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A Novel Approached Based on T-Spherical Fuzzy Schweizer-Sklar Power Heronian Mean Operator for Evaluating Water Reuse Applications under Uncertainty

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Abstract: The T-Spherical Fuzzy set (T-SPHFS) is one of the core simplifications of quite a lot of fuzzy concepts such as fuzzy set (FS), intuitionistic fuzzy set (ITFS), picture fuzzy set (PIFS), Q-rung orthopair fuzzy set (Q-RUOFS), etc. T-SPHFS reveals fuzzy judgment by the degree of positive membership, degree of abstinence, degree of negative membership, and degree of refusal with relaxed conditions, and this is a more powerful mathematical tool to pair with inconsistent, indecisive, and indistinguishable information. In this article, several novel operational laws for T-SPFNs based on the Schweizer–Sklar t-norm (SSTN) and the Schweizer–Sklar t-conorm (SSTCN) are initiated, and some desirable characteristics of these operational laws are investigated. Further, maintaining the dominance of the power aggregation (POA) operators that confiscate the ramifications of the inappropriate data and Heronian mean (HEM) operators that consider the interrelationship among the input information being aggregated, we intend to focus on the T-Spherical fuzzy Schweizer–Sklar power Heronian mean (T-SPHFSSPHEM) operator, the T-Spherical fuzzy Schweizer–Sklar power geometric Heronian mean (T-SPHFSSPGHEM) operator, the T-Spherical fuzzy Schweizer–Sklar power weighted Heronian mean (T-SPHFSSPWHEM) operator, the T-Spherical fuzzy Schweizer–Sklar power weighted geometric Heronian mean (T-SPHFSSPWGHEM) operator, and their core properties and exceptional cases in connection with the parameters. Additionally, deployed on these newly initiated aggregation operators (AOs), a novel multiple attribute decision making (MADM) model is proposed. Then, the initiated model is applied to the City of Penticton (British Columbia, Canada) to select the best choice among the accessible seven water reuse choices to manifest the practicality and potency of the preferred model and a comparison with the proffered models is also particularized.

Keywords: T-SPHFSs; POA operator; HEM operator; MADM; water reuse applications

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1. Introduction

The purification and restoration of non-traditional or polluted water for constructive use are known as water reuse [1]. Water recycling is associated through the use of recovered water, which might help to alleviate water shortage, particularly in anticipation of the recent influence of climate change and increased human activity. Water recycling has become commonplace across the world to address the degradation of water supplies, which has resulted in a reduction in water availability. Water reuse application evaluation is a weight-replacement method that leads to the most optimal water reuse application selection. As a result, the evaluation entails examining a variety of social, technological, economic,

cultural, ecological, and technical criteria in order to guarantee that decisions are made in a sustainable manner [2]. The difficulty in evaluating water reuse applications is that they are diverse in character and may have competing requirements. In the evaluation of water reuse applications, ambiguity occurs where decision makers are unsure about which option they prefer based on specific criteria or even how much they like a certain option. Additionally, there might be some inadequate or inaccurate data about the alternatives' performance in terms of criterion. For tackling such contradictory and ambiguous situations, fuzzy set theory is a particularly valuable strategy.

Fuzzy set (FS) was instigated by Zadeh [3] as a contraption for depicting and transferring precariousness and ambiguity. Since its commencement, FS has managed remarkable consideration from intellectuals around the globe, who calculated its real and theoretical features. The T-Spherical Fuzzy set [4] (T-SPHFS) is the latest generalization of FS and a dominant tool for handling ambiguous, conflicting, and unclear data.

In this article, the T-Spherical fuzzy set-based framework is initiated for water reuse application evaluation. The main aim of the research consists of advancement of the operational laws of T-Spherical fuzzy numbers, improvement in aggregation operators, and proposes an advanced multiple attribute decision-making model based on these improved operational laws and advanced aggregation operators. The research objectives could be explained briefly as follows. When combining water reuse with multiple attribute decision making, uncertainty is an inescapable and predictable part of the evaluation process. To handle uncertain, inconsistent, and unpredictable information, the T-Spherical fuzzy set is one of the most prominent tools. So, the first task of this research is to initiate novel operational laws for the T-Spherical fuzzy set on Schweizer–Sklar t-norm and t-conorm and discussed its core properties.

Aggregation operators (AOs) play a decisive part in the decision-making (DM) process. Scholars have been considerably more conscious of information AOs in recent years, and they have begun to extend many AOs, for instance, the ordinary AOs conferred by Xu [5] and Xu and Yager [6] may aggregate a series of real numbers into a single real number. Several researchers have expanded these traditional AOs in recent years; for instance, Liu et al. [7] initiated the T-SPHFP Muirhead mean operator and applied them to solve MA group DM (MAGDM) problems. The features of various AOs vary. Some AOs, like the Bonferroni mean (BOM) [8], Heronian mean (HEM) [9], Muirhead mean (MUM) operators [10], and Maclaurin symmetric mean (MCSM) operator [11], took into account the correlation between input arguments.

The second task of this research is to merge the power average operator with the Heronian mean operator to initiate T-Spherical fuzzy power Heronian mean operators that have the advantages of removing the bad effect of awkward data and consider the correlation among input data at the same time. Furthermore, based on these aggregation operators, a multiple attribute decision-making model is initiated and applied to solve the water reuse application evaluation [12] problem to show the practicality and effectiveness of the initiated multiple attribute decision-making model.

2. Materials

2.1. Literature Review

High population growth and industrialization in the colonial era have resulted in a slew of environmental issues, prominent among them the deterioration of water resources and the resulting water supply constraints. Developed nations, on average, have greater freshwater resources than emerging and poor ones. However, in industrialized nations, pollution, over-exploitation, degradation, and intense rivalry between agriculture and industry have resulted in larger urban water supply difficulties. The process of weighing options and selecting the best appropriate water reuse application is known as water reuse application evaluation. Water reuse applications are challenging to assess since they are varied in nature and may have competing requirements. To achieve sustainable judgments, the review will now include a study of many factors involving social, scientific, financial,

political, geological, and engineering factors. When evaluating water reuse applications, ambiguity arises when decision-makers are undecided about which choice they prefer based on a given criterion or even how much they favor a certain alternative. To deal with such a situation, the fuzzy set theory is the most appropriate set.

Zadeh [3] was the founder of the Fuzzy set (FS) as a mechanism for illustrating and reassigning unevenness and vagueness. Since its origination, FS has managed notable contemplation from scholars throughout the world, who premeditated its actual and theoretical features. Some of the latest research efforts on the theory and applications of FSs have been launched in economics and business [12–14], genetic algorithms [15,16], and supply chain management [17,18], etc. Following the introduction of the concept of FS, a number of extensions of FSs were expected, such as interval-valued FS [19], which elucidated the membership degree is a subset of $[0, 1]$, and Atanassov's ITFS [20], which elucidated the membership degree (MED) and non-membership degree (NOMD) by a single number in the $[0, 1]$, with the total of these two degrees having to be less than or equal to one. As a result, ITFS explains uncertainty and unreliability in greater depth than FS. The appealing scenario arises when such an entity's MED and NOMD are in the $[0, 1]$, but somehow the total of two such functions exceeds one. The standard ITFS fails to manage such types of data under such scenarios. To address the scenario described earlier, Yager [21–23] suggested Pythagorean fuzzy sets (PytFSs), which might be considered an augmentation of ITFSs. The main distinction amid both PytFS and ITFS would be that in PytFS, the addition of squares of the MED and NOMD will be less than or equal to one, whereas in IFS, the sum of MED and NOMD would be less than or equal to one. After the commencement of PytFS, a variety of studies have been carried out by numerous investigators, such as distance measure [24–27] and correlation coefficient [28,29]. Zhang and Xu [30] conveyed the ample mathematical representation for PytFS and anticipated the initiative of PytF number (PytFN), then they also anticipated a MADM algorithm deployed on PytF TOPSIS to pair with PytFNs. Some other studies about PytFS have been conducted by different investigators and give their applications in different areas [31–33]. Q-RUOFS [34] is another conception of ITFS and PytFS. Q-RUOFS is a more dominant and reliable set to pact with unclear and conflicting information.

In various areas, ITFS, PytFS and Q-RUOFS have been addressed. Unfortunately, there have been some circumstances for which IFS, PyFS, and Q-ROFS are not compatible. For example, the members of three departments, namely the department of mathematics, department of physics, and department of information technology, vote for the selection of the dean post among one of these departments. From these faculties, 100 teachers (Assistant professors and associate professors) are chosen for voting, and 1 professor from each department is chosen for the selection of the dean post. Thirty teachers support professors from the department of mathematics, forty teachers oppose professors from the department of mathematics, twenty teachers abstain from voting, and ten teachers decline to vote. ITFS, PytFS, and Q-RUOFS fail to cope with these sort of data. Cuong and Kreinovich [35] initiated and termed picture fuzzy set (PIFS), another conception of ITFS and FS to cope with such information. Furthermore, there will be instances whenever an entity's MED, neutral MED, and NMED are permissible from a unit interval, yet the sum of these three degrees is greater than one. Under such cases, the traditional PFS is unable to manage the data. As a result, dealing with such data necessitates the use of a more sophisticated mathematical tool. For this intention, Mahmood et al. created the spherical fuzzy set (SPHFS) and T-SPHFS, which is a further expansion of PytFS, Q-RUOFS, and PIFS. T-SPHFS has the same structure as PIFS, with the exception that the total of q^{th} power of these three degrees must be less than or equal to one. As a result, T-SPHFS is a more dominant tool for handling ambiguous, conflicting, and unclear data.

Aggregation operators (AOs) are one of the core concepts in multiple attribute decision-making process. The AOs may aggregate a set of real numbers into a single real number. The characteristics of various AOs vary. Some AOs, such as the PA operator provided by Yager [36], have the capacity of removing the bad effect of awkward data from the

final ranking results and has been expanded on by many investigators from everywhere in the globe to cope with diverse settings, such as Xu [37] who provided the ITF power aggregation operator and applied these to MAGDM, which can mitigate the effects of bad data. Garg et al. [38] extend the crisp PA operator to T-SPHF environment and apply them to solve MADM problems. Some AOs, like the BOM, HEM, and MUM operators and the MCSM operator, took into account the correlation between input arguments. BOM and HEM can consider the correlation among two input arguments while MCSM and MUM can consider correlation among any number of input arguments. These AOs were then expanded to handle a variety of fuzzy situations [39–43].

Most AOs are proposed for aggregating T-SPHFNs utilizing algebraic T-norm (TNO) and T-conorm (TCNO). Currently, Ashraf et al. [44] provided various spherical fuzzy dombi (SPHFD) AOs deployed on Dombi [45] TNO and TCNO and applied them to handle MADM problems under SPHF environment. Archimedean TNO (ATNO) and Archimedean TCNO (ATCNO) are simplifications of Dombi TNO, Dombi TCNO, and other TNO and TCNO, like algebraic, Einstein, Hamacher, and Frank, etc. Dombi TNO and TCNO outperform generic TNO and TCNO on a generic parameter, allowing for more flexibility in the input dataset. Similar to the above TNO and TCNO, Schweizer–Sklar (SS) TNO (SSTNO) and Schweizer–Sklar TCN (SSTCNO) [46] are meticulous cases from the ATNO and ATCNO. SSTNO and SSTCNO are also more flexible and superior to the previous procedures because they have a parameter that may be changeable. Nonetheless, the most common SS research focused on determining the basic theory and forms of SSTNO and SSTCNO [47,48]. SS operations were recently fused with interval-valued ITFS (IVITFS) and ITFS and predicted power weighted averaging (WA)/weighted geometric (WG) AGOs by Liu et al. [49] and Zhang [50]. Wang and Liu [51] predicted a MSM operator for ITFS recognized on SS operational laws (ALs). Liu et al. [52] further offered SS ALs for single-valued neutrosophic numbers and initiated various SS prioritized AGOs for handling MADM problems under an SVN environment. Presently, Zhang et al. [53] predicted some MUM operators for SVNS recognized on SS ALs and applied them to handle MADM problems. Nagarajan et al. [54] proposed few SS ALs for interval neutrosophic set (INS) by captivating the variable parameter from $[0, +\infty]$. They also projected a few WA/WG AOs deployed on these SS ALs for IN numbers. Rong, Y. et al. [55] anticipated a novel MAGDM method based on SS ALs and improved the COPRAS method.

2.2. Theoretical Fundamentals

Since 2018, it has been noted that research on different structures of fuzzy MADM AOs based on SS ALs have been released at a quick pace.

It is clear from Table 1 that no authors have sought to define SS ALs for T-SPHFS and combine them with power HEM operator to cope with T-SPHF data. As a result, we put forward:

Table 1. Offers the latest literature of different structures of fuzzy MADM AOs based on SS ALs.

Authors	Different Structures of Fuzzy MADM AOs Based on SS ALs
"Liu and Wang" [49] (2018)	IVIFSSPWA and IVIFSSPWG operators
"Zhang" [50] (2018)	IFSSWA operator
"Wang and Liu" [51] (2019)	IFSSMSM operator
"Liu et al." [52] (2019)	SVNSSPrWA and SVNSSPrWG operators
"Zhang et al." [53] (2019)	SVNSSMM and SVNSSDMM operators
"Nagarajan et al." [54] (2019)	INSSWA and INSSWG operators
"Rong et al." [55] (2020)	Improved COPRAS method

(1) In dealing with MADM challenges, T-SPHFNs outperform ITFS, PytFS, Q-RUOFS and, PIFS in displaying speculative data by detecting the positive MED, abstinence MD, and negative MED.

(2) The SS operations are far more versatile and, by a variable parameter, superior to the previous procedures.
(3) Fortunately, there are a variety of MADM issues in which the attributes are interrelated, and a number of current AGOs can only mitigate such situations when attributes have been in the form of actual numbers or in the form of other fuzzy structures.
(4) There are currently no AGOs in place to handle the MADM issues under T-SPHF information provided on SSTNO and SSTCNO. We integrated the combination of PA and HM operators with SS operations to tackle MADM issues using T-SPF information in response to this constraint.

As a result of major influences of the previous studies, the following are the priorities and offerings of this work:

(1) Initiating novel SS ALs for T-SPHFNs, discussing its basic properties, and deploying it on the SS ALs anticipating T-SPHFSS power Heronian mean operators, T-SPHFSS power geometric Heronian mean operators, and its weighted form.
(2) Inspecting its basic properties and special cases of these initiating AOs.
(3) Anticipating a MADM model deployed on these initiating AOs.
(4) Applying a MADM model to assess water reuse applications.
(5) Verifying the initiated approach' effectiveness and practicality.

To accomplish these intentions, this paper is planned as follows. In part 3, we instigate a number of vital ideas of T-SPHFS and score and accuracy functions, POA, and HEM operators. In part 4, we scrutinize a number of SS ALs for TSPHFNs where the general parameter acquires values from $[-\infty, 0]$. In part 5, we put forward T-SPHFSSPOHEM and T-SPHFSSPOGHEM operators, their weighted forms, and scrutinize a few properties and meticulous cases of the projected AOs. In part 6, we initiate a MADM model recognized on these AOs and applied it to select a water reuse option in the available options to validate the unassailability and compensations of the initiated approach by weighing against other accessible approaches. Lastly, a petite conclusion is prepared in part 7.

3. Methodology

In this part, some core concepts such as T-SPHFS, the HEM operator, POA operator, and their core properties are discussed.

3.1. The T-Spherical Fuzzy Set and Its Operational Laws

Definition 1. *([4]) Let $\overline{\overline{UE}}$ be a universal set. A T-SPHFS is identified and mathematically signified as:*

$$T-SPHFS=\left\{\left\langle \xi,\overline{\overline{pm}}(\xi),\overline{\overline{as}}(\xi),\overline{\overline{nm}}(\xi)\right\rangle \text{ for all } \xi\in\overline{\overline{UE}}\right\} \tag{1}$$

where $\overline{\overline{pm}}(\xi),\overline{\overline{as}}(\xi),\overline{\overline{nm}}(\xi)\in[0,1]$ are, respectively, symbolizing the positive MED (PMED), the abstinence degree (ABD), and negative-membership degree (NMED) with the condition $0\leq\left(\overline{\overline{pm}}(\xi)\right)^{q}+\left(\overline{\overline{as}}(\xi)\right)^{q}+\left(\overline{\overline{nm}}(\xi)\right)^{q}\leq 1$, and the degree of refusal (RED) is symbolized by $HE=\sqrt[q]{1-\left(\overline{\overline{pm}}(\xi)\right)^{q}+\left(\overline{\overline{as}}(\xi)\right)^{q}+\left(\overline{\overline{nm}}(\xi)\right)^{q}}$. For computational ease, we shall designate a T-spherical fuzzy number (T-SPHFN) by the ordered triple $\overline{\overline{tsp}}=\left(\overline{\overline{pm}},\overline{\overline{as}},\overline{\overline{nm}}\right)$.

The ALs for T-SPHFS were classified by Mahmood et al. [28] and are prearranged below:

Definition 2. *([4]) Let $T-SPHFS_1$ and $T-SPHFS_2$ be any two T-SPHFSs. Then*

$$T-SPHFS_1\subseteq T-SPHFS_2 iff\,\overline{\overline{pm}}_1(\xi)\leq\overline{\overline{pm}}_2(\xi),\overline{\overline{as}}_1(\xi)\geq\overline{\overline{as}}_2(\xi),\overline{\overline{nm}}_1(\xi)\geq\overline{\overline{nm}}_2(\xi) for\ all\ \xi\in\overline{\overline{UE}}. \tag{2}$$

$$T-SPHFS_1 = T-SPHFS_2 iff\ T-SPHFS_1 \subseteq T-SPHFS_2 and T-SPHFS_2 \subseteq T-SPHFS_1. \quad (3)$$

$$T-SPHFS_1 \cup T-SPHFS_2 = \left\{ \left\langle \xi, \max\left(\overline{\overline{pm}}_1(\xi), \overline{\overline{pm}}_2(\xi)\right), \min\left(\overline{\overline{as}}_1(\xi), \overline{\overline{as}}_2(\xi)\right), \min\left(\overline{\overline{nm}}_1(\xi), \overline{\overline{nm}}_2(\xi)\right) \right\rangle \text{ for all } \xi \in \overline{\overline{UE}} \right\}. \quad (4)$$

$$T-SPHFS_1 \cap T-SPHFS_2 = \left\{ \left\langle \xi, \min\left(\overline{\overline{pm}}_1(\xi), \overline{\overline{pm}}_2(\xi)\right), \max\left(\overline{\overline{as}}_1(\xi), \overline{\overline{as}}_2(\xi)\right), \max\left(\overline{\overline{nm}}_1(\xi), \overline{\overline{nm}}_2(\xi)\right) \right\rangle \text{ for all } \xi \in \overline{\overline{UE}} \right\}. \quad (5)$$

For comparison of two T-SPFNs tsf_1 and tsf_2, the score, accuracy functions, and judgment rules are illustrated as follows:

$$\widetilde{SRE}(tsf_1) = \overline{\overline{pm}}_1^q(\xi) - \overline{\overline{as}}_1^q(\xi) - \overline{\overline{nm}}_1^q(\xi)/3,\ \widetilde{SRE} \in [-1,1]. \quad (6)$$

$$\widetilde{ACU}(tsf_1) = \overline{\overline{pm}}_1^q(\xi) + \overline{\overline{as}}_1^q(\xi) + \overline{\overline{nm}}_1^q(\xi)/3;\ \widetilde{ACU} \in [0,1]. \quad (7)$$

For judgment of two T-SPHFNs, the judgment rules are listed below.

i. If $\widetilde{SRE}(tsf_1) > \widetilde{SRE}(tsf_2)$, then tsf_1 is finer than tsf_2 and is designated by $tsf_1 > tsf_2$;
ii. If $\widetilde{SRE}(tsf_1) = \widetilde{SRE}(tsf_2)$ and $\widetilde{ACU}(tsf_1) > \widetilde{ACU}(tsf_2)$, then tsf_1 is superior to tsf_2 and is designated by $tsf_1 > tsf_2$;
iii. If $\widetilde{SRE}(tsf_1) = \widetilde{SRE}(tsf_2)$ and $\widetilde{ACU}(tsf_1) = \widetilde{ACU}(tsf_2)$, then tsf_1 is identical to tsf_2 and is designated by $tsf_1 = tsf_2$.

Definition 3. *([7]) Let the T-SPHFNs be* $tsf_1 = \left\langle \overline{\overline{pm}}_1, \overline{\overline{as}}_1, \overline{\overline{nm}}_1 \right\rangle$ *and* $tsf_2 = \left\langle \overline{\overline{pm}}_2, \overline{\overline{as}}_2, \overline{\overline{nm}}_2 \right\rangle$, *and* $q > 0$. *Then the ALs for T-SPFNs are identified as:*

$$tsf_1 \oplus tsf_2 = \left\langle \left(\overline{\overline{pm}}_1^q + \overline{\overline{pm}}_2^q - \overline{\overline{pm}}_1^q \overline{\overline{pm}}_2^q \right)^{\frac{1}{q}}, \overline{\overline{as}}_1 \overline{\overline{as}}_2, \overline{\overline{nm}}_1 \overline{\overline{nm}}_2 \right\rangle; \quad (8)$$

$$tsf_1 \otimes tsf_2 = \left\langle \overline{\overline{pm}}_1 \overline{\overline{pm}}_2, \left(\overline{\overline{as}}_1^q + \overline{\overline{as}}_2^q - \overline{\overline{as}}_1^q \overline{\overline{as}}_2^q \right)^{\frac{1}{q}}, \left(\overline{\overline{nm}}_1^q + \overline{\overline{nm}}_2^q - \overline{\overline{nm}}_1^q \overline{\overline{nm}}_2^q \right)^{\frac{1}{q}} \right\rangle; \quad (9)$$

$$\Im tsf_1 = \left\langle \left(1 - \left(1 - \overline{\overline{pm}}_1^q \right)^{\Im} \right)^{\frac{1}{q}}, \overline{\overline{as}}_1^{\Im}, \overline{\overline{nm}}_1^{\Im} \right\rangle; \Im > 0 \quad (10)$$

$$tsf_1^{\Im} = \left\langle \overline{\overline{pm}}_1^{\Im}, \left(1 - \left(1 - \overline{\overline{as}}_1^q \right)^{\Im} \right)^{\frac{1}{q}}, \left(1 - \left(1 - \overline{\overline{nm}}_1^q \right)^{\Im} \right)^{\frac{1}{q}} \right\rangle; \Im > 0 \quad (11)$$

$$tsf_1^c = \left\langle \overline{\overline{nm}}_1, \overline{\overline{as}}_1, \overline{\overline{pm}}_1 \right\rangle \quad (12)$$

Definition 4. *([7]) Suppose that the two SPFNs be* $tsf_1 = \left\langle \overline{\overline{pm}}_1, \overline{\overline{as}}_1, \overline{\overline{nm}}_1 \right\rangle$ *and* $tsf_2 = \left\langle \overline{\overline{pm}}_2, \overline{\overline{as}}_2, \overline{\overline{nm}}_2 \right\rangle$. *Then, the normalized Hamming distance between* tsf_1 *and* tsf_2 *is labelled as:*

$$\overline{\overline{DIT}}(tsf_1, tsf_2) = \frac{1}{3}\left(\left| \overline{\overline{pm}}_1^q - \overline{\overline{pm}}_2^q \right| + \left| \overline{\overline{as}}_1^q - \overline{\overline{as}}_2^q \right| + \left| \overline{\overline{nm}}_1^q - \overline{\overline{nm}}_2^q \right| \right) \quad (13)$$

3.2. The Power Average (POA) Operator

Yagar [36] initiated the perception of the POA operator that is the key AO. The POA operator reduced various unconstructive effects of pointlessly high or unreasonably low opinions specified by experts. The predictable POA operator may combine a collection of crisp integers where the weighting vector is based only on the input data and is characterized as:

Definition 5. *([36]) Let the group of positive real number represented by $b_i (i = 1, 2, \ldots, \vartheta)$. Then, the POA operator is a function delineated by*

$$PA(b_1, b_2, \ldots., b_\vartheta) = \frac{\sum_{i=1}^{\vartheta} (1 + T(b_i)) b_i}{\sum_{i=1}^{\vartheta} (1 + T(b_i))}, \tag{14}$$

where

$$T(b_i) = \sum_{\substack{j=1 \\ j \neq i}}^{\vartheta} Supr(b_i, b_j) \tag{15}$$

and $Supr(b, \aleph)$ is the support degree for b from $\aleph$, which should gratify the following constrain. (1) $Supr(b, \aleph) \in [0, 1]$, (2) $Supr(b, \aleph) = Supr(\aleph, b)$, (3) $Supr(b, \aleph) \geq Supr(v, u)$, if $|b - \aleph| < |v - u|$.

3.3. Heronian Mean (HEM) Operator

For aggregation, the HEM [9] operator is also a significant AO, which can embody the interrelationships of the input attributes, and is delineated as:

Definition 6. *([9]) Let $G = [0, 1], a, b \geq 0, HEM^{\mathrm{A,B}} : \mathrm{G}^m \to G$, if $HEM^{\mathrm{A,B}}$ satisfies:*

$$HEM^{\mathrm{A,B}}(c_1, c_2, \ldots, c_\vartheta) = \left(\frac{2}{\vartheta^2 + \vartheta} \sum_{i=1}^{\vartheta} \sum_{j=i}^{\vartheta} c_i^A c_j^B \right)^{\frac{1}{A+B}} \tag{16}$$

Then, the function $HEM^{\mathrm{A,B}}$ is alleged to be HEM operator with parameters A, B. The HEM operator must ensure the characteristics of idempotency, boundedness, and monotonicity.

4. Results

4.1. Schweizer–Sklar ALs for T-SPHFNs

In this subpart, the SS ALs are initiated for T-SPHFNs deployed on SSTNO and SSTCNO, and various core properties of SS ALs for T-SPHFNs are investigated.

The SSTNO and SSTCNO [48] are identified as:

$$\widetilde{TNO}_{SS}(\overline{\overline{vo}}, \overline{\overline{uo}}) = \left(\overline{\overline{vo}}^{\gamma} + \overline{\overline{uo}}^{\gamma} - 1 \right)^{\frac{1}{\gamma}} \text{ and } \widetilde{TCNO}^{*}_{SS}(\overline{\overline{vo}}, \overline{\overline{uo}}) = 1 - \left(\left(1 - \overline{\overline{vo}}\right)^{\gamma} + \left(1 - \overline{\overline{uo}}\right)^{\gamma} - 1 \right)^{\frac{1}{\gamma}}$$

where $\gamma < 0, \overline{\overline{vo}}, \overline{\overline{uo}} \in [0, 1]$.

Additionally, when $\gamma = 0$, we have $\widetilde{TNO}_{\zeta}(\overline{\overline{vo}}, \overline{\overline{uo}}) = \overline{\overline{vo}} \cdot \overline{\overline{uo}}$ and $\widetilde{TCNO}^{*}_{\zeta}(\overline{\overline{vo}}, \overline{\overline{uo}}) = \overline{\overline{vo}} + \overline{\overline{uo}} - \overline{\overline{vo}}.\overline{\overline{uo}}$. That is, SS TNO and SS TCNO condense to algebraic TNO and TCNO.

Now, deployed on SSTNO $\widetilde{TNO}_{\zeta}(\overline{\overline{vo}}, \overline{\overline{uo}})$ and SSTCNO $\widetilde{TCNO}^{*}_{\zeta}(\overline{\overline{vo}}, \overline{\overline{uo}})$, we can permit the subsequent definition about SS ALs of T-SPHFNs.

Definition 7. *Let the three SPHFNs be $tsf = \langle \overline{\overline{pm}}, \overline{\overline{as}}, \overline{\overline{nm}} \rangle, tsf_1 = \langle \overline{\overline{pm}}_1, \overline{\overline{as}}_1, \overline{\overline{nm}}_1 \rangle$, and $tsf_2 = \langle \overline{\overline{pm}}_2, \overline{\overline{as}}_2, \overline{\overline{nm}}_2 \rangle$. Then, the SS ALs for T-SPHFNs are defined as:*

$$tsf_1 \oplus tsf_2 = \left\langle \left(1 - \left(\left(1 - \overline{\overline{pm}}_1^{q}\right)^{\gamma} + \left(1 - \overline{\overline{pm}}_2^{q}\right)^{\gamma} - 1 \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}}, \left(\left(\overline{\overline{as}}_1^{q}\right)^{\gamma} + \left(\overline{\overline{as}}_2^{q}\right)^{\gamma} - 1 \right)^{\frac{1}{q\gamma}}, \left(\left(\overline{\overline{nm}}_1^{q}\right)^{\gamma} + \left(\overline{\overline{nm}}_2^{q}\right)^{\gamma} - 1 \right)^{\frac{1}{q\gamma}} \right\rangle; \tag{17}$$

$$tsf_1 \otimes tsf_2 = \left\langle \left(\left(\overline{\overline{pm}}_1^{q}\right)^{\gamma} + \left(\overline{\overline{pm}}_2^{q}\right)^{\gamma} - 1 \right)^{\frac{1}{q\gamma}}, \left(1 - \left(\left(1 - \overline{\overline{as}}_1^{q}\right)^{\gamma} + \left(1 - \overline{\overline{as}}_2^{q}\right)^{\gamma} - 1 \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}}, \left(1 - \left(\left(1 - \overline{\overline{nm}}_1^{q}\right)^{\gamma} + \left(1 - \overline{\overline{nm}}_2^{q}\right)^{\gamma} - 1 \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}} \right\rangle; \tag{18}$$

$$\xi tsf = \left\langle \left(1 - \left(\xi\left(1 - \overline{\overline{pm}}^{q}\right)^{\gamma} - (\xi - 1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \left(\xi\overline{\overline{as}}^{q\gamma} - (\xi - 1)\right)^{\frac{1}{q\gamma}}, \left(\xi\overline{\overline{nm}}^{q\gamma} - (\xi - 1)\right)^{\frac{1}{q\gamma}} \right\rangle; \tag{19}$$

$$tsf^{\xi} = \left\langle \left(\xi\overline{\overline{pm}}^{q\gamma} - (\xi - 1)\right)^{\frac{1}{q\gamma}}, \left(1 - \left(\xi\left(1 - \overline{\overline{as}}^{q}\right)^{\gamma} - (\xi - 1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \left(1 - \left(\xi\left(1 - \overline{\overline{nm}}^{q}\right)^{\gamma} - (\xi - 1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}} \right\rangle. \tag{20}$$

Likewise, a number of attractive properties of the T-SPHFNs ALs can be easily obtained.

Theorem 1. *Let the three T-SPHFNs be* $tsf = \langle \overline{\overline{pm}}, \overline{\overline{as}}, \overline{\overline{nm}} \rangle, tsf_1 = \langle \overline{\overline{pm}}_1, \overline{\overline{as}}_1, \overline{\overline{nm}}_1 \rangle$ *and* $tsf_2 = \langle \overline{\overline{pm}}_2, \overline{\overline{as}}_2, \overline{\overline{nm}}_2 \rangle$. *Then,*

(1) $tsf_1 \oplus_{SS} tsf_2 = tsf_2 \oplus_{SS} tsf_1$;
(2) $tsf_1 \otimes_{SS} tsf_2 = tsf_2 \otimes_{SS} tsf_1$;
(3) $\lambda(tsf_1 \oplus_{SS} tsf_2) = \lambda tsf_1 \oplus_{SS} \lambda tsf_2,\ \lambda \geq 0$;
(4) $\lambda_1 tsf \oplus_{SS} \lambda_2 tsf = (\lambda_1 + \lambda_2) tsf;\ \lambda_1, \lambda_2 \geq 0$;
(5) $tsf^{\lambda_1} \otimes_{SS} tsf^{\lambda_2} = (tsf)^{\lambda_1 + \lambda_2},\ \lambda_1, \lambda_2 \geq 0$;
(6) $tsf_1^{\lambda} \otimes_{SS} tsf_2^{\lambda} = (tsf_1 \otimes_{SS} tsf_2)^{\lambda},\ \lambda \geq 0$.

4.2. The T-Spherical Fuzzy Schweizer-Sklar Power Heronian Mean Operators

The T-SPHFSSPOHEM and T-SPHFSSPOGHEM Operators

In this sub-subpart, we initiate various new AOs under the T-SPHF environment by merging the POA operator with the HEM operator and the GHEM, respectively.

Definition 8. *Let* $tsf_i = \langle \overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i \rangle\ (i = 1, 2, \ldots, l)$ *be a group of T-SPHFNs, and then the T-SPHFSSPOHEM operator is explained as follows:*

$$T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)} \left(\sum_{\substack{i,j=1 \\ j=i}}^{l} \left(\frac{l(1+T(tsf_i))}{\sum_{h=1}^{l}(1+T(tsf_h))} tsf_i \right)^{A} \otimes_{SS} \left(\frac{l\left(1+T\left(tsf_j\right)\right)}{\sum_{h=1}^{l}(1+T(tsf_h))} tsf_j \right)^{B} \right) \right)^{\frac{1}{A+B}} \tag{21}$$

where $A, B \geq 0, T(tsf_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} \mathrm{supr}\left(tsf_i, tsf_j\right), \mathrm{supr}\left(tsf_i, tsf_j\right) = 1 - \overline{\overline{DIT}}\left(tsf_i, tsf_j\right)$, *and* $\overline{\overline{DIT}}\left(tsf_i, tsf_j\right)$ *can be computed by Equation (13).*

Let $\mathfrak{v}_i = \frac{(1+T(tsf_i))}{\sum_{h=1}^{l}(1+T(tsf_h))}$, *then the definition of T-SPHFSSPOHEM operator is equivalent to the subsequent form:*

$$T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)} \left(\sum_{\substack{i,j=1 \\ j=i}}^{l} \left(l\mathfrak{v}_i tsf_i\right)^{A} \otimes_{SS} \left(l\mathfrak{v}_j tsf_j\right)^{B} \right) \right)^{\frac{1}{A+B}} \tag{22}$$

Theorem 2. *Let* $A \geq 0,\ B \geq 0$ *and* A, B *take no more than one value of 0 at a time,* $tsf_i = \langle \overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i \rangle$ *be a group of T-SPHFNs. Then, utilizing the T-SPHFSSPOHEM operator, their fused values are also T-SPHFN, and*

$$T-SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l)$$

$$= \left\langle \left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}}, \right.$$

$$\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \tag{23}$$

$$\left.\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{u}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle.$$

Proof. From the Schweizer–Sklar operational laws for T-SPHFS, we have

$$l\mathfrak{v}_i tsf_i = \left\langle \left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{q\gamma}}, \left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{q\gamma}}\right\rangle;$$

and

$$\left(l\mathfrak{v}_i tsf_i\right)^{A} = \left\langle \left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{q\gamma}}, \left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.$$

$$\left.\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle.$$

Similarly,

$$l\mathfrak{v}_j tsf_j = \left\langle \left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{q\gamma}}, \left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{q\gamma}}\right\rangle;$$

and

$$\left(l\mathfrak{v}_j tsf_j\right)^{B} = \left\langle \left(B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{q\gamma}}, \left(1-\left(B\left(1-\left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.$$

$$\left.\left(1-\left(B\left(1-\left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle.$$

$$\left(l\mathfrak{v}_i tsf_i\right)^{A} \otimes_{SS} \left(l\mathfrak{v}_j tsf_j\right)^{B} =$$

$$\left\langle \left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(l_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{q\gamma}}, \right.$$

$$\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma}-(l_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},$$

$$\left.\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle,$$

$$\sum_{\substack{i,j=1\\ j=i}}^{l} \left(l\mathfrak{v}_i tsf_i\right)^A \otimes_{SS} \left(l\mathfrak{v}_j tsf_j\right)^B$$

$$= \left\langle \left(1 - \left(\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\left(1 - \overline{\overline{pm}}_i^{q}\right)^{\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\left(1 - \overline{\overline{pm}}_j^{q}\right)^{\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma} - \left(\frac{l(l+1)}{2} - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.$$

$$\left(\sum_{\substack{i,j=1\\ j=i}}^{u} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma} - \left(\frac{l(l+1)}{2} - 1\right)\right)^{\frac{1}{q\gamma}},$$

$$\left.\left(\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma} - \left(\frac{l(l+1)}{2} - 1\right)\right)^{\frac{1}{q\gamma}}\right\rangle.$$

$$\frac{2}{l(l+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{l} \left(l\mathfrak{v}_i sft_i\right)^A \otimes_{SS} \left(l\mathfrak{v}_j sft_j\right)^B\right)$$

$$= \left\langle \left(1 - \left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\left(1 - \overline{\overline{pm}}_i^{q}\right)^{\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\left(1 - \overline{\overline{pm}}_j^{q}\right)^{\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.$$

$$\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{q\gamma}},$$

$$\left.\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{u} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{q\gamma}}\right\rangle.$$

$$\left(\frac{2}{l(l+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{l} \left(l\mathfrak{v}_i tsf_i\right)^A \otimes_{SS} \left(l\mathfrak{v}_j tsf_j\right)^B\right)\right)^{\frac{1}{A+B}}$$

$$= \left\langle \left(\frac{1}{A+B}\left(1 - \left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\left(1 - \overline{\overline{pm}}_i^{q}\right)^{\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\left(1 - \overline{\overline{pm}}_j^{q}\right)^{\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma} - \left(\frac{1}{A+B} - 1\right)\right)^{\frac{1}{q\gamma}}, \right.$$

$$\left(1 - \left(\frac{1}{A+B}\left(1 - \left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma} - \left(\frac{1}{A+B} - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},$$

$$\left.\left(1 - \left(\frac{1}{A+B}\left(1 - \left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l} \left(1 - \left(A\left(1 - \left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma} - \left(l\mathfrak{v}_i - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} + B\left(1 - \left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma} - \left(l\mathfrak{v}_j - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma} - A - B + 1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma} - \left(\frac{1}{A+B} - 1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle.$$

Hence,

$$
\begin{aligned}
&T-SPFHSSPOHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l)\\
&=\left\langle\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}},\right.\\
&\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{as}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{as}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\\
&\left.\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{nm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{nm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle.
\end{aligned}
$$

□

Now, we confer various core properties of the anticipated T-SPHFSSPOHEM operator.

Theorem 3. (*Idempotency*) *Let $A \geq 0,\ B \geq 0$ and A, B take no more than one value of 0 at a time, $tsf_i = \langle\overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i\rangle (i = 1,2,\ldots,l)$ be a group of T-SPHFNs, and $tsf_i = \langle\overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i\rangle = tsf = \langle\overline{\overline{pm}}, \overline{\overline{as}}, \overline{\overline{nm}}\rangle (i = 1,2,\ldots,l)$. Then*

$$T-SPHFSSPOHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l) = tsf = \langle\overline{\overline{pm}}, \overline{\overline{as}}, \overline{\overline{nm}}\rangle \tag{24}$$

Proof. Since, $tsf_i = \langle\overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i\rangle = tsf = \langle\overline{\overline{pm}}, \overline{\overline{as}}, \overline{\overline{nm}}\rangle (i = 1,2,\ldots,l)$, we have

$$\mathrm{supr}\left(tsf_g, tsf_z\right) = 1\ (\forall\ g,z = 1,2,\ldots,l)\ \mathrm{so}\ \mathfrak{v}_g = \frac{1}{u}(\forall\ g = 1,2,\ldots,u)\ \mathrm{and}\ l\mathfrak{v}_g = l\frac{1}{l} = 1(\forall g = 1,2,\ldots,l)$$

Then

$$
\begin{aligned}
&T-SPHFSSPOHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l) = T-SPHFSSPOHEM^{A,B}(tsf,tsf,\ldots,tsf)\\
&=\left\langle\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{u}\left(1-\left(A\left(1-\left(\left(1-\overline{\overline{pm}}^{q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(\left(1-\overline{\overline{pm}}^{q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}},\right.\\
&\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\overline{\overline{as}}^{q}\right)^{\gamma}+B\left(1-\overline{\overline{as}}^{q}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\\
&\left.\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\overline{\overline{nm}}^{q}\right)^{\gamma}+B\left(1-\overline{\overline{nm}}^{q}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle,
\end{aligned}
$$

$$= \left\langle \left(\frac{1}{A+B}\left((A+B)\overline{\overline{pm}}^{q\gamma} - A - B + 1 \right) - \frac{1}{A+B} + 1 \right)^{\frac{1}{q\gamma}}, \left(1 - \left(\frac{1}{A+B}\left((A+B)\left(1 - \overline{\overline{as}}^{q} \right)^{\gamma} - A - B + 1 \right) - \frac{1}{A+B} + 1 \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}}, \right.$$
$$\left. \left(1 - \left(\frac{1}{A+B}\left((A+B)\left(1 - \overline{\overline{nm}}^{q} \right)^{\gamma} - A - B + 1 \right) - \frac{1}{A+B} + 1 \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}} \right\rangle,$$
$$= \langle \overline{\overline{pm}}, \overline{\overline{as}}, \overline{\overline{nm}} \rangle = tsf \ \square$$

Theorem 4. ***(Commutativity)*** *Let* $(tsf', tsf'_2, \ldots, tsf'_l)$ *be any permutation of* $(tsf_1, tsf_2, \ldots, tsf_l)$*. Then*

$$T - SPHFSSPOHEM^{A,B}\left(tsf'_1, tsf'_2, \ldots, tsf'_l\right) = T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \tag{25}$$

Proof. Since $(tsf', tsf'_2, \ldots, tsf'_l)$ is any permutation of $(tsf_1, tsf_2, \ldots, tsf_l)$. SO,

$$T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(u+1)} \left(\sum_{\substack{i,j=1 \\ j=i}}^{l} \left(l\mho_i tsf_i \right)^A \otimes_{SS} \left(l\mho_j tsf_j \right)^B \right) \right)^{\frac{1}{A+B}},$$
$$= \left(\frac{2}{l(l+1)} \left(\sum_{\substack{i,j=1 \\ j=i}}^{l} \left(l\mho_i tsf'_i \right)^A \otimes_{SS} \left(l\mho_j tsf'_j \right)^B \right) \right)^{\frac{1}{A+B}},$$
$$= T - SPHFSSPOHEM^{A,B}(tsf'_1, tsf'_2, \ldots, tsf'_l).$$

□

Theorem 5. ***(Boundedness)*** *Let* $tsf_i = \langle \overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i \rangle (i = 1, 2, \ldots, l)$ *be a group of T-SPHFNs,* $tsf^- = \langle \text{mini}_i \overline{\overline{pm}}_i, \text{maxi}_i \overline{\overline{as}}_i, \text{maxi}_i \overline{\overline{nm}}_i \rangle$ *and* $tsf^+ = \langle \text{maxi}_i \overline{\overline{pm}}_i, \text{mini}_i \overline{\overline{as}}_i, \text{mini}_i \overline{\overline{nm}}_i \rangle$*. Then*

$$tsf^- \leq T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \leq tsf^+ \tag{26}$$

Proof. By the comparison method in Definition 2, we have $tsf_i \geq tsf'_i$, then deployed on Theorems 3 and 4, we have

$$T - SPHFSSPHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \geq T - SPHFSSPOHEM^{A,B}\left(tsf_1^-, tsf_2^-, \ldots, tsf_l^-\right) = tsf^-$$

Similarly, we can have

$$T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \leq T - SPHFSSPOHEM^{A,B}\left(tsf_1^+, tsf_2^+, \ldots, tsf_l^+\right) = tsf^+.$$

So, we have

$$T - SPHFSSPOHEM^{A,B}(tsf_1^-, tsf_2^-, \ldots, tsf_l^-) \leq T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \leq T - SPHFSSPOHEM^{A,B}(tsf_1^+, tsf_2^+, \ldots, tsf_l^+)$$

Hence,

$$tsf^- \leq T - SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \leq tsf^+$$

□

By allocating different values to the parameters γ, q, A *and* B, various special cases of the T-SPHFSSPOHEM operator can be obtained, which can be expressed as given below:

Case 1. If $\gamma = 0$, then the T-SPHFSSPOHEM operator degenerates to the T-SPHFPOHEM operator, which can be expressed as follows:

$$T-SPHFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left((l\mho_i tsf_i)^A \otimes \left(l\mho_j tsf_j\right)^B \right) \right)^{\frac{1}{A+B}}$$

$$= \left\langle \left(\left(1 - \prod_{\substack{i,j=1 \\ j=i}}^{u} \left(1 - \left(1 - \left(1 - \overline{\overline{pm}}_i^q \right)^{l_i} \right)^A \left(1 - \left(1 - \overline{\overline{pm}}_j^q \right)^{l_j} \right)^B \right)^{\frac{2}{l(l+1)}} \right)^{\frac{1}{q}} \right)^{\frac{1}{A+B}}, \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(1 - \left(\overline{\overline{as}}_i \right)^{ql\mho_i} \right)^A \left(1 - \left(\overline{\overline{as}}_j \right)^{ql\mho_j} \right)^B \right)^{\frac{2}{l(l+1)}} \right)^{\frac{1}{A+B}} \right)^{\frac{1}{q}}, \right.$$

$$\left. \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ j=i}}^{u} \left(1 - \left(1 - \left(\overline{\overline{nm}}_i \right)^{ql\mho_i} \right)^A \left(1 - \left(\overline{\overline{nm}}_j \right)^{ql\mho_j} \right)^B \right)^{\frac{2}{l(l+1)}} \right)^{\frac{1}{A+B}} \right)^{\frac{1}{q}} \right\rangle. \tag{27}$$

Case 2. If $q = 1$, then the T-SPHFSSPOHEM operator degenerates to the picture fuzzy Schweizer–Sklar power Heronian mean (PIFSSPOHEM) operator, which can be expressed as follows:

$$PIFSSPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{u} \left((l\mho_i tsf_i)^A \otimes_{SS} \left(l\mho_j tsf_j\right)^B \right) \right)^{\frac{1}{A+B}}$$

$$= \left\langle \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\mho_i \left(1 - \overline{\overline{pm}}_i \right)^\gamma - (l\mho_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma + B \left(1 - \left(l\mho_j \left(1 - \overline{\overline{pm}}_j \right)^\gamma - (lm_j - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}}, \right.$$

$$1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\mho_i \overline{\overline{as}}_i^\gamma - (l\mho_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma + B \left(1 - \left(l\mho_j \overline{\overline{as}}_j^\gamma - (l\mho_j - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}}, \right.$$

$$\left. 1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\mho_i \overline{\overline{nm}}_i^\gamma - (l\mho_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma + B \left(1 - \left(l\mho_j \overline{\overline{nm}}_j^\gamma - (l\mho_j - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}} \right\rangle. \tag{28}$$

Case 3. If $q = 2$, then the T-SPHFSSPOHEM operator degenerates to the Spherical fuzzy Schweizer–Sklar power Heronian mean (SPFSSPOHEM) operator, which can be expressed as follows:

$$SPFSSPHM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left((l\mathfrak{v}_i tsf_i)^A \otimes_{SS} (l\mathfrak{v}_j tsf_j)^B \right) \right)^{\frac{1}{A+B}}$$

$$= \left\langle \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \left(1 - \overline{\overline{pm}}_i^2 \right)^\gamma - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma + B \left(1 - \left(l\mathfrak{v}_j \left(1 - \overline{\overline{pm}}_j^2 \right)^\gamma - (l\mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{2\gamma}}, \right.$$

$$\left(1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \overline{\overline{as}}_i^{2\gamma} - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma + B \left(1 - \left(l\mathfrak{v}_j \overline{\overline{as}}_j^{2\gamma} - (l\mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{2}}, \tag{29}$$

$$\left. \left(1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \overline{\overline{nm}}_i^{2\gamma} - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma + B \left(1 - \left(l\mathfrak{v}_j \overline{\overline{nm}}_j^{2\gamma} - (l\mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{2}} \right\rangle.$$

Case 4. If $B \to 0$, then the T-SPHFSSPOHEM operator degenerates to the T-SPHF descending POA operator, which can be expressed as follows:

$$T - SPFSSPHM^{A,B}(tsf_1, tsf_2, \ldots, tsf_n) = \lim_{B \to 0} \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left((l\mathfrak{v}_i tsf_i)^A \otimes_{SS} (l\mathfrak{v}_j tsf_j)^B \right) \right)^{\frac{1}{A+B}}$$

$$= \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left((l+1-i)(l\mathfrak{v}_i tsf_i)^A \right) \right)^{\frac{1}{A}}$$

$$= \left\langle \left(\frac{1}{A} \left(1 - \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left((l+1-i) \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \left(1 - \overline{\overline{pm}}_i^q \right)^\gamma - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - (A-1) \right)^{\frac{1}{\gamma}} \right)^\gamma \right) - \left((2l-i) \left(\frac{2}{l(l+1)} \right) - 1 \right) \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A} - 1 \right) \right)^{\frac{1}{q\gamma}}, \right.$$

$$\left(1 - \left(\frac{1}{A} \left(1 - \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left((l+1-i) \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \overline{\overline{as}}_i^{q\gamma} - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - (A-1) \right)^{\frac{1}{\gamma}} \right)^\gamma \right) - \left((2l-i) \left(\frac{2}{l(l+1)} \right) - 1 \right) \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}}, \tag{30}$$

$$\left. \left(1 - \left(\frac{1}{A} \left(1 - \left(\frac{2}{l(l+1)} \sum_{i=1}^{n} \left((l+1-i) \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \overline{\overline{nm}}_i^{q\gamma} - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - (A-1) \right)^{\frac{1}{\gamma}} \right)^\gamma \right) - \left((2l-i) \left(\frac{2}{l(l+1)} \right) - 1 \right) \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{A} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}} \right\rangle.$$

Case 5. If $A \to 0$, then the T-SPHFSSPOHEM operator degenerates to the T-SPF ascending POA operator, which can be expressed as follows:

$$T - SPHFSSPOHEM^{0,B}(tsf_1, tsf_2, \ldots, tsf_n) = \lim_{A \to 0} \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left((l\mathfrak{v}_i tsf_i)^A \otimes_{SS} (l\mathfrak{v}_j tsf_j)^B \right) \right)^{\frac{1}{A+B}}$$

$$= \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left(i (l\mathfrak{v}_i tsf_i)^B \right) \right)^{\frac{1}{B}}$$

$$= \left\langle \left(\frac{1}{B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left(i \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \left(1 - \overline{\overline{pm}}_i^q \right)^\gamma - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - (A-1) \right)^{\frac{1}{\gamma}} \right)^\gamma \right) - \left((l+i-1) \left(\frac{2}{l(l+1)} \right) - 1 \right) \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{B} - 1 \right) \right)^{\frac{1}{q\gamma}}, \right.$$

$$\left(1 - \left(\frac{1}{B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left(i \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \overline{\overline{as}}_i^{q\gamma} - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - (A-1) \right)^{\frac{1}{\gamma}} \right)^\gamma \right) - \left((l+i-1) \left(\frac{2}{l(l+1)} \right) - 1 \right) \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}}, \tag{31}$$

$$\left. \left(1 - \left(\frac{1}{B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{i=1}^{l} \left(i \left(1 - \left(A \left(1 - \left(l\mathfrak{v}_i \overline{\overline{nm}}_i^{q\gamma} - (l\mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^\gamma - (A-1) \right)^{\frac{1}{\gamma}} \right)^\gamma \right) - \left((l+i-1) \left(\frac{2}{l(l+1)} \right) - 1 \right) \right)^{\frac{1}{\gamma}} \right)^\gamma - \left(\frac{1}{B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}} \right\rangle.$$

Case 6. If $B \to 0$, and $\text{supr}\left(tsf_i, tsf_j\right) = h(h \in [0,1])(\forall i \neq j)$, then the T-SPHFSSPOHEM operator degenerates to the T-SPHF linear descending WA operator, which can be expressed as follows:

$$T-SPHFSSPOHEM^{A,0}(tsf_1,tsf_2,\ldots,tsf_n)=\lim_{B\to 0}\left(\begin{array}{c}\frac{2}{l(l+1)}\sum\limits_{\substack{i,j=1\\ j=1}}^{l}(l\mathfrak{w}_i tsf_i)^A\otimes\left(l\mathfrak{w}_j tsf_j\right)^B\end{array}\right)^{\frac{1}{A+B}}$$
$$=\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}(l+1-i)tsf_i^A\right)^{\frac{1}{A}}$$
$$=\langle\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\overline{\overline{pm}}_i^{q\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{q\gamma}},\right.$$
$$\left(1-\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\tag{32}$$
$$\left.\left(1-\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\rangle$$

Certainly, the importance degree of T-SPHFNs $tsf_i^A(i=1,2,\ldots.,l)$ is $(l,l-1,\ldots.,1)$.

Case 7. If $A\to 0$, and $\mathrm{supr}\left(tsf_i,tsf_j\right)=h(h\in[0,1])(\forall i\neq j)$, then the T-SPHFSSPOHEM operator degenerates to the T-SPHF linear ascending WA operator, which can be expressed as follows:

$$T-SPFSSPHM^{0,B}(tsf_1,tsf_2,\ldots,tsf_n)=\lim_{A\to 0}\left(\begin{array}{c}\frac{2}{l(l+1)}\sum\limits_{\substack{i,j=1\\ j=1}}^{l}(l\mathfrak{w}_i tsf_i)^A\otimes\left(l\mathfrak{w}_j tsf_j\right)^B\end{array}\right)^{\frac{1}{A+B}}$$
$$=\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left(itsf_i^B\right)\right)^{\frac{1}{B}}$$
$$=\langle\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((i)\left(1-\left(B\overline{\overline{pm}}_i^{q\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{q\gamma}},\right.$$
$$\left(1-\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((i)\left(1-\left(B\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\tag{33}$$
$$\left.\left(1-\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((i)\left(1-\left(B\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\rangle.$$

Case 8. If $A=B=\frac{1}{2}$, and $\mathrm{supr}\left(tsf_i,tsf_j\right)=h(h\in[0,1])(\forall i\neq j)$, then the T-SPHFSSPOHEM operator degenerates to the T-SPHF basic HEM operator, which can be expressed as follows:

$$T-SPHFSSPOHEM^{\frac{1}{2},\frac{1}{2}}(tsf_1,tsf_2,\ldots,tsf_n)=\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=1}}^{l}\left(\left(tsf_i\right)^{\frac{1}{2}}\otimes\left(tsf_j\right)^{\frac{1}{2}}\right)$$
$$=\langle\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\frac{1}{2}\overline{\overline{pm}}_i^{q\gamma}+\frac{1}{2}\overline{\overline{pm}}_j^{q\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{q}}\right),\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\frac{1}{2}\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}+\frac{1}{2}\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{q\gamma}},\tag{34}$$
$$\left.\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\frac{1}{2}\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}+\frac{1}{2}\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)\right\rangle.$$

Case 9. If $A=B=1$, and $\mathrm{supr}\left(tsf_i,tsf_j\right)=h(h\in[0,1])(\forall i\neq j)$, then the T-SPHFSSPOHEM operator degenerates to the T-SPHF linear HEM operator, which can be expressed as follows:

$$T-SPHFSSPOHEM^{1,1}(tsf_1,tsf_2,\ldots,tsf_n)=\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(tsf_i\otimes tsf_j\right)\right)^{\frac{1}{2}}$$

$$=\left\langle\left(\frac{1}{2}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\overline{\overline{pm}}_i^{q\gamma}+\overline{\overline{pm}}_j^{q\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{q\gamma}}\right)+\frac{1}{2}\right),\left(1-\left(\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}+\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}+\frac{1}{2}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right),\right.$$

$$\left.\left(1-\left(\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}+\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}+\frac{1}{2}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle. \tag{35}$$

Definition 9. *Let* $tsf_i=\langle\overline{\overline{pm}}_i,\overline{\overline{as}}_i,\overline{\overline{nm}}_i\rangle$ $(i=1,2,\ldots.,l)$ *be a group of T-SPHFNs, and then the T-SPHF Schweizer–Sklar power geometric Heronian mean (T-SPHFSSPOGHEM) operator is explained as follows:*

$$T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_n)=\frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{\frac{l(1+T(tsf_j))}{\sum_{z=1}^{l}(1+T(tsf_z))}}\oplus_{SS}Btsf_j^{\frac{l(1+T(tsf_j))}{\sum_{z=1}^{l}(1+T(tsf_z))}}\right)\right)^{\frac{2}{l(l+1)}} \tag{36}$$

where $A,B\geq 0, T(tsf_i)=\sum_{\substack{j=1\\ j\neq i}}^{l}\operatorname{supr}\left(tsf_i,tsf_j\right), \operatorname{supr}\left(tsf_i,tsf_j\right)=1-\overline{\overline{DIT}}\left(tsf_i,tsf_j\right)$, *and* $\overline{\overline{DIT}}\left(tsf_i,tsf_j\right)$ *can be computed by Equation (13).*

Let $_i=\frac{(1+T(tsf_i))}{\sum_{z=1}^{l}(1+T(tsf_z))}$, *then the definition of T-SPHFSSPOGHEM operator is equivalent to the following form:*

$$T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l)=\frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus_{SS}Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}} \tag{37}$$

Theorem 6. *Let* $A\geq 0$, $B\geq 0$ *and* A,B *take no more than one value of 0 at a time,* $tsf_i=\langle\overline{\overline{pm}}_i,\overline{\overline{as}}_i,\overline{\overline{nm}}_i\rangle$ *be a group of T-SPHFNs. Then, utilizing the T-SPHFSSPOGHEM operator, their fused values are also T-SPHFN, and*

$$
\begin{aligned}
&T-SPHFSSPOGHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_n)\\
&= \left\langle \left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.\\
&\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}}\right.\\
&\left.\left.\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}}\right) \right\rangle.
\end{aligned}
\tag{38}
$$

Proof. From the Schweizer–Sklar operational laws for T-SPFS, we have

$$
tsf_i^{l_i} = \left\langle \left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{q\gamma}-(l_i-1)\right)^{\frac{1}{q\gamma}}, \left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}} \right\rangle,
$$

and

$$
\begin{aligned}
Atsf_i^{l_i} = \left\langle \left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{tsf}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.\\
\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{q\gamma}}, \left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{q\gamma}} \left.\right\rangle.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&tsf_j^{l_j} = \left\langle \left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{q\gamma}}, \left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}} \right\rangle,\\
&Bsf_j^{l_j} = \left\langle \left(1-\left(B\left(1-\left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.\\
&\left(B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{q\gamma}}, \left(B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{q\gamma}} \left.\right\rangle.\\
&Atsf_i^{l_i} \oplus_{SS} Btsf_j^{l_j} =\\
&\left\langle \left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.\\
&\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{q\gamma}}\\
&\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{q\gamma}} \left.\right\rangle.
\end{aligned}
$$

Therefore,

$$\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus_{SS}Btsf_j^{l_j}\right)$$

$$=\left\langle\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\tfrac{l(l+1)}{2}-1\right)\right)^{\frac{1}{q\gamma}},\right.$$

$$\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\tfrac{l(l+1)}{2}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\right.$$

$$\left.\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\tfrac{l(l+1)}{2}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle.$$

$$\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus_{SS}Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}$$

$$=\left\langle\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{2}{l(l+1)}}\right.,$$

$$\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{2}{l(l+1)}}\right)^{\frac{1}{q}},$$

$$\left.\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{2}{l(l+1)}}\right)^{\frac{1}{q}}\right\rangle.$$

$$\tfrac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus_{SS}Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}$$

$$=\left\langle\left(1-\left(\tfrac{1}{A+B}\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{2}{l(l+1)}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right.,$$

$$\left(\tfrac{1}{A+B}\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{2}{l(l+1)}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q\gamma}},$$

$$\left.\left(\tfrac{1}{A+B}\left(1-\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{w}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{w}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{w}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{w}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{2}{l(l+1)}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q\gamma}}\right\rangle.$$

Hence,

$$T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l)$$

$$=\left\langle\left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{q\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\right.$$

$$\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j^{q}\right)^{\gamma}-(n\Omega_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}},\right.$$

$$\left.\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j^{q}\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{q\gamma}}\right\rangle.$$

□

Now we confer several desirable properties of the anticipated T-SPHFSSPOGHEM operator.

Theorem 7. ***(Idempotency)*** *Let $A\geq 0$, $B\geq 0$ and A,B take no more than one value of 0 at a time, $tsf_i=\left\langle\overline{\overline{pm}}_i,\overline{\overline{as}}_i,\overline{\overline{nm}}_i\right\rangle(i=1,2,\ldots,l)$ be a group of T-SPHFNs, and $tsf_i=\left\langle\overline{\overline{pm}}_i,\overline{\overline{as}}_i,\overline{\overline{nm}}_i\right\rangle=tsf=\left\langle\overline{\overline{pm}},\overline{\overline{as}},\overline{\overline{nm}}\right\rangle(i=1,2,\ldots,l)$. Then*

$$T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l)=tsf=\left\langle\overline{\overline{pm}},\overline{\overline{as}},\overline{\overline{nm}}\right\rangle \tag{39}$$

Theorem 8. ***(Commutativity)*** *Let $(tsf'_1,tsf'_2,\ldots,tsf'_l)$ be any permutation of $(tsf_1,tsf_2,\ldots,tsf_l)$. Then*

$$T-SPHFSSPOGHEM^{A,B}\left(tsf'_1,tsf'_2,\ldots,tsf'_l\right)=T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l) \tag{40}$$

Theorem 9. ***(Boundedness)*** *Let $tsf_i=\left\langle\overline{\overline{pm}}_i,\overline{\overline{as}}_i,\overline{\overline{nm}}_i\right\rangle(i=1,2,\ldots,l)$ be a group of T-SPHFNs, $tsf^-=\left\langle\mathrm{mini}_i\overline{\overline{pm}}_i,\mathrm{maxi}_i\overline{\overline{as}}_i,\mathrm{maxi}_i\overline{\overline{nm}}_i\right\rangle$ and $tsf^+=\left\langle\mathrm{maxi}_i\overline{\overline{pm}}_i,\mathrm{mini}_i\overline{\overline{as}}_i,\mathrm{mini}_i\overline{\overline{nm}}_i\right\rangle$. Then*

$$tsf^-\leq T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l)\leq tsf^+ \tag{41}$$

The proofs of these Theorems are the same as the proofs of the Theorems for T-SPHFSSPOHEM operators. Hence, these are committed here.

By allocating distinct values of the parameters γ,q,A *and* B, various special cases of the T-SPHFSSPOGHEM operator can be obtained, which can be expressed as given below:

Case 1. If $\gamma=0$, then the T-SPHFSSPOGHEM operator degenerates to the T-SPFPGHM operator, which can be expressed as follows:

$$T-SPHFSSPOGHEM^{A,B}(tsf_1,tsf_2,\ldots,tsf_l)=\frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}$$

$$=\left\langle\left(1-\left(1-\prod_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(1-\left(\overline{\overline{pm}}_i\right)^{ql_i}\right)^{A}\left(1-\left(\overline{\overline{pm}}_j\right)^{ql_j}\right)^{B}\right)^{\frac{2}{l(l+1)}}\right)^{\frac{1}{A+B}}\right)^{\frac{1}{q}},\right. \tag{42}$$

$$\left.\left(\left(1-\prod_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(1-\left(1-\overline{\overline{as}}_i^{q}\right)^{l_i}\right)^{A}\left(1-\left(1-\overline{\overline{as}}_j^{q}\right)^{l_j}\right)^{B}\right)^{\frac{2}{l(l+1)}}\right)^{\frac{1}{q}}\right)^{\frac{1}{A+B}},\left(\left(1-\prod_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(1-\left(1-\overline{\overline{nm}}_i^{q}\right)^{l_i}\right)^{A}\left(1-\left(1-\overline{\overline{nm}}_j^{q}\right)^{l_j}\right)^{B}\right)^{\frac{2}{l(l+1)}}\right)^{\frac{1}{q}}\right)^{\frac{1}{A+B}}\right\rangle.$$

Case 2. If $q = 1$, then the T-SPHFSSPOGHEM operator degenerates to the picture fuzzy Schweizer–Sklar power Heronian mean (PIFSSPOGHEM) operator, which can be expressed as follows:

$$
\begin{aligned}
&T-SPHFSSPOGHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l)\\
&= \left\langle \left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right),\right.\\
&\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}},\\
&\left.\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right\rangle.
\end{aligned}
\tag{43}
$$

Case 3. If $q = 2$, then the T-SPHFSSPOGHEM operator degenerates to the Spherical fuzzy Schweizer-Sklar power Heronian mean (SPHFSSPOGHEM) operator, which can be expressed as follows:

$$
\begin{aligned}
&T-SPHFSSPOGHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l)\\
&= \left\langle \left(1-\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{2\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\overline{\overline{pm}}_j^{2\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{2}}\right.,\\
&\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^2\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{as}}_j^2\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{2\gamma}},\\
&\left.\left(\frac{1}{A+B}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^2\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}+B\left(1-\left(l\mathfrak{v}_j\left(1-\overline{\overline{nm}}_j^2\right)^{\gamma}-(l\mathfrak{v}_j-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-A-B+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}-\left(\frac{1}{A+B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{2\gamma}}\right\rangle.
\end{aligned}
\tag{44}
$$

Case 4. If $B \to 0$, then the T-SPHFSSPOGHEM operator degenerates to the T-SPHF descending PG operator, which can be expressed as follows:

$$
\begin{aligned}
&T-SPHFSSPOGHEM^{A,0}(tsf_1, tsf_2, \ldots, tsf_n) = \lim_{B\to 0}\frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}\\
&= \frac{1}{A}\left(\prod_{i=1}^{l}\left(A\Theta_i^{l_i}\right)^{(l+1-i)}\right)^{\frac{2}{l(l+1)}}\\
&= \left(1-\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right.,\\
&\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^q\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{q\gamma}}\\
&\left.\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^q\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{q\gamma}}\right\rangle.
\end{aligned}
\tag{45}
$$

Case 5. If $A \to 0$, then the T-SPHFSSPOGHEM operator degenerates to the T-SPHF ascending power geometric operator, which can be expressed as follows:

$$T-SPFSSPGHM^{0,B}(tsf_1,tsf_2,\ldots,tsf_n)=\lim_{A\to 0}\frac{1}{B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}$$

$$=\frac{1}{B}\left(\prod_{i=1}^{l}\left(Btsf_i^{l_i}\right)^{i}\right)^{\frac{2}{l(l+1)}}$$

$$=\left\langle\left(1-\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left(i\left(1-\left(B\left(1-\left(l\mathfrak{v}_i\overline{\overline{pm}}_i^{q\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\right.\right.$$

$$\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left(i\left(1-\left(B\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{as}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{q\gamma}}\right.$$

$$\left.\left.\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left(i\left(1-\left(B\left(1-\left(l\mathfrak{v}_i\left(1-\overline{\overline{nm}}_i^{q}\right)^{\gamma}-(l\mathfrak{v}_i-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{q\gamma}}\right\rangle.\right.\right. \tag{46}$$

Case 6. If $B\to 0$, and $\mathrm{supr}\left(tsf_i,tsf_j\right)=h(h\in[0,1])(\forall i\neq j)$, then the T-SPHFSSPOGHEM operator relapses to the T-SPF linear descending WGA operator, which can be expressed as follows:

$$T-SPHFSSPOGHEM^{A,0}(tsf_1,tsf_2,\ldots,tsf_n)=\lim_{B\to 0}\frac{1}{B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}$$

$$=\frac{1}{A}\left(\prod_{i=1}^{l}\left(Atsf_i\right)^{(l+1-i)}\right)^{\frac{2}{l(l+1)}}$$

$$=\left\langle\left(1-\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\right.\right.$$

$$\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\overline{\overline{as}}_i^{q\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{q\gamma}},$$

$$\left.\left.\left(\frac{1}{A}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((l+1-i)\left(1-\left(A\overline{\overline{nm}}_i^{q\gamma}-(A-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((2l-i)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{A}-1\right)\right)^{\frac{1}{q\gamma}}\right\rangle.\right. \tag{47}$$

Certainly, the importance degree of T-SPHN $Atsf_i(i=1,2,\ldots.,l)$ is $(l,l-1,\ldots.,1)$.

Case 7. If $A\to 0$, and $\mathrm{supr}\left(tsf_i,tsf_j\right)=h(h\in[0,1])(\forall i\neq j)$, then the T-SPHFSSPOGHEM operator relapses to the T-SPHF linear ascending WG operator, which can be expressed as follows:

$$T-SPHFSSPOGHEM^{0,B}(tsf_1,tsf_2,\ldots,tsf_l)=\lim_{A\to 0}\frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i}\oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}}$$

$$=\frac{1}{B}\left(\prod_{i=1}^{l}\left(Btsf_i\right)^{i}\right)^{\frac{2}{l(l+1)}}$$

$$=\left\langle\left(1-\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((i)\left(1-\left(B\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}},\right.\right.$$

$$\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{n}\left((i)\left(1-\left(B\overline{\overline{as}}_i^{q\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{q\gamma}},$$

$$\left.\left.\left(\frac{1}{B}\left(1-\left(\frac{2}{l(l+1)}\sum_{i=1}^{l}\left((i)\left(1-\left(B\overline{\overline{nm}}_i^{q\gamma}-(B-1)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\left((l+i-1)\left(\frac{2}{l(l+1)}\right)-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\left(\frac{1}{B}-1\right)\right)^{\frac{1}{q\gamma}}\right\rangle.\right. \tag{48}$$

Case 8. If $A = B = \frac{1}{2}$, and $\sup r\left(tsf_i, tsf_j\right) = h(h \in [0,1])(\forall i \neq j)$, then the T-SPHFSSPOGHEM operator degenerates to the T-SPHF geometric Heronian mean operator, which can be expressed as follows:

$$\begin{aligned} T-SPFSSPGHM^{\frac{1}{2},\frac{1}{2}}(tsf_1, tsf_2, \ldots, tsf_n) &= \frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i} \oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}} \\ &= \left(\frac{1}{2}\prod_{\substack{i,j=1\\ j=i}}^{l}\left(tsf_i \oplus tsf_j\right)\right)^{\frac{2}{l(l+1)}} \\ &= \left\langle\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\frac{1}{2}\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}+\frac{1}{2}\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{q\gamma}}, \left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\frac{1}{2}\overline{\overline{as}}_i^{q\gamma}+\frac{1}{2}\overline{\overline{as}}_j^{q\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right. \\ &\left.\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\frac{1}{2}\overline{\overline{nm}}_i^{q\gamma}+\frac{1}{2}\overline{\overline{nm}}_j^{q\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}\right\rangle \end{aligned} \tag{49}$$

Case 9. If $A = B = 1$, and $\sup r\left(tsf_i, tsf_j\right) = h(h \in [0,1])(\forall i \neq j)$, then the T-SPHFSSPOGHEM operator relapses to the T-SPHF basic geometric Heronian mean operator, which can be expressed as follows:

$$\begin{aligned} T-SPHFSSPOGHEM^{1,1}(tsf_1, tsf_2, \ldots, tsf_n) &= \frac{1}{A+B}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(Atsf_i^{l_i} \oplus Btsf_j^{l_j}\right)\right)^{\frac{2}{l(l+1)}} \\ &= \frac{1}{2}\left(\prod_{\substack{i,j=1\\ j=i}}^{l}\left(tsf_i \oplus tsf_j\right)\right)^{\frac{2}{l(l+1)}} \\ &= \left\langle\left(1-\left(\left(\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\left(1-\overline{\overline{pm}}_i^{q}\right)^{\gamma}+\left(1-\overline{\overline{pm}}_j^{q}\right)^{\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}+\frac{1}{2}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{q}}, \right.\right. \\ &\left.\left(\frac{1}{2}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{l}\left(1-\left(\overline{\overline{as}}_i^{q\gamma}+\overline{\overline{as}}_j^{q\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}+\frac{1}{2}\right)^{\frac{1}{q\gamma}}, \right. \\ &\left.\left.\left(\frac{1}{2}\left(1-\left(\frac{2}{l(l+1)}\sum_{\substack{i,j=1\\ j=i}}^{n}\left(1-\left(\overline{\overline{nm}}_i^{q\gamma}+\overline{\overline{nm}}_j^{q\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{\gamma}+\frac{1}{2}\right)^{\frac{1}{q\gamma}}\right\rangle. \end{aligned} \tag{50}$$

4.3. T-SPHFSSWPOHEM and T-SPHFSSWPOGHEM Operators

In this subpart, we initiate the T-SPHFSSWPOHEM operator and the T-SPHFSWPOGHEM operator by taking the importance of the attributes.

Definition 10. *Let* $A, B \geq 0, tsf_i = \langle\overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i\rangle(i = 1, 2, \ldots, l)$ *be a group of T-SPHFNs and* $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_n)^T$ *be the importance degree of* $tsf_i(i = 1, 2, \ldots, l)$*, where* $\varsigma_i \geq 0$ *and* $\sum_{i=1}^{l}\varsigma_i = 1$*. The T-SPHFSSWPOHEM operator is explained as:*

$$T-SPHFSSWPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)}\left(\sum_{\substack{i,j=1\\ j=i}}^{l}\left(\frac{l\varsigma_i\left(1+T\left(tsf_i\right)\right)}{\sum_{z=1}^{l}\varsigma_z\left(1+T\left(tsf_z\right)\right)}tsf_i\right)^{A} \otimes_{SS}\left(\frac{l\varsigma_j\left(1+T\left(tsf_j\right)\right)}{\sum_{z=1}^{l}\varsigma_z\left(1+T\left(tsf_z\right)\right)}tsf_j\right)^{B}\right)\right)^{\frac{1}{A+B}} \tag{51}$$

where $A, B \geq 0, T(tsf_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} \text{supr}(tsf_i, tsf_j), \text{supr}(tsf_i, tsf_j) = 1 - \overline{\overline{DIT}}(tsf_i, tsf_j)$, *and* $\overline{\overline{DIT}}(tsf_i, tsf_j)$ *can be computed by Equation (13).*

Let $_i = \frac{(1+T(tsf_i))}{\sum_{z=1}^{l}(1+T(tsf_z))}$, *then the definition of T-SPHFSSWPOHEM operator is equivalent to the following form:*

$$T - SPHFSSWPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \left(\frac{2}{l(l+1)} \left(\sum_{\substack{i,j=1 \\ j=i}}^{l} (l\varsigma_i \mathfrak{v}_i tsf_i)^A \otimes_{SS} \left(l_j \varsigma_j tsf_j\right)^B \right) \right)^{\frac{1}{A+B}} \tag{52}$$

Theorem 10. *Let* $A \geq 0,\ B \geq 0$ *and* A, B *take no more than one value of 0 at a time,* $tsf_i = \langle \overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i \rangle$ *be a group of T-SPFNs. Then, utilizing the T-SPHFSSWPOHEM operator, their fused values are also T-SPHFN, and*

$$\begin{aligned}
&T - SPHFSSWPOHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \\
&= \Bigg\langle \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\varsigma_i \mathfrak{v}_i \left(1 - \overline{\overline{pm}}_i^{\eta} \right)^{\gamma} - (l\varsigma_i \mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} + B \left(1 - \left(l\varsigma_j \mathfrak{v}_j \left(1 - \overline{\overline{pm}}_j^{\eta} \right)^{\gamma} - (l\varsigma_j \mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\eta\gamma}}, \\
&\left(1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\varsigma_i \mathfrak{v}_i \overline{\overline{as}}_i^{\eta\gamma} - (l\varsigma_i \mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} + B \left(1 - \left(l\varsigma_j \mathfrak{v}_j \overline{\overline{as}}_j^{\eta\gamma} - (l\varsigma_j \mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{\eta}}, \\
&\left(1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\varsigma_i \mathfrak{v}_i \overline{\overline{nm}}_i^{\eta\gamma} - (l\varsigma_i \mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} + B \left(1 - \left(l\varsigma_j \mathfrak{v}_j \overline{\overline{nm}}_j^{\eta\gamma} - (l\varsigma_j \mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{\eta}} \Bigg\rangle.
\end{aligned} \tag{53}$$

Definition 11. *Let* $A, B \geq 0, tsf_i = \langle \overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i \rangle (i = 1, 2, \ldots, l)$ *be a group of T-SPHFNs and* $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_n)^T$ *be the importance degree of* $tsf_i (i = 1, 2, \ldots, l)$, *where* $\varsigma_i \geq 0$ *and* $\sum_{i=1}^{n} \varsigma_i = 1$. *The T-SPHFSSWPOGHEM operator is explained as:*

$$T - SPHFSSWPOGHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) = \frac{1}{A+B} \left(\prod_{\substack{i,j=1 \\ j=i}}^{l} \left(Atsf_i^{\frac{l\varsigma_i(1+T(tsf_i))}{\sum_{z=1}^{l} \varsigma_z(1+T(tsf_z))}} \oplus_{SS} Btsf_j^{\frac{l\varsigma_j(1+T(tsf_j))}{\sum_{z=1}^{l} \varsigma_z(1+T(tsf_z))}} \right) \right)^{\frac{2}{l(l+1)}} \tag{54}$$

where $A, B \geq 0, T(tsf_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} \text{supr}(tsf_i, tsf_j), \text{supr}(tsf_i, tsf_j) = 1 - \overline{\overline{DST}}(tsf_i, tsf_j)$, *and* $\overline{\overline{DST}}(tsf_i, tsf_j)$ *can be computed by Equation (13).*

Let $_i = \frac{(1+T(tsf_i))}{\sum_{z=1}^{l}(1+T(tsf_z))}$, *then the definition of T-SPFSSWPGHM operator is equivalent to the following form:*

$$T - SPFSSWPOGHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_n) = \frac{1}{A+B} \left(\prod_{\substack{i,j=1 \\ j=i}}^{l} \left(Atsf_i^{l\varsigma_i \mathfrak{v}_i} \oplus_{SS} Btsf_j^{l\varsigma_j \mathfrak{v}_j} \right) \right)^{\frac{2}{l(l+1)}} \tag{55}$$

Theorem 11. *Let $A \geq 0$, $B \geq 0$ and A, B take no more than one value of 0 at a time, $tsf_i = \langle \overline{\overline{pm}}_i, \overline{\overline{as}}_i, \overline{\overline{nm}}_i \rangle$ be a group of T-SPHFNs. Then, utilizing the T-SPHFSSWPOGHEM operator, their fused values are also T-SPHFN, and*

$$
\begin{aligned}
&T-SPHFSSWPOGHEM^{A,B}(tsf_1, tsf_2, \ldots, tsf_l) \\
&= \left\langle 1 - \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\varsigma_i \mathfrak{v}_i \overline{\overline{pm}}_i^{q\gamma} - (l\varsigma_i \mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} + B \left(1 - \left(l\varsigma_j \mathfrak{v}_j \overline{\overline{pm}}_j^{q\gamma} - (l\varsigma_j \mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right) - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{q}}, \right. \\
&\left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\varsigma_i \mathfrak{v}_i \left(1 - \overline{\overline{as}}_i^{q} \right)^{\gamma} - (l\varsigma_i \mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} + B \left(1 - \left(l\varsigma_j \mathfrak{v}_j \left(1 - \overline{\overline{as}}_j^{q} \right)^{\gamma} - (l\varsigma_j \mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right) - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{q\gamma}} \right. \\
&\left. \left. \left(\frac{1}{A+B} \left(1 - \left(\frac{2}{l(l+1)} \sum_{\substack{i,j=1 \\ j=i}}^{l} \left(1 - \left(A \left(1 - \left(l\varsigma_i \mathfrak{v}_i \left(1 - \overline{\overline{nm}}_i^{q} \right)^{\gamma} - (l\varsigma_i \mathfrak{v}_i - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} + B \left(1 - \left(l\varsigma_j \mathfrak{v}_j \left(1 - \overline{\overline{nm}}_j^{q} \right)^{\gamma} - (l\varsigma_j \mathfrak{v}_j - 1) \right)^{\frac{1}{\gamma}} \right)^{\gamma} - A - B + 1 \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right) - \left(\frac{1}{A+B} - 1 \right) \right)^{\frac{1}{q\gamma}} \right\rangle.
\end{aligned}
\tag{56}
$$

The proofs of Theorems 10 and 11 are the same as Theorems 2 and 6. Therefore, here we omit their proofs.

5. The Approach to Solve MADM Problems

In this part, we initiate an innovative process for MADM problems, which is based on the initiated T-SPHFSSWPOHEM and T-SPHFSSWPOGHEM. The main goal of the MADM problem is to choose the best option between several available options. The procedure of the initiated process can be articulated as follows:

Let $VS = \{VS_1, VS_2, \ldots, VS_m\}$ and $RE = \{RE_1, RE_2, \ldots, RE_n\}$ be groups of alternatives/options and attributes/criteria, respectively. Let the weight vector of each attribute RE_h is CE_h where $h = 1, 2, \ldots, l$ and $\sum_{h=1}^{l} CE_h = 1$. The assessment information is provided in the form of T-SPHFNs. So, the T-SPHF decision matrix (DM) $MT = [\Theta_{ih}]$ in which $tsf_{ih} = \left(tsf_{ih}^1, tsf_{ih}^2, \ldots, tsf_{ih}^l \right) \in T - SPHFS(CE_h)$ where $h = 1, 2, \ldots, l$ and $i = 1, 2 \ldots, m$ as follows:

$$
\begin{bmatrix}
Alternatives \backslash Attributes & RE_1 & RE_2 & \ldots\ldots & RE_l \\
VS_1 & \Theta_{11} & \Theta_{12} & \ldots\ldots & \Theta_{1l} \\
VS_2 & \Theta_{21} & \Theta_{22} & \ldots\ldots & \Theta_{2l} \\
. & . & . & \ldots\ldots & . \\
. & . & . & & . \\
VS_m & \Theta_{m1} & \Theta_{m2} & \ldots. & \Theta_{ml}
\end{bmatrix}
$$

The initiated process is in accord with the following steps:

Step 1. Normalize the DM. In this step, we put up the T-SPHF DM. For illustration, for all $i = 1, \ldots, m$ and $h = 1, \ldots, l$, we calculate

$$\begin{bmatrix} Alternatives/attributes & RE_1 & RE_2 & \ldots\ldots & RE_l \\ VS_1 & RE_{1h}^1 & RE_{1h}^2 & \ldots\ldots & RE_{lh}^l \\ VS_2 & RE_{2h}^1 & RE_{2h}^2 & \ldots\ldots & RE_{lh}^l \\ . & . & . & . & . \\ . & . & . & . & . \\ VS_m & RE_{mh}^1 & RE_{mh}^2 & \ldots\ldots & RE_{mh}^l \end{bmatrix},$$

$$\begin{bmatrix} Alternatives/attributes & RE_1 & RE_2 & \ldots\ldots & RE_l \\ VS_1 & tsf_{1h}^1 & tsf_{1h}^2 & \ldots\ldots & tsf_{lh}^l \\ VS_2 & tsf_{2h}^1 & tsf_{2h}^2 & \ldots\ldots & tsf_{lh}^l \\ . & . & . & . & . \\ . & . & . & . & . \\ VS_m & tsf_{mh}^1 & tsf_{mh}^2 & \ldots\ldots & tsf_{mh}^l \end{bmatrix}.$$

where

$$tsf_{ih}^g = \frac{RE_{ih}^g}{\sum_{g=1}^{l} RE_{ih}^g}, (g = 1, \ldots, l) \tag{57}$$

Then the T-SPHF DM $MT = [\Theta_{ih}]$ in which $tsf_{ih} = \left(tsf_{ih}^1, tsf_{ih}^2, \ldots, tsf_{ih}^n\right) \in T - SPHFS(CE_h)$ is a T-SPHF DM, where $h = 1, \ldots, l$ and $i = 1, \ldots, m$. This conveys that VS_i is a T-SPHFS on the attributes set $RE = \{RE_1, RE_2, \ldots, RE_l\}$.

Step 2. Determine the Support Degree by utilizing the formula:

$$Sup(tsf_{ih}, tsf_{ik}) = 1 - \overline{\overline{DIT}}(tsf_{ih}, tsf_{ik}); \tag{58}$$

where $\overline{\overline{DST}}(tsf_{ih}, tsf_{ik})$ symbolized as the distance measure among two T-SPFNs tsf_{ih} and tsf_{ik} provided in Equation (13).

Step 3. Determine the weighted Sup $T(tsf_{ih})$ of the T-SPHFN tsf_{ih} by the other T-SPHFNs $tsf_{ik}(h, k = 1, \ldots, l)$ by the formula:

$$T(tsf_{ih}) = \sum_{\substack{j=1 \\ k \neq h}}^{q} Sup(tsf_{ih}, tsf_{ik}); (h, k = 1, \ldots, l). \tag{59}$$

Step 4. Determine the weight $\mho_h$ associated with T-SPFN tsf_h, by utilizing the formula:

$$\mho_h = \frac{l\varsigma_h(1 + T(tsf_h))}{\sum_{h=1}^{l}(1 + T(tsf_{ih}))}, i = 1, \ldots, m; h = 1, 2, \ldots, l. \tag{60}$$

Step 5. Determine the overall assessment value of every alternative tsf_i by utilizing the formula:

$$T - SPHFSSWPOHEM^{A,B}(tsf_{i1}, tsf_{i2}, \ldots, tsf_{il}) \tag{61}$$

or

$$T - SPHFSSWPOGHEM^{A,B}(tsf_{i1}, tsf_{i2}, \ldots, tsf_{il}) \tag{62}$$

Step 6. Find out the score values $\widetilde{SRE}(tsf_i)$ of the collective T-SPFNs $tsf_i (i = 1, 2, \ldots, m)$ to rank the alternatives $VS_i (i = 1, 2, \ldots, m)$ and then pick the finest one(s). If the score values of T-PFNs are the same, then we determine the accuracy values and rank the alternatives according to their accuracy values and pick the finest one(s).

6. Numerical Example

In this subpart, we utilize the initiated algorithm based on these newly initiated aggregation operators for assessing water reuse applications. These data are taken from reference [4].

The suggested algorithm is applied for the City of Penticton (CoP) in British Columbia (BC), Canada. Residents, businesses, institutions, and industries all use public water. The wastewater is processed and treated using biological nutrient removal equipment at an automated wastewater treatment facility. Half of the filtered water is reused, while the rest is released into a river. There are seven alternatives (Water reuses) such as VS_1 toilet flushing (TF), VS_2 watering vegetables in the garden (VW), VS_3 watering flowers in the garden, VS_4 agricultural irrigation (AI), VS_5 hydration of public parks (PPW), VS_6 watering of golf courses (GCW), and VS_7 drinking water (DW). These seven alternatives are assessed by using five attributes namely, RE_1 public acceptability, RE_2 freshwater saving, RE_3 life cycle cost, RE_4 human health risk, and RE_4 governments' policies, and the data are given in Tables 2 and 3. The weight of the attributes are $\varsigma_j = (0.2, 0.2, 0.2, 0.2, 0.2)$. The local governments' policies are assessed by utilizing a linguistic term set. The corresponding T-SPFNs of linguistic terms are given in Table 3.

Table 2. Data of public acceptability and fresh water saving.

Alternatives\Attributes	RE_1 *Public Acceptability*			RE_2 *Fresh Water Saving*		
	Agree	**Neutrality**	**Disagree**	**Low**	**Medium**	**High**
TF VS_1	80	9	11	428.8	536	643.2
VW VS_2	63.5	13	23.5	2624.8	3281	3937.2
FW VS_3	84.5	10	5.5	3192.5	3990.6	4788.8
AI VS_4	74.5	10	15.5	3192.5	3990.6	4788.8
PPW VS_5	85.5	8	6.5	886.3	1107.9	1329.5
GCW VS_6	88.5	7	4.5	361.8	452.3	542.7
DW VS_7	24	14	62	3192.5	3990.6	4788.8

Table 3. Data of life cycle cost, human health risk, and government policies.

Alternatives\Attributes	RE_3 *Life Cycle Cost*			RE_4 *Human Health Risk*			RE_5 *Governments Polices*
	Agree	**Neutrality**	**Disagree**	**Low**	**Medium**	**High**	
TF VS_1	1,555,358	1,944,198	2,333,038	7.1×10^{-12}	7.51×10^{-12}	8.30×10^{-12}	$M\ \langle 0.5, 0.5, 0.5\rangle$
VW VS_2	1,637,219	20,46,524	2,455,829	1.83×10^{-11}	1.89×10^{-11}	2.03×10^{-11}	$L\ \langle 0.2, 0.7, 0.7\rangle$
FW VS_3	834,019	1,042,524	1,251,028	1.78×10^{-11}	1.84×10^{-11}	1.99×10^{-11}	$H\ \langle 0.8, 0.2, 0.2\rangle$
AI VS_4	146,660	183,326	219,991	9.07×10^{-12}	1.0×10^{-11}	1.26×10^{-11}	$M\ \langle 0.5, 0.5, 0.5\rangle$
PPW VS_5	635,529	794,411	953,293	9.34×10^{-12}	9.77×10^{-12}	1.07×10^{-11}	$H\ \langle 0.8, 0.2, 0.2\rangle$
GCW VS_6	78,219	97,774	117,328	8.43×10^{-12}	8.87×10^{-12}	9.83×10^{-12}	$M\ \langle 0.5, 0.5, 0.5\rangle$
DW VS_7	1,197,674	1,497,092	1,796,511	2.76×10^{-8}	4.01×10^{-8}	1.00×10^{-7}	$VL\ \langle 0.10, 0.10, 0.90\rangle$

Now, we will utilize the suggested algorithm to solve the assessment of the water reuse application. The following steps should be followed.

Step 1. Discover the T-SPF decision matrix $MT = \left[\Theta_{ij}\right]\ i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 5$ by exploiting Equation (57), listed in Tables 4 and 5.

Table 4. T-SPF decision matrix MT.

Alternatives\Attributes	RE_1	RE_2	RE_3
VS_1	$\langle 0.80, 0.09, 0.110\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$	$\langle 0.0350, 0.4386, 0.5263\rangle$
VS_2	$\langle 0.6350, 0.1300, 0.2350\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$
VS_3	$\langle 0.8450, 0.1000, 0.0550\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$
VS_4	$\langle 0.7450, 0.1000, 0.1550\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$
VS_5	$\langle 0.8550, 0.0800, 0.0650\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$
VS_6	$\langle 0.8850, 0.0700, 0.0450\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$
VS_7	$\langle 0.2400, 0.1400, 0.6200\rangle$	$\langle 0.2666, 0.3334, 0.4000\rangle$	$\langle 0.2667, 0.3333, 0.4000\rangle$

Table 5. T-SPF decision matrix MT.

Alternatives\Attributes	RE_4	RE_5
VS_1	$\langle 0.3099, 0.3278, 0.3623\rangle$	$\langle 0.3333, 0.3333, 0.3333\rangle$
VS_2	$\langle 0.4651, 0.4830, 0.0519\rangle$	$\langle 0.1250, 0.4375, 0.4375\rangle$
VS_3	$\langle 0.3173, 0.3280, 0.3547\rangle$	$\langle 0.6667, 0.1667, 0.1667\rangle$
VS_4	$\langle 0.2864, 0.3158, 0.3979\rangle$	$\langle 0.3333, 0.3333, 0.3333\rangle$
VS_5	$\langle 0.3133, 0.3277, 0.3589\rangle$	$\langle 0.6667, 0.1667, 0.1667\rangle$
VS_6	$\langle 0.3107, 0.3269, 0.3623\rangle$	$\langle 0.3333, 0.3333, 0.3333\rangle$
VS_7	$\langle 0.1646, 0.2391, 0.5963\rangle$	$\langle 0.0909, 0.0909, 0.8182\rangle$

Step 2. Discover the support degrees $Sup\left(\Theta^i{}_{ij}, \Theta^i{}_{ih}\right), i = 1, 2, \ldots, 7, j, h = 1, 2, \ldots, 5$, by exploiting Equation (58), and we have

$$S^1_{12} = S^1_{21} = 0.7267, S^1_{13} = S^1_{31} = 0.6373, S^1_{14} = S^1_{41} = 0.7458, S^1_{15} = S^1_{51} = 0.7564, S^1_{23} = S^1_{32} = 0.9106,$$
$$S^1_{24} = S^1_{42} = 0.9809, S^1_{25} = S^1_{52} = 0.9704, S^1_{34} = S^1_{43} = 0.8915, S^1_{35} = S^1_{53} = 0.8810, S^1_{45} = S^1_{45} = 0.9870,$$
$$S^2_{12} = S^2_{21} = 0.8230, S^2_{13} = S^2_{31} = 0.8230, S^2_{14} = S^2_{41} = 0.8481, S^2_{15} = S^2_{51} = 0.7672, S^2_{23} = S^2_{32} = 1.000,$$
$$S^2_{24} = S^2_{42} = 0.8584, S^2_{25} = S^2_{52} = 0.9443, S^2_{34} = S^2_{43} = 0.8584, S^2_{35} = S^2_{53} = 0.9443, S^2_{45} = S^2_{45} = 0.8562,$$
$$S^3_{12} = S^3_{21} = 0.6997, S^3_{13} = S^3_{31} = 0.6997, S^3_{14} = S^3_{41} = 0.7221, S^3_{15} = S^3_{51} = 0.8960, S^3_{23} = S^3_{32} = 1.0000,$$
$$S^3_{24} = S^3_{42} = 0.9776, S^3_{25} = S^3_{52} = 0.8037, S^3_{34} = S^3_{43} = 0.9776, S^3_{35} = S^3_{53} = 0.8037, S^3_{45} = S^3_{45} = 0.8261,$$
$$S^4_{12} = S^4_{21} = 0.7597, S^4_{13} = S^4_{31} = 0.7597, S^4_{14} = S^4_{41} = 0.7677, S^4_{15} = S^4_{51} = 0.7893, S^4_{23} = S^4_{32} = 1.0000,$$
$$S^4_{24} = S^4_{42} = 0.9920, S^4_{25} = S^4_{52} = 0.9704, S^4_{34} = S^4_{43} = 0.9920, S^4_{35} = S^4_{53} = 0.9704, S^4_{45} = S^4_{45} = 0.9708,$$
$$S^5_{12} = S^5_{21} = 0.6932, S^5_{13} = S^5_{31} = 0.6932, S^5_{14} = S^5_{41} = 0.7138, S^5_{15} = S^5_{51} = 0.8895, S^5_{23} = S^5_{32} = 1.0000,$$
$$S^5_{24} = S^5_{42} = 0.9794, S^5_{25} = S^5_{52} = 0.8040, S^5_{34} = S^5_{43} = 0.9794, S^5_{35} = S^5_{53} = 0.8037, S^5_{45} = S^5_{45} = 0.8243,$$
$$S^6_{12} = S^6_{21} = 0.6746, S^6_{13} = S^6_{31} = 0.6746, S^6_{14} = S^6_{41} = 0.6940, S^6_{15} = S^6_{51} = 0.7042, S^6_{23} = S^6_{32} = 1.0000,$$
$$S^6_{24} = S^6_{42} = 0.9805, S^6_{25} = S^6_{52} = 0.9704, S^6_{34} = S^6_{43} = 0.9805, S^6_{35} = S^6_{53} = 0.9704, S^6_{45} = S^6_{45} = 0.9870,$$
$$S^7_{12} = S^7_{21} = 0.8902, S^7_{13} = S^7_{31} = 0.8902, S^7_{14} = S^7_{41} = 0.9677, S^7_{15} = S^7_{51} = 0.8848, S^7_{23} = S^7_{32} = 1.0000,$$
$$S^7_{24} = S^7_{42} = 0.9022, S^7_{25} = S^7_{52} = 0.7750, S^7_{34} = S^7_{43} = 0.9022, S^7_{35} = S^7_{53} = 0.7750, S^7_{45} = S^7_{45} = 0.8728.$$

Step 3. Exploiting Equation (59), to discover the weighted support Sup $T\left(RE_{ij}\right)$. For simplicity, we indicate $T\left(RE_{ij}\right)$ by T_{ij}, we have

$$T_{11} = 2.8663, T_{12} = 3.5886, T_{13} = 3.3204, T_{14} = 3.6053, T_{15} = 3.5948,$$
$$T_{21} = 3.2612, T_{22} = 3.6257, T_{23} = 3.6257, T_{24} = 3.4211, T_{25} = 3.5120,$$
$$T_{31} = 3.0174, T_{32} = 3.4809, T_{33} = 3.4809, T_{34} = 3.5034, T_{35} = 3.3295,$$
$$T_{41} = 3.0763, T_{42} = 3.7220, T_{43} = 3.7220, T_{44} = 3.7224, T_{45} = 3.7008,$$
$$T_{51} = 2.9897, T_{52} = 3.4763, T_{53} = 3.4763, T_{54} = 3.4969, T_{55} = 3.3212,$$
$$T_{61} = 2.7474, T_{62} = 3.6255, T_{63} = 3.6255, T_{64} = 3.6421, T_{65} = 3.6320,$$
$$T_{71} = 3.6329, T_{72} = 3.5673, T_{73} = 3.5673, T_{74} = 3.6448, T_{75} = 3.3075.$$

Step 4. Exploiting Equation (60), to discover weight Ω_j (j = 1,2, … ,5), we have

$$\Omega_{11} = 0.8797, \Omega_{12} = 1.0440, \Omega_{13} = 0.9830, \Omega_{14} = 1.0478, \Omega_{15} = 1.0454,$$
$$\Omega_{21} = 0.9492, \Omega_{22} = 1.0304, \Omega_{23} = 1.0304, \Omega_{24} = 0.9849, \Omega_{25} = 1.0051,$$
$$\Omega_{31} = 0.9209, \Omega_{32} = 1.0272, \Omega_{33} = 1.0272, \Omega_{34} = 1.0323, \Omega_{35} = 0.9925,$$
$$\Omega_{41} = 0.8883, \Omega_{42} = 1.0291, \Omega_{43} = 1.0291, \Omega_{44} = 1.0291, \Omega_{45} = 1.0244,$$
$$\Omega_{51} = 0.9167, \Omega_{52} = 1.0285, \Omega_{53} = 1.0285, \Omega_{54} = 1.0333, \Omega_{55} = 0.9929,$$
$$\Omega_{61} = 0.8413, \Omega_{62} = 1.0384, \Omega_{63} = 1.0384, \Omega_{64} = 1.0421, \Omega_{65} = 1.0398,$$
$$\Omega_{71} = 1.0196, \Omega_{72} = 1.0051, \Omega_{73} = 1.0051, \Omega_{74} = 1.0222, \Omega_{75} = 0.9480.$$

Step 5. Exploiting Equation (61), to discover the overall assessment values of each alternative VS_i, $i = 1, 2, \ldots, 7$, we have $(A = 2, B = 3, \gamma = -2 \; and \; q = 2)$,

$$VS_1 = \langle 0.3807, 0.0453, 0.1154 \rangle, VS_2 = \langle 0.3411, 0.0987, 0.1324 \rangle,$$
$$VS_3 = \langle 0.5126, 0.0370, 0.1317 \rangle, VS_4 = \langle 0.3771, 0.0589, 0.1155 \rangle,$$
$$VS_5 = \langle 0.5197, 0.0226, 0.1330 \rangle, VS_6 = \langle 0.4862, 0.0479, 0.1192 \rangle,$$
$$VS_7 = \langle 0.1979, 0.1159, 0.3747 \rangle.$$

or

Exploiting Equation (61), to discover the overall assessment values of each alternative VS_i, $i = 1, 2, \ldots, 7$, we have $(A = 2, B = 3, \gamma = -2 \; and \; q = 2)$,

$$VS_1 = \langle 0.1056, 0.3031, 0.3426 \rangle, VS_2 = \langle 0.2943, 0.3416, 0.3069 \rangle,$$
$$VS_3 = \langle 0.1759, 0.2445, 0.2716 \rangle, VS_4 = \langle 0.1299, 0.2822, 0.3335 \rangle,$$
$$VS_5 = \langle 0.2070, 0.2412, 0.2737 \rangle, VS_6 = \langle 0.1047, 0.2824, 0.3142 \rangle,$$
$$VS_7 = \langle 0.3889, 0.2171, 0.5854 \rangle.$$

Step 6. Utilizing Equation (6) given in Definition (2), to discover the score values of overall T-SPFNs VS_i, $i = 1, 2, \ldots, 7$, to rank the alternatives, we have

$$Scr(VS_1) = 0.0432, Scr(VS_2) = 0.0297, Scr(VS_3) = 0.0813, Scr(VS_4) = 0.0418,$$
$$Scr(VS_5) = 0.0840, Scr(VS_6) = 0.0733, Scr(VS_7) = -0.0382.$$

or

$$Scr(VS_1) = -0.0660, Scr(VS_2) = -0.0414, Scr(VS_3) = -0.0342, Scr(VS_4) = -0.0580,$$
$$Scr(VS_5) = -0.0301, Scr(VS_6) = -0.0558, Scr(VS_7) = -0.0796.$$

According to score values, the ranking orders are $VS_5 > VS_3 > VS_6 > VS_1 > VS_4 > VS_2 > VS_7$ and $VS_5 > VS_3 > VS_2 > VS_6 > VS_4 > VS_1 > VS_7$**.** Hence the best one is VS_5 hydration of public parks, and the worst one is VS_7 drinking water (DW).

6.1. Effect of Parameters

6.1.1. Effect of the Parameters A, B on Ranking Order Utilizing T-SPFSSWHM and T-SPFSSPWGHM Operators

In this subpart, the effect of the parameters A, B utilizing T-SPFSSPWHM and T-SPFSSPWGHM operators are investigated, and the values of the parameters $q = 2$ and $\gamma = -2$ are fixed. The score values and ranking orders for distinct values of the parameters A, B utilizing T-SPFSSPWHM and T-SPFSSPWGHM operators are given in Tables 6 and 7. From Table 6, one can notice that the ranking order is slightly different for different values of the parameter A, B, although the best and the worst alternative remains the same. From Table 6, we also noticed that when the values of the parameters enlarge, the score values of the alternatives decline. Similarly, From Table 7, one can notice that the ranking order is different for different values of the parameter A, B. When $A, B = 1$, the best one is VS_3, while the worst alternative remains the same. From Table 7 we also noticed that when the values of the parameters increase, the score values of the alternatives increase.

Table 6. Effect of the parameters A, B on final ranking order utilizing the T-SPFSSPWHM operator.

Parameters	Score Values	Ranking Order
$A, B = 1$	$Scr(VS_1) = 0.0296, Scr(VS_2) = 0.0186, Scr(VS_3) = 0.0850,$ $Scr(VS_4) = 0.0126, Scr(VS_5) = 0.0901, Scr(VS_6) = 0.0874,$ $Scr(VS_7) = -0.0872.$	$VS_5 > VS_6 > VS_3 > VS_1 > VS_2 > VS_4 > VS_7$
$A = 3, B = 2$	$Scr(VS_1) = -0.0721, Scr(VS_2) = -0.0774, Scr(VS_3) = -0.0118,$ $Scr(VS_4) = -0.0682, Scr(VS_5) = -0.0091, Scr(VS_6) = -0.0317,$ $Scr(VS_7) = -0.1550.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$A = 5, B = 4$	$Scr(VS_1) = -0.0502, Scr(VS_2) = -0.0521, Scr(VS_3) = 0.0083,$ $Scr(VS_4) = -0.0464, Scr(VS_5) = 0.0106, Scr(VS_6) = -0.0137,$ $Scr(VS_7) = -0.1343.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$A = 10, B = 8$	$Scr(VS_1) = -0.0544, Scr(VS_2) = -0.0551, Scr(VS_3) = 0.0030,$ $Scr(VS_4) = -0.0485, Scr(VS_5) = 0.0049, Scr(VS_6) = -0.0206,$ $Scr(VS_7) = -0.1367.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$A = 20, B = 15$	$Scr(VS_1) = -0.0669, Scr(VS_2) = -0.0676, Scr(VS_3) = -0.0103,$ $Scr(VS_4) = -0.0588, Scr(VS_5) = -0.0087, Scr(VS_6) = -0.0346,$ $Scr(VS_7) = -0.1468.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 30, B = 25$	$Scr(VS_1) = -0.0524, Scr(VS_2) = -0.0516, Scr(VS_3) = 0.0040,$ $Scr(VS_4) = -0.0449, Scr(VS_5) = 0.0055, Scr(VS_6) = -0.0208,$ $Scr(VS_7) = -0.1338.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 50, B = 48$	$Scr(VS_1) = -0.0240, Scr(VS_2) = -0.0186, Scr(VS_3) = 0.0326,$ $Scr(VS_4) = -0.0208, Scr(VS_5) = 0.0342, Scr(VS_6) = 0.0104,$ $Scr(VS_7) = -0.1123.$	$VS_5 > VS_3 > VS_6 > VS_2 > VS_4 > VS_1 > VS_7$
$A = 70, B = 50$	$Scr(VS_1) = -0.0764, Scr(VS_2) = -0.0768, Scr(VS_3) = -0.0209,$ $Scr(VS_4) = -0.0664, Scr(VS_5) = -0.0196, Scr(VS_6) = -0.0455,$ $Scr(VS_7) = -0.1546.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$A = 150, B = 120$	$Scr(VS_1) = -0.0621, Scr(VS_2) = -0.0615, Scr(VS_3) = -0.0067,$ $Scr(VS_4) = -0.0525, Scr(VS_5) = -0.0055, Scr(VS_6) = -0.0319,$ $Scr(VS_7) = -0.1416.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$

Table 7. Effect of the parameters A, B on final ranking order utilizing the T-SPFSSPWHM operator.

Parameters	Score Values	Ranking Order
$A, B = 1$	$Scr(VS_1) = -0.0697, Scr(VS_2) = -0.0531, Scr(VS_3) = -0.0041,$ $Scr(VS_4) = -0.0311, Scr(VS_5) = -0.0042, Scr(VS_6) = -0.0262,$ $Scr(VS_7) = -0.1240.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 3, B = 2$	$Scr(VS_1) = -0.0099, Scr(VS_2) = -0.0042, Scr(VS_3) = 0.0417,$ $Scr(VS_4) = 0.0043, Scr(VS_5) = 0.0418, Scr(VS_6) = 0.0114,$ $Scr(VS_7) = -0.0836.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 5, B = 4$	$Scr(VS_1) = -0.0237, Scr(VS_2) = -0.0175, Scr(VS_3) = 0.0299,$ $Scr(VS_4) = -0.0073, Scr(VS_5) = 0.0300, Scr(VS_6) = -0.0001,$ $Scr(VS_7) = -0.0939.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 10, B = 8$	$Scr(VS_1) = -0.0206, Scr(VS_2) = -0.0152, Scr(VS_3) = 0.0345,$ $Scr(VS_4) = -0.0053, Scr(VS_5) = 0.0347, Scr(VS_6) = 0.0030,$ $Scr(VS_7) = -0.0919.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 20, B = 15$	$Scr(VS_1) = -0.0116, Scr(VS_2) = -0.0076, Scr(VS_3) = 0.0440,$ $Scr(VS_4) = 0.0014, Scr(VS_5) = 0.0444, Scr(VS_6) = 0.0112,$ $Scr(VS_7) = -0.0859.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 30, B = 25$	$Scr(VS_1) = -0.0209, Scr(VS_2) = -0.0161, Scr(VS_3) = 0.0363,$ $Scr(VS_4) = -0.0061, Scr(VS_5) = 0.0366, Scr(VS_6) = 0.0035,$ $Scr(VS_7) = -0.0931.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 50, B = 48$	$Scr(VS_1) = -0.0430, Scr(VS_2) = -0.0318, Scr(VS_3) = 0.0226,$ $Scr(VS_4) = -0.0181, Scr(VS_5) = 0.0228, Scr(VS_6) = -0.0092,$ $Scr(VS_7) = -0.1049.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 70, B = 50$	$Scr(VS_1) = -0.0039, Scr(VS_2) = -0.0016, Scr(VS_3) = 0.0526,$ $Scr(VS_4) = 0.0072, Scr(VS_5) = 0.0531, Scr(VS_6) = 0.0189,$ $Scr(VS_7) = -0.0811.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$A = 150, B = 120$	$Scr(VS_1) = -0.0126, Scr(VS_2) = -0.0098, Scr(VS_3) = 0.0451,$ $Scr(VS_4) = -0.0004, Scr(VS_5) = 0.0456, Scr(VS_6) = 0.0112,$ $Scr(VS_7) = -0.0885.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$

6.1.2. Effect of the Parameter γ on Final Ranking Orders Utilizing T-SPFSSPWHM and T-SPFSSPWGHM Operators

In this subpart, the effect of the parameter γ on final ranking orders utilizing T-SPFSSPWHM and T-SPFSSPWGHM operators are investigated, and the value of other parameters $A = 2$, $B = 3$ *and* $q = 2$ are fixed. The score values and ranking orders for different values of the parameter γ utilizing T-SPFSSWPHM and T-SPFSSPWGHM operators are given in Tables 8 and 9. One can notice from Table 8 that for distinct values of the parameter γ the ranking order is different. That is, utilizing the T-SPFSSPWHM

operator for distinct values of γ the best alternative is either VS_5 or VS_6 while the worst alternative remain the same which is VS_7. We can also notice that when the values of the parameter decrease, the score values of the alternative increase. Similarly, from Table 9, we can see that for distinct values of the parameter γ the ranking order is different from the ranking order obtained for $\gamma = -2$. That is, utilizing the T-SPFSSPWGHM operator for distinct values of γ, the best alternative is VS_3 while the worst alternative remains the same, which is VS_7. We can also notice that when the values of the parameter decline, the score values of the alternative decline.

Table 8. Effect of the parameter γ on final ranking orders utilizing the T-SPFSSPWHM operator.

Parameter	Score Values	Ranking Order
$\gamma = -4$	$Scr(VS_1) = 0.0510, Scr(VS_2) = 0.0311, Scr(VS_3) = 0.0892,$ $Scr(VS_4) = 0.0382, Scr(VS_5) = 0.0934, Scr(VS_6) = 0.1085,$ $Scr(VS_7) = -0.0548.$	$VS_6 > VS_5 > VS_3 > VS_1 > VS_4 > VS_2 > VS_7$
$\gamma = -8$	$Scr(VS_1) = 0.0970, Scr(VS_2) = 0.0462, Scr(VS_3) = 0.1728,$ $Scr(VS_4) = 0.0806, Scr(VS_5) = 0.1806, Scr(VS_6) = 0.1634,$ $Scr(VS_7) = -0.0545.$	$VS_6 > VS_5 > VS_3 > VS_1 > VS_4 > VS_2 > VS_7$
$\gamma = -16$	$Scr(VS_1) = 0.1789, Scr(VS_2) = 0.0812, Scr(VS_3) = 0.2101,$ $Scr(VS_4) = 0.1434, Scr(VS_5) = 0.2170, Scr(VS_6) = 0.2042,$ $Scr(VS_7) = -0.0440.$	$VS_5 > VS_3 > VS_6 > VS_1 > VS_4 > VS_2 > VS_7$
$\gamma = -20$	$Scr(VS_1) = 0.1890, Scr(VS_2) = 0.0886, Scr(VS_3) = 0.2169,$ $Scr(VS_4) = 0.1527, Scr(VS_5) = 0.2208, Scr(VS_6) = 0.2243,$ $Scr(VS_7) = -0.0435.$	$VS_6 > VS_5 > VS_3 > VS_1 > VS_4 > VS_2 > VS_7$
$\gamma = -30$	$Scr(VS_1) = 0.1977, Scr(VS_2) = 0.1056, Scr(VS_3) = 0.2224,$ $Scr(VS_4) = 0.1612, Scr(VS_5) = 0.2308, Scr(VS_6) = 0.2444,$ $Scr(VS_7) = -0.0401.$	$VS_6 > VS_5 > VS_3 > VS_1 > VS_4 > VS_2 > VS_7$
$\gamma = -50$	$Scr(VS_1) = 0.2016, Scr(VS_2) = 0.1170, Scr(VS_3) = 0.2292,$ $Scr(VS_4) = 0.1666, Scr(VS_5) = 0.2371, Scr(VS_6) = 0.2543,$ $Scr(VS_7) = -0.0386.$	$VS_6 > VS_5 > VS_3 > VS_1 > VS_4 > VS_2 > VS_7$
$\gamma = -100$	$Scr(VS_1) = 0.2042, Scr(VS_2) = 0.1234, Scr(VS_3) = 0.2322,$ $Scr(VS_4) = 0.1703, Scr(VS_5) = 0.2388, Scr(VS_6) = 0.2578,$ $Scr(VS_7) = -0.0364.$	$VS_6 > VS_5 > VS_3 > VS_1 > VS_4 > VS_2 > VS_7$

Table 9. Effect of the parameter γ on final ranking orders utilizing the T-SPFSSPWGHM operator.

Parameter	Score Values	Ranking Order
$\gamma = -4$	$Scr(VS_1) = -0.0705, Scr(VS_2) = -0.0648, Scr(VS_3) = -0.0367,$ $Scr(VS_4) = -0.0558, Scr(VS_5) = -0.03669, Scr(VS_6) = -0.0522,$ $Scr(VS_7) = -0.1338.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$\gamma = -8$	$Scr(VS_1) = -0.0685, Scr(VS_2) = -0.0680, Scr(VS_3) = -0.0303,$ $Scr(VS_4) = -0.0485, Scr(VS_5) = -0.0305, Scr(VS_6) = -0.0453,$ $Scr(VS_7) = -0.1722.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$\gamma = -16$	$Scr(VS_1) = -0.0748, Scr(VS_2) = -0.0883, Scr(VS_3) = -0.0356,$ $Scr(VS_4) = -0.0499, Scr(VS_5) = -0.0359, Scr(VS_6) = -0.0474,$ $Scr(VS_7) = -0.2128.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$\gamma = -20$	$Scr(VS_1) = -0.0803, Scr(VS_2) = -0.0924, Scr(VS_3) = -0.0388,$ $Scr(VS_4) = -0.0514, Scr(VS_5) = -0.0391, Scr(VS_6) = -0.0491,$ $Scr(VS_7) = -0.2214.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$\gamma = -30$	$Scr(VS_1) = -0.0997, Scr(VS_2) = -0.1012, Scr(VS_3) = -0.0453,$ $Scr(VS_4) = -0.0546, Scr(VS_5) = -0.0455, Scr(VS_6) = -0.0525,$ $Scr(VS_7) = -0.2335.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$\gamma = -50$	$Scr(VS_1) = -0.1237, Scr(VS_2) = -0.1116, Scr(VS_3) = -0.0526,$ $Scr(VS_4) = -0.0584, Scr(VS_5) = -0.0528, Scr(VS_6) = -0.0566,$ $Scr(VS_7) = -0.2424.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
$\gamma = -100$	$Scr(VS_1) = -0.1415, Scr(VS_2) = -0.1225, Scr(VS_3) = -0.05923,$ $Scr(VS_4) = -0.0620, Scr(VS_5) = -0.05925, Scr(VS_6) = -0.0610,$ $Scr(VS_7) = -0.2500.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$

6.1.3. Effect of the Parameter q on Final Ranking Orders Utilizing T-SPFSSPWHM and T-SPFSSPWGHM Operators

In this subpart, the effect of the parameter q on final ranking orders utilizing T-SPFSSPWHM and T-SPFSSPWGHM operators are investigated, and the value of other parameters $A = 2$, $B = 3$ *and* $\gamma = -2$ are fixed. The score values and ranking orders for different values of the parameter q utilizing T-SPFSSWPHM and T-SPFSSPWGHM operators are given in Table 10. One can notice from Table 10, that is, for distinct values of the parameter, q, the ranking order is totally different. That is, utilizing T-SPFSSPWHM and T-SPFPWGHM operators for distinct values of q, the best alternative is either VS_1, VS_5 or VS_6 while the worst alternative is VS_1, VS_4 *and* VS_7. The reason behind these different ranking orders is that these AOs are more flexible due to consisting of general parameters. Therefore, the MADM model based on these aggregation operators is more flexible. Hence, the decision-maker may choose the values of these parameters according to the actual needs of the situations.

Table 10. Effect of the parameter q on final ranking orders utilizing T-SPFSSPWHM and T-SPFSSPWGHM operators.

Parameter	Score Values Utilizing T-SPFSSPWHM Operator	Score Values Utilizing T-SPFSSPWHM Operator	Ranking Orders
$q = 1$	$Scr(VS_1) = 0.0835, Scr(VS_2) = 0.0020,$ $Scr(VS_3) = 0.1545, Scr(VS_4) = 0.0516,$ $Scr(VS_5) = 0.1254, Scr(VS_6) = 0.0384,$ $Scr(VS_7) = -0.1155.$	$Scr(VS_1) = -0.1446, Scr(VS_2) = -0.1786,$ $Scr(VS_3) = -0.0558, Scr(VS_4) = -0.1185,$ $Scr(VS_5) = -0.0550, Scr(VS_6) = -0.1009,$ $Scr(VS_7) = -0.2521.$	$VS_3 > VS_5 > VS_1 > VS_4 > VS_6 > VS_2 > VS_7$ and $VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
$q = 3$	$Scr(VS_1) = -0.0278, Scr(VS_2) = -0.0302,$ $Scr(VS_3) = -0.2872, Scr(VS_4) = -0.3415,$ $Scr(VS_5) = -0.2812, Scr(VS_6) = -0.3259,$ $Scr(VS_7) = -0.1611.$	$Scr(VS_1) = 0.2333, Scr(VS_2) = -0.0180,$ $Scr(VS_3) = 0.0740, Scr(VS_4) = 0.1601,$ $Scr(VS_5) = 0.0749, Scr(VS_6) = 0.2939,$ $Scr(VS_7) = -0.0786.$	$VS_1 > VS_2 > VS_7 > VS_5 > VS_3 > VS_6 > VS_4$ and $VS_6 > VS_1 > VS_4 > VS_5 > VS_3 > VS_2 > VS_7$
$q = 5$	$Scr(VS_1) = 0.0120, Scr(VS_2) = 0.0425,$ $Scr(VS_3) = 0.0915, Scr(VS_4) = 0.0723,$ $Scr(VS_5) = 0.0926, Scr(VS_6) = 0.0891,$ $Scr(VS_7) = 0.0153.$	$Scr(VS_1) = 0.1296, Scr(VS_2) = -0.0015,$ $Scr(VS_3) = 0.0329, Scr(VS_4) = 0.0278,$ $Scr(VS_5) = 0.0332, Scr(VS_6) = 0.2233,$ $Scr(VS_7) = -0.0640.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_2 > VS_7 > VS_1$ and $VS_6 > VS_1 > VS_5 > VS_3 > VS_4 > VS_2 > VS_7$
$q = 7$	$Scr(VS_1) = 0.0808, Scr(VS_2) = 0.0777,$ $Scr(VS_3) = 0.0897, Scr(VS_4) = 0.0810,$ $Scr(VS_5) = 0.0907, Scr(VS_6) = 0.0915,$ $Scr(VS_7) = -0.0145.$	$Scr(VS_1) = 0.0156, Scr(VS_2) = 0.0154,$ $Scr(VS_3) = 0.0877, Scr(VS_4) = 0.0337,$ $Scr(VS_5) = 0.1071, Scr(VS_6) = 0.2859,$ $Scr(VS_7) = -0.0504.$	$VS_6 > VS_5 > VS_3 > VS_4 > VS_1 > VS_2 > VS_7$ and $VS_6 > VS_5 > VS_3 > VS_4 > VS_1 > VS_2 > VS_7$
$q = 9$	$Scr(VS_1) = 0.0814, Scr(VS_2) = 0.0788,$ $Scr(VS_3) = 0.0856, Scr(VS_4) = 0.0802,$ $Scr(VS_5) = 0.0863, Scr(VS_6) = 0.0880,$ $Scr(VS_7) = 0.0310.$	$Scr(VS_1) = 0.0347, Scr(VS_2) = -0.0347,$ $Scr(VS_3) = 0.0239, Scr(VS_4) = 0.2086,$ $Scr(VS_5) = 0.0424, Scr(VS_6) = 0.1413,$ $Scr(VS_7) = -0.0450.$	$VS_6 > VS_5 > VS_3 > VS_4 > VS_1 > VS_2 > VS_7$ and $VS_4 > VS_6 > VS_5 > VS_3 > VS_1 > VS_2 > VS_7$

6.1.4. Comparison with Existing Approaches

In this subpart, we compare our produced MADM model, which is based on these developed novel AGOs, to some current techniques, such as the approach initiated by Garg et al. [38] and the MADM model initiated by Tahir et al. [4]. The score values and ranking orders are given in Table 11. From Table 11, one can notice that the ranking order obtained from these approaches and the proposed approaches are the same, except the approach developed on the T-SPFWG operator. This shows that the developed MADM decision making model is valid. The initiated MADM model has some advantages over the existing approaches.

(1) The anticipated MADM model is based on the newly initiated aggregation operators. That is, these aggregation operators are proposed utilizing SS ALs for T-SPFNs, which consist of general parameters that make the decision-making process more flexible. Meanwhile, the existing MADM models are based on the aggregation operators, which are initiated utilizing algebraic ALs.

(2) The existing aggregation operators have the characteristic that they can only remove the effect of awkward data by utilizing power weight vector, while the anticipated aggregation operators have the ability to remove the effect of awkward data as well as consider the interrelationship among the input data at the same time.

(3) The other advantage of the anticipated AOs is that it consists of general parameters, which make the decision-making process more flexible. Therefore, the initiated AOs are more practical and comparative in their utilization while solving MADM models under T-SPF information.

Table 11. Comparison with existing approaches.

Aggregation Operators	Score Values	Ranking Order
T-SPFPWA operator [38]	$Scr(VS_1) = 0.2335, Scr(VS_2) = 0.1975,$ $Scr(VS_3) = 0.2881, Scr(VS_4) = 0.2201,$ $Scr(VS_5) = 0.2902, Scr(VS_6) = 0.2704,$ $Scr(VS_7) = 0.0716.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
T-SPFPGWA operator [38]	$Scr(VS_1) = -0.3280, Scr(VS_2) = -0.3291,$ $Scr(VS_3) = -0.2458, Scr(VS_4) = -0.2913,$ $Scr(VS_5) = -0.2461, Scr(VS_6) = -0.2852,$ $Scr(VS_7) = -0.3938.$	$VS_3 > VS_5 > VS_6 > VS_4 > VS_2 > VS_1 > VS_7$
T-SPFWA operator [4]	$Scr(VS_1) = 0.0211, Scr(VS_2) = 0.0036,$ $Scr(VS_3) = 0.0806, Scr(VS_4) = 0.0128,$ $Scr(VS_5) = 0.0834, Scr(VS_6) = 0.0656,$ $Scr(VS_7) = -0.0971.$	$VS_5 > VS_3 > VS_6 > VS_1 > VS_4 > VS_2 > VS_7$
T-SPFWG operator [4]	$Scr(VS_1) = -0.0262, Scr(VS_2) = -0.0268,$ $Scr(VS_3) = 0.0285, Scr(VS_4) = -0.0111,$ $Scr(VS_5) = 0.0321, Scr(VS_6) = 0.0059,$ $Scr(VS_7) = -0.1271.$	$VS_5 > VS_3 > VS_6 > VS_4 > VS_1 > VS_2 > VS_7$
In this article	$Scr(VS_1) = 0.0432, Scr(VS_2) = 0.0297,$ $Scr(VS_3) = 0.0813, Scr(VS_4) = 0.0418,$ $Scr(VS_5) = 0.0840, Scr(VS_6) = 0.0733,$ $Scr(VS_7) = -0.0382.$	$VS_5 > VS_3 > VS_6 > VS_1 > VS_4 > VS_2 > VS_7$
In this article	$Scr(VS_1) = -0.0660, Scr(VS_2) = -0.0414,$ $Scr(VS_3) = -0.0342, Scr(VS_4) = -0.0580,$ $Scr(VS_5) = -0.0301, Scr(VS_6) = -0.0558,$ $Scr(VS_7) = -0.0796.$	$VS_5 > VS_3 > VS_2 > VS_6 > VS_4 > VS_1 > VS_7$

7. Conclusions

One of several implementations of multi-criteria decision-making (MCDM) problems is indeed the assessment of water reuse strategies. Water reuse is a potential method for boosting the urban supply of water, particularly in light of the changing standards such as climate change and increased human activity. A cost-effective, long-term water reuse application should pose an admissible health risk to customers. Data collection is frequently coupled with difficulties of ambiguity, hesitation, and parameterization, making water reuse application evaluation difficult. In this article, a T-Spherical fuzzy set-based decision support is initiated to offer an efficient approach to explain ambiguity, hesitation, and uncertainty. The contribution of this article is fourfold. Firstly, the novel SS ALs for T-SPFN are initiated and some the vital characteristic of these ALs are investigated. Secondly, based on these novel SS ALs, some T-SPFSSPHM operators such as the T-Spherical fuzzy Schweizer–Sklar power Heronian mean operator, the T-Spherical fuzzy Schweizer–Sklar power geometric Heronian mean operator, the T-Spherical fuzzy Schweizer–Sklar power weighted Heronian mean operator, the T-Spherical fuzzy Schweizer–Sklar power weighted geometric Heronian mean operator, and their vital properties and special cases with respect to the parameters are discussed. By giving specific values to the general parameters, we can observe that some of the existing AOs are special cases of these newly initiated AOs. These AOs have advantages over the existing AOs. These AOs have the capacity to remove the influence of awkward data and consider the interrelationship among the input data at the same time, while the existing aggregation operators for T-SPFS can remove the effect of awkward data or create interrelationships among input data. Thirdly, based on these AOs, a MADM model is anticipated. Lastly, the anticipated model is applied to select the best option in water reuse from the available options.

In future, we will apply the anticipated approach to some new applications, such as supply chain management [5,9], public transportation [24], traffic control [54], digital twin model [56], and so on, or extend the anticipated model to some more extended form of T-SPFSs.

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