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Smooth Information Criterion for Regularized Estimation of Item Response Models

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Abstract: Item response theory (IRT) models are frequently used to analyze multivariate categorical data from questionnaires or cognitive test data. In order to reduce the model complexity in item response models, regularized estimation is now widely applied, adding a nondifferentiable penalty function like the LASSO or the SCAD penalty to the log-likelihood function in the optimization function. In most applications, regularized estimation repeatedly estimates the IRT model on a grid of regularization parameters λ. The final model is selected for the parameter that minimizes the Akaike or Bayesian information criterion (AIC or BIC). In recent work, it has been proposed to directly minimize a smooth approximation of the AIC or the BIC for regularized estimation. This approach circumvents the repeated estimation of the IRT model. To this end, the computation time is substantially reduced. The adequacy of the new approach is demonstrated by three simulation studies focusing on regularized estimation for IRT models with differential item functioning, multidimensional IRT models with cross-loadings, and the mixed Rasch/two-parameter logistic IRT model. It was found from the simulation studies that the computationally less demanding direct optimization based on the smooth variants of AIC and BIC had comparable or improved performance compared to the ordinarily employed repeated regularized estimation based on AIC or BIC.

Keywords: regularized estimation; item response models; smooth information criterion; differential item functioning; multidimensional item response model; Rasch model; SCAD penalty

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1. Introduction

Item response theory (IRT; [1–5]) modeling is a class of statistical models that analyze discrete multivariate data. In these models, a vector $\boldsymbol{X} = (X_1, \dots, X_I)$ of I discrete variables X_i ($i = 1, \dots, I$; also referred to as items) is summarized by a unidimensional or multidimensional factor variable $\boldsymbol{\theta}$. In this article, we confine ourselves to dichotomous random variables $X_i \in \{0, 1\}$.

The multivariate distribution for the vector $\boldsymbol{X} \in \{0, 1\}^I$ in the IRT model is defined as

$$\mathsf{P}(\boldsymbol{X} = \boldsymbol{x}; \boldsymbol{\gamma}) = \int \prod_{i=1}^{I} \mathsf{P}(X_i = x_i | \boldsymbol{\theta}; \boldsymbol{\gamma}_i) f(\boldsymbol{\theta}; \boldsymbol{\beta}) \, \mathrm{d}\boldsymbol{\theta} \,, \tag{1}$$

where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_I, \boldsymbol{\beta})$ is the vector of model parameters. The vector $\boldsymbol{\gamma}_i$ contains item parameters of item i, while $\boldsymbol{\beta}$ parametrizes the density f of the factor variable $\boldsymbol{\theta}$. Note that (1) includes a local independence assumption. That is, the items X_i are conditionally independent given the factor variable $\boldsymbol{\theta}$. The function $\boldsymbol{\theta} \mapsto \mathsf{P}(X_i = x_i | \boldsymbol{\theta}; \boldsymbol{\gamma}_i)$ is also referred to as the item response function (IRF; [6–8]). The two-parameter logistic (2PL) model [9] uses the IRF $\theta \mapsto \Psi(a_i\theta - b_i)$, where Ψ denotes the logistic distribution function.

Now, assume that N independent replications of $\boldsymbol{X}$ are available. The parameter vector $\boldsymbol{\gamma}$ from these observations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ can be estimated by minimizing the negative log-likelihood function

$$l(\boldsymbol{\gamma}) = -\sum_{n=1}^{N} \log \mathsf{P}(\boldsymbol{X} = \boldsymbol{x}_n; \boldsymbol{\gamma}) , \tag{2}$$

where the parameter vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_H)$ contains H components that have to be estimated.

In various applications, the IRT model (1) is not identified or includes too many parameters, making the interpretation difficult. To this end, some sparsity structure [10] on model parameters $\boldsymbol{\gamma}$ is imposed. Regularized estimation as a machine learning technique is employed in IRT models to make estimation feasible [11–13]. More formally, sparsity structure on $\boldsymbol{\gamma}$ is imposed by replacing the negative log-likelihood function with a negative regularized log-likelihood function

$$l_{\text{reg}}(\boldsymbol{\gamma}; \lambda) = l(\boldsymbol{\gamma}) + N \sum_{h=1}^{H} \iota_h \mathcal{P}(\gamma_h, \lambda) , \tag{3}$$

where ι_h is an indicator variable for the parameter γ_h that takes values 0 or 1. The indicator ι_h equals 1 if γ_h is regularized (i.e., the sparsity structure assumption applies to this parameter), while it is 0 if γ_h should not be regularized. Let $H_1 = \sum_{h=1}^{H} \iota_h$ be the number of regularized parameters and $H_0 = H - H_1$ is the number of nonregularized model parameters. The regularized negative log-likelihood function l_{reg} defined in (3) includes a penalty function $\mathcal{P}$ that decodes the assumptions about sparsity. For a scalar parameter x, the least absolute shrinkage and selection operator (LASSO; [14]) penalty is a popular penalty function used in regularization, and it is defined as

$$\mathcal{P}_{\text{LASSO}}(x, \lambda) = \lambda |x| , \tag{4}$$

where λ is a nonnegative regularization parameter that controls the extent of sparsity in the obtained parameter estimate. It is well-known that the LASSO penalty introduces bias in estimated parameters. To circumvent this issue, the smoothly clipped absolute deviation (SCAD; [15]) penalty has been proposed.

$$\mathcal{P}_{\text{SCAD}}(x, \lambda) = \begin{cases} \lambda |x| & \text{if } |x| < \lambda \\ -(x^2 - 2a\lambda |x|^2 + \lambda^2)(2(a-1))^{-1} & \text{if } \lambda \le |x| \le a\lambda \\ (a+1)\lambda^2 & \text{if } |x| > a\lambda \end{cases} \tag{5}$$

In many studies, the recommended value of $a = 3.7$ (see [15]) has been adopted (e.g., [10,16]). Note that $\mathcal{P}_{\text{SCAD}}$ has the property of the LASSO penalty around zero, but has zero derivatives for x values that strongly differ from zero.

A parameter estimate $\widehat{\boldsymbol{\gamma}}$ of the regularized IRT model is defined as an estimator defined as the minimizer of l_{reg}

$$\widehat{\boldsymbol{\gamma}}(\lambda) = \arg\min_{\boldsymbol{\gamma}} l_{\text{reg}}(\boldsymbol{\gamma}; \lambda) . \tag{6}$$

Note that the penalty function $\mathcal{P}$ involves a fixed tuning parameter λ. Hence, the parameter estimate $\widehat{\boldsymbol{\gamma}}(\lambda)$ depends on λ. A crucial issue of the LASSO and the SCAD penalty functions is that they are nondifferentiable functions because the function $x \mapsto |x|$ is nondifferentiable. Hence, particular estimation techniques for nondifferentiable optimization problems must be applied [14,17,18]. As an alternative, the nondifferentiable optimization function can be replaced by a differentiable approximation [19–22]. For example, the absolute value function $x \mapsto |x|$ in the SCAD penalty can be replaced with $x \mapsto \sqrt{x^2 + \varepsilon}$ for a sufficiently small ε such as $\varepsilon = 0.001$. Using differentiable approximations has the advantage that ordinary gradient-based optimizers can be utilized.

In practice, the estimation of the regularized IRT model is carried out on a grid of T values of λ in a grid $\Lambda = \{\lambda_1, \dots, \lambda_T\}$. For each value of the tuning parameter λ_t, a parameter estimate $\widehat{\boldsymbol{\gamma}}(\lambda_t)$ is obtained. A final parameter estimate $\widehat{\boldsymbol{\gamma}}$ is obtained by minimizing an information criterion

$$IC(\widehat{\boldsymbol{\gamma}}(\lambda)) = 2l(\widehat{\boldsymbol{\gamma}}(\lambda)) + K_N \left(H_0 + \sum_{h=1}^{H} \iota_h \mathbf{1}(\widehat{\gamma}_h(\lambda) \neq 0) \right) , \tag{7}$$

where the factor K_N is chosen as $K_N = 2$ for the Akaike information criterion (AIC; [23]) and $K_N = \log N$ for the Bayesian information criterion (BIC; [24]) (see [25]).

If the regularized likelihood function is evaluated with differentiable approximations, there are no regularized parameters that exactly equal zero (in contrast to special-purpose optimizers for regularized estimation; [17]). Hence, estimated parameters γ_h are counted as zero if they do not exceed a fixed threshold τ (such as 0.001, 0.01, or 0.02) in its absolute value. Hence, the approximated information criterion is computed as

$$IC(\widehat{\boldsymbol{\gamma}}(\lambda)) = 2l(\widehat{\boldsymbol{\gamma}}(\lambda)) + K_N \left(H_0 + \sum_{h=1}^{H} \iota_h \mathbf{1}(|\widehat{\gamma}_h(\lambda)| > \tau) \right) . \tag{8}$$

The final estimator of $\boldsymbol{\gamma}$ is defined as

$$\widehat{\boldsymbol{\gamma}}_{IC} = \widehat{\boldsymbol{\gamma}}(\widehat{\lambda}_{\text{opt}}) \text{ with } \widehat{\lambda}_{\text{opt}} = \arg\min_{\lambda \in \Lambda} IC(\widehat{\boldsymbol{\gamma}}(\lambda)) . \tag{9}$$

Depending on the chosen value of K_N, the regularized parameter estimate can be based on the AIC and BIC.

The ordinary estimation approach to regularized estimation described above has the computational disadvantage that it requires a sequential fitting of models on the grid Λ of the regularization parameter λ. This approach is referred to as an indirect optimization approach because it first minimizes a criterion function (i.e., the regularized likelihood function) with respect to $\boldsymbol{\gamma}$ for a fixed value of λ and optimizes a second criterion (i.e., the AIC or BIC) in the second step. O'Neill and Burke [26] proposed an estimation approach to regularized estimation that directly minimizes a smooth version of the BIC (i.e., smooth Bayesian information criterion, SBIC) for regression models. This direct estimation approach has been successfully implemented for structural equation models [21,27]. For these models, the optimization based on SBIC had similar, if not better, performance than the ordinary estimation of regularized models based on the AIC and BIC. In this paper, we explore whether the smooth information criteria SBIC and the smooth Akaike information criterion (SAIC) also hold promise for various applications in IRT models. Using a computationally cheaper alternative for regularized estimation is probably even more important for IRT models than for structural equation models because IRT models are more difficult to estimate and more computationally demanding. To the best of our knowledge, this is the first attempt at using smoothed information criteria in IRT models.

The rest of this paper is structured as follows. The optimization using smooth information criteria is outlined in Section 2. Afterward, three applications of regularized IRT models are investigated in three simulation studies. Section 3 presents Simulation Study 1, which studies regularized estimation for differential item functioning. Section 4 presents Simulation Study 2, which investigates the regularized estimation of multidimensional IRT models. The last Simulation Study 3 in Section 5 is devoted to regularized estimation of the mixed Rasch/2PL model. Finally, this study closes with a discussion in Section 6.

2. Smooth Information Criterion

In theory, a parameter estimate $\widehat{\boldsymbol{\gamma}}$ for $\boldsymbol{\gamma}$ of the IRT model may be obtained by directly minimizing an information criterion

$$\widehat{\boldsymbol{\gamma}} = \arg\min_{\boldsymbol{\gamma}} \left\{ 2l(\boldsymbol{\gamma}) + K_N \left(H_0 + \sum_{h=1}^{H} \iota_h \mathbf{1}(\gamma_h \neq 0) \right) \right\} . \tag{10}$$

The optimization function in (10) can be interpreted as a regularized log-likelihood function with an L_0 penalty [28,29]. Obviously, the indicator function $\mathbf{1}$ in (10) counts the number of regularized parameters that differ from zero. Researchers O'Neill and Burke [26] proposed substituting the indicator function with a suitable differentiable approximation $\mathcal{N}_\varepsilon$. To this end, a smooth information criterion, such as the SAIC and the SBIC, is obtained. In more detail, the differentiable approximation $\mathcal{N}_\varepsilon$ for $\mathbf{1}$ is defined as

$$\mathcal{N}_\varepsilon(x) = \frac{x^2}{x^2 + \varepsilon} , \tag{11}$$

where $\varepsilon > 0$ is a sufficiently small tuning parameter, such as $\varepsilon = 0.001$. The function $\mathcal{N}_\varepsilon$ takes values close to zero for x arguments close to 0 and approaches 1 if $|x|$ moves away from 0. A smoothed information criterion $SIC(\boldsymbol{\gamma})$ (abbreviated as SIC) can be defined as

$$SIC(\boldsymbol{\gamma}) = 2l(\boldsymbol{\gamma}) + K_N \left(H_0 + \sum_{h=1}^{H} \iota_h \mathcal{N}_\varepsilon(\gamma_h) \right) . \tag{12}$$

We obtain the SAIC for the choices of K_N in (12) of $K_N = 2$ and the SBIC for $K_N = \log(N)$.

Hence, the minimization problem (10) can be replaced by

$$\widehat{\boldsymbol{\gamma}} = \arg\min_{\boldsymbol{\gamma}} SIC(\boldsymbol{\gamma}) = \arg\min_{\boldsymbol{\gamma}} \left\{ 2l(\boldsymbol{\gamma}) + K_N \left(H_0 + \sum_{h=1}^{H} \iota_h \mathcal{N}_\varepsilon(\gamma_h) \right) \right\} . \tag{13}$$

The optimization function in (13) directly minimizes a smoothed version of the information criterion.

3. Simulation Study 1: Differential Item Functioning

In the first Simulation Study 1, the assessment of differential item functioning (DIF; [30–32]) is considered as an example. DIF occurs in datasets with multiple groups if item parameters are not invariant (i.e., they are not equal) across groups. In this study, the case of two groups in the unidimensional 2PL model is treated. The IRF is given by

$$\mathsf{P}(X_i = 1 | G = g, \theta) = \Psi(a_i\theta - b_i - \delta_i \mathbf{1}(G = 2)) \text{ for } g = 1, 2 , \tag{14}$$

where δ_i indicates the DIF in item intercepts, which is also referred to as uniform DIF. The item parameters of item X_i are given by $\boldsymbol{\gamma}_i = (a_i, b_i, \delta_i)$. The mean and the standard deviation of θ in the first group are fixed for identification reasons to 0 and 1, respectively. Then, the mean μ_2 and the standard deviation σ_2 of θ in the second group can be estimated.

It has been pointed out that additional assumptions about DIF effects δ_i must be imposed for model identification [33–35]. Assuming a sparsity structure on the DIF effects may be one plausible option. To this end, DIF effects δ_i ($i = 1, \ldots, I$) are regularized in the optimization based on the regularized log-likelihood function (3) or the minimization of the SIC (13). Regularized estimation of DIF in IRT models has been widely discussed in the literature [36–42].

3.1. Method

In this simulation study, we use a data-generating model (DGM) similar to the one used in the simulation study in [38]. The factor variable θ was assumed to be univariate normally distributed. We fixed the mean μ_1 and the standard deviation σ_1 of the factor variable θ in the first group to 0 and 1, respectively. The factor variable θ had a mean μ_2 of 0.5 and a standard deviation σ_2 of 0.8 in the second group. In total, $I = 25$ items were used in this simulation study.

We now describe the item parameters used for the IRF defined in (14). The common item discriminations a_i of the 25 items were chosen as 1.3, 1.4, 1.5, 1.7, 1.6, 1.3, 1.4, 1.5, 1.7, 1.6, 1.3, 1.4, 1.5, 1.7, 1.6, 1.3, 1.4, 1.5, 1.7, 1.6, 1.3, 1.4, 1.5, 1.7, and 1.6. The item difficulties b_i were chosen as −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, and −2.0. The DIF effects δ_i were zero for the first 15 items. Items 16 to 25 had non-zero DIF effects.

In the condition of small DIF effects (see [38]), we chose δ_i values of −0.60, 0.60, −0.65, 0.70, 0.65, −0.70, 0.60, −0.65, 0.70, and −0.65 for Items 16 to 25. In the condition of large DIF effects, we multiplied these effects by 2. These two conditions are referred to as balanced DIF conditions because the DIF effects δ_i average to zero. In line with other studies, we also considered unbalanced DIF [43], in which we took absolute DIF effects in the small DIF and large DIF conditions. In the unbalanced DIF conditions, all DIF effects δ_i were assumed positive and did not average to zero. The item parameters can also be found at https://osf.io/ykew6 (accessed on 2 April 2024).

Moreover, we varied the sample size N in this simulation study by 500, 1000, and 2000. There were $N/2$ subjects in each of the two groups.

The regularized 2PL model with DIF was estimated with the regularized likelihood function using the SCAD penalty on a nonequidistant grid of 37 λ values between 0.0001 and 1 (see the R simulation code at https://osf.io/ykew6; accessed on 2 April 2024). We approximated the nondifferentiable SCAD penalty function by its differentiable approximating function using the tuning $\varepsilon = 0.001$. We saved parameter estimates that minimized AIC and BIC. Item parameters that did not exceed the threshold $\tau = 0.02$ in its absolute value were regularized. In the direct minimization of SAIC and SBIC, we tried the values 0.01, 0.001, and 0.0001 of the tuning parameters ε. It was found that $\varepsilon = 0.001$ performed best, which is the reason why we only reported this solution.

As the outcome of the simulation study, we studied (average) absolute bias and (average) root mean square error (RMSE) of model parameter estimates as well as type-I error rates and power rates. Absolute bias and RMSE were computed for estimates of distribution parameters μ_2 and σ_2. Moreover, absolute bias and RMSE were computed for all estimates of DIF effects δ_i. Formally, let γ_h be the hth parameter ($h = 1, \ldots, H$) in the model parameter vector $\boldsymbol{\gamma}$. Let $\widehat{\gamma}_{hr}$ be the parameter estimate of γ_h in replication r ($r = 1, \ldots, R$). The absolute bias (abias) of the parameter estimate $\widehat{\gamma}_h$ was computed as

$$\text{abias}(\widehat{\gamma}_h) = \left| \frac{1}{R} \sum_{r=1}^{R} \widehat{\gamma}_{hr} - \gamma_h \right| . \tag{15}$$

The RMSE was computed as

$$\text{RMSE}(\widehat{\gamma}_h) = \sqrt{\frac{1}{R} \sum_{r=1}^{R} (\widehat{\gamma}_{hr} - \gamma_h)^2} . \tag{16}$$

The average absolute bias and average RMSE were computed for DIF effects with true values of 0 (i.e., DIF effects for Items 1 to 15; non-DIF items) and for DIF effects different from 0 (i.e., DIF effects for Items 16 to 25; DIF items). The (average) type-I error rates was assessed for non-DIF items as the proportion of events in which an estimated DIF effect differed from zero (i.e., it exceeded the threshold $\tau = 0.02$ in its absolute value). The (average) power rates were determined for DIF items accordingly. More formally, the type-I error rate or power rate (abbreviated as "rate" in (17)) was determined by

$$\text{rate}(\widehat{\gamma}_h) = 100 \cdot \frac{1}{R} \sum_{r=1}^{R} \mathbf{1}(|\widehat{\gamma}_{hr}| > \tau) . \tag{17}$$

Absolute bias values smaller than 0.03 were classified as acceptable in this simulation study. Moreover, type-I error rates smaller than 10.0 and power rates larger than 80.0 were seen as satisfactory.

In total, $R = 750$ replications were conducted in each of the 2 (small vs. large DIF) $\times$ 2 (balanced vs. unbalanced DIF) $\times$ 3 (sample size) = 12 cells of the simulation study. The entire simulation study was conducted with the R [44] statistical software. The estimation of the regularized IRT model was carried out using the `sirt::xxirt()` function in the R package sirt [45]. Replication material for the simulation study can be found at https://osf.io/ykew6 (accessed on 2 April 2024).

3.2. Results

Table 1 displays the average absolute bias and the average RMSE of model parameters as a function of the extent of DIF and sample size N for balanced and unbalanced DIF. It turned out that the mean μ_2 and the standard deviation σ_2 of the second group were unbiasedly estimated in the balanced DIF condition. Moreover, while DIF effects for non-DIF items were unbiasedly estimated, DIF effects were biased for moderate sample sizes (i.e., for $N = 500$ and 1000). In general, there was a similar behavior of regularized estimation based on AIC and BIC compared to its smooth competitors SAIC and SBIC. However, smooth information criteria had some advantages in smaller samples with respect to the RMSE. Note that SAIC was the frontrunner in all balanced DIF conditions regarding the RMSE of the estimate of μ_2.

Table 1. Simulation Study 1: (Average) absolute bias and average root mean square error (RMSE) of model parameters as a function of the extent of differential item functioning (DIF) and sample size N for balanced and unbalanced DIF.

			(Average) Absolute Bias				(Average) RMSE			
Par	**DIF**	***N***	**AIC**	**SAIC**	**BIC**	**SBIC**	**AIC**	**SAIC**	**BIC**	**SBIC**
			Balanced DIF							
μ_2	small	500	0.001	0.004	0.000	0.003	0.104	0.090	0.113	0.104
		1000	0.002	0.000	0.001	0.001	0.069	0.064	0.075	0.070
		2000	0.001	0.001	0.000	0.000	0.048	0.046	0.046	0.047
	large	500	0.005	0.000	0.007	0.004	0.101	0.093	0.096	0.098
		1000	0.003	0.004	0.001	0.001	0.071	0.066	0.067	0.068
		2000	0.002	0.001	0.002	0.002	0.050	0.049	0.049	0.049
σ_2	small	500	0.002	0.002	0.007	0.001	0.071	0.070	0.070	0.070
		1000	0.001	0.001	0.002	0.001	0.046	0.046	0.046	0.046
		2000	0.001	0.001	0.001	0.002	0.032	0.032	0.032	0.032
	large	500	0.003	0.001	0.000	0.002	0.068	0.067	0.067	0.067
		1000	0.002	0.003	0.002	0.002	0.045	0.044	0.044	0.044
		2000	0.001	0.001	0.001	0.001	0.035	0.035	0.035	0.035
δ_i (no DIF)	small	500	0.006	0.006	0.003	0.004	0.216	0.187	0.108	0.148
		1000	0.005	0.003	0.002	0.002	0.139	0.113	0.061	0.069
		2000	0.002	0.002	0.001	0.001	0.098	0.073	0.032	0.028
	large	500	0.003	0.005	0.003	0.002	0.201	0.188	0.082	0.142
		1000	0.006	0.004	0.001	0.002	0.140	0.115	0.047	0.067
		2000	0.006	0.002	0.001	0.001	0.098	0.073	0.031	0.030
δ_i (DIF)	small	500	0.025	0.024	**0.182**	**0.077**	0.349	0.330	0.496	0.398
		1000	0.006	0.008	**0.062**	**0.041**	0.211	0.213	0.315	0.276
		2000	0.003	0.003	0.010	0.011	0.137	0.135	0.155	0.157
	large	500	0.026	0.024	0.022	0.022	0.311	0.302	0.340	0.311
		1000	0.017	0.017	0.015	0.015	0.212	0.207	0.210	0.208
		2000	0.007	0.004	0.004	0.003	0.149	0.146	0.146	0.146

Table 1. *Cont.*

Par	DIF	N	(Average) Absolute Bias				(Average) RMSE			
			AIC	SAIC	BIC	SBIC	AIC	SAIC	BIC	SBIC
			Unbalanced DIF							
μ_2	small	500	**0.093**	**0.099**	**0.115**	**0.091**	0.157	0.139	0.163	0.144
		1000	**0.050**	**0.047**	**0.049**	**0.033**	0.111	0.087	0.110	0.085
		2000	0.025	0.008	0.013	0.004	0.069	0.048	0.072	0.048
	large	500	**0.053**	**0.072**	0.024	0.020	0.145	0.122	0.143	0.100
		1000	0.024	0.030	0.003	0.002	0.084	0.076	0.077	0.068
		2000	0.004	0.012	0.001	0.000	0.053	0.059	0.048	0.048
σ_2	small	500	0.004	0.004	0.001	0.003	0.067	0.067	0.067	0.067
		1000	0.000	0.000	0.001	0.000	0.048	0.048	0.048	0.048
		2000	0.000	0.000	0.001	0.000	0.032	0.031	0.031	0.031
	large	500	0.000	0.001	0.000	0.001	0.065	0.065	0.064	0.065
		1000	0.000	0.001	0.001	0.000	0.046	0.046	0.045	0.045
		2000	0.001	0.002	0.001	0.001	0.031	0.032	0.031	0.032
δ_i (no DIF)	small	500	**0.114**	**0.108**	**0.035**	**0.062**	0.296	0.251	0.170	0.207
		1000	**0.072**	**0.055**	0.028	0.016	0.207	0.150	0.143	0.095
		2000	**0.037**	0.013	0.015	0.001	0.122	0.079	0.098	0.023
	large	500	**0.079**	**0.114**	**0.032**	0.030	0.255	0.236	0.206	0.144
		1000	**0.038**	**0.051**	0.005	0.003	0.133	0.145	0.072	0.065
		2000	0.010	0.025	0.001	0.001	0.055	0.107	0.023	0.030
δ_i (DIF)	small	500	**0.185**	**0.221**	**0.399**	**0.262**	0.405	0.409	0.553	0.462
		1000	**0.089**	**0.111**	**0.158**	**0.117**	0.270	0.285	0.368	0.319
		2000	**0.036**	0.010	0.020	0.010	0.167	0.135	0.179	0.151
	large	500	**0.075**	**0.112**	**0.037**	0.029	0.339	0.312	0.354	0.303
		1000	**0.036**	**0.050**	0.005	0.004	0.212	0.199	0.207	0.196
		2000	0.011	0.020	0.004	0.004	0.138	0.144	0.137	0.138

Note. Par = parameter; μ_2 = mean of θ in second group; σ_2 = standard deviation of θ in second group; δ_i (no DIF) = DIF parameters with zero population values; δ_i (DIF) = DIF parameters with non-zero population values; Absolute bias values larger than 0.03 are printed in bold font.

In the unbalanced DIF condition, estimated group means and DIF effects were generally biased. However, the bias decreased with increased sample size and was smaller with large instead of small DIF effects. SBIC was the frontrunner on five out of six conditions for estimates of μ_2 with respect to the RMSE. Only for $N = 500$ and small DIF, SAIC outperformed the other estimators.

Table 2 presents average type-I error and power rates for DIF effects of non-DIF and DIF items. It is evident that AIC and SAIC had inflated type-I error rates. Moreover, BIC and SBIC had acceptable type-I error rates. However, SBIC had an inflated type-I error rate for $N = 500$ in the unbalanced DIF condition with a small DIF. Overall, the power rates of regularized estimators AIC and BIC performed similarly to their smooth alternatives SAIC and SBIC. However, SBIC slightly outperformed BIC in terms of power rates.

Table 2. Simulation Study 1: Type-I error rate and power rate for DIF effects δ_i as a function of the extent of differential item functioning (DIF) and sample size N for balanced and unbalanced DIF.

DIF	N	Type-I Error Rate				Power Rate			
		AIC	SAIC	BIC	SBIC	AIC	SAIC	BIC	SBIC
		Balanced DIF							
small	500	**17.0**	**13.7**	2.1	6.2	83.6	85.8	**52.7**	**73.3**
	1000	**14.4**	8.2	1.4	2.1	97.0	96.1	81.5	87.3
	2000	**14.4**	6.2	0.7	0.5	99.9	99.8	97.6	97.5

Table 2. *Cont.*

DIF	N	Type-I Error Rate AIC	SAIC	BIC	SBIC	Power Rate AIC	SAIC	BIC	SBIC
large	500	**15.3**	**14.2**	1.2	5.8	99.7	99.8	97.5	99.4
	1000	**15.0**	9.0	0.8	2.0	100	100	99.9	100
	2000	**14.5**	6.2	0.6	0.6	100	100	100	100
		Unbalanced DIF							
small	500	**25.8**	**23.8**	4.8	**11.3**	**68.9**	**65.6**	**30.6**	**54.0**
	1000	**21.0**	**15.8**	5.1	3.7	89.4	85.4	**71.3**	**79.9**
	2000	**13.6**	9.1	2.7	0.4	97.7	99.6	95.9	98.1
large	500	**14.5**	**28.9**	3.4	6.3	98.0	99.0	96.7	98.9
	1000	9.6	**22.4**	0.6	2.0	99.8	100	99.7	100
	2000	3.4	**21.5**	0.3	0.6	100	100	100	100

Note. Type-I error rates larger than 10.0 and power rates smaller than 80.0 are printed in bold font.

4. Simulation Study 2: Multidimensional Logistic Item Response Model

In this Simulation Study 2, the multidimensional logistic IRT model [46] with cross-loadings is studied. That is, each item X_i is allocated to a primary dimension θ_d. However, it could be that this item also loads on other dimensions than the primary dimension (i.e., the target factor variable). Formally, the IRF of the multidimensional logistic IRT model is given by

$$P(X_i = 1|\boldsymbol{\theta}) = \Psi\left(\sum_{d=1}^{D} a_{id}\theta_d - b_i\right), \tag{18}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_D)$. All item discriminations a_{id} are regularized in the estimation, except those that load on the primary dimension. The means and standard deviations of factor variables θ_d are fixed at 0 and 1 for identification reasons, respectively. The correlations between the dimensions can be estimated.

The regularized estimation of this model has been discussed in Refs. [47–49]. To ensure the identifiability of the model parameter, a sparse loading structure for item discriminations a_{id} is imposed. That is, most item discriminations are (approximately) zero in the DGM. Only a few loadings are allowed to differ from 0. Notably, regularized estimation of factor models can be regarded as an alternative to rotation methods in exploratory factor analysis [50,51].

4.1. Method

In this simulation study, we used a DGM with $I = 20$ items and $D = 2$ factor variables θ_1 and θ_2. The first 10 items loaded on the first dimension, while Items 11 to 20 loaded on the second dimension. The factor variable (θ_1, θ_2) was bivariate normally distributed with standardized normally distributed components and a fixed correlation ρ of 0.5.

Moreover, we specified five cross-loadings. Items 2 and 6 had a cross-loading of size δ on the second dimension, while Items 13, 16, and 19 had a cross-loading of size δ on the first dimension. The DGM is visualized in Figure 1. In more detail, the loading matrix $\boldsymbol{A}$ that contains the item discriminations a_{id} (see (18)) is given by

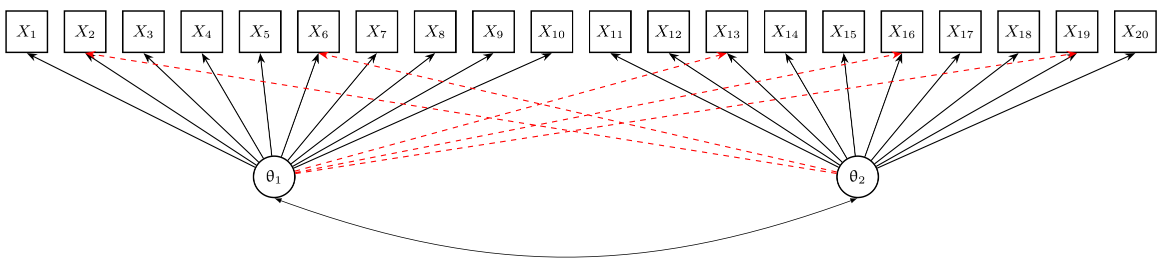

Figure 1. Simulation Study 2: Data-generating model with $I = 20$ items X_i ($i = 1, \ldots, 20$) and two factor variables θ_1 and θ_2. Cross-loadings are depicted by red dashed lines.

$$
\boldsymbol{A} = \begin{pmatrix} 0.6 & 0 \\ 0.8 & \delta \\ 1.0 & 0 \\ 1.4 & 0 \\ 1.2 & 0 \\ 0.6 & \delta \\ 0.8 & 0 \\ 1.0 & 0 \\ 1.4 & 0 \\ 1.2 & 0 \\ 0 & 0.6 \\ 0 & 0.8 \\ \delta & 1.0 \\ 0 & 1.4 \\ 0 & 1.2 \\ \delta & 0.6 \\ 0 & 0.8 \\ 0 & 1.0 \\ \delta & 1.4 \\ 0 & 1.2 \end{pmatrix} . \tag{19}
$$

The size of the cross-loading δ was chosen as 0.3, indicating a small cross-loading), or 0.5, indicating a large cross-loading. The item difficulties b_i (see (18)) of the 20 items were -0.8, 0.4, 1.2, 2.0, -2.0, -0.8, 0.4, 1.2, 2.0, -2.0, -0.8, 0.4, 1.2, 2.0, -2.0, -0.8, 0.4, 1.2, 2.0, and -2.0. The item parameters can also be found at https://osf.io/ykew6 (accessed on 2 April 2024).

We varied the sample size N as 500, 1000, and 2000, which may be interpreted as a small, moderate, and large sample size.

Like in Simulation Study 1, we compared the performance of regularized estimation based on AIC and BIC with the smooth alternatives SAIC and SBIC. A nonequidistant grid of 37 λ values between 0.0001 and 1 was chosen (see the R simulation code at https://osf.io/ykew6; accessed on 2 April 2024). The optimization functions were specified with the same tuning parameters for differentiable approximations as in Simulation Study 1 (see Section 3.1). (Average) absolute bias and (average) RMSE of model parameters, as well as type-I error rates and power rates for cross-loadings, were assessed for the four estimation methods.

In total, $R = 750$ replications were conducted in each of the 2 (small vs. large cross-loadings) $\times$ 3 (sample size) $= 6$ cells of the simulation study. The whole simulation study was conducted using the statistical software R [44]. The estimation of the regularized multidimensional logistic IRT model was carried out using the `sirt::xxirt()` function in the R package sirt [45]. Replication material for this simulation study can also be found at https://osf.io/ykew6 (accessed on 2 April 2024).

4.2. Results

Table 3 reports the (average) absolute bias and (average) RMSE of estimated model parameters. It turned out that the factor correlation ρ was biased for small and moderate sample sizes of $N = 500$ and $N = 1000$. The bias was reduced with larger cross-loadings in large sample sizes. However, a notable bias was even present for a large sample size $N = 2000$ if the BIC or SBIC was used. However, AIC and SAIC outperformed the other criteria for estimates of ρ with respect to bias and RMSE. Interestingly, the RMSE of SAIC was substantially smaller compared to AIC for the factor correlation ρ, as well as for true zero cross-loadings (i.e., rows "$CL = 0$" in Table 3) and non-zero cross-loadings (i.e., rows "$CL \neq 0$" in Table 3).

Table 3. Simulation Study 2: (Average) absolute bias and average root mean square error (RMSE) of model parameters as a function of the size of cross-loadings and sample size N.

Par	CL	N	Absolute Bias				RMSE			
			AIC	SAIC	BIC	SBIC	AIC	SAIC	BIC	SBIC
ρ	small	500	**0.064**	**0.054**	**0.104**	**0.070**	0.151	0.102	0.134	0.110
		1000	**0.040**	**0.070**	**0.088**	**0.085**	0.157	0.096	0.124	0.104
		2000	0.015	0.018	**0.059**	**0.065**	0.077	0.052	0.082	0.081
	large	500	**0.055**	**0.062**	**0.136**	**0.073**	0.167	0.109	0.179	0.117
		1000	0.029	**0.054**	**0.074**	**0.057**	0.144	0.100	0.124	0.099
		2000	0.010	0.014	0.016	0.014	0.062	0.047	0.058	0.051
$CL = 0$	small	500	**0.041**	0.016	0.015	0.008	0.243	0.113	0.130	0.095
		1000	0.025	0.028	0.008	0.014	0.188	0.119	0.092	0.088
		2000	0.011	0.006	0.006	0.003	0.102	0.062	0.049	0.034
	large	500	**0.044**	0.016	0.024	0.011	0.282	0.119	0.181	0.109
		1000	0.027	0.028	0.013	0.015	0.183	0.127	0.096	0.095
		2000	0.009	0.009	0.005	0.002	0.094	0.067	0.049	0.030
$CL \neq 0$	small	500	**0.102**	**0.141**	**0.238**	**0.177**	0.287	0.287	0.306	0.296
		1000	**0.075**	**0.118**	**0.211**	**0.192**	0.237	0.235	0.288	0.273
		2000	0.021	**0.033**	**0.132**	**0.154**	0.140	0.146	0.231	0.243
	large	500	**0.078**	**0.130**	**0.295**	**0.166**	0.341	0.353	0.452	0.379
		1000	**0.036**	**0.066**	**0.151**	**0.107**	0.228	0.245	0.336	0.291
		2000	0.011	0.009	0.025	0.028	0.123	0.116	0.161	0.161

Note. Par = parameter; ρ = correlation between factors θ_1 and θ_2; $CL = 0$ = cross-loading with zero population value; $CL \neq 0$ = cross-loading with non-zero population value;; Absolute bias values larger than 0.03 are printed in bold font.

Table 4 shows type-I error rates and power rates of estimated cross-loadings. It is evident that AIC had inflated type-I error rates, while type-I error rates of SAIC, BIC, and SBIC were acceptable. Importantly, there were low power rates for BIC and SBIC, in particular for small cross-loadings. The SAIC estimation method may be preferred if the goal is detecting non-zero cross-loadings.

Table 4. Simulation Study 2: Type-I error rate and power rate for cross-loadings as a function of the size of cross-loadings and sample size N.

CL	N	Type-I Error Rate				Power Rate			
		AIC	SAIC	BIC	SBIC	AIC	SAIC	BIC	SBIC
small	500	**16.7**	5.1	2.1	2.5	**41.2**	**30.6**	**9.4**	**22.4**
	1000	**18.7**	8.8	2.0	3.2	**58.0**	**46.5**	**17.9**	**24.3**
	2000	**15.1**	6.2	1.7	0.7	86.5	82.4	**44.0**	**38.3**
large	500	**18.5**	4.8	3.1	2.8	**68.5**	**59.5**	**27.3**	**52.0**
	1000	**17.0**	10.0	2.5	3.2	87.4	81.0	**59.2**	**70.6**
	2000	**13.3**	7.8	1.3	0.7	98.6	98.7	92.3	92.6

Note. CL = size of cross-loadings; Type-I error rates larger than 10.0 and power rates smaller than 80.0 are printed in bold font.

5. Simulation Study 3: Mixed Rasch/2PL Model

Recently, a mixed Rasch/2PL model [52] (see also [53]) received some attention. The idea of this unidimensional IRT model is to find items that conform to the Rasch model [54], while there can be a subset of items that follow the more complex 2PL model [9]. The IRF of this model is given by

$$\mathrm{P}(X_i = 1|\theta) = \Psi(\exp(\alpha_i)\theta - b_i) . \quad (20)$$

Note that the IRF in (20) is just a reparametrized 2PL model with item discriminations $a_i = \exp(\alpha_i)$. Hence, $\alpha_i = \log(a_i)$ are the logarithms of item discriminations a_i. The case

$\alpha_i = 0$ corresponds to the Rasch model because $a_i = \exp(\alpha_i) = 1$, while $\alpha_i \neq 0$ results in item discriminations a_i different from 1. The mean of the factor variable θ is fixed to 0, while the standard deviation σ should be estimated.

In order to achieve identifiability of the model parameters, a sparsity structure of the logarithms of item discriminations α_i is imposed. Hence, the majority of items is assumed to follow the Rasch model. Again, the sparsity structure is directly implemented in a regularized estimation of the mixed Rasch/2PL model.

5.1. Method

In this simulation study, we used $I = 20$ items for the DGM of the mixed Rasch/2PL model. The factor variable θ was assumed to be normally distributed with a zero mean and a standard deviation $\sigma = 1.2$. The item difficulties b_i (see the IRF in (20)) of the 20 items were chosen as −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, −2.0, −0.8, 0.4, 1.2, 2.0, and −2.0. The first 14 items followed the Rasch model (i.e., $\alpha_i = 0$ for $i = 1, \ldots, 15$). Items 15 to 20 followed the 2PL model and had α_i values that equaled $-\delta$, δ, $-\delta$, δ, δ, and $-\delta$. The size of δ controlled the deviation from the Rasch model. We either chose δ as $\log(1.4) = 0.336$ and $\log(2) = 0.693$, indicating small and large deviations from the Rasch model. Moreover, we manipulated the direction of the deviation from the Rasch model. While the previously described conditions had α_i that canceled out on average and resulted in a balanced deviation from the Rasch model (i.e., there was an equal number of items that are smaller and larger than 1, respectively), we also specified an unbalanced deviation from the Rasch model in which Items 15 to 20 all had the value δ. In this condition, we also studied small (i.e., $\delta = 0.336$) and large (i.e., $\delta = 0.693$) deviations from the Rasch model. Hence, in the case of unbalanced deviations from the Rasch model, items had either discriminations of 1 or larger than 1. The item parameters can also be found at https://osf.io/ykew6 (accessed on 2 April 2024).

Like in the other two simulation studies, we varied the sample size N as 500, 1000, and 2000.

Again, like in Simulation Study 1 and Simulation Study 2, we compared the performance of regularized estimation based on AIC and BIC with the smooth alternatives SAIC and SBIC. A nonequidistant grid of 33 λ values between 0.001 and 1 was chosen (see the R simulation code at https://osf.io/ykew6; accessed on 2 April 2024). The optimization functions were specified with the same tuning parameters for differentiable approximations as in Simulation Study 1 (see Section 3.1). (Average) absolute bias and (average) RMSE of model parameters σ and α_i ($i = 1, \ldots, I$), as well as type-I error rates and power rates for logarithms of item discriminations, were assessed.

Overall, $R = 750$ replications were conducted in each of the 2 (small vs. large deviations) $\times$ 2 (balanced vs. unbalanced deviations) $\times$ 3 (sample size) = 12 cells of the simulation study. This simulation study was also executed using the statistical software R [44]. Like in the other two simulation studies, the regularized multidimensional logistic IRT model was estimated using the `sirt::xxirt()` function in the R package sirt [45]. Replication material for this simulation study can also be found at https://osf.io/ykew6 (accessed on 2 April 2024).

5.2. Results

Table 5 contains the (average) absolute bias and (average) RMSE for the estimated model parameters. Notably, there was a different pattern of findings in the conditions of balanced and unbalanced deviations from the Rasch model. In general, SAIC performed well for the estimation of σ, except for small balanced deviations from the Rasch model with a sample size of $N = 500$. In most of the conditions, estimation based on SBIC performed similarly, if not better, than BIC for the estimation of σ in terms of RMSE.

Table 5. Simulation Study 3: (Average) absolute bias and average root mean square error (RMSE) of model parameters as a function of the sample size N and the size and the extent and direction of deviations from the Rasch model

			(Average) Absolute Bias				(Average) RMSE			
Par	**Dev**	N	**AIC**	**SAIC**	**BIC**	**SBIC**	**AIC**	**SAIC**	**BIC**	**SBIC**
			Balanced deviations from the Rasch model							
σ	small	500	**0.053**	**0.038**	0.004	0.010	0.099	0.082	0.070	0.072
		1000	**0.062**	0.005	0.012	0.004	0.107	0.054	0.051	0.056
		2000	**0.109**	0.001	0.025	0.009	0.167	0.032	0.044	0.037
	large	500	0.018	0.014	0.004	0.006	0.069	0.065	0.064	0.064
		1000	0.016	0.002	0.002	0.002	0.050	0.044	0.041	0.041
		2000	0.007	0.000	0.000	0.000	0.035	0.033	0.030	0.030
$\alpha_i = 0$	small	500	**0.035**	0.017	0.001	0.002	0.124	0.091	0.048	0.047
		1000	**0.048**	0.003	0.002	0.001	0.119	0.060	0.027	0.020
		2000	**0.096**	0.001	0.003	0.000	0.166	0.042	0.023	0.003
	large	500	0.018	0.012	0.002	0.002	0.101	0.082	0.046	0.044
		1000	0.017	0.003	0.001	0.000	0.066	0.058	0.023	0.014
		2000	0.007	0.001	0.000	0.000	0.033	0.040	0.010	0.003
$\alpha_i \neq 0$	small	500	**0.071**	**0.070**	**0.127**	**0.137**	0.224	0.239	0.283	0.289
		1000	**0.081**	0.024	**0.075**	**0.101**	0.184	0.154	0.212	0.237
		2000	**0.120**	0.003	**0.072**	**0.045**	0.199	0.085	0.159	0.156
	large	500	0.014	0.017	0.015	0.018	0.207	0.215	0.230	0.231
		1000	0.019	0.006	0.006	0.008	0.139	0.136	0.138	0.143
		2000	0.004	0.004	0.003	0.004	0.095	0.095	0.093	0.094
			Unbalanced deviations from the Rasch model							
σ	small	500	0.008	**0.032**	**0.084**	**0.083**	0.076	0.081	0.110	0.109
		1000	0.015	0.008	0.028	**0.055**	0.051	0.050	0.061	0.078
		2000	0.011	0.003	0.002	0.015	0.036	0.032	0.031	0.040
	large	500	0.001	0.001	0.001	0.001	0.069	0.062	0.061	0.060
		1000	0.005	0.001	0.000	0.001	0.046	0.043	0.040	0.040
		2000	0.002	0.003	0.002	0.002	0.034	0.032	0.030	0.029
$\alpha_i = 0$	small	500	0.009	0.005	0.008	0.004	0.099	0.078	0.057	0.047
		1000	0.014	0.003	0.002	0.001	0.068	0.056	0.027	0.019
		2000	0.010	0.002	0.001	0.000	0.044	0.042	0.015	0.005
	large	500	0.003	0.003	0.001	0.001	0.105	0.079	0.046	0.041
		1000	0.004	0.001	0.001	0.000	0.069	0.053	0.025	0.013
		2000	0.003	0.001	0.001	0.000	0.047	0.039	0.014	0.002
$\alpha_i \neq 0$	small	500	0.029	**0.069**	**0.194**	**0.205**	0.180	0.207	0.281	0.289
		1000	0.012	0.014	**0.069**	**0.145**	0.103	0.116	0.182	0.240
		2000	0.012	0.001	0.002	**0.037**	0.068	0.067	0.075	0.129
	large	500	0.008	0.006	0.007	0.005	0.129	0.126	0.127	0.127
		1000	0.011	0.007	0.006	0.006	0.091	0.089	0.089	0.089
		2000	0.003	0.003	0.003	0.003	0.062	0.061	0.061	0.061

Note. Par = parameter; Dev = size of deviation from the Rasch model; σ = standard deviation of factor variable θ; $\alpha_i = 0$ = logarithm of item discriminations with zero population value; $\alpha_i \neq 0$ = logarithm of item discriminations with non-zero population value;; Absolute bias values larger than 0.03 are printed in bold font.

Table 6 displays type-I error rates and power rates for estimated logarithms of item discriminations. In contrast to the estimation based on the AIC, SAIC had acceptable type-I error rates. Moreover, power rates for detecting deviations from the Rasch model were much higher for SAIC then BIC or SBIC.

Table 6. Simulation Study 3: Type-I error rate and power rate for logarithm of item discriminations as a function of the sample size N and the size and the extent and direction of deviations from the Rasch model.

		Type-I Error Rate				Power Rate			
Dev	N	**AIC**	**SAIC**	**BIC**	**SBIC**	**AIC**	**SAIC**	**BIC**	**SBIC**
		Balanced deviations from the Rasch model							
small	500	**14.6**	7.7	1.3	1.2	**68.8**	**61.6**	**40.6**	**39.6**
	1000	**18.5**	6.5	0.7	0.3	**77.6**	85.5	**63.6**	**55.9**
	2000	**36.4**	7.7	0.6	0.0	**78.8**	98.7	**75.5**	**79.8**
large	500	**11.4**	6.6	1.3	1.2	97.9	95.8	93.8	93.6
	1000	9.3	6.6	0.6	0.2	99.7	99.8	99.4	98.9
	2000	4.4	7.4	0.1	0.0	100	100	100	100
		Unbalanced deviations from the Rasch model							
small	500	**11.4**	5.7	1.8	1.1	80.4	**69.5**	**30.9**	**29.9**
	1000	**10.8**	5.9	0.8	0.3	97.9	94.0	**72.5**	**50.9**
	2000	8.3	7.7	0.4	0.1	99.9	99.9	98.2	87.3
large	500	**13.7**	5.9	1.2	1.0	100	100	99.9	99.9
	1000	**12.1**	5.8	0.7	0.2	100	100	100	100
	2000	**10.9**	7.3	0.4	0.0	100	100	100	100

Note. Dev = size of deviation from the Rasch model; Type-I error rates larger than 10.0 and power rates smaller than 80.0 are printed in bold font.

6. Discussion

In this article, we compared the ordinarily employed indirect regularized estimation based on a grid of regularization parameters λ with a subsequent discrete minimization of AIC and BIC with a direct minimization of smooth information criteria SAIC and SBIC [26] for the estimation of regularized item response models. It turned out that the direct SIC-based estimation methods resulted in comparable, in many cases, or better performance than the indirect regularization estimation methods based on AIC and BIC. This is remarkable because SIC-based minimization is computationally much simpler, and ordinary gradient-based optimization routines can be utilized.

We studied the performance of SAIC and SBIC in three simulation studies that focus on differential item functioning, (semi-)exploratory multidimensional IRT models, and model choice between the Rasch model and the 2PL model. These three cases frequently appear in applications of regularized IRT models, which is why we chose these settings for our work.

In this article, we confined ourselves to analyzing dichotomous item responses and continuous factor variables. Future research could investigate the application of these techniques to polytomous item response, count item response data [55], or cognitive diagnostic models that involve multivariate binary factor variables [56]. More generally, smooth information criteria can be used in all modeling approaches that involve regularized estimation. In the field of econometrics or social science, possible applications could be (generalized) linear regression models [57], regularized panel models [58], or regularized estimation for analyzing heterogeneous treatment effects [59].

Notably, we did not investigate the estimation of standard errors in this article. Future research may investigate this with an application of the Huber–White variance estimation formula [60,61] applied to the subset of parameters that resulted in non-zero values [62].

Finally, two different targets in the analysis of item response models should be distinguished in regularized estimation. First, the selection or detection of non-zero effects like cross-loadings or DIF effects may be the focus. For this goal, model selection based on information criteria can prove helpful in order to control type-I error rates. Second, if the focus lies on structural parameters (such as group means or factor correlations), choosing a parsimonious model that tries to penalize the number of estimated parameters, like in information criteria, may not be beneficial in terms of bias and variability of structural parameters [21]. It can be advantageous to use a sufficiently small regularization parameter λ to ensure the empirical identifiability of the model but not to focus on effect selection if structural parameters are of interest [63]. In this sense, sparsity in effects is imposed in a defensive way.

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Data Availability Statement: Supplementary material for the simulation studies can be found at https://osf.io/ykew6 (accessed on 2 April 2024).

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

2PL	two-parameter logistic
AIC	Akaike information criterion
BIC	Bayesian information criterion
DGM	data-generating model
DIF	differential item functioning
IRF	item response function
IRT	item response theory
LASSO	least absolute shrinkage and selection operator
ML	maximum likelihood
RMSE	root mean square error
SAIC	smooth Akaike information criterion
SBIC	smooth Bayesian information criterion
SCAD	smoothly clipped absolute deviation
SIC	smooth information criterion

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