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Local Convergence Analysis of an Eighth Order Scheme Using Hypothesis Only on the First Derivative

Ioannis K. Argyros ¹, Ramandeep Behl ^{2,*} and Sandile S. Motsa ²

¹ Department of Mathematics Sciences Lawton, Cameron University, Lawton, OK 73505, USA; iargyros@cameron.edu

² School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Pietermaritzburg 3209, South Africa; sandilemotsa@gmail.com

* Correspondence: ramanbeh187@yahoo.in; Tel.: +27-62-092-3095 or +91-98-1468-0508

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Abstract: In this paper, we propose a local convergence analysis of an eighth order three-step method to approximate a locally unique solution of a nonlinear equation in a Banach space setting. Further, we also study the dynamic behaviour of that scheme. In an earlier study, Sharma and Arora (2015) did not discuss these properties. Furthermore, the order of convergence was shown using Taylor series expansions and hypotheses up to the fourth order derivative or even higher of the function involved which restrict the applicability of the proposed scheme. However, only the first order derivatives appear in the proposed scheme. To overcome this problem, we present the hypotheses for the proposed scheme maximum up to first order derivative. In this way, we not only expand the applicability of the methods but also suggest convergence domain. Finally, a variety of concrete numerical examples are proposed where earlier studies can not be applied to obtain the solutions of nonlinear equations on the other hand our study does not exhibit this type of problem/restriction.

Keywords: Kung-Traub method; local convergence; divided difference; Banach space; Lipschitz constant; radius of convergence

1. Introduction

Numerical analysis is a wide-ranging discipline having close connections with mathematics, computer science, engineering and the applied sciences. One of the most basic and earliest problems of numerical analysis are concerned with efficiently and accurately finding the approximate locally unique solution x^* of an equation of the form

$$F(x) = 0, \tag{1}$$

where F is a Fréchet differentiable operator defined on a convex subset \mathbb{D} of \mathbb{X} with values in \mathbb{Y} , where \mathbb{X} and \mathbb{Y} are Banach spaces.

Analytical methods for solving such equations are almost non-existent for obtaining the exact numerical values of the required roots. Therefore, it is only possible to obtain approximate solutions and one has to be satisfied with approximate solutions up to any specified degree of accuracy, by relying on numerical methods which are based on iterative procedures. Therefore, worldwide, researchers resort to an iterative method and a plethora of iterative methods has been proposed [1–15]. While using these iterative methods, researchers face the problems of slow

convergence, non-convergence, divergence, inefficiency or failure (for detail please see Traub [14] and Petković et al. [11]).

The convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iterative procedures. On the other hand, the local convergence is based on the information around a solution, to find estimates of the radii of convergence balls. A very important problem in the study of iterative procedures is the convergence domain. Therefore, it is very important to propose the radius of convergence of the iterative methods.

We study the local convergence analysis of a three step method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned}
 y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
 z_n &= \phi_4(x_n, y_n), \\
 x_{n+1} &= \phi_8(x_n, y_n, z_n) \\
 &= z_n - (2[z_n, y_n; F] - [z_n, x_n; F])^{-1} (F'(x_n) - [y_n, x_n, F] + [z_n, y_n; F])F(x_n)^{-1}F(z_n),
 \end{aligned}
 \tag{2}$$

where $x_0 \in \mathbb{D}$ is an initial point, $[\cdot, \cdot; F] : \mathbb{D}^2 \rightarrow L(\mathbb{X})$, ϕ_4 is any two-point optimal fourth-order scheme and $\phi_8(x, y, z) := z - (2[z, y; F] - [z, x; F])^{-1} (F'(x) - [y, x; F] + [z, y; F])F'(x)^{-1}F(z)$. The eighth order of convergence of method Equation (2) was shown in [13] when $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ and $[x, y; F] = \frac{F(x)-F(y)}{x-y}$ for $x \neq y$ and $[x, x; F] = F'(x)$. That is when $[\cdot, \cdot; F]$ is a divided difference of first order of operator F [4,5]. The local convergence was shown using Taylor series expansions and hypotheses reaching up to the fifth order derivative. The hypotheses on the derivatives of F limit the applicability of method Equation (2). As a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $\mathbb{D} = [-\frac{1}{\pi}, \frac{2}{\pi}]$ by

$$F(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} .$$

Then for $x \neq 0$, we have

$$F'(x) = 2x^2 - x^3 \cos\left(\frac{1}{x}\right) + 3x^2 \log(\pi^2 x^2) + 5x^4 \sin\left(\frac{1}{x}\right),$$

$$F''(x) = -8x^2 \cos\left(\frac{1}{x}\right) + 2x(5 + 3 \log(\pi^2 x^2)) + x(20x^2 - 1) \sin\left(\frac{1}{x}\right)$$

and

$$F'''(x) = \frac{1}{x} \left[(1 - 36x^2) \cos\left(\frac{1}{x}\right) + x \left(22 + 6 \log(\pi^2 x^2) + (60x^2 - 9) \sin\left(\frac{1}{x}\right) \right) \right].$$

One can easily see that the function F''' is unbounded on \mathbb{D} at the point $x = 0$. Hence, the results in [13] cannot be applied to show the convergence of method Equation (2) or its special cases requiring hypotheses on the fifth derivative of function F or higher. Notice that, in particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [1–15]. These results show that initial guesses should be close to the required root for the convergence of the corresponding methods. However, how close should an initial guess should be for the convergence of the corresponding method to be possible? These local results give no information on the radius of the ball convergence for the corresponding method. The same technique can be used for other methods.

In the present study, we expand the applicability of method given by Equation (2) using only hypotheses on the first order derivative of function F . We also propose the computable radii of convergence and error bounds based on the Lipschitz constants. We further present the range of initial guesses x^* that tell us how close the initial guesses should be required to guarantee the convergence

of the method Equation (2). This problem was not addressed in [13]. The advantages of our approach are similar to the ones already mentioned for the method described by Equation (2).

The rest of the paper is organised as follows: in Section 2, we present the local convergence analysis of scheme Equation (2). Section 3 is devoted to the numerical examples which demonstrate our theoretical results. Finally, the conclusion is given in Section 4.

2. Local Convergence: One Dimensional Case

In this section, we define some scalar functions and parameters to study the local convergence of method Equation (2).

Let $K_0 > 0, K_1 > 0, K > 0, L_0 > 0, L > 0, M \geq 1, \lambda > 1$ be given constants. Let us also assume $g_2 : \left[0, \frac{1}{L_0}\right) \rightarrow \mathbb{R}$, be a nondecreasing and continuous function. Define functions g_1 and p in the interval $\left[0, \frac{1}{L_0}\right)$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0t)}, p(t) = (K_0g_2(t)t^{\lambda-1} + K_1)t$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0 + L}.$$

Notice that the function g_1 is a monotonically increasing function on the interval $[0, 1)$. We have $g_1(r_1) = 1$ and for each $t \in [0, r_1) : 0 \leq g_1(t) < 1$. Further, define function $h_2 : \left[0, \frac{1}{L_0}\right) \rightarrow \mathbb{R}$ by $h_2(t) = g_2(t)t^{\lambda-1} - 1$.

Suppose that

$$\begin{aligned} g_2(t)t^{\lambda-1} &< 1, \text{ for each } [0, r_1), \\ h_2(l) &> 0, \\ \text{as } t \rightarrow l &< r_1 \text{ for some } l > 0. \end{aligned} \tag{3}$$

Then, we have $h_2(0) = -1 < 0$. By Equation (3) and the intermediate value theorem, function h_2 has zeros in the interval $(0, l)$. Further, let r_2 be the smallest such zero. Further, define functions q and h_q on the interval $\left[0, \frac{1}{L_0}\right)$ by $q(t) = p(t) + 2 [K_0g_2(t)t^\lambda + K_1g_1(t)t]$ and $h_q(t) = q(t) - 1$.

Using $h_q(0) = -1 < 0$ and Equation (3), we deduce that function h_q has a smallest zero denoted by r_q .

Finally, define functions g_3 and h_3 on the interval $[0, r_q)$ by

$$g_3(t) = \left(1 + \frac{(K + (L_0 + K_1)t + K_0g_1(t)t)M}{(1 - L_0t)(1 - q(t))}\right) g_2(t)t^{\lambda-1}$$

and

$$h_3(t) = g_3(t) - 1.$$

Then, we get $h_3(0) = -1$ and $h_3(t) \rightarrow +\infty$ as $t \rightarrow r_q^-$. Denote by r_3 the smallest zero of function h_3 on the interval $(0, r_q^-)$. Define

$$r = \min\{r_1, r_2, r_3\}. \tag{4}$$

Then, we have that

$$0 < r \leq r_1 < \frac{1}{L_0} \tag{5}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t), \tag{6}$$

$$0 \leq p(t), \tag{7}$$

$$0 \leq q(t) < 1, \tag{8}$$

$$0 \leq g_2(t) < 1 \tag{9}$$

and

$$0 \leq g_3(t) < 1. \tag{10}$$

$U(\gamma, s)$ and $\bar{U}(\gamma, s)$ stand, respectively for the open and closed balls in \mathbb{X} with center $\gamma \in \mathbb{X}$ and radius $s > 0$.

Next, we present the local convergence analysis of scheme Equation (2) using the preceding notations.

Theorem 1. *Let us consider $F : \mathbb{D} \subset \mathbb{X} \rightarrow \mathbb{Y}$ be a Fréchet differentiable operator. Let us also assume $[\cdot, \cdot; F] : \mathbb{D}^2 \rightarrow L(\mathbb{X})$ be a divided difference of order one. Suppose there exist $x^* \in \mathbb{D}$, a neighborhood \mathbb{D}_0 of \mathbb{D} , $L_0 > 0$, $\lambda > 1$ such that Equation (3) holds and for each $x \in \mathbb{D}_0$*

$$F(x^*) = 0, \quad F'(x)^{-1} \in L(\mathbb{Y}, \mathbb{X}) \tag{11}$$

$$\|z(x) - x^*\| \leq g_2(\|x - x^*\|)\|x - x^*\|^\lambda \tag{12}$$

and

$$\|F'(x^*)^{-1} (F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|, \tag{13}$$

where $z(x) = \phi_4(x, x - F'(x)^{-1}F(x))$.

Moreover, suppose that there exist $K_0 > 0$, $K_1 > 0$, $K > 0$, $L > 0$ and $M \geq 1$ such that for each $x, y \in U(x^*, \frac{1}{L_0}) \cap \mathbb{D}_0$

$$\|F'(x^*)^{-1} ([x, y; F] - F'(x^*))\| \leq K_0\|x - x^*\| + K_1\|y - x^*\|, \tag{14}$$

$$\|F'(x^*)^{-1}[x, y; F]\| \leq K, \tag{15}$$

$$\|F'(x^*)^{-1} (F'(x) - F'(y))\| \leq L\|x - y\|, \tag{16}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M, \tag{17}$$

and

$$\bar{U}(x^*, r) \subseteq \mathbb{D}_0, \tag{18}$$

where the radius of convergence r is defined by Equation (4). Then, the sequence $\{x_n\}$ generated by method Equation (2) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \dots$ and converges to x^* . Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{19}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{20}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{21}$$

where the “ g ” functions are defined by previously. Furthermore, for $T \in [r, \frac{2}{L_0})$, the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap \mathbb{D}_0$.

Proof of Theorem 1. We shall show that estimates Equations (19)–(21) with the help of mathematical induction. By hypotheses $x_0 \in U(x^*, r) - \{x^*\}$, Equations (5) and (13), we get that

$$\|F'(x^*)^{-1} (F'(x_0) - F'(x^*))\| \leq L_0 \|x_0 - x^*\| < L_0 r < 1. \tag{22}$$

It follows from Equation (22) and the Banach Lemma on invertible operators [4,12] that $F'(x_0)^{-1} \in L(\mathbb{Y}, \mathbb{X})$, y_0 is well defined and

$$\|F'(x_0)^{-1} F'(x^*)\| \leq \frac{1}{1 - L_0 \|x_0 - x^*\|}. \tag{23}$$

Using the first substep of method Equation (2) for $n = 0$, Equations (4), (6), (11), (16) and (23), we get in turn

$$\begin{aligned} \|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1} F(x_0)\| \\ &\leq \|F'(x_0)^{-1} F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1} (F'(x^* + \theta(x_0 - x^*)) - F'(x_0)) (x_0 - x^*) d\theta \right\| \\ &\leq \frac{L \|x_0 - x^*\|^2}{1 - L_0 \|x_0 - x^*\|} = g_1(\|x_0 - x^*\|) \|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < r, \end{aligned} \tag{24}$$

which shows Equation (19) and $y_0 \in U(x^*, r)$. Then, from Equations (3) and (12), we see that (20) follows for $n = 0$. Hence, $z_0 \in U(x^*, r)$. Next, we shall show that $(2[z_0, y_0; F] - [z_0, x_0; F])^{-1} \in L(\mathbb{Y}, \mathbb{X})$. Using Equations (12), (14) and the strict monotonicity of q , we obtain in turn that

$$\begin{aligned} &\|F'(x^*)^{-1} [2([z_0, y_0; F] - F'(x^*)) - ([z_0, x_0; F] - F'(x^*))]\| \\ &\leq 2 \|F'(x^*)^{-1} ([z_0, y_0; F] - F'(x^*))\| + \|F'(x^*)^{-1} ([z_0, x_0; F] - F'(x^*))\| \\ &\leq 2 (K_0 \|z_0 - x^*\| + K_1 \|y_0 - x^*\|) + p(\|x_0 - x^*\|) \\ &\leq 2 (K_0 g_2(\|x_0 - x^*\|) \|x_0 - x^*\|^\lambda + K_1 g_1(\|x_0 - x^*\|) \|x_0 - x^*\|) + p(\|x_0 - x^*\|) \\ &= q(\|x_0 - x^*\|) < q(r) < 1. \end{aligned} \tag{25}$$

That is

$$\|(2[z_0, y_0; F] - [z_0, x_0; F])^{-1} F'(x^*)\| \leq \frac{1}{1 - q(\|x_0 - x^*\|)}. \tag{26}$$

Hence, x_1 is well defined by the third substep of method Equation (2) for $n = 0$. We can write by Equation (11)

$$F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*)) (x_0 - x^*) d\theta. \tag{27}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta \|x_0 - x^*\| < r$, for all $\theta \in (0, 1)$. Hence, we have that $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Then, by Equations (17) and (27) we get that

$$\|F'(x^*)^{-1} F(x_0)\| = \left\| \int_0^1 F'(x^*)^{-1} F'(x^* + \theta(x_0 - x^*)) (x_0 - x^*) d\theta \right\| \leq M \|x_0 - x^*\|. \tag{28}$$

We also have that by replacing x_0 by z_0 in Equation (28) that

$$\|F'(x^*)^{-1} F(z_0)\| \leq M \|z_0 - x^*\|, \tag{29}$$

since $z_0 \in U(x^*, r)$.

Then, using the last substep of method Equation (2) for $n = 0$, Equations (4), (10), (13)–(15), (19), (20) (for $n = 0$), (23), (26) and (29), we obtain that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|z_0 - x^*\| + \left\| F'(x_0)^{-1}F'(x^*) \right\| \left[\left\| F'(x^*)^{-1}([y_0, x_0; F] - F'(x^*)) \right\| \right. \\ &\quad \left. + \left\| F'(x^*)^{-1}(F'(x_0) - F'(x^*)) \right\| + \left\| F'(x^*)^{-1}[z_0, x_0; F] \right\| \right] \\ &\quad \times \left\| (2[z_0, y_0; F] - [z_0, x_0; F])^{-1}F'(x^*) \right\| \left\| F'(x^*)^{-1}F(z_0) \right\| \\ &\leq \|z_0 - x^*\| + \frac{(K + L_0\|x_0 - x^*\| + K_0\|y_0 - x^*\| + K_1\|x_0 - x^*\|)M\|z_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)(1 - q(\|x_0 - x^*\|))} \\ &\leq \left(1 + \frac{(K + (L_0 + K_1)\|x_0 - x^*\| + K_0g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)M}{(1 - L_0\|x_0 - x^*\|)(1 - q(\|x_0 - x^*\|))} \right) \|z_0 - x^*\| \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{30}$$

which shows Equation (21) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0 by x_n, y_n, z_n in the preceding estimates we arrive at Equations (19)–(21) for all $n = 0, 1, 2, \dots$. Using the monotonicity of g_3 on the interval $[0, r]$, $x_0 \in U(x^*, r)$ and (21), we have with $c = g_3(\|x_0 - x^*\|) \in [0, 1)$

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\| \leq \dots \leq c^n\|x_0 - x^*\|, \tag{31}$$

where $r < 1$. This yields $\lim_{n \rightarrow \infty} x_n = x^*$ and $x_{n+1} \in U(x^*, r)$.

Finally, to show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ be such that $F(y^*) = 0$. Set $Q = \int_0^1 F'(x^* + \theta(y^* - x^*)) d\theta$. Then, using Equation (13), we get that

$$\left\| F'(x^*)^{-1}(Q - F'(x^*)) \right\| \leq L_0 \int_0^1 \theta \|x^* - y^*\| d\theta \leq \frac{L_0}{2} T < 1. \tag{32}$$

Hence, $Q^{-1} \in L(\mathbb{Y}, \mathbb{X})$. Then, in view of the identity $F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. It is worth noticing that in view of Equations (12), (21), (31) and the definition of functions $g_i, i = 1, 2, 3$ the convergence order of method Equation (2) is at least λ . \square

Remark 1

(a) In view of Equation (13) and the estimate

$$\begin{aligned} \left\| F'(x^*)^{-1}F'(x) \right\| &= \left\| F'(x^*)^{-1}(F'(x) - F'(x^*)) + F'(x^*) \right\| \\ &\leq 1 + \left\| F'(x^*)^{-1}(F'(x) - F'(x^*)) \right\| \\ &\leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition Equation (17) can be dropped and M can be replaced by

$$M = M(t) = 1 + L_0t$$

or $M = 2$, since $t \in [0, \frac{1}{L_0})$.

(b) The results obtained here can be used for operators F satisfying the autonomous differential equation [4,5] of the form

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^* . As an example, let $F(x) = e^x + 2$. Then, we can choose $P(x) = x - 2$.

- (c) The radius r_1 was shown in [4,5] to be the convergence radius for Newton’s method under conditions (11) and (12). It follows from Equation (4) and the definition of r_1 that the convergence radius r of the method Equation (2) cannot be larger than the convergence radius r_1 of the second order Newton’s method. As already noted in [4,5] r_1 is at least the convergence radius given by Rheinboldt [12]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \rightarrow \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L} \rightarrow 0.$$

That is our convergence radius r is at most three times larger than Rheinboldt’s. The same value for r_R given by Traub [14].

- (d) We shall show how to define function g_2 and l appearing in condition Equation (3) for the method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= \phi_4(x_n, y_n) := y_n - [y_n, x_n; F]^{-1}F'(x_n)[y_n, x_n; F]^{-1}F(y_n) \\ x_{n+1} &= \phi_8(x_n, y_n, z_n). \end{aligned} \tag{33}$$

Clearly method Equation (33) is a special case of method Equation (2). If $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, then method Equation (33) reduces to Kung-Traub method [14]. We shall follow the proof of Theorem 1 but first we need to show that $[y_n, x_n; F]^{-1} \in L(\mathbb{Y}, \mathbb{X})$. By assuming the hypotheses of Theorem 1, we get that

$$\begin{aligned} \left\| F'(x^*)^{-1} ([y_n, x_n; F] - F'(x^*)) \right\| &\leq K_0 \|y_n - x^*\| + K_1 \|x_n - x^*\| \\ &\leq (K_0 g_1(\|x_n - x^*\|) + K_1) \|x_n - x^*\| \\ &= p_0(\|x_n - x^*\|), \end{aligned}$$

The function $h_{p_0}(t) = p_0(t) - 1$, where $p_0(t) = (K_0 g_1(t) + K_1)t$, has a smallest zero denoted by r_{p_0} in the interval $(0, \frac{1}{L_0})$. Set $l = r_{p_0}$. Then, we have from the last substep of method Equation (33) that

$$\begin{aligned} \|z_n - x^*\| &\leq \|y_n - x^*\| + \left\| [y_n, x_n; F]^{-1}F'(x^*) \right\| \left\| F'(x^*)^{-1}F(x^*) \right\| \\ &\quad \times \left\| [y_n, x_n; F]^{-1}F'(x^*) \right\| \left\| F'(x^*)^{-1}F(y_n) \right\| \\ &\leq \|y_n - x^*\| + \frac{M^2}{(1 - p_0(\|x_n - x^*\|))^2} \|y_n - x^*\| \\ &\leq \left(1 + \frac{M^2}{(1 - p_0(\|x_n - x^*\|))^2} \right) g_1(\|x_n - x^*\|) \|x_n - x^*\| \\ &= \frac{1}{2} \left(1 + \frac{M^2}{(1 - p_0(\|x_n - x^*\|))^2} \right) \frac{L \|x_n - x^*\|^2}{1 - L_0 \|x_n - x^*\|}. \end{aligned} \tag{34}$$

It follows from Equation (34) that $\lambda = 2$ and

$$g_2(t) = \frac{1}{2} \frac{L}{1 - L_0 t} \left(1 + \frac{M^2}{(1 - p_0(t))^2} \right). \tag{35}$$

Then, the convergence radius is given by

$$r = \min\{r_1, r_2, r_{p_0}, r_3\}. \tag{36}$$

3. Numerical Example and Applications

In this section, we will verify the validity and effectiveness of our theoretical results which we have proposed in Section 2 based on the scheme proposed by Sharma et al. [13]. For this purpose, we shall choose a variety of nonlinear equations and systems of nonlinear equations which are mentioned in the following examples including motivational example. At this point, we will chose the following methods

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - (2[y_n, x_n; F] - F'(x_n))^{-1}F(y_n), \\ x_{n+1} = \phi_8(x_n, y_n, z_n), \end{cases} \tag{37}$$

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - (2[y_n, x_n; F]^{-1} - F'(x_n)^{-1})F(y_n), \\ x_{n+1} = \phi_8(x_n, y_n, z_n) \end{cases} \tag{38}$$

and

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - (3I - 2F'(x_n)^{-1}[y_n, x_n, F])F'(x_n)^{-1}F(y_n), \\ x_{n+1} = \phi_8(x_n, y_n, z_n) \end{cases} \tag{39}$$

denoted by M_1, M_2 and M_3 , respectively.

First of all, we require the initial guesses x_0 which gives the guarantee for convergence of the iterative methods. Therefore, we shall calculate the values of $r_R, r_1, r_2, r_q, r_3, r_{p_0}$ and r to find the range of convergence domain, which are displayed in the Tables 1–4 up to 5 significant digits. However, we have the values of these constants up to several number of significant digits. Then, we will also verify the theoretical order of convergence of these methods for scalar equations on the basis of the results obtained from computational order of convergence and $\left| \frac{e_{n+1}}{e_n^p} \right|$. In the Tables 5 and 6, we display the number of iteration indexes (n), approximated zeros (x_n), residual errors of the corresponding function ($|F(x_n)|$), errors $|e_n|$ (where $e_n = x_n - x^*$), $\left| \frac{e_{n+1}}{e_n^p} \right|$ and the asymptotic errors constant $\eta = \lim_{n \rightarrow \infty} \left| \frac{e_{n+1}}{e_n^p} \right|$. In addition, We will use the formulas proposed by Sánchez et al. in [7] to calculate the computational order of convergence (COC), which is given by

$$\rho = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}}, \quad \text{for each } n = 0, 1, 2, \dots \tag{40}$$

or the approximate computational order of convergence (ACOC) [7]

$$\rho^* = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \dots \tag{41}$$

Table 1. Different values of parameters which satisfy Theorem 1.

r_R	r_1	r_2	r_3	r_q	r_{p_0}	r
0.017098	0.017098	0.0061188	0.0077833	0.0080573	0.019592	0.0061188

Table 2. Different values of parameters which satisfy Theorem 1.

r_R	r_1	r_2	r_3	r_q	r_{p_0}	r
0.22856	0.22856	0.089459	0.086166	0.11255	0.26191	0.086166

Table 3. Different values of parameters which satisfy Theorem 1.

r_R	r_1	r_2	r_3	r_q	r_{p_0}	r
0.37258	0.38269	0.15120	0.13166	0.19038	0.44149	0.13166

Table 4. Different values of parameters which satisfy Theorem 1.

r_R	r_1	r_2	r_3	r_q	r_{p_0}	r
0.0075648	0.0075648	0.0027072	0.0035090	0.0035649	0.0086685	0.0027072

Table 5. Convergence behavior of different cases on Example 1.

Methods	n	$\ F(x_n)\ $	$\ e_n\ $	ζ	$\left\ \frac{e_n}{e_{n-1}} \right\ $	η
M_1	0	6.1e-2	5.6e-3			
	1	1.6e-17	1.8e-18		0.00006065014094	2.153729894e+49
	2	1.0e-93	9.1e-95	4.9289	2.314984786e+12	
	3	1.4e-514	1.3e-515	5.5155	2.153729894e+49	
M_2	0	6.1e-2	5.6e-3			
	1	2.3e-13	2.1e-14		0.00002153200773	4.146304252e-6
	2	2.3e-59	2.1e-60	4.0264	0.00001076045277	
	3	8.6e-244	7.9e-245	4.0090	4.146304252e-6	
M_3	0	6.1e-2	5.6e-3			
	1	3.6e-13	3.3e-14		0.00003395540062	9.787194795e-6
	2	2.2e-58	2.0e-59	4.0278	0.00001654426326	
	3	1.7e-239	1.5e-240	4.0050	9.787194795e-6	

Table 6. Convergence behavior of different cases on Example 2.

Methods	n	x_n	$ F(x_n) $	$ e_n $	ζ	$\left \frac{e_n}{e_{n-1}} \right $	η
M_1	0	0.65	2.3e-1	6.5e-2			
	1	0.714805912187027	6.5e-10	1.8e-10		0.5649113510	0.3156709071
	2	0.714805912362778	1.1e-78	2.9e-79	8.0295	0.3156709076	
	3	0.714805912362778	1.1e-629	1.5e-629	8.0000	0.3156709071	
M_2	0	0.65	2.3e-1	6.5e-2			
	1	0.714805910701679	6.1e-9	1.7e-9		5.339230124	2.311637753
	2	0.714805912362778	4.9e-70	1.3e-70	8.0479	2.311637800	
	3	0.714805912362778	8.8e-559	2.4e-559	8.0000	2.311637753	
M_3	0	0.65	2.3e-1	6.5e-2			
	1	0.714805907218054	1.9e-8	5.1e-9		16.53656565	6.144786331
	2	0.714805912362778	1.1e-65	3.0e-66	8.0606	6.144786786	
	3	0.714805912362778	1.5e-523	4.2e-524	8.0000	6.144786331	

We calculate the computational order of convergence, asymptotic errors constant and other constants up to several numbers of significant digits (minimum 1000 significant digits) to minimise the round off errors.

In the context of systems of nonlinear equations, we also consider a nonlinear systems in Example 3 to check the proposed theoretical results for nonlinear systems. In this regard, we displayed the number of iteration indexes (n), residual errors of the corresponding function ($\|F(x_n)\|$), errors $\|e_n\|$ (where $e_n = x_n - x^*$), $\left\| \frac{e_n}{e_{n-1}^p} \right\|$ and the asymptotic errors constant $\eta = \lim_{n \rightarrow \infty} \left\| \frac{e_n}{e_{n-1}^p} \right\|$ in the Tables 7 and 8.

Table 7. Convergence behavior of different cases on Example 3.

Methods	n	$\ F(x_n)\ $	$\ e_n\ $	ζ	$\left\ \frac{e_n}{e_{n-1}^p} \right\ $	η
M_1	0	1.7e-1	1.6e-1			
	1	7.2e-12	7.2 e-12		0.00001412834175	0.001157325063
	2	3.0e-93	3.0e-93	7.8569	0.0004283379739	
	3	7.9e-744	7.9e-744	7.9947	0.001157325063	
M_2	0	1.7e-1	1.6e-1			
	1	1.9e-5	1.9e-5		0.02642235158	0.2812313492
	2	2.8e-20	2.8e-20	3.7651	0.2223712525	
	3	1.7e-79	1.7e-79	3.9931	0.2812313492	
M_3	0	1.7e-1	1.6e-1			
	1	2.8e-5	2.5e-5		0.03967215686	0.4999499551
	2	2.5e-19	2.5e-19	3.7373	0.3860243040	
	3	1.9e-75	1.9e-75	3.9920	0.4999499551	

Table 8. Convergence behavior of different cases on Example 4.

Methods	n	x_n	$ F(x_n) $	$ e_n $	ζ	$\left \frac{e_n}{e_{n-1}^p} \right $	η
M_1	0	0.317	3.0e-4	1.3e-3			
	1	0.318309886183791	3.8e-18	1.6e-17		1.881774410e+6	1.746669349e+6
	2	0.318309886183791	2.1e-129	8.7e-129	8.0023	1.746669349e+6	
	3	0.318309886183791	1.4e-1019	6.0e-1019	8.0000	1.746669349e+6	
M_2	0	0.317	3.0e-4	1.3e-3			
	1	0.318309886183790	5.5e-17	2.3e-16		2.711175817e+7	2.406103772e+7
	2	0.318309886183791	5.3e-119	2.2e-118	8.0041	2.406103772e+7	
	3	0.318309886183791	3.5e-935	1.5e-934	8.0000	2.406103772e+7	
M_3	0	0.317	3.0e-4	1.3e-3			
	1	0.318309886183790	1.7e-16	7.2 e-16		8.333410395e+7	7.204280059e+7
	2	0.318309886183791	1.3e-114	5.3e-114	8.0052	7.204280059e+7	
	3	0.318309886183791	1.1e-899	4.7e-899	8.0000	7.204280059e+7	

As we mentioned in the earlier paragraph, we calculate the values of all the constants and functional residuals up to several numbers of significant digits but, due to the limited paper space, we display the values of x_n up to 15 significant digits. Further, the values of other constants, namely, ζ (COC) up to 5 significant digits and the values $\left| \frac{e_n}{e_{n-1}^p} \right|$, η and $\left\| \frac{e_n}{e_{n-1}^p} \right\|$ are up to 10 significant digits. Furthermore, the residual errors in the function/systems of nonlinear functions ($|F(x_n)|$ or $\|F(x_n)\|$) and the errors $|e_n|$ or $\|e_n\|$ are displayed up to 2 significant digits with exponent power which are mentioned in the following tables corresponding to the test function. However, a minimum of 1000 significant digits are available with us for every value.

Furthermore, we consider the approximated zero of test functions when the exact zero is not available, which is corrected up to 1000 significant digits to calculate $\|x_n - x^*\|$. During the current numerical experiments with programming language Mathematica (Version 9), all computations have been done with multiple precision arithmetic, which minimises round-off errors.

Further, we use $\lambda = 2$ and function g_2 defined by Equation (35) in all the examples.

Example 1. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^3$, $\mathbb{D} = \bar{U}(0, 1)$. Define F on \mathbb{D} for $v = (x, y, z)^T$ by

$$F(v) = (10x + \sin(x + y) - 1, 8y - \cos^2(z - y) - 1, 12z + \sin z - 1)^T. \tag{42}$$

Then the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} 10 + \cos(x + y) & \cos(x + y) & 0 \\ 0 & 8 + \sin 2(z - y) & \sin 2(z - y) \\ 0 & 0 & 12 + \cos z \end{bmatrix}.$$

Then, for $x^* = (0.068978\dots, 0.246442\dots, 0.076929\dots)$, we get $L = L_0 = 38.9911$, $K = M = 168.962$ and $K_0 = K_1 = \frac{L_0}{2}$. We calculate the radius of convergence, the residual errors of the corresponding function ($\|F(x_n)\|$) (by considering initial approximation (0.072, 0.25, 0.080) in the methods M_1 , M_2 and M_3), errors $\|e_n\|$ (where $e_n = x_n - x^*$), $\left\| \frac{e_n}{e_{n-1}^p} \right\|$ and the asymptotic errors constant $\eta = \lim_{n \rightarrow \infty} \left\| \frac{e_n}{e_{n-1}^p} \right\|$, which are displayed in the Tables 1 and 5.

Example 2. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $\mathbb{D} = [-1, 1]$ and define function F on \mathbb{D} by

$$F(x) = e^x - 4x^2. \tag{43}$$

Then, for $x^* = 0.714806\dots$, we obtain that $L_0 = L = 2.91681$, $M = K = 1.82827$ and $K_0 = K_1 = \frac{L_0}{2}$. Hence, we obtain all the values of constants and residual errors in the functions, etc., which we have described earlier in the Tables 2 and 6.

Example 3. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}^3$, $\mathbb{D} = \bar{U}(0, 1)$. Define F on \mathbb{D} for $v = (x, y, z)^T$ by

$$F(v) = \left(e^x - 1, \frac{e - 1}{2}y^2 + y, z \right)^T. \tag{44}$$

Then the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0, 0, 0)$, $F'(x^*) = F'(x^*)^{-1} = \text{diag}\{1, 1, 1\}$, $L_0 = e - 1 < L = 1.78957239$, $K_0 = K_1 = \frac{e-1}{2}$ and $K = M = 1.7896$. Hence, we calculate the radius of convergence, the residual errors of the corresponding function ($\|F(x_n)\|$) (by considering initial approximation (0.094, 0.094, 0.095) in the methods M_1 , M_2 and M_3), errors $\|e_n\|$ (where $e_n = x_n - x^*$) and other constants, which are given in the Tables 3 and 7.

Example 4. Returning back to the motivational example in the introduction of this paper, we have $L = L_0 = \frac{2}{2\pi+1}(80 + 16\pi + (11 + 12 \log 2)\pi^2)$, $M = K = 2$ and $K_0 = K_1 = \frac{L_0}{2}$. We display all the values of constants and residual errors in the functions which we have described earlier in the Tables 4 and 8.

3.1. Results and Discussion

Sharma and Arora’s study was only valid for simple roots of the scalar equation. The order of convergence was shown using Taylor series expansions, and hypotheses up to the fourth

order derivative (or even higher) of the function involved, which restricts the applicability of the proposed scheme. However, only the first order derivatives appear in the proposed scheme. In order to overcome this problem, we propose the hypotheses only up to the first order derivative. the applicability of the proposed scheme can be seen in the Examples 1, 3 and motivational example, where earlier studies were not successful. We also provide the radius of convergence for the considered test problem which gives the guarantee for the convergence.

In addition, we have also calculated residual error in the each corresponding test function and the difference between the exact zero and approximated zero. It is straightforward to say from the Tables 5–8 that the mentioned methods have smaller residual error in each corresponding test function and a smaller difference error between the exact and approximated zero. So, we can say that these methods converge faster towards the exact root. Moreover, the mentioned methods also have simple asymptotic error constant in most of the test functions which can be seen in the Tables 5–8. The dynamic study of iterative methods via basins of attraction also confirm the faster convergence. However, one can find different behaviour of our methods when he/she considers some different nonlinear equations. The behaviour of the iterative methods mainly depends on the body structure of the iterative method, considered test function, initial guess and programming software, etc.

4. Basin of Attractions

In this section, we further investigate some dynamical properties of the attained simple root finders in the complex plane by analysing the structure of their basins of attraction. It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterised by the iterative method applied to a fixed polynomial $p(z) \in \mathbb{C}$, see, e.g., [6,16,17]. The aim herein is to use the basin of attraction as another way for comparing the iterative methods.

From the dynamical point of view, we consider a rectangle $A = [-3, 3] \times [-3, 3] \in \mathbb{C}$ with a 400×400 grid, and we assign a color to each point $z_0 \in A$ according to the simple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each method is considered to be 100. In this way, we distinguish the attraction basins by their colours for different iterative methods.

Test Problem 1. Let $p_1(z) = (z^4 + 1)$, having simple zeros $\{-0.707107 - 0.707107i, -0.707107 + 0.707107i, 0.707107 - 0.707107i, 0.707107 + 0.707107i\}$. It is straightforward to see from Figure 1 that the method M_1 is the best method in terms of less chaotic behaviour to obtain the solutions. Further, M_1 also has the largest basins for the solution and is faster in comparison to all the mentioned methods namely, M_2 and M_3 .

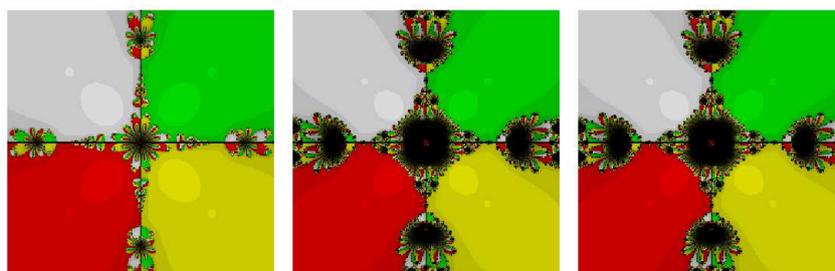


Figure 1. The methods M_1 , M_2 and M_3 , respectively for test problem 1.

Test Problem 2. Let $p_2(z) = (z^3 + 2z - 1)$, having simple zeros $\{-0.0992186 - 2.24266i, -0.0992186 + 2.24266i, 0.198437\}$. Based on Figure 2, it is observed that all the methods, namely, M_1 , M_2 and M_3 have almost zero non convergent points. However, method M_1 has a larger and brighter basin of attraction as compared to the methods M_2 and M_3 . Further, the dynamics behavior of the method M_1 on the boundary points is less chaotic than other methods M_2 and M_3 .

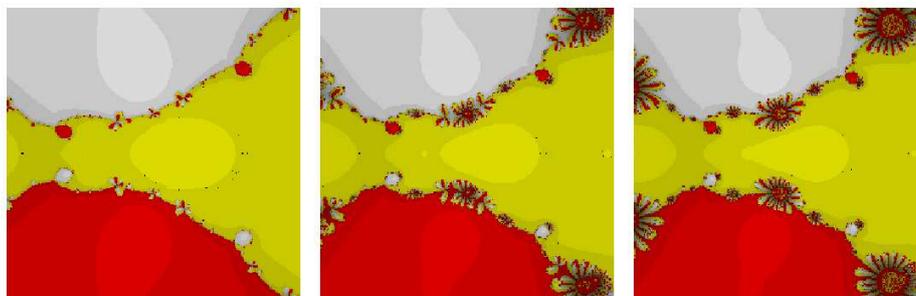


Figure 2. The methods M_1 , M_2 and M_3 , respectively for test problem 2.

Test Problem 3. Let $p_3(z) = (z^6 + z)$, having simple zeros $\{-0.809017 - 0.587785i, -0.809017 + 0.587785i, 0, 0.309017 - 0.951057i, 0.309017 + 0.951057i, 1\}$. It is concluded on the basis of Figure 3, that the method M_1 has a much lower number of non convergent points as compared to M_2 and M_3 (in fact we can say that method M_1 has almost zero non convergent points in this region). Further, the dynamics behaviors of the methods M_2 and M_3 are shown to be very chaotic on the boundary points.

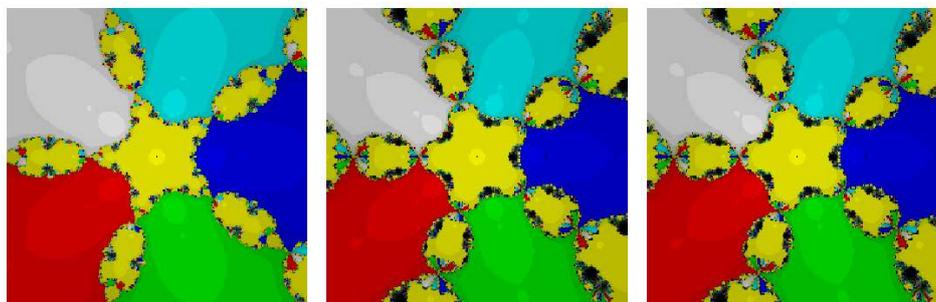


Figure 3. The methods M_1 , M_2 and M_3 , respectively for test problem 3.

5. Conclusions

Commonly, researchers have mentioned that the initial guess should be close to the required root for the granted convergence of their proposed schemes for solving nonlinear equations. However, how close should the initial guess be if it is required to guarantee the convergence of the proposed method? We propose the computable radius of convergence and errors bound by using Lipschitz conditions in this paper. Further, we also reduce the hypotheses from fourth order derivative of the involved function to only first order derivative. It is worth noting that the method in Equation (2) does not change if we use the conditions of Theorem 1 instead of the stronger conditions proposed by Sharma and Arora (2015). Moreover, to obtain the errors bound in practice and order of convergence, we can use the computational order of convergence which is defined in numerical Section 3. Therefore, we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative.

In an earlier study, Sharma and Arora mentioned that their work is valid only for \mathbb{R} . However, in our study we have shown that this scheme will work in any space. The order of convergence

for the proposed scheme may be unaltered or reduce in another space because it depends upon the space and function (for details please see the numerical section). Finally, on account of the results obtained in Section 3, it can be concluded that the proposed study not only expands the applicability but also gives the computable radius of convergence and errors bound by the scheme given by Sharma and Arora (2015), for obtaining simple roots of nonlinear equations as well as systems of nonlinear equations.

Author Contributions: Selection of the proposed scheme and the local convergence analysis (pages 1–8) given by Ioannis K. Argyros. Numerical examples and applications, results and discussion, Basin of attraction and concluding remarks presented by Ramandeep Behl and Sandile S. Motsa.

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