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Application of Doubly Connected Dominating Sets to Safe Rectangular Smart Grids

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Abstract: Smart grids, together with the Internet of Things, are considered to be the future of the electric energy world. This is possible through a two-way communication between nodes of the grids and computer processing. It is necessary that the communication is easy and safe, and the distance between a point of demand and supply is short, to reduce the electricity loss. All these requirements should be met at the lowest possible cost. In this paper, we study a two-dimensional rectangular grid graph which is considered to be a model of a smart grid; nodes of the graph represent points and devices of the smart grid, while links represent possible ways of communication and energy transfer. We consider the problem of choosing the lowest possible number of locations (nodes, points) of the grid which could serve as energy sources (or a source of different resources) to other nodes in such a way that we ensure reduction in electricity loss and provide safe communication and resistance to failures and increases in energy demand.Therefore, we study minimum doubly connected dominating sets in grid graphs. We show that the proposed solutions are the best possible in terms of the number of source points for the case of narrow grid graphs and we give upper and lower bounds for the case of wide grid graphs.

Keywords: smart grids; security; communication; doubly connected domination number; rectangular grid graphs

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Citation: Cyman, J.; Raczek, J. Application of Doubly Connected Dominating Sets to Safe Rectangular Smart Grids. *Energies* **2022**, *15*, 2969. https://doi.org/10.3390/en15092969

Academic Editors: Mostafa Fouda, Mohamed Ibrahem and Ahmed F. Zobaa

Received: 17 February 2022
Accepted: 17 April 2022
Published: 19 April 2022

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1. Introduction

Smart grids are similar to conventional power grids, but they introduce two-way communication so that electricity and information can be exchanged between electrical utilities and a customer. Controls, computers, new technologies and tools can work together to make a smart grid not only safe, but also more efficient, reliable and more environmentally friendly. For example, they can integrate renewable resources of energy, such as wind and solar sources, to provide electricity for plug-in electric cars. In this way, smart grids, with the help of smart meters, can easily manage our electricity needs at a lower cost.

Since the amount of electricity provided by renewable resources of energy (especially wind and solar sources) often relies on weather conditions, a smart grid needs to analyze data and optimize the use of the energy to keep up with constantly changing energy demands. By deferring the electricity usage to off-peak hours, we can reduce operating costs. Usually, the power we are using right now was generated just a second ago, many kilometers away. At each instance, the amount of energy generated must equal the consumption of the entire grid, so a smart grid should manage electricity production and consumption in real-time.

Because of the tasks and requirements for smart grids, the two-way communication between nodes should be immediate, easy and safe, and the distance between a point of demand and supply should be short to reduce the electricity loss and time propagation.

In addition, the smart grid should be resistant to temporary increased demand for electricity (peaks) and malfunctions. All these requirements should be met at the lowest possible cost. These demands inspired the authors to study a two-dimensional rectangular grid graph which is considered to be a model of a smart grid; nodes of the graph represent points and devices of the smart grid, while links represent possible ways of communication and energy transfer. Then we studied the problem of choosing the lowest possible number of locations (nodes, points) of the grid which could serve as special nodes (for example energy sources) to other nodes in such a way that each such ordinary node is at an immediate neighbourhood of a source node and there is a path between any two source nodes, completely contained in links between the source nodes, and there is also a similar path between any two ordinary nodes, completely contained in links between them. The first requirement ensures the reduction in loss of electricity, while the last two provide safe communication and resistance to failures and increases in energy demand. We show that the proposed solutions are the best possible in terms of the number of source points. To achieve this aim, we apply graph theory and study doubly connected domination number in grid graphs. The novelty of the paper includes the possibility of applying the doubly connected domination in smart grids modeled as grid graphs. Other new results concern exact values for the number of nodes in the smallest doubly connected sets for the case of narrow grids, and upper and lower bounds for this number in other cases.

The paper is organized as follows. In Section 2, we present studies related to smart grids and domination in grid graphs. In Section 3, all necessary definitions and notions are introduced. The main results of the paper are presented in Section 4, which is divided into two subsections: first, we study narrow grid graphs, and afterwards, wider grid graphs. The last Section 5 is devoted to discussion of the obtained results, applications and ideas for future research.

2. Related Work

Graph theory has provided a solid mathematical basis for developing new tools for complex networks analysis. Many of these tools have enormous potential to be implemented in power systems as well [1]. Graph coloring, and its variants, find application in channel assignment, for example, see [2,3]. In [4], the authors present a structured approach to creating integrated infrastructure models for smart grids using graph theory. In [1], several chapters are devoted to the use of graph theory, in particular graph symmetry and centrality, for the complex networks of charging stations.

The notion of domination in graphs has also been widely studied, since domination in graphs helps to solve problems related to localization of different resources and devices. For example, in [5], the authors studied minimum distance dominating sets in complex networks. In [6,7], dominating sets were used to make such safe networks. Power domination is a type of domination problem defined and studied with immediate application to power lines. It helps to solve the problem of monitoring an electric power system by placing as few measurement devices as possible (see [8–11]). Minimum connected dominating sets apply in ad hoc networks to find a virtual backbone, which helps achieve efficient broadcasting in these networks [12]. This type of domination is studied in grid graphs in [13].

In this paper, we study doubly minimum connected dominating sets in grid graphs with application to smart grids. This type of domination set was first defined in 2006 [14], and further studied by [15–19]. In these studies, the authors investigated the computational complexity of the graph parameter, basic properties and bounds on the doubly connected domination number, the influence of removing a link on this parameter and studied this number in coronas of graphs and lexicographic products. Nevertheless, this type of domination parameter has still not been thoroughly studied and there is still a great deal to discover. Until now, nobody has studied minimum doubly connected dominating sets in grid graphs, nor their application to smart grids.

A rectangular grid graph is a graph whose drawing, embedded in two-dimensional Euclidean space, forms regular rectangles. Grid graphs can also be described as Cartesian products of two paths (see Section 3 for detailed definitions). This regularity of the structure of the grid graph implies that the structures of many variants of minimum dominating sets also create repeated patterns, which are symmetric in many cases [20,21]. Since grid graphs can serve as a model for a sensor's localization in a network, geographical coordinates on a map or coordinates in a two-dimensional Cartesian plane, these graphs have potential practical applications.

Our paper is devoted to patterns that occur while constructing minimum doubly connected dominating sets. Since both the minimum dominating set and its complement have to be connected, the results may look like a labyrinth or a bark beetle path in a board. The basic pattern that is constructed for smaller grids is repeated for bigger ones, bringing the idea of symmetrical arrangements.

3. Preliminaries

A simple graph (or graph for short) $G = (V, E)$ consists of a non-empty finite set V of elements called nodes and a set E of two-element subsets of elements of V, which are called links. If $\{u, v\}$ is a link, then we will write uv for short. This definition of a graph can be regarded as a mathematical model of any network or grid in which links reflect the two-way communication channels between nodes.

A Cartesian product of graphs G_1 and G_2 is the graph $G = G_1 \square G_2$, where the set of nodes is the Cartesian product of $V(G_1)$ and $V(G_2)$, that is $V(G) = V(G_1) \times V(G_2)$ and two nodes (a, b) and (c, d) are adjacent if, and only if, $a = c$ and $bd \in E(G_2)$ or if $b = d$ and $ac \in E(G_1)$.

A path graph (or a path) is a graph $P_n = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\}$. A rectangular grid graph is a Cartesian product of two paths, say P_s and P_l. For this kind of graph, we can give a more straightforward definition. For two positive integers s, l, a grid graph $G[s, l]$ is a graph with node set $V(G[s, l]) = \{(i, j) : i = 1, \ldots, s; j = 1, \ldots, l\}$ and two nodes $(a, b), (c, d) \in V(G[s, l])$ are adjacent if, and only if, $|a - c| + |b - d| = 1$.

In Figure 1, we can see a graph $G[6, 5]$. Nodes are labeled by their coordinates. Note that each inner node is of degree 4, that is, it has 4 neighbours. The nodes on the border of the grid are of degree 3, while four nodes in the corners are of degree 2. By C_m, we denote the nodes in the column $m, m = 1, \ldots, s$ of the grid graph, that is $C_m = \{(m, j) : j = 1, \ldots, l\}$ and by R_n the nodes in the row $n, n = 1, \ldots, l$, that is $R_n = \{(i, n) : i = 1, \ldots, s\}$.

Let $D \subseteq V(G)$ be a subset of the set of nodes of a graph G. The subgraph induced by D is a graph $G[D] = (D, E_D)$, where E_D is the set of these links whose both ends belong to D. A subgraph $G[D]$ is connected if any two nodes of D can be connected by a path completely contained in $G[D]$.

In Figure 2, the set $D_1 = \{(1, 2), (2, 2), (1, 3), (2, 3)\}$ induces a connected subgraph, while the set $D_2 = \{(4, 4), (4, 5), (6, 4)\}$ induces a disconnected subgraph. There is no path between nodes $(4, 4)$ and $(6, 4)$ that is contained in the subgraph induced by D_2.

A set $D \subseteq V(G)$ is a dominating set of a graph G if every node in $V(G) - D$ is adjacent to at least one node in D. A set $D \subseteq V(G)$ is a doubly connected dominating set of G (MDCDS) if it is dominating and the induced subgraph $G[D]$ and $G[V - D]$ are connected. The doubly connected domination number $\gamma_c^c(G)$ is the cardinality of a smallest doubly connected dominating set of G. Although determining the minimum doubly connected domination number is NP-hard for many graph classes (see [14]), in this paper, we give exact values of this number for narrow rectangular grid graphs, namely $G[s, l]$ for $l = 1, \ldots, 5$, and we give upper and lower bounds for $l \geq 6$.

For two nodes x and y, let $d_G(x, y)$ denote the distance between x and y in G, that is the length (number of links) of a shortest path between x and y. If D is a set of nodes of G and x is a node of G, then the distance from x to D, denoted by $d_G(x, D)$, is the shortest distance from x to a node of D. For example, in Figure 2 the distance between nodes $(3, 2)$ and $(5, 3)$ is 3.

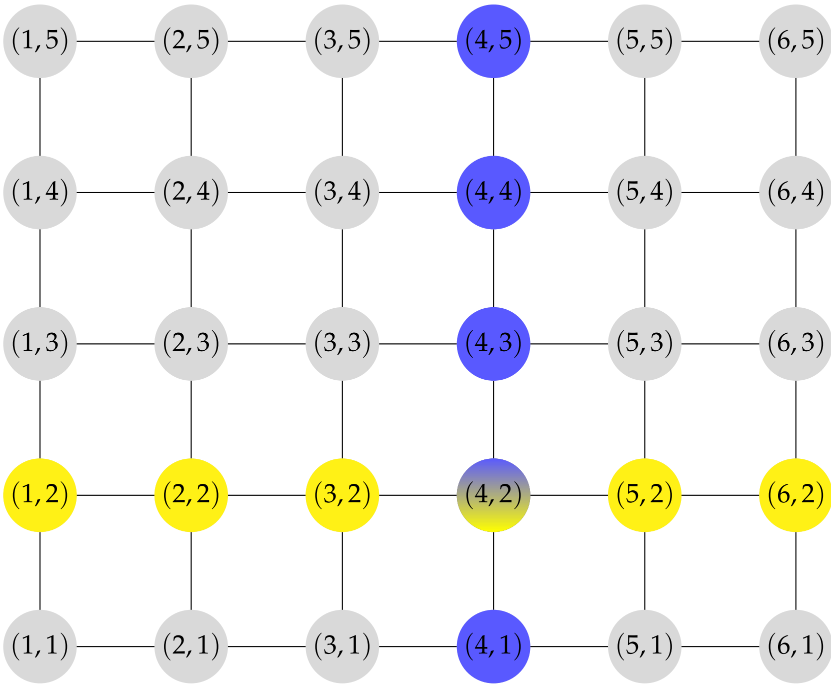

Figure 1. A grid graph $G[6,5]$. Column C_4 is denoted by blue, row R_2 is denoted by yellow.

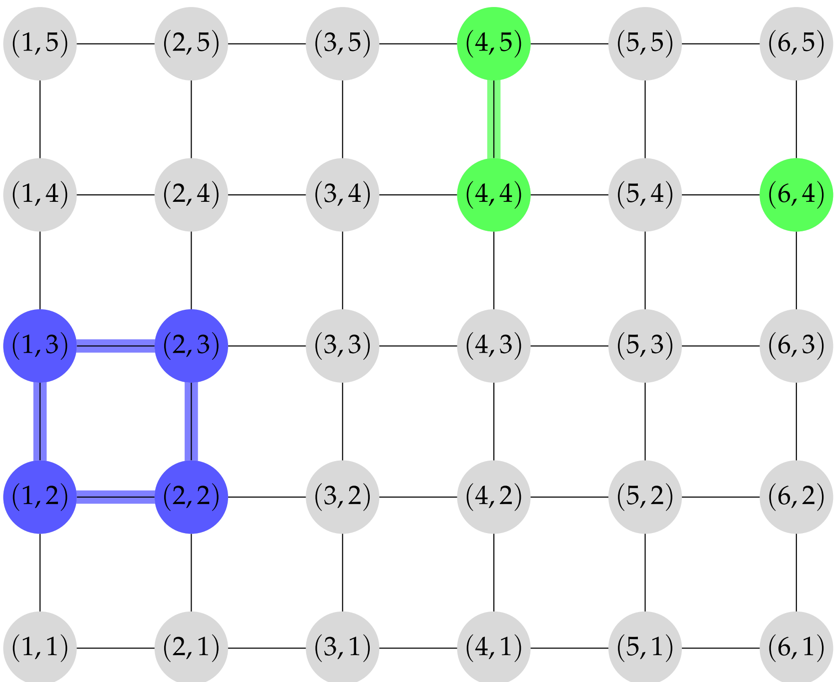

Figure 2. The blue subgraph is connected, while the green one is disconnected.

4. Results

In this section, we study the minimum doubly connected dominating sets in grid graphs. We start by analyzing the cases of narrow grids, then we move on to wider grids. At the end, we generalize our results. Without loss of generality, for each case of a grid graph $G[s,l]$, we assume $s \geq l$.

4.1. Results for Narrow Grids

We begin this section with the following lemma, which says that if two nodes lying on the border of a grid do not belong to a minimum doubly connected dominating set, then they belong to the same column or to the same row.

Lemma 1. *Let D be a minimum doubly connected dominating set in $G[s,l]$. If $u, v \in (C_1 \cup C_s \cup R_1 \cup R_l) - D$, then exactly one possibility is true:*
- $u, v \in C_1 - D$;
- $u, v \in C_s - D$;
- $u, v \in R_1 - D$;
- $u, v \in R_l - D$.

Proof. Let D be a minimum doubly connected dominating set in $G[s,l]$ and let $u, v \in (C_1 \cup C_s \cup R_1 \cup R_l) - D$. We proceed to the proof by contradiction, so suppose the thesis is false. Without loss of generality, let $u \in C_1 - D$ and $v \in (C_s \cup R_1 \cup R_l) - (D \cup C_1)$.

Since $G[V-D]$ is connected, there exists a $(u-v)$-path such that each node of the path is contained in $V-D$. Since $G[D]$ is also connected, the (u,v)-path contains the node $(1,1)$ or $(1,l)$. Without loss of generality, we assume that $(1,1)$ belongs to the $(u-v)$-path. However, this situation is possible only if $v = (1,1)$ and $v \in R_1 - D$, which contradicts the assumption. Therefore $u, v \in R_1 - D$. □

Lemma 1 implies that except for perhaps the corner nodes, three out of four border rows and columns are contained in each MDCDS.

Our next results determine exact values of the doubly connected domination number of grids $G[s,1]$, $G[s,2]$, G[s,3], $G[s,4]$ and $G[s,5]$.

Proposition 1. *Let $s \geq 2$. Then*

$$\gamma_c^c(G[s,1]) = s - 1.$$

Proof. Since $G[s,1]$ is a path, the result follows by [14]. □

Proposition 2. *Let $s \geq 2$. Then*

$$\gamma_c^c(G[s,2]) = s.$$

Proof. Clearly, R_1 is a doubly connected dominating set of $G[s,2]$. Hence, $\gamma_c^c(G[s,2]) \leq s$.

Now let D be a MDCDS of $G[s,2]$. Since $V = R_1 \cup R_2$, Lemma 1 implies that $|V-D| \leq s$. Therefore, $\gamma_c^c(G[s,2]) = s$. □

Figure 3 is an example of a smallest doubly connected dominating set of $G[11,2]$: the set of all black nodes form a MDCDS of this grid graph. This is also true for the set of all white nodes. Note that this is exactly half of the nodes of the grid.

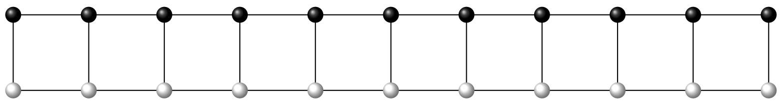

Figure 3. A minimum doubly connected dominating set of $G[11,2]$ formed by the set of all black nodes.

Proposition 3. *Let $s \geq 3$. Then*

$$\gamma_c^c(G[s,3]) = 2s.$$

Proof. Clearly, $R_2 \cup R_3$ is a doubly connected dominating set of $G[s,3]$, so $\gamma_c^c(G[s,3]) \leq 2s$.

Let D be a minimum doubly connected dominating set of $G[s,3]$. Then $|D| \leq 2s$ and thus without loss of generality either $(V-D) \cap C_1 \neq \emptyset$ or $(V-D) \cap R_1 \neq \emptyset$.

In the first case, Lemma 1 implies that $|D| \geq s-1+s-1+1 = 2s-1$. Moreover, one more node is needed to dominate $(1,2)$, so $|D| \geq 2s$.

In the second case, Lemma 1 implies that $|D| \geq s+1+1 = s+2$. If $|(V-D) \cap R_1| = t$, then, since D is doubly connected and dominating, at least $t-2$ more nodes of R_2 belong to D to dominate R_1. Thus, $|D| \geq s+2+(s-t)+(t-2) = 2s$. □

Figure 4 is an example of a grid $G[11,3]$ in which the smallest doubly connected dominating set consists of all black nodes. Note that two thirds of all nodes are needed to form an MDCDS in this grid.

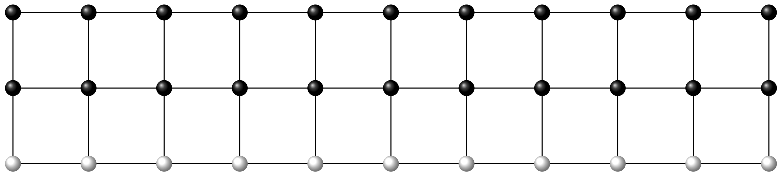

Figure 4. A minimum doubly connected dominating set of $G[11,3]$ formed by the set of all black nodes.

Proposition 4. *Let $s \geq 4$. Then*

$$\gamma_c^c(G[s,4]) = 2s+2.$$

Proof. Clearly, $R_1 \cup R_4 \cup C_1$ is a doubly connected dominating set of $G[s,4]$, so $\gamma_c^c(G[s,4]) \leq 2s+2$.

Let D be a minimum doubly connected dominating set of $G[s,4]$. Then $|D| \leq 2s+2$ and thus, without loss of generality, either $(V-D) \cap C_1 \neq \emptyset$ or $(V-D) \cap R_1 \neq \emptyset$.

In the first case, Lemma 1 implies that $|D| \geq s-1+s-1+2 = 2s$. Moreover, because of $(1,2)$ and $(1,3)$, two more nodes must be added to D to make D connected and dominating, so $|D| \geq 2s+2$.

In the second case, Lemma 1 implies that $|D| \geq s+2+2 = s+4$. If $|(V-D) \cap R_1| = t$, then at least $t-2$ more nodes of R_2 belong to D to dominate R_1. Thus, $|D| \geq s+4+(s-t)+(t-2) = 2s+2$. □

Figure 5 is an example of a smallest doubly connected dominating set of $G[11,4]$.

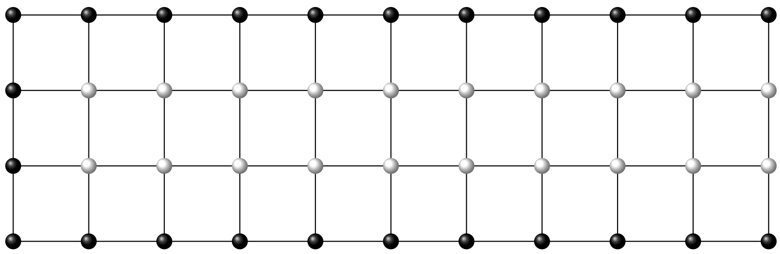

Figure 5. A minimum doubly connected dominating set of $G[11,4]$ formed by the set of all black nodes.

Proposition 5. *Let $s \geq 5$. Then*

$$\gamma_c^c(G[s,5]) = 2s+4.$$

Proof. Clearly, $(R_2 \cup R_5 \cup C_1 \cup C_s) - (\{(2,2),(s,1)\})$ is a doubly connected dominating set of $G[s,5]$, so $\gamma_c^c(G[s,5]) \leq 2s+4$.

Let D be a minimum doubly connected dominating set of $G[s,5]$. Then $|D| \leq 2s+4$ and thus, without loss of generality, either $(V-D) \cap C_1 \neq \emptyset$ or $(V-D) \cap R_1 \neq \emptyset$.

In the first case Lemma 1 implies that $A = (R_1 \cup R_5 \cup C_s) - C_1 \subseteq D$, so $|D| \geq 2s+1$. However, elements of the set $\{(i,3) : i = 1,2,\ldots,s-2\}$ are not dominated by any node of A, so at least four (or more) nodes must be added to D to make D connected and dominating. This implies that $|D| > 2s + 4$, which contradicts $|D| \leq 2s + 4$. Therefore this is not the case.

In the second case, Lemma 1 implies that $|D| \geq s + 3 + 3 = s + 6$. If $|(V - D) \cap R_1| = t$, then at least $t - 2$ more nodes of R_2 belong to D to dominate R_1. Thus, $|D| \geq s + 6 + (s - t) + (t - 2) = 2s + 4$. □

Figure 6 is an example of a smallest doubly connected dominating set of $G[11, 5]$.

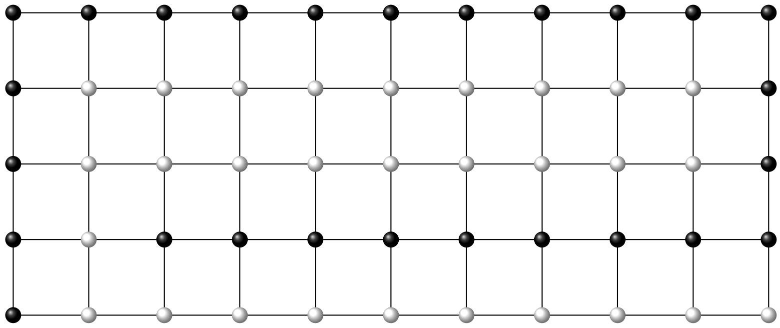

Figure 6. A minimum doubly connected dominating set of $G[11, 5]$ formed by the set of all black nodes.

4.2. Results for Wide Grids

In this part, we focus on grid graphs $G[s, l]$ with $s \geq l \geq 6$. Some general results are valid also for narrow graphs and this is indicated in the assumptions of the theorems. The previous results are used in the inductive proofs as the basic step of the induction. The results are divided and studied depending on the remainder when dividing l by 3. The first results are devoted to upper bounds on the doubly connected domination number, while the last are devoted to lower bounds on this number. It is worth noting that the minimum doubly connected domination number lies in between the upper and the lower bounds.

First, we present the theorems; their proofs can be found below.

Theorem 1. *Let $5 \leq l \leq s$ and let $l = 3k + 2$ for some positive integer k. Then*

$$\gamma_c^c(G[s, l]) \leq \frac{(l-2)s}{3} + l + s - 1.$$

Theorem 2. *Let $3 \leq l \leq s$ and let $l = 3k$ for some positive integer k. Then*

$$\gamma_c^c(G[s, l]) \leq \frac{ls}{3} + l + s - 3.$$

Theorem 3. *Let $s \geq l \geq 3$ and let $l = 3k + 1$ for some positive integer k. Then*

$$\gamma_c^c(G[s, l]) \leq \frac{(l-1)s}{3} + l + s - 2.$$

Theorem 4. *For $G[s, l]$, where $6 \leq l \leq s$,*

$$\gamma_c^c(G[s, l]) \geq \left\lceil \frac{(l-2)s}{3} + s + \frac{2}{3}l - \frac{2}{3} \right\rceil.$$

Proof of Theorem 1. Proposition 5 implies that the result is true for $l = 5$.

We continue the proof by induction on the positive integer $l \equiv 2 \mod 3$. Let $l \geq 8$ and assume that for $l' = l - 3$, $\gamma_c^c(G[s, l']) \leq \frac{(l'-2)s}{3} + l' + s - 1$. Observe that

$$G[s, l'] = G[s, l] - \{(i, j) : i = 1, 2, \ldots, s; j = l - 2, l - 1, l\}.$$

We extend D' into a doubly connected dominating set of $G[s,l]$.

By Lemma 1, either $R_1 \subset D'$ or $R_{l'} \subset D'$. Without loss of generality, we assume that $R_{l'} \subset D'$. Now, we add to D' all nodes of degree 2 or 3 belonging to $G[s,l]$ and not belonging to $G[s,l']$, and we remove one node of R_{l-3} in order to make $V - D'$ connected. We may remove either $(2, l-3)$ or $(s-1, l-3)$, because in all other cases D' is neither dominating nor connected. Denote by D the set obtained from D' after the modifications. Clearly D is a doubly connected dominating set of $G[s,l]$. Moreover,

$$\begin{aligned}
\gamma_c^c(G[s,l]) &\leq |D| = |D'| + 2 + s + 2 - 1 \\
&= \frac{(l'-2)s}{3} + l' + s - 1 + s + 3 \\
&= \frac{(l-5)s}{3} + (l-3) + s - 1 + s + 3 \\
&= \frac{(l-2)s}{3} + l + s - 1.
\end{aligned}$$

See an example of $G[23,20]$ and its doubly connected dominating set in Figure 7. $\square$

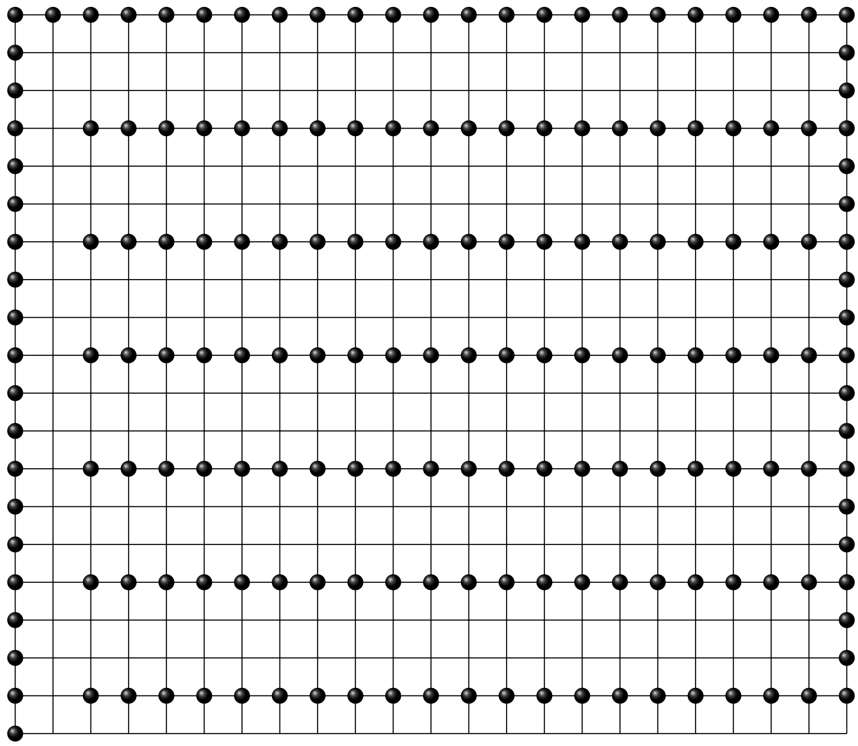

Figure 7. $G[23,20]$ and a doubly connected dominating set denoted in black. Nodes not belonging to the doubly connected dominating set are not marked and located at the intersections of the links without the black marks.

Proof of Theorem 2. Proposition 3 implies that the result is true for $l = 3$.

We continue the proof by induction on the positive integer $l \equiv 0 \mod 3$. Let $l \geq 6$ and assume that for $l' = l - 3$, $\gamma_c^c(G[s,l']) \leq \frac{l's}{3} + l' + s - 3$. Observe that

$$G[s,l'] = G[s,l] - \{(i,j) : i = 1,2,\ldots,s; j = l-2, l-1, l\}.$$

We extend D' into a doubly connected dominating set of $G[s,l]$ in the same way as in the proof of Theorem 1. Again we obtain that $|D| = |D'| + s + 3$, so

$$\begin{aligned}\gamma_c^c(G[s,l]) &\leq |D| = |D'| + s + 3\\ &= \frac{l's}{3} + l' + s - 3 + s + 3\\ &= \frac{(l-3)s}{3} + (l-3) + s + s\\ &= \frac{ls}{3} + l + s - 3.\end{aligned}$$

See an example of $G[23,21]$ and a doubly connected dominating set in Figure 8. □

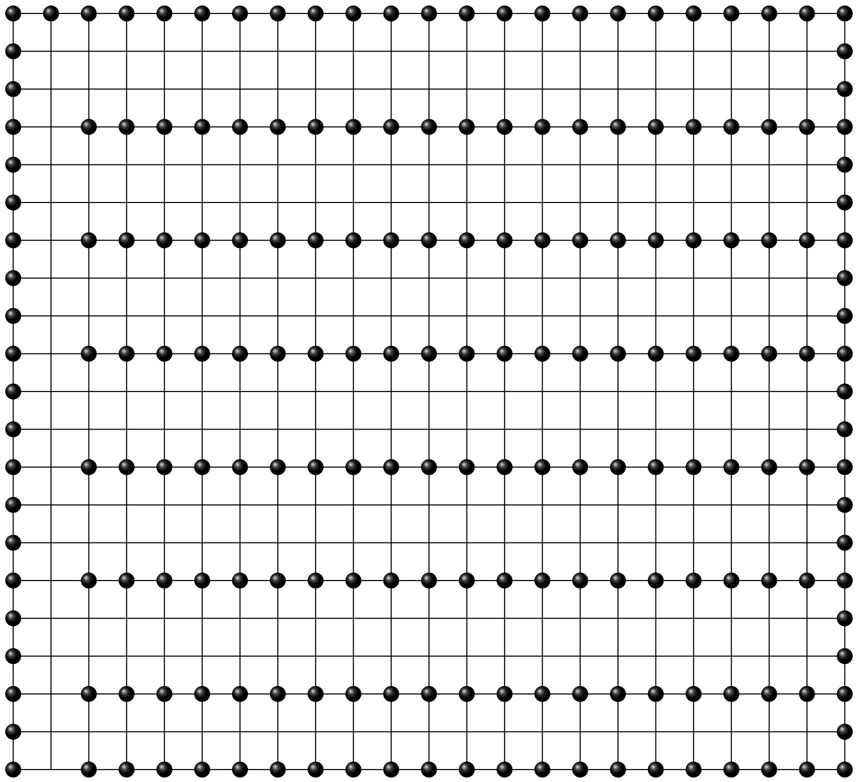

Figure 8. $G[23,21]$ and a doubly connected dominating set. Nodes not belonging to the doubly connected dominating set are not marked and located at the intersections of the links without the black marks.

Proof of Theorem 3. Proposition 4 implies that the result is true for $l = 4$.

We continue the proof by induction on the positive integer $l \equiv 1 \mod 3$. Let $l \geq 7$ and assume that for $l' = l - 3$, $\gamma_c^c(G[s,l']) \leq \frac{l's}{3} + l' + s - 3$. Again

$$G[s,l'] = G[s,l] - \{(i,j) : i = 1,2,\ldots,s; j = l-2, l-1, l\}.$$

We add new nodes to D' to make it a doubly connected dominating set of $G[s,l]$ in the same way as in the proof of Theorem 1. Again we obtain that $|D| = |D'| + s + 3$, so

$$\begin{aligned} \gamma_c^c(G[s,l]) &\leq |D| = |D'| + s + 3 \\ &= \frac{(l'-1)s}{3} + l' + s - 2 + s + 3 \\ &= \frac{(l-4)s}{3} + (l-3) + s + s + 1 \\ &= \frac{(l-1)s}{3} + l + s - 2. \end{aligned}$$

See an example of $G[23,22]$ and a doubly connected dominating set in Figure 9. □

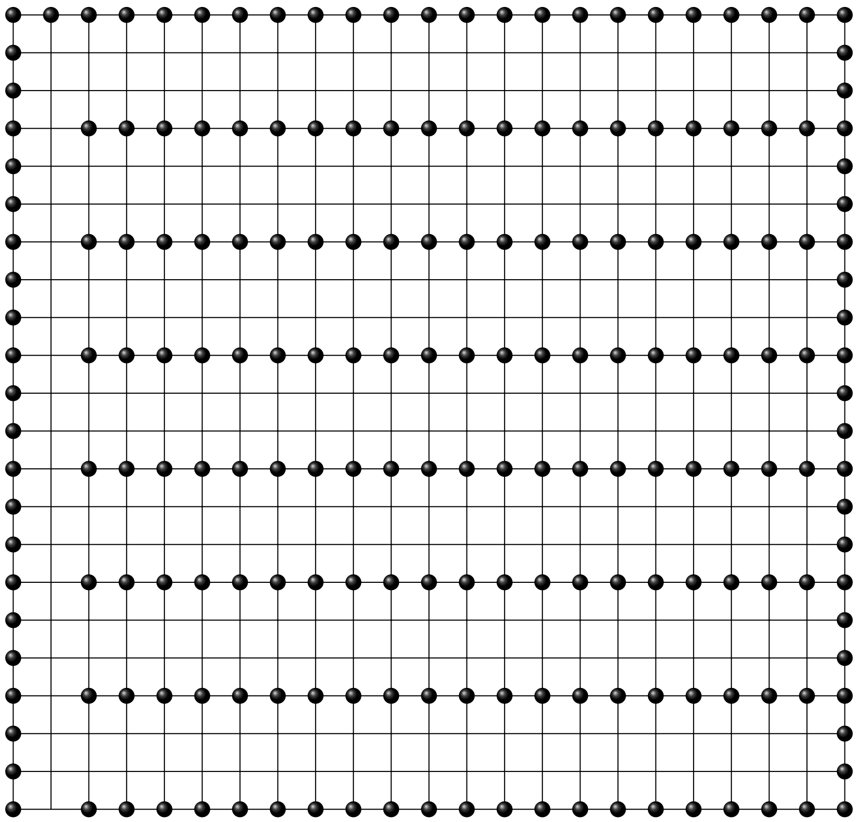

Figure 9. $G[23,22]$ and its doubly connected dominating set. Nodes not belonging to the doubly connected dominating set are not marked and located at the intersections of the links without the black marks.

Proof of Theorem 4. Let $G[s,l]$, where $6 \leq l \leq s$, be a grid graph and let D be a doubly connected dominating set of $G[s,l]$.

Since the maximum degree among nodes of $G[s,l]$ is 4, D may dominate at most $4|D|$ nodes of $V - D$. However, D is connected, so we conclude that D may dominate at most $4|D| - 2(|D| - 1) = 2|D| + 2$ nodes of $V - D$. Moreover, accordingly to the Lemma 1, D contains at least three nodes of degree 2 and at least $s - 2 + 2(l-2) = s + 2l - 6$ nodes of degree 3. For this reason, D may dominate at most

$$2|D| + 2 - 6 - (s + 2l - 6) = 2|D| - s - 2l + 2$$

nodes of $V - D$. Therefore

$$sl = |V| = |D| + |V - D| \leq |D| + 2|D| - s - 2l + 2 = 3|D| - s - 2l + 2$$

and hence
$$|D| \geq \frac{sl + s + 2l - 2}{3}.$$

Since D is a doubly connected dominating set of $G[s,l]$, this inequality must be true also for the smallest such a set, so
$$\gamma_c^c(G[s,l]) \geq \left\lceil \frac{(l-2)s}{3} + s + \frac{2}{3}l - \frac{2}{3} \right\rceil,$$

since the minimum doubly connected domination number is an integer. □

5. Conclusions and Discussion

This paper contributes to filling the gap in studying and inventing safe and reliable smart grids. For this reason, we introduce and study a rectangular grid graph as a model for smart grids. We present exact values for the doubly connected domination numbers for narrow grid graphs and straightforward formulas for the upper and lower bounds for this number in wide grid graphs. We also give examples of (minimum) doubly connected dominating sets for some of them. These examples give an idea of how doubly connected dominating sets may look in any given grid graph. It will be interesting to find and prove in the future the exact values for the doubly connected domination number in wide grid graphs; however, the authors believe that the values of this number are very close to those presented in this paper for the upper bounds.

Doubly connected dominating sets might be used to help locate facilities in smart grids. For example, in the nodes belonging to the minimum doubly connected dominating set, we may locate energy sources (or batteries). In this way, each energy consumer is directly connected to a source of energy. Moreover, the subgraph induced by the set of source nodes is connected, so the sources may communicate and even transfer energy with each other without intermediation of the consumer nodes. Similarly, the subgraph induced by the set of consumer nodes is connected. Due to this, consumers may exchange information, such as reconciling electricity usage graphics for sustainable energy use and for supporting green sources of energy. This can be performed without involving source nodes. Hence, the significance of the proposed research refers to safety, economy, ecology and reliability of the future energetic world.

In the future, it will be worth considering minimum doubly connected dominating sets in other types of grid graphs, for example in triangular or hexagonal grids.

Author Contributions: Conceptualization, J.C. and J.R.; methodology, J.C.; investigation, J.C. and J.R.; resources, J.R.; writing—original draft preparation, J.R.; writing—review and editing, J.C. and J.R.; visualization, J.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Ministry Subsidy for Research for the Gdańsk University of Technology.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Abbreviations

The following abbreviations are used in this manuscript:

MDCDS Minimum Doubly Connected Dominating Set

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