSgEgA`	—	0.2	0.2	—	0.1	7.7873×10^{-11}	102	0.007282	10^{-10}
`Algo1`	1	0.2	0.2	0.16	0.1	7.1827×10^{-11}	66	0.000688	10^{-10}

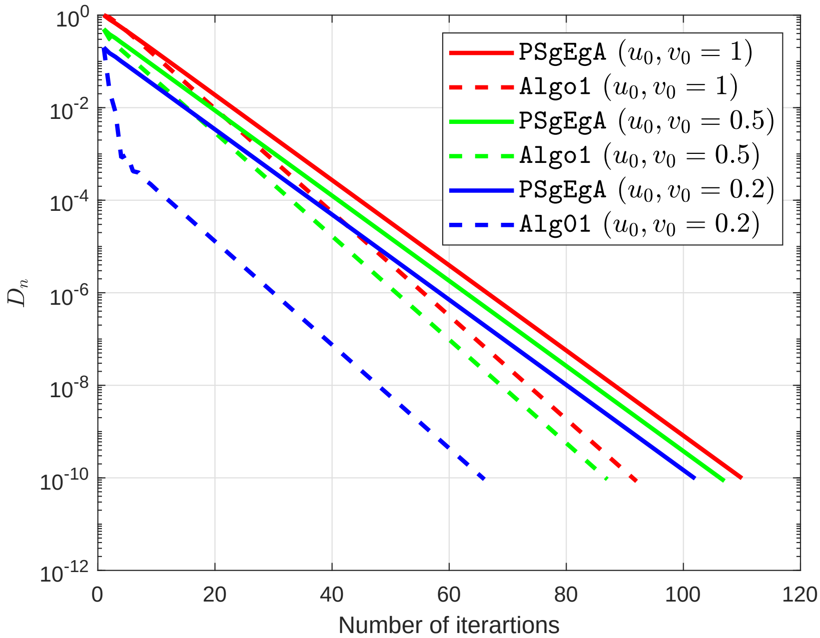

Figure 5. Experiment in Section 6.2.2: Comparison of Algorithm 1 with Algorithm 3.1 in [35,36].

6.3. Nash–Cournot Oligopolistic Equilibrium Model

Consider a Nash–Cournot oligopolistic equilibrium model [19] based on n companies that manufacture the same commodity. Each company produces u_i amount of commodity and u denotes a vector whose entries u_i. The price function for each company i is defined by $P_i(S) = \phi_i - \psi_i S$, where $S = \sum_{i=1}^{m} u_i$ and $\phi_i > 0$, $\psi_i > 0$. Now, consider a profit function for each company i are $F_i(u) = P_i(S)u_i - t_i(u_i)$, where $t_i(u_i)$ is the value tax and fee for producing u_i. Let $K_i = [u_i^{\min}, u_i^{\max}]$ is the set of action of each company i and accumulated actions for whole model taken the form as $K := K_1 \times K_2 \times \cdots \times K_n$. In addition, each company wants to get peak revenue on the assertion that the output of the other companies is an input parameter. The strategy being used to deal with this sort of model mainly focuses on the well-known Nash equilibrium idea. A point $u^* \in K = K_1 \times K_2 \times \cdots \times K_n$ is equilibrium point of the model if

$$F_i(u^*) \geq F_i(u^*[u_i]), \ \forall u_i \in K_i, \ \forall i = 1, 2, \cdots, n,$$

with vector $u^*[u_i]$ denote a vector achievement from u^* by considering u_i^* with u_i. Let $f(u,v) := \varphi(u,v) - \varphi(u,u)$ with $\varphi(u,v) := -\sum_{i=1}^{n} F_i(u[v_i])$ and the problem of determine the Nash equilibrium point is

$$\text{Find } u^* \in K : f(u^*, v) \geq 0, \ \forall v \in K.$$

Next, the bifunction f is written as

$$f(u,v) = \langle Pu + Qv + q, v - u \rangle,$$

where $q \in \mathbb{R}^m$ and the matrices P, Q are

$$Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \ P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 3 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

with $q = (1, 2, -1, 2, -1)^T$ and$K = \{u \in \mathbb{R}^5 : -2 \leq u_i \leq 5\}$. During this example, we use the values of the parameters $u_{-1} = (1, 2, 1, 2, 0)^T$, $u_0 = (1, 3, 1, 1, 2)^T$ and $v_0 = (1, 2, 1, 1, 2)^T$.

6.3.1. Algorithm 2. Behaviour for Different Step-Size Sequences λ_n:

The class of step-size sequences $\{\lambda_n\}$ used in the experiments are:

(I) $\lambda_n = \frac{1}{(n+2)^q}$, $q \in \{1.0; 0.8; 0.5; 0.3; 0.1\}$;
(II) $\lambda_n = \frac{1}{(\log(n+3))^q}$, $q \in \{7; 5; 3; 2; 0.5\}$.

Figures 6 and 7 describe the numerical results for Algorithm 2 (`Algo2`) by using the above define classes of step-size sequences.

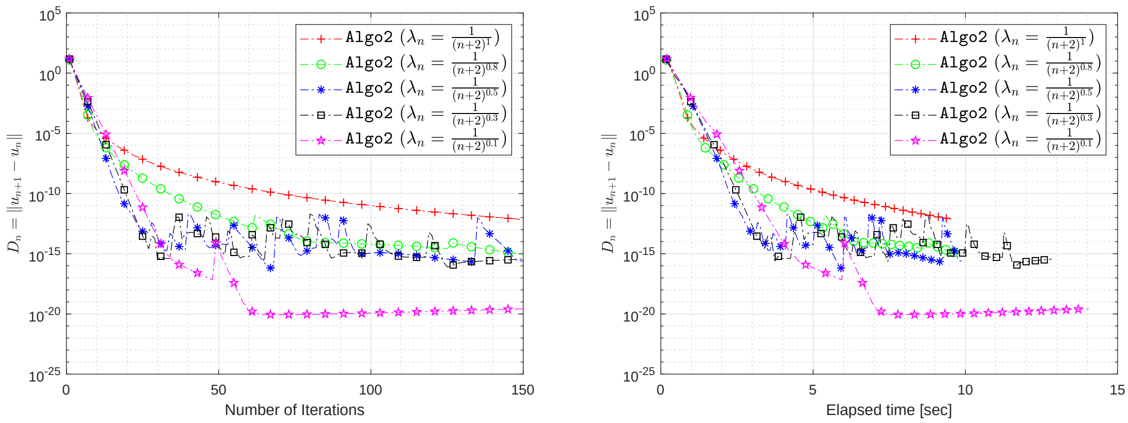

Figure 6. Experiment in Section 6.3.1: Algorithm 2 behaviour with respect to step-size sequences $\lambda_n = \frac{1}{(n+2)^q}$.

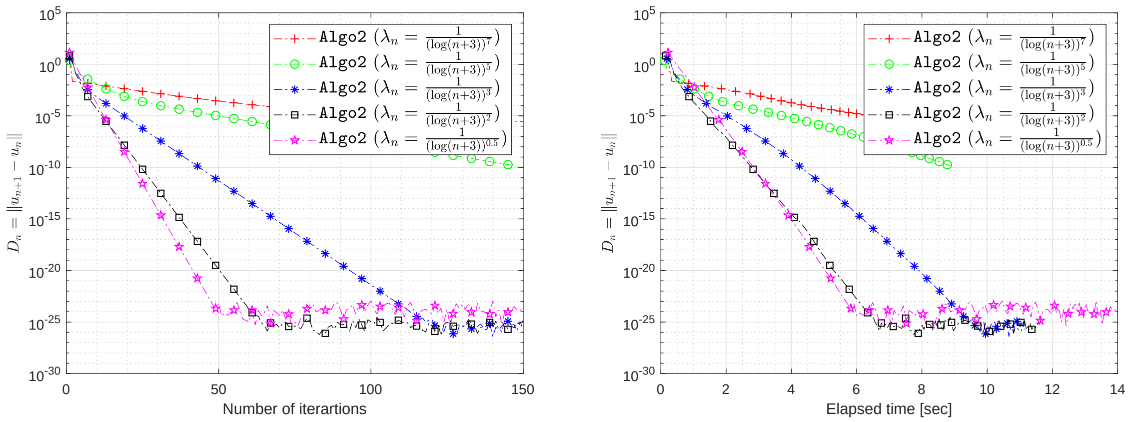

Figure 7. Experiment in Section 6.3.1: Algorithm 2 behaviour with respect to step-size sequences $\lambda_n = \frac{1}{(\log(n+3))^q}$.

6.3.2. Algorithm 2. Comparison with Existing Algorithms

Figure 8 describes the numerical results of Algorithm 2 (`Algo2`) using the stepsize sequences $\lambda_n = \frac{1}{n+1}$.

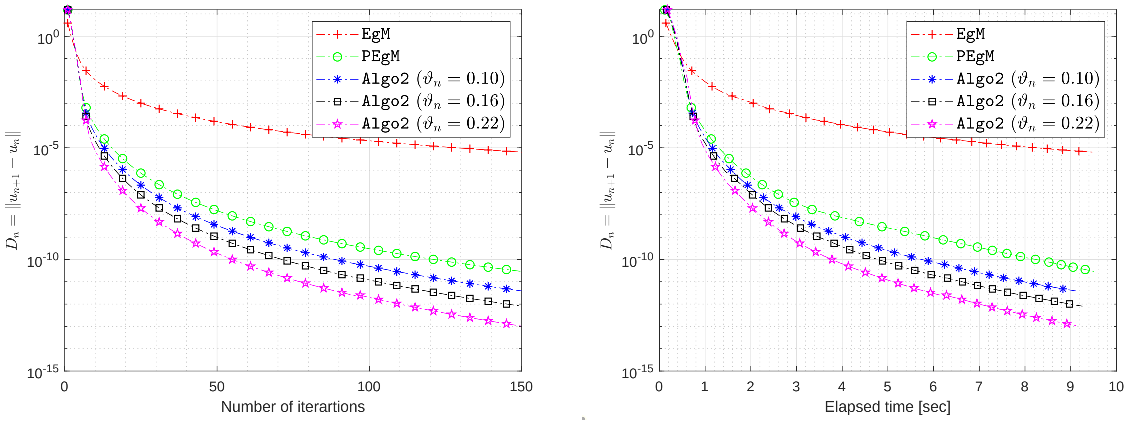

Figure 8. Experiment in Section 6.3.2: Comparison of Algorithm 2 with Algorithm 1 (EgM) in [23] and Algorithm 3.1 (PEgM) in [45].

Discussion About Numerical Experiments: We have the following observations regarding the above-mentioned experiments:

(1) Figures 1 and 4 and Tables 3 and 6 reported results for Algorithm 1 while using different values for ϑ_n. From these results, we can see that the value of θ_n nearer the upper bound value $\sqrt{5} - 2$ is more appropriate and enhances the effectiveness of the suggested algorithms.

(2) It can also be acknowledged that the efficiency of the algorithm depends on the complexity of the problem and tolerance of the error term. More time and a significant number of iterations are required in the case of large-scale problems. In this situation, we can see that the certain value of the step-size enhances the performance of the algorithm and boosts the convergence rate.

(3) From Figure 5 and Table 7, it can also be noted that the choice of the initial points and the complexity of the bifunction affect the performance of algorithms in terms of the number of iterations and time of execution in seconds.

(4) We have the following observation from Figure 3 and Figures 6–8 with Table 5.

   (i) No previous information of Lipschitz-constant L_1, L_2 is required for running algorithms on Matlab.

   (ii) In fact, the convergence rate of algorithms depends entirely on the convergence rate of step-size sequences λ_n.

   (iii) The convergence rate of the iterative sequence often depends on the complexity of the problem as well as on the size of the problem.

   (iv) Due to the variable step-size sequence, a specific step-size value that is not appropriate for the current iteration of the method often causes inconsistency and a hump in the behavior of the iterative sequence.

7. Conclusions

Two different approaches are proposed in this paper to deal with two families of equilibrium problems. The first algorithm is an inertial two-step step proximal-like method that generates a weak converging iterative sequence and it can solve pseudomonoton equilibrium problems. In addition, we use the diminishing and non-summable step-size sequence for the second algorithm to achieve the strong convergence. The key advantage of the second algorithm is that iterative sequences have been developed with no prior knowledge of a strong pseudomonotonicity and Lipschitz-type constants of a bifunction. Numerical findings were mentioned to show the numerical efficiency of algorithms as compared to other algorithms. Such numerical studies imply that the inertial effects normally enhance the effectiveness of the iterative sequence in this context.

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References

1. Blum, E. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **1994**, *63*, 123–145.
2. Fan, K. *A Minimax Inequality and Applications, INEQUALITIES III*; Shisha, O., Ed.; Academic Press: New York, NY, USA, 1972.
3. Biegler, L.T. *Nonlinear Programming: Concepts, Algorithms, and Applications to Chemical Processes*; SIAM-Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 2010; Volume 10.
4. Dafermos, S. Traffic Equilibrium and Variational Inequalities. *Transp. Sci.* **1980**, *14*, 42–54. [CrossRef]
5. Ferris, M.C.; Pang, J.S. Engineering and Economic Applications of Complementarity Problems. *SIAM Rev.* **1997**, *39*, 669–713. [CrossRef]
6. Nagurney, A. *Network Economics: A Variational Inequality Approach*; Springer: Dordrecht, The Netherlands, 1993. [CrossRef]
7. Patriksson, M. *The Traffic Assignment Problem: Models and Methods*; Courier Dover Publications: Mineola, NY, USA, 2015.
8. Cournot, A.A. *Recherches sur les principes mathématiques de la théorie des richesses*; Wentworth Press Hachette: Paris, France, 1838.
9. Arrow, K.J.; Debreu, G. Existence of an Equilibrium for a Competitive Economy. *Econometrica* **1954**, *22*, 265. [CrossRef]
10. Nash, J.F. 5. Equilibrium Points in n-Person Games. In *The Essential John Nash*; Nasar, S., Ed.; Princeton University Press: Princeton, NJ, USA, 2002; pp. 49–50. [CrossRef]
11. Nash, J. Non-Cooperative Games. *Ann. Math.* **1951**, *54*, 286. [CrossRef]
12. Muu, L.D.; Oettli, W. Convergence of an adaptive penalty scheme for finding constrained equilibria. *Nonlinear Anal. Theory, Methods Appl.* **1992**, *18*, 1159–1166. [CrossRef]
13. Moudafi, A. Proximal point algorithm extended to equilibrium problems. *J. Nat. Geom.* **1999**, *15*, 91–100.
14. Mastroeni, G. On auxiliary principle for equilibrium problems. In *Equilibrium Problems and Variational Models*; Springer: Berlin, Germany, 2003; pp. 289–298.
15. Martinet, B. Brève communication. Régularisation d'inéquations variationnelles par approximations successives. *Rev. Française D'informatique Et De Rech. Opérationnelle. Série Rouge* **1970**, *4*, 154–158. [CrossRef]
16. Rockafellar, R.T. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **1976**, *14*, 877–898. [CrossRef]
17. Konnov, I. Application of the Proximal Point Method to Nonmonotone Equilibrium Problems. *J. Optim. Theory Appl.* **2003**, *119*, 317–333. [CrossRef]
18. Flåm, S.D.; Antipin, A.S. Equilibrium programming using proximal-like algorithms. *Math. Program.* **1996**, *78*, 29–41. [CrossRef]
19. Quoc Tran, D.; Le Dung, M.N.V.H. Extragradient algorithms extended to equilibrium problems. *Optimization* **2008**, *57*, 749–776. [CrossRef]
20. Quoc, T.D.; Anh, P.N.; Muu, L.D. Dual extragradient algorithms extended to equilibrium problems. *J. Glob. Optim.* **2011**, *52*, 139–159. [CrossRef]

21. Lyashko, S.I.; Semenov, V.V. A New Two-Step Proximal Algorithm of Solving the Problem of Equilibrium Programming. In *Optimization and Its Applications in Control and Data Sciences*; Springer International Publishing: New York, NY, USA, 2016; pp. 315–325. [CrossRef]
22. Anh, P.N.; Hai, T.N.; Tuan, P.M. On ergodic algorithms for equilibrium problems. *J. Glob. Optim.* **2015**, *64*, 179–195. [CrossRef]
23. Hieu, D.V. New extragradient method for a class of equilibrium problems in Hilbert spaces. *Appl. Anal.* **2017**, *97*, 811–824. [CrossRef]
24. ur Rehman, H.; Kumam, P.; Cho, Y.J.; Yordsorn, P. Weak convergence of explicit extragradient algorithms for solving equilibirum problems. *J. Inequalities Appl.* **2019**, *2019*. [CrossRef]
25. Anh, P.N.; An, L.T.H. The subgradient extragradient method extended to equilibrium problems. *Optimization* **2012**, *64*, 225–248. [CrossRef]
26. Ur Rehman, H.; Kumam, P.; Je Cho, Y.; Suleiman, Y.I.; Kumam, W. Modified Popov's explicit iterative algorithms for solving pseudomonotone equilibrium problems. *Optimization Methods and Software* **2020**, 1–32. [CrossRef]
27. Vinh, N.T.; Muu, L.D. Inertial Extragradient Algorithms for Solving Equilibrium Problems. *Acta Math. Vietnam.* **2019**, *44*, 639–663. [CrossRef]
28. ur Rehman, H.; Kumam, P.; Kumam, W.; Shutaywi, M.; Jirakitpuwapat, W. The Inertial Sub-Gradient Extra-Gradient Method for a Class of Pseudo-Monotone Equilibrium Problems. *Symmetry* **2020**, *12*, 463. [CrossRef]
29. Hieu, D.V. An inertial-like proximal algorithm for equilibrium problems. *Math. Methods Oper. Res.* **2018**, *88*, 399–415. [CrossRef]
30. ur Rehman, H.; Kumam, P.; Abubakar, A.B.; Cho, Y.J. The extragradient algorithm with inertial effects extended to equilibrium problems. *Comput. Appl. Math.* **2020**, *39*. [CrossRef]
31. Hieu, D.V.; Cho, Y.J.; bin Xiao, Y. Modified extragradient algorithms for solving equilibrium problems. *Optimization* **2018**, *67*, 2003–2029. [CrossRef]
32. ur Rehman, H.; Kumam, P.; Argyros, I.K.; Alreshidi, N.A.; Kumam, W.; Jirakitpuwapat, W. A Self-Adaptive Extra-Gradient Methods for a Family of Pseudomonotone Equilibrium Programming with Application in Different Classes of Variational Inequality Problems. *Symmetry* **2020**, *12*, 523. [CrossRef]
33. ur Rehman, H.; Kumam, P.; Argyros, I.K.; Deebani, W.; Kumam, W. Inertial Extra-Gradient Method for Solving a Family of Strongly Pseudomonotone Equilibrium Problems in Real Hilbert Spaces with Application in Variational Inequality Problem. *Symmetry* **2020**, *12*, 503. [CrossRef]
34. Muu, L.D.; Quoc, T.D. Regularization Algorithms for Solving Monotone Ky Fan Inequalities with Application to a Nash-Cournot Equilibrium Model. *J. Optim. Theory Appl.* **2009**, *142*, 185–204. [CrossRef]
35. Kassay, G.; Hai, T.N.; Vinh, N.T. Coupling popov's algorithm with subgradient extragradient method for solving equilibrium problems. *J. Nonlinear Convex Anal.* **2018**, *19*, 959–986.
36. Liu, Y.; Kong, H. The new extragradient method extended to equilibrium problems. *Rev. De La Real Acad. De Cienc. Exactas, Físicas Y Naturales. Ser. A. Matemáticas* **2019**, *113*, 2113–2126. [CrossRef]
37. Goebel, K.; Reich, S. Uniform convexity. In *Hyperbolic Geometry, and Nonexpansive*; Marcel Dekker, Inc.: New York, NY, USA, 1984.
38. Bianchi, M.; Schaible, S. Generalized monotone bifunctions and equilibrium problems. *J. Optim. Theory Appl.* **1996**, *90*, 31–43. [CrossRef]
39. Tiel, J.V. *Convex Analysis: An Introductory Text*, 1st ed.; Wiley: New York, NY, USA, 1984.
40. Ofoedu, E. Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space. *J. Math. Anal. Appl.* **2006**, *321*, 722–728. [CrossRef]
41. Heinz, H.; Bauschke, P.L.C. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2nd ed.; CMS Books in Mathematics; Springer International Publishing: New York, NY, USA, 2017.
42. Attouch, F.A.H. An Inertial Proximal Method for Maximal Monotone Operators via Discretization of a Nonlinear Oscillator with Damping. *Set Valued Var. Anal.* **2001**, *9*, 3–11. [CrossRef]
43. Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–598. [CrossRef]

44. Maiorano, A.; Song, Y.; Trovato, M. Dynamics of non-collusive oligopolistic electricity markets. In Proceedings of the 2000 IEEE Power Engineering Society Winter Meeting, Conference Proceedings (Cat. No.00CH37077), Singapore, 23–27 January 2000. [CrossRef]
45. Hieu, D.V. Convergence analysis of a new algorithm for strongly pseudomontone equilibrium problems. *Numer. Algorithms* **2017**, *77*, 983–1001. [CrossRef]