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Local Convergence Analysis of a One Parameter Family of Simultaneous Methods with Applications to Real-World Problems

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Abstract: In this paper, we provide a detailed local convergence analysis of a one-parameter family of iteration methods for the simultaneous approximation of polynomial zeros due to Ivanov (Numer. Algor. 75(4): 1193–1204, 2017). Thus, we obtain two local convergence theorems that provide sufficient conditions to guarantee the Q-cubic convergence of all members of the family. Among the other contributions, our results unify the latest such kind of results of the well known Dochev–Byrnev and Ehrlich methods. Several practical applications are further given to emphasize the advantages of the studied family of methods and to show the applicability of the theoretical results.

Keywords: simultaneous methods; local convergence; error estimates; basins of attraction; polynomial zeros; real-world applications

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1. Introduction

Let $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ be a complex polynomial of degree $n \geq 2$. In 1891, Weierstrass [1] established and studied the first iteration method for the simultaneous determination of all zeros of f. The *Weierstrass method* is quadratically convergent and defined as follows:

$$x^{(k+1)} = x^{(k)} - W(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where Weierstrass iteration function $W \colon \mathcal{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is given by $W(x) = (W_1(x), \ldots, W_n(x))$ with

$$W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i}(x_i - x_j)} \quad (i = 1, \ldots, n), \tag{2}$$

where $a_0 \in \mathbb{C}$ is the leading coefficient of f. Here and throughout the whole paper, $\mathcal{D}$ shall denote the set of all vectors in $\mathbb{C}^n$ with pairwise distinct components, that is, the set

$$\mathcal{D} = \left\{ x \in \mathbb{C}^n \ : \ x_i \neq x_j \ \text{ whenever } \ i \neq j \right\}. \tag{3}$$

The first semilocal convergence theorem about the method (1) is due to Weierstrass himself in the mentioned paper [1], and the first local convergence theorem in the literature was proved by Dochev [2] in 1962. Thereafter, many papers have been dedicated to the convergence of Weierstrass method, until 2016, when Proinov [3] proved local and semilocal convergence theorems that generalize and improve all previous results about this method. For more detailed historical survey about the Weierstrass method, we refer the reader to the monograph [4] and the papers [3,5].

The second simultaneous method in the literature is due to Bulgarian mathematicians Dochev and Byrnev [6]. The *Dochev–Byrnev method* is cubically convergent and can be defined by the following fixed point iteration:

$$x^{(k+1)} = T(x^{(k)}) \qquad k = 0, 1, 2, \ldots, \tag{4}$$

where the iteration function T is defined in $\mathbb{C}^n$ by $T(x) = (T_1(x), \ldots, T_n(x))$ with

$$T_i(x) = x_i - \frac{f(x_i)}{g'(x_i)}\left(2 - \frac{f'(x_i)}{g'(x_i)} + \frac{1}{2}\frac{f(x_i)}{g'(x_i)}\frac{g''(x_i)}{g'(x_i)}\right) \tag{5}$$

and the polynomial g is defined by

$$g(z) = c_o \prod_{j=1}^{n}(z - x_j). \tag{6}$$

In 1972, Prešić [7] rediscovered the Dochev–Byrnev method (4) by defining its iteration function in the following equivalent form:

$$T_i(x) = x_i - W_i(x)\left(1 - \sum_{j \neq i} \frac{W_j(x)}{x_i - x_j}\right), \tag{7}$$

where W is the *Weierstrass correction* (2). In 1974, Milovanović [8] gave an elegant derivation of (4) while, in 1983, the method (4) was again rediscovered by Tanabe [9], where it became widely known under the name *Tanabe's method* (see, e.g., [4] and references therein). In fact, the equivalence of the iteration functions (5) and (7) was proved only in 2016 by Proinov [10] (Theorem 4.1). In the same paper [10], Proinov proved a semilocal convergence theorem that generalizes and improves all previous such results about this method. On the other hand, the local convergence of the Dochev–Byrnev method (4) was firstly studied by Semerdzhiev and Pateva [11] and then by Semerdzhiev [12]. In 1982, Kyurkchiev [13] (see also [14] (Theorem 19.1)) improved the results of the above mentioned authors while, in 2022, Proinov and Marcheva [15] stated local and semilocal convergence results of a family of iteration methods with an accelerated convergence that contains the Dochev-Byrnev method. Thus, they improved all local convergence results of this method and put an end to this research direction.

The third simultaneous method in the literature is due to Ehrlich [16] in 1967 and was rediscovered by Börsch-Supan [17] in 1970. In 1982, Werner [18] proved that these two methods are equivalent (see also Carstensen [19]). It is known that *Ehrlich's method* is cubically convergent and can be defined by the following iteration:

$$x^{(k+1)} = \Phi(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{8}$$

where the iteration function $\Phi\colon \mathscr{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is defined by $\Phi(x) = (\Phi_1(x), \ldots, \Phi_n(x))$

$$\Phi_i(x) = x_i - \frac{1}{\dfrac{f'(x_i)}{f(x_i)} - \displaystyle\sum_{j \neq i}^{n} \frac{1}{x_i - x_j}} = x_i - \frac{W_i(x)}{1 + \displaystyle\sum_{j \neq i} \frac{W_j(x)}{x_i - x_j}}. \tag{9}$$

In 2016, Proinov [10] and recently Proinov and Vasileva [20] proved semilocal and local convergence results that generalize, improve and complement all of the existing results about Ehrlich's method (8).

In 2017, Ivanov [21] constructed a one-parameter family of simultaneous methods that includes, as special cases, Weierstrass' method (1), Dochev–Byrnev's method (4) and Ehrlich's method (8), and obtained a semilocal convergence result that unifies the above mentioned such results about Dochev–Byrnev's and Ehrlich's methods. *Ivanov's family of simultaneous methods* can be defined in $\mathbb{C}^n$ by the following iteration:

$$x^{(k+1)} = T_\alpha(x^(k)), \qquad k = 0, 1, 2, \ldots, \tag{10}$$

where the iteration function $T_\alpha\colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$T_\alpha(x) = (T_1(x), \ldots, T_n(x)) \text{ with } T_i(x) = x_i - W_i(x) \frac{1 + (\alpha - 1) \sum_{j \neq i} \dfrac{W_j(x)}{x_i - x_j}}{1 + \alpha \sum_{j \neq i} \dfrac{W_j(x)}{x_i - x_j}}. \tag{11}$$

Obviously, the domain of the iteration function T_α is the set

$$D = \left\{ x \in \mathcal{D} : 1 + \alpha \sum_{j \neq i} \frac{W_j(x)}{x_i - x_j} \neq 0, \; i = 1, \ldots, n \right\}. \tag{12}$$

As mentioned, the iteration (10) includes Weierstrass' method (1) for $\alpha \to \infty$, Dochev–Byrnev's method (4) for $\alpha = 0$ and Ehrlich's method (8) for $\alpha = 1$.

In this paper, we prove two kinds of local convergence theorems (Theorems 3 and 4) that supply sufficient conditions to guarantee the Q-cubic convergence of the family (10), along with a priori and a posteriori error estimates and an estimate of the asymptotic error constant. Moreover, our results unify the recent results of Proinov and Marcheva [15] and Proinov and Vasileva [20] regarding Dochev–Byrnev and Ehrlich's methods. Further, we conduct some experiments in order to emphasize the convergence behavior of the methods (10) and to show their applicability to some important real-world problems. At the end of our study, we compare Dochev–Byrnev's and Ehrlich's methods with each other and with some other members of the family (10) on the basis of their stability ([22]) and the obtained theoretical result.

2. Notations and Preliminaries

To make the paper self-contained, in this section, we provide some previous results that play a crucial role in our study.

Throughout the present study, $\mathbb{C}[z]$ denotes the ring of the univariate polynomials over $\mathbb{C}$, the vector space $\mathbb{R}^n$ is endowed with the standard coordinate-wise ordering $\preceq$ defined by

$$x \preceq y \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for each } i = 1, \ldots, n$$

and the space $\mathbb{C}^n$ is equipped with p-norm $\|\cdot\|_p$ for some $1 \leq p \leq \infty$ and with the vector norm $\|\cdot\|$ defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \text{and} \quad \|x\| = (|x_1|, \ldots, |x_n|).$$

In addition, we define the functions $d\colon \mathbb{C}^n \to \mathbb{R}^n$ and $\delta\colon \mathbb{C}^n \to \mathbb{R}_+$ by

$$d(x) = (d_1(x), \ldots, d_n(x)) \text{ with } d_i(x) = \min_{j \neq i} |x_i - x_j| \text{ and } \delta(x) = \min_{i \neq j} d_i(x) \tag{13}$$

and, for two vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{R}^n$, we use the denotation x/y for the vector in $\mathbb{R}^n$ defined by

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \cdots, \frac{|x_n|}{y_n} \right)$$

provided that y has only nonzero components. For a given number p ($1 \leq p \leq \infty$), we define the number q by

$$1 \leq q \leq \infty \quad \text{with} \quad 1/p + 1/q = 1$$

and, for a given $n \in \mathbb{N}$, we use the denotations:

$$a = (n-1)^{1/q}, \quad b = 2^{1/q}. \tag{14}$$

Note that $1 \leq a \leq n-1$ and $1 \leq b \leq 2$. For a non-negative integer k and $r \geq 1$, $S_k(r)$ stands for the sum

$$S_k(r) = \sum_{0 \leq j < k} r^j.$$

In addition, we assume that $0^0 \equiv 1$ and we denote by $\mathbb{R}_+$ the set of non-negative numbers and, with J, some interval in $\mathbb{R}_+$ containing 0. Finally, a vector $\xi \in \mathbb{C}^n$ is called *root-vector* of f if

$$f(z) = a_0 \prod_{i=1}^{n} (z - \xi_i) \quad \text{for all } z \in \mathbb{C}.$$

Definition 1 ([23,24])**.** *Let J be an interval on $\mathbb{R}_+$ containing 0. A function $\varphi \colon J \to \mathbb{R}_+$ is said to be quasi-homogeneous of exact degree $m \geq 0$ if it satisfies the following two conditions:*

(i) $\quad \varphi(\lambda t) \leq \lambda^m \varphi(t) \quad$ *for all* $\lambda \in [0,1]$ *and* $t \in J$*;*

(ii) $\quad \displaystyle\lim_{t \to 0^+} \frac{\varphi(t)}{t^m} \neq 0.$

The next definitions outline two important classes of iteration functions that will be used onwards.

Definition 2 ([24] (Definition 9))**.** *A function $F \colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ is said to be an iteration function of first kind at a point $\xi \in \mathcal{D}$ if there is a quasi-homogeneous function $\phi \colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$ such that, for each vector $x \in \mathbb{C}^n$ with $E(x) \in J$, the following conditions hold:*

$$x \in D \quad \text{and} \quad \| F(x) - \xi \| \preceq \phi(E(x)) \, \| x - \xi \|, \tag{15}$$

where the function $E \colon \mathbb{C}^n \to \mathbb{R}_+$ is defined by

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p \quad (1 \leq p \leq \infty) \tag{16}$$

with d defined by (13)*. The function ϕ is said to be the* control function *of F.*

Definition 3 ([24] (Definition 10))**.** *A function $F \colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ is called an iteration function of second kind at a point $\xi \in \mathbb{C}^n$ if there is a nonzero quasi-homogeneous function $\beta \colon J \to \mathbb{R}_+$ such that, for each vector $x \in \mathcal{D}$ with $E(x) \in J$, the following conditions hold:*

$$x \in D \quad \text{and} \quad \| F(x) - \xi \| \preceq \beta(E(x)) \, \| x - \xi \|, \tag{17}$$

where the function $E \colon \mathcal{D} \to \mathbb{R}_+$ is defined by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \quad (1 \leq p \leq \infty). \tag{18}$$

The function β is called the control function of F.

To prove our local convergence result of the first kind, we shall use the following result of Proinov [24].

Theorem 1 ([24] (Theorem 3))**.** *Suppose $T\colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ and $\xi \in \mathbb{C}^n$ is a fixed point of T with pairwise distinct components. Let T be an iteration function of first kind at $\xi \in \mathbb{C}^n$ with control function $\phi\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$, and let $x^{(0)} \in \mathbb{C}^n$ be an initial guess of ξ such that*

$$E(x^{(0)}) \in J \quad \text{and} \quad \phi(E(x^{(0)})) < 1.$$

Then, the Picard iteration $x^{(k+1)} = T(x^{(k)})$, $k = 0, 1, 2, \ldots$, is well defined and converges to ξ with Q-order $r = m + 1$ and with the following error estimates for all $k \geq 0$:

$$\|x^{(k)} - \xi\| \preceq \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \text{and} \quad \|x^{(k+1)} - \xi\| \preceq \lambda^{r^k} \|x^{(k)} - \xi\|,$$

where $\lambda = \phi(E(x^{(0)}))$. In addition, the following estimate of the asymptotic error constant holds:

$$\limsup_{k\to\infty} \frac{\|x^{(k+1)} - \xi\|_p}{\|x^{(k)} - \xi\|_p^r} \leq \frac{1}{\delta(\xi)^m} \lim_{t\to 0^+} \frac{\phi(t)}{t^m}. \tag{19}$$

To prove our local convergence result of the second kind, we need the following general local convergence result of Proinov [24]:

Theorem 2 ([24] (Theorem 4))**.** *Let $T\colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function of second kind at a point $\xi \in \mathbb{C}^n$ with a nonzero control function $\beta\colon J \to \mathbb{R}_+$ of exact degree $m \geq 0$ and let $x^{(0)} \in \mathcal{D}$ be an initial guess satisfying*

$$E(x^{(0)}) \in J \quad \text{and} \quad \Psi(E(x^{(0)})) \geq 0,$$

where the function $\Psi\colon J \to \mathbb{R}$ is defined by

$$\Psi(t) = 1 - bt - \beta(t)(1 + bt) \tag{20}$$

with $b = 2^{1/q}$. Then, the vector ξ is a fixed point of T with distinct components and the Picard iteration $x^{(k+1)} = T(x^{(k)})$, $k = 0, 1, 2, \ldots$, is well defined and converges to ξ with Q-order $r = m + 1$ and with the following error estimates for all $k \geq 0$:

$$\|x^{(k)} - \xi\| \preceq \theta^k \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \text{and} \quad \|x^{(k+1)} - \xi\| \preceq \theta \lambda^{r^k} \|x^{(k)} - \xi\|$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ and the functions ψ and ϕ are defined by

$$\psi(t) = 1 - bt(1 + \beta(t)) \quad \text{and} \quad \phi(t) = \beta(t)/\psi(t). \tag{21}$$

We end this section with several useful technical lemmas.

Lemma 1 ([25])**.** *Let x, y, ξ be three vectors in $\mathbb{C}^n$, and let ξ have pairwise distinct components. If $\|y - \xi\| \preceq \alpha \|x - \xi\|$ for some $\alpha \geq 0$, then, for all $i, j = 1, \ldots, n$, we have*

$$|x_i - y_j| \geq \left(1 - (1 + \alpha^q)^{1/q} \left\|\frac{x - \xi}{d(\xi)}\right\|_p\right) |\xi_i - \xi_j|.$$

Lemma 2 ([25])**.** *Let x, y, ξ be three vectors in $\mathbb{C}^n$ and x have pairwise distinct components. If $\|y - \xi\| \preceq \alpha \|x - \xi\|$ for some $\alpha \geq 0$, then, for all $i \neq j$, we have*

$$|x_i - y_j| \geq \left(1 - (1 + \alpha) \left\|\frac{x - \xi}{d(x)}\right\|_p\right) |x_i - x_j|.$$

The following lemma provides an inequality that plays an important role in our study.

Lemma 3. *Let $x, y \in \mathbb{C}^n$ be vectors with pairwise distinct components, $\lambda \geq 0$ and $1 \leq p \leq \infty$. Then, for all $i \neq j$, we have*

$$|x_i - y_j| \geq \left(1 - (1+\lambda^q)^{1/q} \left\| \frac{x-y}{d(y)} \right\|_p \right) |x_i - x_j|.$$

Proof. Set $E(x,y) = \|(x-y)/d(y)\|_p$. The claimed inequality is obvious if $E(x,y) \geq 1/(1+\lambda^q)^{1/q}$. Let $E(x,y) < 1/(1+\lambda^q)^{1/q}$. Then, from the triangle inequality and Lemma 1 with $\xi = y$ and $\alpha = 0$, we obtain

$$\begin{aligned} |x_i - x_j| &= |x_i - y_j + y_j - x_j| \leq |x_i - y_j| \left(1 + \frac{|x_j - y_j|}{|x_i - y_j|}\right) \leq |x_i - y_j| \left(1 + \frac{|x_j - y_j|}{(1 - E(x,y))d_j(y)}\right) \\ &\leq |x_i - y_j| \left(1 + \frac{E(x,y)}{1 - E(x,y)}\right) = \frac{|x_i - y_j|}{1 - E(x,y)} \leq \frac{|x_i - y_j|}{1 - (1+\lambda^q)^{1/q} E(x,y)} \end{aligned}$$

which completes the proof. □

3. Convergence Analysis

In this section, we prove two local convergence theorems about the family (10) to supply sufficient conditions that guarantee the Q-cubic convergence of all members of the family and provide a priori and a posteriori error estimates together with an estimate of the asymptotic error constant of the methods.

The next lemma gives a useful representation of the iteration function (11).

Lemma 4. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ with only simple roots in $\mathbb{C}$, $\xi \in \mathbb{C}^n$ be a root-vector of f and $T_\alpha \colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ be the iteration function defined by (11). Suppose $x \in D$ is a vector with distinct components such that $f(x_i) \neq 0$. Then, the following relation holds:*

$$T_i(x) - \xi_i = \sigma_i (x_i - \xi_i) \tag{22}$$

where σ_i is defined by

$$\sigma_i = \frac{(1-\alpha)C_i^2 - (1+\alpha C_i)\, B_i}{(1-B_i)(1+\alpha C_i)} = \frac{(1-\alpha)[(A_i - 1)^2 - A_i^2\, B_i] - \alpha A_i\, B_i}{1 - \alpha + \alpha A_i (1 - B_i)} \tag{23}$$

with

$$A_i = \prod_{j \neq i} \left(\frac{x_i - \xi_j}{x_i - x_j} \right), \quad B_i = (x_i - \xi_i) \sum_{j \neq i} \frac{x_j - \xi_j}{(x_i - \xi_j)(x_i - x_j)} \quad \textit{and} \quad C_i = \sum_{j \neq i} \frac{W_j(x)}{x_i - x_j}. \tag{24}$$

Proof. We start with the proof of the first equality of (23). Thus, using the known relation (see, e.g., [20] (Lemma 2))

$$\frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i}^{n} \frac{1}{x_i - x_j} = \frac{1 - B_i}{x_i - \xi_i} \tag{25}$$

together with the second equality of (9), we obtain

$$\frac{W_i(x)}{x_i - x_j} = \frac{1 + C_i}{1 - B_i}\,. \tag{26}$$

From this and (11), we obtain

$$\begin{aligned} T_i(x) - \xi_i &= \left(1 - \frac{1 + C_i}{1 - B_i}\,\frac{1 - C_i + \alpha\, C_i}{1 + \alpha\, C_i}\right)(x_i - \xi_i) \\ &= \frac{(1 + \alpha\, C_i)(1 - B_i) - 1 + (1 - \alpha)\, C_i^2 - \alpha\, C_i}{(1 - B_i)(1 + \alpha\, C_i)}\,(x_i - \xi_i) \\ &= \sigma_i(x_i - \xi_i) \end{aligned}$$

which proves the first equality of (23). The second equality of (23) follows immediately from the relation $C_i = A_i(1 - B_i) - 1$, which, in turn, follows from (26) and the known identity $A_i = W_i(x)/(x_i - \xi_i)$ (see [26] (Equation (28))). □

3.1. Local Convergence of the First Kind

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ with n simple roots and let $\xi \in \mathbb{C}^n$ be a root-vector of f. In this section, we study the convergence of the iteration (10) regarding the function of initial conditions $E\colon \mathbb{C}^n \to \mathbb{R}_+$ defined by (16).

Let a and b be the numbers (14). For the needs of the main result, we define the real functions $\gamma, \eta, \mu\colon [0, 1/b) \to [1, \infty)$ by

$$\gamma(t) = \left(1 + \frac{a\,t}{(n-1)(1 - bt)}\right)^{n-1}, \quad \eta(t) = \frac{a\,t^2}{(1 - t)(1 - bt)} \quad \text{and} \quad \mu(t) = (\gamma(t) - 1)^2 + \gamma(t)^2\,\eta(t). \tag{27}$$

Using these functions, we define ϕ_α by

$$\phi_\alpha(t) = \begin{cases} \dfrac{|1 - \alpha|\,\mu(t) + |\alpha|\,\gamma(t)\,\eta(t)}{|1 - \alpha| - |\alpha|\,\gamma(t)\,(1 + \eta(t))} & \text{for } Re(\alpha) < 1/2, \\[2ex] \dfrac{|\alpha|\,\nu(t)^2 + \eta(t)\,(1 + |\alpha|\,\nu(t))}{(1 - \eta(t))(1 - |\alpha|\,\nu(t))} & \text{for } Re(\alpha) = 1/2, \\[2ex] \dfrac{|1 - \alpha|\,\mu(t) + |\alpha|\,\eta(t)\,(1 - t)^{n-1}}{|\alpha|\,(1 - \eta(t))(1 - t)^{n-1} - |1 - \alpha|} & \text{for } Re(\alpha) > 1/2, \end{cases} \tag{28}$$

where ν is defined by

$$\nu(t) = \frac{a\,\gamma(t)\,t}{1 - b\,t}. \tag{29}$$

The following lemma shows that T_α is an iteration function of first kind at a point ξ.

Lemma 5. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ that has only simple zeros in $\mathbb{C}$, and let $\xi \in \mathbb{C}^n$ be a root-vector of f. Then, $T_\alpha\colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ defined by (11) is an iteration function of first kind at ξ with control function ϕ_α defined by (28).*

Proof. First, let us define the functions g_α, h_α and l_α by

$$\begin{aligned} g_\alpha(t) &= |1 - \alpha| - |\alpha|\,\gamma(t)\,(1 + \eta(t)), \\ h_\alpha(t) &= (1 - \eta(t))(1 - |\alpha|\,\nu(t)), \\ l_\alpha(t) &= |\alpha|\,(1 - \eta(t))(1 - t)^{n-1} - |1 - \alpha|. \end{aligned} \tag{30}$$

Now, let $x \in \mathbb{C}^n$ be such that $E(x) < 1/b$ with $g_\alpha(t) > 0$ for $Re(\alpha) < 1/2$, $h_\alpha(t) > 0$ for $Re(\alpha) = 1/2$ or $l_\alpha(t) > 0$ for $Re(\alpha) > 1/2$. According to Definition 2 and Lemma 4, we ought to prove that

$$x \in D \quad \text{and} \quad |\sigma_i| \leq \phi_\alpha(E(x)) \quad \text{for each} \quad i = 1, \ldots, n, \tag{31}$$

where the number σ_i is defined by (23).

We divide the proof into two cases.

Case 1. *Let $Re(\alpha) = 1/2$. Note that, in this case, we have $|\alpha| = |1-\alpha|$. Thus, from the first identity of (23), the triangle inequality and the following inequalities:*

$$|B_i| \le \frac{a\,E(x)^2}{(1-E(x))(1-bE(x))} = \eta(E(x)) \quad \text{and} \quad |C_i| \le \frac{a\gamma(E(x))E(x)}{1-bE(x)} = \nu(E(x)) \tag{32}$$

we obtain

$$|\sigma_i| \le \frac{|1-\alpha|\,|C_i|^2 + |1+\alpha C_i|\,|B_i|}{|1-B_i|\,|1+\alpha C_i|} \le \frac{|\alpha|\,\nu(E(x))^2 + \eta(E(x))\,(1+|\alpha|\,\nu(E(x)))}{h_\alpha(E(x))} = \phi_\alpha(E(x))$$

which completes the proof of this case. Observe that the first inequality of (32) can be found in [24] (Lemma 4) whereas the second one follows from Lemma 1 and the known inequalities (see, e.g., [26] (Equation (30)))

$$|W_j| \le \gamma(E(x))|x_j - \xi_j| \quad \text{and} \quad \sum_{j\neq i} \frac{x_j - \xi_j}{x_i - x_j} \le \frac{aE(x)}{1-bE(x)}.$$

Case 2. *Let $Re(\alpha) \neq 1/2$. If $Re(\alpha) < 1/2$, then, from the first inequality of (32) and $|A_i| \le \gamma(E(x))$ (see the last identity in the proof of Lemma 4 and the penultimate inequality in the previous case), we obtain*

$$|\sigma_i| \le \frac{|1-\alpha|(|A_i-1|^2 + |A_i|^2\,|B_i|) + |\alpha|\,|A_i|\,|B_i|}{|1-\alpha| - |\alpha|\,|A_i|\,|1-B_i|} \le \frac{|1-\alpha|\,\mu(E(x)) + |\alpha|\,\gamma(E(x))\,\eta(E(x))}{g_\alpha(E(x))} = \phi_\alpha(E(x)). \tag{33}$$

If $Re(\alpha) > 1/2$, we use Lemma 3 with $y = \xi$ and $\lambda = 0$ to obtain the inequality

$$|A_i| = \prod_{j\neq i} \left| \frac{x_i - \xi_j}{x_i - x_j} \right| \ge (1-E(x))^{n-1}$$

which, together with the above used upper estimates of $|A_i|$ and $|B_i|$, leads us to

$$\begin{aligned} |\sigma_i| &\le \frac{|1-\alpha|(|A_i-1|^2 + |A_i|^2\,|B_i|) + |\alpha|\,|A_i|\,|B_i|}{|\alpha|\,|A_i|\,|1-B_i| - |1-\alpha|} \\ &\le \frac{|1-\alpha|\,\mu(E(x)) + |\alpha|\,\eta(E(x))(1-E(x))^{n-1}}{l_\alpha(E(x))} = \phi_\alpha(E(x)). \end{aligned} \tag{34}$$

This completes the proof of the lemma. □

The following is our first main result in this paper.

Theorem 3. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \ge 2$ with only simple zeros and $\xi \in \mathbb{C}^n$ be a root-vector of f. Suppose that $x^{(0)} \in \mathbb{C}^n$ is an initial approximation satisfying*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(\xi)} \right\|_p < \frac{1}{b} \quad \text{and} \quad \phi_\alpha(E(x^{(0)})) < 1, \tag{35}$$

where the function ϕ_α is defined by (28). Then, the iteration (10) is well defined and Q-cubically convergent to ξ with the following error estimates for all $k \ge 0$:

$$\|x^{(k+1)} - \xi\| \preceq \lambda^{3^k}\,\|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{\frac{3^k-1}{2}}\,\|x^{(0)} - \xi\|, \tag{36}$$

where $\lambda = \phi(E(x^0))$. In addition, we have the following estimate of the asymptotic error constant:

$$\limsup_{k\to\infty} \frac{\|x^{(k+1)} - \xi\|_p}{\|x^{(k)} - \xi\|_p^3} \le \begin{cases} \dfrac{a\,(|1-\alpha| + |\alpha|)}{(|1-\alpha| - |\alpha|)\,\delta(\xi)^2} & \text{for } Re(\alpha) < 1/2, \\ \dfrac{a}{\delta(\xi)^2} & \text{for } Re(\alpha) = 1/2, \\ \dfrac{a\,(|1-\alpha| + |\alpha|)}{(|\alpha| - |1-\alpha|)\,\delta(\xi)^2} & \text{for } Re(\alpha) > 1/2. \end{cases} \tag{37}$$

Proof. The proof straightforwardly follows from Lemma 5 and Theorem 1. □

3.2. Local Convergence of the Second Kind

In the present section, we study the convergence of the iteration (10) under the function of initial conditions $E\colon \mathcal{D} \subset \mathbb{C}^n \to \mathbb{R}_+$ defined by (18).

Let a and b be the numbers (14). For the needs of the main result, we define the real functions $\gamma, \eta, \nu, \mu\colon [0,1) \to [1,\infty)$ by

$$\gamma(t) = \left(1 + \frac{a\,t}{n-1}\right)^{n-1}, \quad \eta(t) = \frac{a\,t^2}{1-t}, \quad \nu(t) = a\,t\,\gamma(t) \quad \text{and} \quad \mu(t) = (\gamma(t)-1)^2 + \gamma(t)^2\,\eta(t). \tag{38}$$

Using these functions, we define β_α by

$$\beta_\alpha(t) = \begin{cases} \dfrac{|1-\alpha|\,\mu(t) + |\alpha|\,\gamma(t)\,\eta(t)}{|1-\alpha| - |\alpha|\,\gamma(t)\,(1+\eta(t))} & \text{for } Re(\alpha) < 1/2, \\ \dfrac{|\alpha|\,\nu(t)^2 + \eta(t)\,(1 + |\alpha|\,\nu(t))}{(1-\eta(t))(1-|\alpha|\,\nu(t))} & \text{for } Re(\alpha) = 1/2, \\ \dfrac{|1-\alpha|\,\mu(t) + |\alpha|\,\eta(t)\,(1-t)^{n-1}}{|\alpha|\,(1-\eta(t))(1-t)^{n-1} - |1-\alpha|} & \text{for } Re(\alpha) > 1/2. \end{cases} \tag{39}$$

The following lemma shows that T_α is an iteration function of second kind at a point ξ.

Lemma 6. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \ge 2$ that has only simple zeros in $\mathbb{C}$, and $\xi \in \mathbb{C}^n$ be a root-vector of f. Then, $T_\alpha\colon D \subset \mathbb{C}^n \to \mathbb{C}^n$ defined by (11) is an iteration function of second kind at ξ with control function β_α defined by (39).*

Proof. The proof is performed the same way as the one of Lemma 5. One just has to use the inequalities

$$|B_i| \le \frac{a\,E(x)^2}{1-E(x)} = \eta(E(x)) \quad \text{and} \quad |C_i| \le a\gamma(E(x))E(x) = \nu(E(x))$$

instead of (32). The first of these inequalities can be found in [24] (Lemma 6), whereas the second one follows from Lemma 1 and the known inequalities

$$|W_j| \le \gamma(E(x))|x_j - \xi_j| \quad \text{and} \quad \sum_{j\ne i} \frac{x_j - \xi_j}{x_i - x_j} \le aE(x).$$

□

Now, we are ready to introduce our second main result.

Theorem 4. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ with only simple zeros and $\xi \in \mathbb{C}^n$ be a root-vector of f. Suppose that $x^{(0)} \in \mathbb{C}^n$ is an initial approximation satisfying*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(x^{(0)})} \right\|_p < 1 \quad \text{and} \quad \Psi_\alpha(E(x^{(0)})) \geq 0, \tag{40}$$

where the function $\Psi_\alpha \colon J \to \mathbb{R}$ is defined by

$$\Psi_\alpha(t) = 1 - bt - \beta_\alpha(t)(1 + bt) \tag{41}$$

with β_α defined by (39). Then, the iteration (10) is well defined and Q-cubically convergent to ξ with the following error estimates for all $k \geq 0$:

$$\|x^{(k)} - \xi\| \preceq \theta^k \lambda^{(3^k - 1)/2} \|x^{(0)} - \xi\| \quad \text{and} \quad \|x^{(k+1)} - \xi\| \preceq \theta \lambda^{3^k} \|x^{(k)} - \xi\|$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ and ψ and ϕ are defined by

$$\psi(t) = 1 - bt(1 + \beta_\alpha(t)) \quad \text{and} \quad \phi(t) = \beta_\alpha(t)/\psi(t). \tag{42}$$

Proof. The conclusions of the theorem follow directly from Lemma 6 and Theorem 2. □

4. Computational Aspects and Applications

In the present section, we show the practical applicability of the theoretical results obtained in the previous section and then apply several particular members of the family (10) to solve some important real-world problems. For convenience, we shall consider the case $p = \infty$.

4.1. Comparative Analysis

Here, we use Theorem 3 to compare the convergence domains and the error estimates of some particular methods of the family (10).

Define the function ϕ_α by (28) in the case $p = \infty$; that is, with γ, η, ν and μ defined by

$$\gamma(t) = \left(1 + \frac{t}{1 - 2t}\right)^{n-1}, \quad \eta(t) = \frac{(n-1)\,t^2}{(1-t)(1-2t)}, \quad \nu(t) = \frac{(n-1)\,\gamma(t)\,t}{1 - 2\,t}$$

and

$$\mu(t) = (\gamma(t) - 1)^2 + \gamma(t)^2\,\eta(t).$$

Evidently, the initial condition (35) can be presented in the form $E(x_0) < R$, where R is the unique solution of the equation $\phi_\alpha(t) = 1$ in the interval $(0, 1/2)$. Thus, the bigger the number R, the larger the convergence domain and the better the error estimates of the respective method.

Further on, we compare Dochev–Byrnev's method (4) ($\alpha = 0$) and Ehrlich's method (8) ($\alpha = 1$) with each other and with three other members of the family (10), obtained for $\alpha = 1/2$ and for the complex numbers $\alpha = \alpha 1 = 0.722 + 0.126i$ and $\alpha = \alpha 2 = 0.238 - 0.004i$, which were randomly chosen from the rectangles

$$\{z \in \mathbb{C}\colon 0.5 < Re(z) \leq 1 \text{ and } |Im(z)| \leq 1\} \quad \text{and} \quad \{z \in \mathbb{C}\colon -1 \leq Re(z) < 0.5 \text{ and } |Im(z)| \leq 1\}.$$

Note that the last three values of α were chosen to cover the three different cases in (28).

The graphs of the functions ϕ_0, $\phi_{1/2}$, ϕ_1, $\phi_{\alpha 1}$ and $\phi_{\alpha 2}$, obtained for $n = 5$, are depicted in Figure 1. It can be clearly seen from the figure that Ehrlich's method (8) has the largest convergence domain and the best error estimates among the considered methods. However, despite the randomness of the choice, the method obtained for $\alpha = \alpha 1$ has a very large convergence domain and relatively good error estimates.

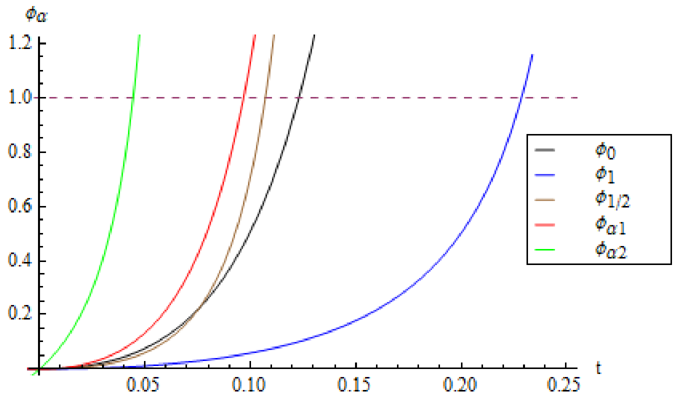


Figure 1. Graph of the functions ϕ_0, $\phi_{1/2}$, ϕ_1, $\phi_{\alpha 1}$ and $\phi_{\alpha 2}$ for $n = 5$.

4.2. Applications to Some Real-World Problems

In this part, some computer implementations are performed to show the applicability of four particular methods of the family (10) to some important real-world problems, as well as to emphasize the convergence behavior of the chosen methods.

Suppose that $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 2$ and $x^{(0)} \in \mathbb{C}^n$ is an initial approximation with pairwise distinct coordinates. In order to compute all of the zeros of f simultaneously, in the examples below, we apply four members of the family (10), namely the ones obtained for $\alpha = 0$ (Dochev–Burnev's method), $\alpha = 1$ (Ehrlich's method), $\alpha = 1/2$ and $\alpha = 0.766 + 0.484i$. The penultimate value of α was chosen because of its 'border' role in the proof of the theoretical results, whereas the last one was randomly chosen from the square

$$\{z \in \mathbb{C} \colon |Re(z)| \leq 1 \quad \text{and} \quad |Im(z)| \leq 1\}.$$

In the performed examples, we used the following theorem of [27] as a stopping criterion.

Theorem 5 ([27] (Theorem 5.1))**.** *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ and $(x^{(k)})_{k=0}^{\infty}$ be an iterative sequence in $\mathbb{C}^n$ consisting of vectors with pairwise distinct components. Then, for every $k \geq 0$, there exists a root-vector $\xi \in \mathbb{C}^n$ of f such that*

$$E_f(x^{(k)}) < \tau = \frac{1}{(1 + \sqrt{n-1})^2}$$

implies

$$\| x^{(k)} - \xi \|_\infty \leq \varepsilon_k = \alpha(E_f(x^{(k)}))\, \|W_f(x^{(k)})\|_\infty$$

where the function E_f is defined by $E_f(x) = \|W(x)/d(x)\|_\infty$ and $\alpha \colon [0, \tau] \to \mathbb{R}_+$ is defined by

$$\alpha(t) = \frac{2}{1 - (n-2)t + \sqrt{(1-(n-2)t)^2 - 4t}}. \tag{43}$$

In 2000, Weerakoon and Fernando [28] (Definition 2.2) gave a definition for the *computational order of convergence (COC)* of an iterative sequence. A main drawback of this definition is the involvement of the exact roots of f. In order to avoid this problem, Cordero and Torregrosa [29], and later on Grau-Sánchez et al. [30] (Definition 2), used some *approximations* of COC that do not involve the roots of f. Following these ideas, together with Theorem 5, we made use of the following practically applicable COC (see also [25]):

$$r_k = \frac{\ln(\varepsilon_{k+1}/\varepsilon_k)}{\ln(\varepsilon_k/\varepsilon_{k-1})}. \tag{44}$$

In the considered examples, we calculated the smallest k that satisfies the following stopping criterion:

$$E_f(x^{(k)}) < \tau \quad \text{and} \quad \varepsilon_k < 10^{-10} \tag{45}$$

In the tables below, for any of the tested polynomials, we exhibit the values of k, $E_f(x^{(k)})$, τ, ε_k, ε_{k+1} and r_k with at least six decimal digits.

Example 1 (Quarter car suspension model)**.** *The shock absorber is an element of the suspension system that is also used to control the transient behavior of the vehicle mass and the suspension mass (see Pulvirenti [31], Konieczny [32]). It is one of the most complex parts of the suspension system due to its nonlinear behavior. In fact, the damping force of the dampers is described by an asymmetric nonlinear hysteresis loop (Liu [33]). For the purposes of the present example, the vehicle characteristics are represented by a two-degrees-of-freedom quarter-car model to study the damper effect based on linear and nonlinear damping characteristics. Since simpler models such as linear and piecewise linear models are ineffective in describing the behavior of the damper, it is essential to develop a damper model that describes the nonlinear hysteresis characteristics of the damper such as a polynomial model. The equations of the motion of the masses are given as follows:*

$$\begin{aligned} &m_s \ddot{x}_s + k_s(x_s - x_u) + F = 0 \\ &m_u \ddot{x}_u - k_s(x_s - x_u) - k_t(x_r - x_u) - F = 0, \end{aligned} \tag{46}$$

where m_s and m_u are over-sprung and under-sprung masses, x_s and x_u are the displacement variables for m_s and m_u, respectively, x_r describes disturbances from road bumps and k_s and k_t are coefficients connected to the stiffness of the suspension's system spring and the tire stiffness. The damper force F in (46) can be fitted by the following polynomial (see Barethiye [34]):

$$f(z) = -77.14\, z^4 + 23.14\, z^3 + 342.7\, z^2 + 956.7\, z + 124.5. \tag{47}$$

For the simultaneous approximation of all of the zeros of f, we apply the above mentioned four members of the family (10) with Aberth's initial approximation $x^{(0)} \in \mathbb{C}^n$ defined by ([35])

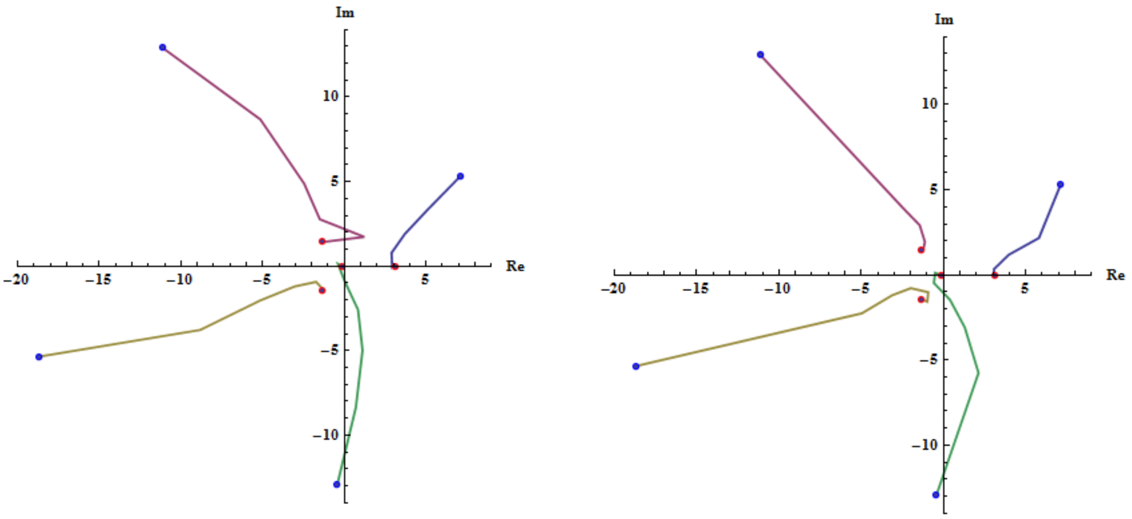
$$x_j^{(0)} = -\frac{a_1}{n} + re^{i\,\theta_j}, \qquad \theta_j = \frac{\pi}{n}\left(2j - \frac{3}{2}\right), \qquad j = 1, \ldots, n, \tag{48}$$

where $r = 14$, a_1 is the second coefficient and n is the degree of the corresponding polynomial.

One can see from Table 1 that, for $\alpha = 1$ (Ehrlich's method), the stopping criterion (45) is satisfied at the eighth iteration with an error estimation of less than 10^{-24}, and, at the ninth step, the roots are found with an accuracy of 10^{-74}. It is also seen that the obtained COC confirms the theoretical order of convergence proven in Section 3. The approximation trajectories for this example calculated for $\alpha = 1$ (Ehrlich's method) and $\alpha = 0.766 + 0.484i$ are plotted in Figure 2a,b, where the blue points are the coordinates of the initial approximation and the red ones are the zeros of f.

Table 1. Numerical results for Example 1.

Method	k	$E_f(x^{(k)})$	τ	ε_k	ε_{k+1}	r_k
$\alpha = 0$	9	2.060×10^{-15}	0.133975	3.841×10^{-15}	3.256×10^{-44}	3.000205
$\alpha = 1$	8	1.546×10^{-25}	–	2.882×10^{-25}	4.487×10^{-75}	3.000012
$\alpha = 1/2$	8	2.224×10^{-15}	–	4.147×10^{-15}	3.057×10^{-44}	3.001956
$\alpha = 0.766 + 0.484i$	9	6.258×10^{-18}	–	1.166×10^{-17}	5.851×10^{-52}	3.000188



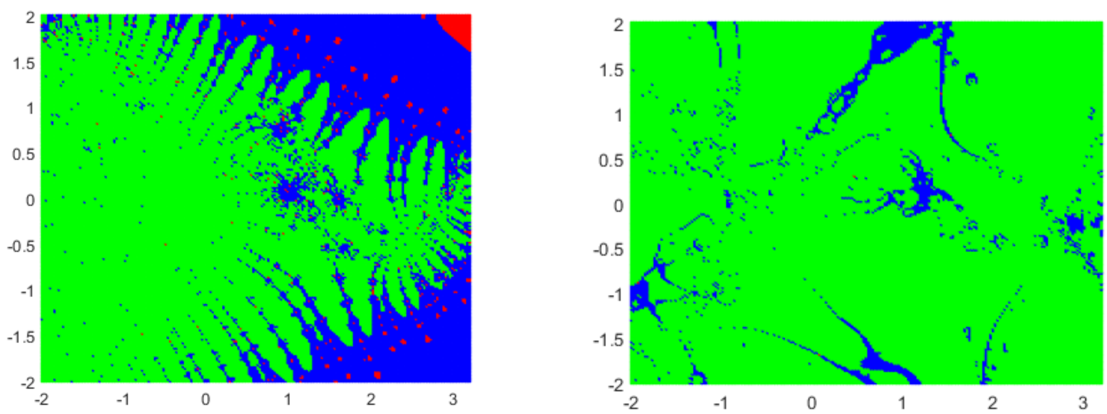
(**a**) (**b**)

Figure 2. Trajectories of approximations for Example 1. (**a**) For $\alpha = 1$ (Ehrlich's method). (**b**) For $\alpha = 0.7669 + 0.4847i$.

In this example, we also compare the stability of the considered four methods (see e.g., [22]). For this purpose, the basins of attraction of the roots of (47) are obtained in the rectangle

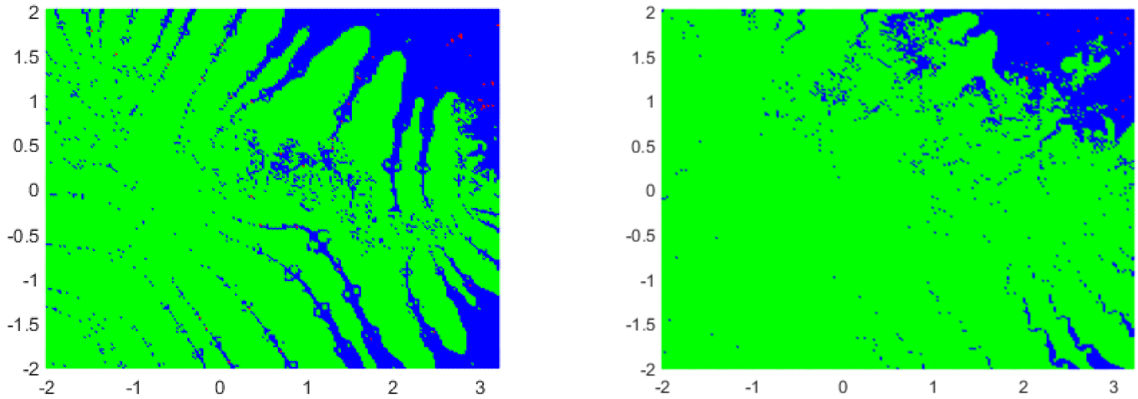
$$\{z \in \mathbb{C}\colon\ -2 < Re(z) \leq 3.2 \text{ and } |Im(z)| \leq 2\}$$

which contains the root-vector $\xi = (-0.136743, -1.32692 - 1.43467i, -1.32692 + 1.43467i, 3.09056)$ of f. The dynamical planes, using a mesh of 200×200 points with k iterations, are obtained and depicted in Figure 3. The sets of the initial approximations that generate iteration sequences to fulfill the accuracy criterion $\max_{1 \leq i \leq 4} |f(x_i^{(k)})| < 10^{-10}$ at the preset iterations ($k = 30$ for Dochev–Byrnev's method ($\alpha = 0$), $k = 10$ for Ehrlich's method ($\alpha = 1$) and $k = 18$ for the methods obtained with $\alpha = 1/2$ and $\alpha = 0.7669 + 0.4847i$) are green-colored, the initial approximations that generate sequences such that $\max_{1 \leq i \leq 4} |f(x_i^{(k)})| \geq 10^{10}$ at the mentioned iterations are red-colored and the rest are in blue. It can be seen from the figure that Ehrlich's method and the random method behave better than the other two methods. It can also be seen that Dochev-Byrnev's method behaves the worst for this polynomial.



(**a**) (**b**)

Figure 3. *Cont.*



(**c**) (**d**)

Figure 3. Basins of attraction for Example 1. (**a**) Dochev–Byrnev's method ($\alpha = 0$). (**b**) Ehrlich's method ($\alpha = 1$). (**c**) The method by $\alpha = 1/2$. (**d**) The method by $\alpha = 0.7669 + 0.4847i$.

Example 2 (Thermo-denaturation of milk proteins)**.** *To study the behavior of the milk polydisperse system and the change in its structure while turning into gel during enzymatic coagulation, Kabadzhov [36] (Ch. 4) sequentially measured the optical density and permeability of natural skimmed fresh milk (NSFM). Note that, during heat-denaturation, the protein molecule passed through several stages (usually four or five) of structural changes before turning into gel, and thence into a food product (cheese, yellow cheese, etc.). In the optical spectra, this is observed as maximums (see Figure 4), which correspond to the maximal values of the heat capacity.*

The endothermic temperature characteristic of the thermo-denaturation process during all stages can be easily and quickly found by means of a non-destructive turbidimetric method performed on a specially constructed fiber-optic measuring device [36] (Figure 3.1a).

Using the OriginePro 8.5 *software product, a correlation analysis of the dependence of the optical density on temperature was made at the used wavelengths in the range from* 650 *nm to* 850 *nm in the temperature interval from* 60° *to* 80°*. The function of the dependence between the optical density and the heating temperature during the kinetic process was approximated by the following polynomial (see Figure 4):*

$$\begin{aligned} f(z) \;=\; & -1.006 \times 10^{-10}\, z^9 - 1.937 \times 10^{-8}\, z^8 + 2.867 \times 10^{-5}\, z^7 - 0.008\, z^6 + 1.286\, z^5 \\ & -117.529\, z^4 + 6749.614\, z^3 - 239750.379\, z^2 + 4.832 \times 10^6 z - 4.241 \times 10^7. \end{aligned} \tag{49}$$

In this example, we applied the above mentioned members of the family (10) *for the simultaneous computing of all zeros of f starting from Aberth's initial approximation* (48) *with $r = 160$.*

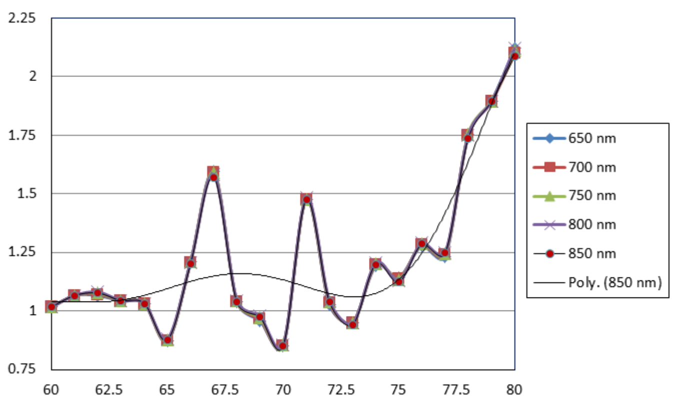
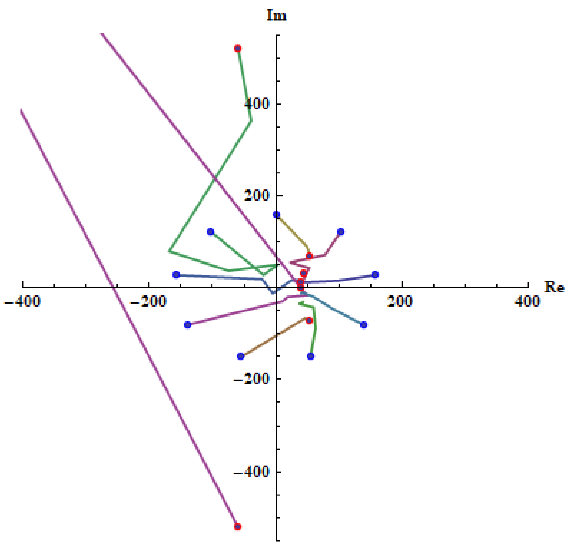


Figure 4. Dependence between the optical density and the temperature.

The data in Table 2 show that Ehrlich's method and the random one give better results than the other two methods. It is interesting to note that, for this example, the random method behaves better than Ehrlich's method and much better than Dochev–Burnev's method. The approximation trajectories for Ehrlich's method and the random method are plotted in Figure 5a,b. Despite the similarity of the convergence behaviors, the trajectories of the approximations of the two methods are highly different. It is worth noting that, in Figure 5a, the coordinate starting from the initial point $-138.564 - 79.999i$ 'jumps away' to the point $-849.322 + 1560.620i$ before turning back towards the root.

Table 2. Numerical results for Example 2.

Method	k	$E_f(x^{(k)})$	τ	ε_k	ε_{k+1}	r_k
$\alpha = 0$	23	7.876×10^{-13}	0.06822	1.003×10^{-11}	1.244×10^{-35}	3.000619
$\alpha = 1$	14	1.043×10^{-20}	–	4.847×10^{-18}	1.107×10^{-59}	3.002078
$\alpha = 1/2$	17	7.045×10^{-13}	–	8.978×10^{-12}	6.633×10^{-36}	3.016032
$\alpha = 0.766 + 0.484i$	13	3.968×10^{-12}	–	5.057×10^{-11}	1.023×10^{-33}	3.002426



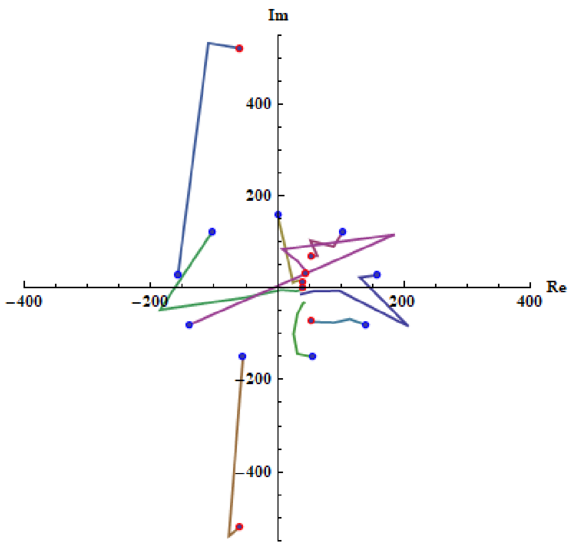


Figure 5. Trajectories of approximations for Example 2. (**a**) For $\alpha = 1$ (Ehrlich's method). (**b**) For $\alpha = 0.7669 + 0.4847i$.

Example 3 (Schrödinger wave equation for a hydrogen atom)**.** *In quantum mechanics, the position of the electron relative to the core has a probability distribution that is related to the solution of the Schrödinger wave equation for a charged particle moving in a Coulomb potential. The famous Schrödinger equation for a single particle of mass μ moving in a central potential can be written in the following form:*

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi - k\frac{e^2}{r}\Psi = \mathcal{E}\Psi\,, \tag{50}$$

where $\mathcal{E}$ is the energy and r is the distance of the electron from the core. In spherical coordinates, Equation (50) takes the form

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] + \frac{e^2}{r}\Psi = \mathcal{E}\Psi. \tag{51}$$

By using some conventional methods, the last equation can be separated into two equations called a radial equation and angular equation. It is well known that the angular equation can also be separated into two equations, one of which leading to the associated Legendre equation (see, e.g., [37]):

$$(1-x^2)f''(x) - 2xf'(x) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)f(x) = 0. \tag{52}$$

In the azimuthally symmetric case $m = 0$, the solution of (52) can be presented in terms of Legendre polynomials. For the purposes of our example, we applied the above mentioned members of the family (10) for the simultaneous computing of all zeros of the polynomial

$$f(z) = 46189\,z^{10} - 109395\,z^8 + 90090\,z^6 - 30030\,z^4 + 3465\,z^2 - 63 \tag{53}$$

that coincide with the zeros of the tenth-order Legendre polynomial. In this example, we used a different kind of initial approximation, namely the following:

$$\begin{aligned} x^{(0)} \quad = \quad & (-3.343 + 2.375i, 2.690 - 0.008i, -3.155 - 1.844i, 3.592 - 1.444i, -2.417 + 0.840i, \\ & 0.843 - 3.729i, -1.468 - 3.630i, 2.031 + 2.411i, -2.295 - 1.281i, -1.212 - 2.508i) \end{aligned}$$

whose coordinates are randomly chosen from the square

$$\{z \in \mathbb{C} \colon |Re(z)| \le 4 \;\; and \;\; |Im(z)| \le 4\}.$$

Despite the 'rough' choice of the initial approximation, it can be seen from Table 3 that, for $\alpha = 1$ (Ehrlich's method), the stopping criterion (45) is satisfied at the 13th iteration with an error estimation smaller than 10^{-18}, and, at the next step, the roots are found with an accuracy of 10^{-55}. It can also be seen that the obtained COC confirms the theoretical order of convergence proven in Section 3. Approximation trajectories for this example calculated for the methods obtained for $\alpha = 1$ (Ehrlich's method) and $\alpha = 0.766 + 0.484i$ are plotted in Figure 6a,b.

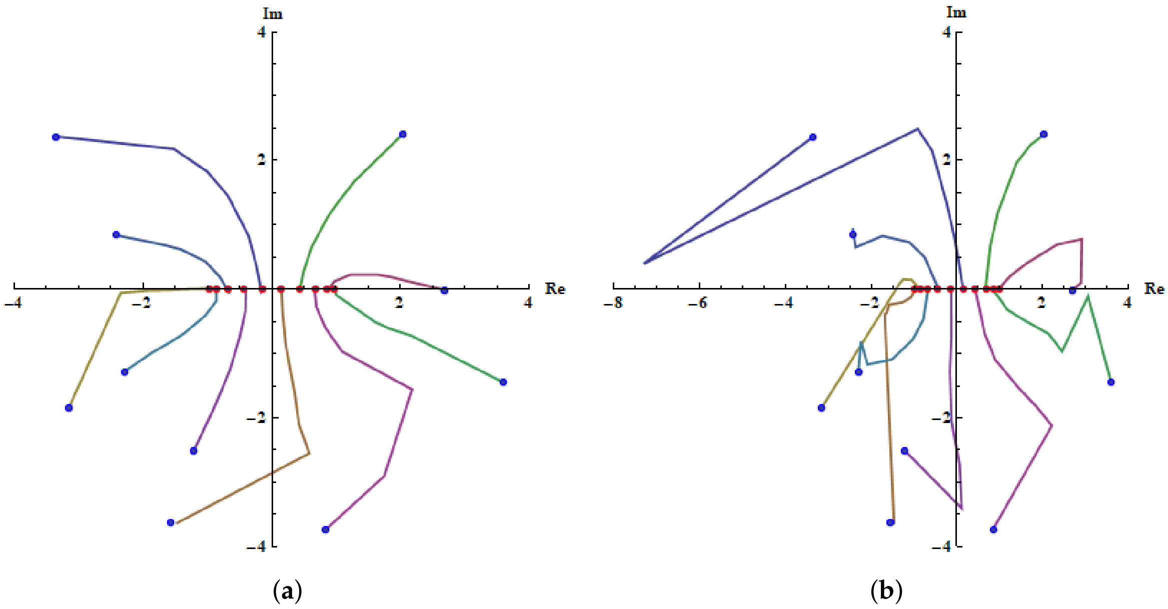


Figure 6. Trajectories of approximations for Example 3. (**a**) For $\alpha = 1$ (Ehrlich's method). (**b**) For $\alpha = 0.7669 + 0.4847i$.

Table 3. Numerical results for Example 3.

Method	k	$E_f(x^{(k)})$	τ	ε_k	ε_{k+1}	r_k
$\alpha = 0$	19	8.233×10^{-9}	0.0625	8.961×10^{-11}	4.148×10^{-26}	2.996272
$\alpha = 1$	13	1.257×10^{-18}	–	1.368×10^{-19}	2.897×10^{-56}	3.000015
$\alpha = 1/2$	17	1.473×10^{-16}	–	3.625×10^{-17}	8.827×10^{-49}	2.999946
$\alpha = 0.766 + 0.484i$	15	1.292×10^{-19}	–	2.152×10^{-20}	1.473×10^{-58}	3.003039

5. Conclusions

A detailed local convergence analysis of the one-parameter family of simultaneous methods (10) has been provided. As a result, two types of local convergence theorems (Theorems 3 and 4) with a priori and a posteriori error estimates and with an assessment of the asymptotic error constant were proven. The obtained theorems unify the best results of their kind regarding the well-known Dochev-Byrnev and Ehrlich's methods. A comparative analysis based on the stability and on Theorem 3 was conducted to show the advantages and disadvantages of several particular members of the family (10). Some real-world applications were given to show the applicability and to emphasize the convergence behaviors of the methods (10).

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