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Two Level Trade Credit Policy Approach in Inventory Model with Expiration Rate and Stock Dependent Demand under Nonzero Inventory and Partial Backlogged Shortages

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Abstract: In present real life situations, the stock and expiration date directly impact on the demand of an item. In this context, this research work develops an inventory model for stock and expiration rate-dependent demand under a two-level trade credit policy. Specifically, the following three situations are studied: (i) trade credit policy without zero ending inventory; (ii) trade credit policy with zero ending inventory; (iii) trade credit policy with partial backlogged shortages. The proposed inventory model is formulated as a non-linear constrained optimization problem. Some theoretical results are derived, and an algorithm is stated in order to solve the proposed inventory model. The main objective of the inventory model is to determine the optimal cycle length, the optimal ending inventory level, and the optimal number of units displayed which maximize the total profit. Some numerical examples are solved. Finally, a sensitivity analysis is done with the aim to see the impacts of a variation of the input parameters on the decision variables and the total profit.

Keywords: inventory; stock and expiration rate-dependent demand; partially backlogged shortage; two-level trade credit policy

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1. Introduction

The first inventory model was introduced by Harris [1] in 1913. This inventory model is well known as the economic order quantity (EOQ) inventory model. It was created based on a few basic and very restrictive assumptions. After that, a lot of research work has been done on Harris [1]'s inventory model by considering several realistic situations. The EOQ inventory model assumes that the demand is constant and the shortage is not permitted. However, in real life, the demands are not always fixed and in fact depend on several factors, such as the time, the price, the stock, the expiration date, and the attractive discount rate, among others aspects. For example, the expiration date (the freshness of the product) has a huge impact on the demand. Another important factor that increases the demand of the products is the displayed stock. The items with an expiration date deteriorate through time and the deterioration rate of these is directly proportional to the maximum lifetime. In this context, Chen and Teng [2] determined retailers' optimal ordering policy for fixed lifetime deteriorating items under trade credit policy. Wu et al. [3] investigated the effects of trade credit financing in a lot-size model for deteriorating items with expiration dates. Teng et al. [4] developed an inventory model with an advance payment scheme for deteriorating items with expiration dates. In the same year, Sarkar et al. [5] considered a variable backorder, inspection process, and discount policy in a supply chain model for fixed lifetime products. Later, Tiwari et al. [6] formulated an inventory model with partial backlogged shortages and bi-level trade credit financing for deteriorating items with expiration dates. Uthayakumar and Tharani [7] included advance payments and trade credit financing in an inventory model with shortages for deteriorating items with expiration dates. Recently, Liao et al. [8] introduced a supply chain model for non-instantaneous deteriorating items with expiration dates under a hybrid payment policy.

Perhaps the displayed stock concepts in the inventory area were first introduced by Levin et al. [9]. They showed that the displayed stock influences the customer to buy more products. Silver and Peterson [10] presented an inventory model showing that sales and the inventory level are proportional to the displayed stock. Baker and Urban [11] proposed an inventory model with a specific demand function as the power function of the stock level. Thereafter, Mandal and Phaujdar [12] derived an inventory model with a linear form of the stock-dependent demand. Urban [13] studied an economic order quantity (EOQ) inventory model without ending the zero stock model with a displayed stock dependent demand. Additionally, Urban [13]'s inventory model was generalized in Urban and Baker [14] by considering that the demand is a function of time, stock, and price. After that, many inventory models have been done based on demand as a function of one or several factors. For example, see the following research works: Dye and Ouyang [15], Soni and Shah [16], Yang et al. [17], Teng et al. [18], Zhou et al. [19], Soni [20], Zhong and Zhou [21], Wu et al. [3], Min et al. [22], Panda et al. [23] and others. For instance, Panda et al. [23] investigated a two-warehouse inventory model with partial backlogging and trade credit policy for deteriorating items with price- and stock-dependent demand. Das et al. [24] analyzed a production inventory model with a replacement period, stock, and price-dependent demand. Banu et al. [25] examined the trade credit period and stock-dependent demand rate in a supply chain model with variable imperfect production rates.

Permissible delay in payment is one of the most important and popular strategies in the business world. With this strategy, the suppliers attract more customers, and therefore more sales of the product. It is worth mentioning that due to this strategy, businesses at present are more competitive and pose a greater challenge. Basically, in a two-level trade credit policy, the suppliers provide the facility of a delay in payment (credit time) to retailers with some terms and conditions. The retailers then accept this and also give the facility of a delay in payment (credit time) to his/her customers. All this is done with the aim of attracting more customers, and consequently selling more products. The delay in payment (credit time) facility consists of giving to a buyer the payment facility of paying the purchased amount up to a certain period. On the one hand, no interest is charged by the vendor to his/her buyer if the payment is done up to the credit period. On the other hand, an interest is charged by the vendor to his/her buyer when the credit period is over. Recently, several researchers have performed several kinds of research by including a two-level trade credit concept. To the best of our knowledge, Goyal [26] developed an economic order quantity (EOQ) inventory model under permissible delay in payments. After that, Aggarwal and Jaggi [27] extended Goyal [26]'s inventory model by considering that the items deteriorate with an exponential rate. On the other hand, Jamal et al. [28] also extended Goyal [26]'s inventory model by considering shortages. Chung [29] stated a theorem for Goyal [26]'s inventory model. Teng [30] introduced an inventory model with trade credit financing by distinguishing the difference between unit cost and unit price, and then calculated the interest earned by the sales revenue of items. All the previous mentioned inventory models are related to trade credit financing for stock control for a single echelon of the chain. Thereafter, Huang [31] extended the inventory model with the trade credit for a supply chain system with both an upstream and a downstream credit. Teng and Goyal [32] modified the assumption of Huang [31]'s inventory model by introducing the concept that the retailer obtains its revenue from N to $N + T$, not from 0 to T. They also stated that the unit selling price is higher than the unit purchased cost. At the same time, Teng [33] obtained the optimal ordering policy for the retailer to deal with credit risk clients, as well as good credit clients. Min et al. [22] worked on an inventory model under stock dependent demand and two-level trade credit. In this line of research, there exist the following papers in the area of the two-level trade credit policy such as Kreng and Tan [34], Teng and Lou [35], Mahata [36], Chung and Cárdenas-Barrón [37], Chung et al. [38], Wu et al. [3], Chen et al. [39], Shah and Cárdenas-Barrón [40] and others. For example, Jaggi et al. [41] evaluated the effect of credit financing and two storage facilities in a non-instantaneous deteriorating inventory model with price-dependent demand. Mohanty et al. [42] developed an inventory model with trade-credit financing and a preservation facility for deteriorating items under a random planning horizon. Aliabadi et al. [43] solved a non-instantaneous deteriorating inventory problem with credit period and carbon emissions dependent demand by using a geometric programming approach. Liao et al. [44] addressed an EOQ inventory model with a delay in payment policy for non-instantaneous deteriorating items with the aim of finding an optimal ordering policy. Shaikh et al. [45] considered price-sensitive demand, inflation, and reliability in a production inventory model for deteriorating items under trade credit policy. Das et al. [46] took into account preservation facility and trade credit financing in a non-instantaneous deteriorating inventory model. Table 1 presents a comparison between some published research works and the proposed inventory model.

Table 1. A brief comparison of the published research works with the proposed inventory model.

Author (s)	Type of Model	Product Lifetime/ Expiration Date	Demand Dependent on	Partial/Complete Backlogging	Trade Credit Policy
Chen and Teng [2]	Purchase	√	Constant		√
Wu et al. [3]	Purchase	√	Expiration rate		√
Teng et al. [4]	Purchase	√	Constant	√	
Sarkar [5]	Supply chain	√	Constant	√	
Tiwari et al. [6]	Purchase	√	Selling price	√	√
Uthayakumar and Tharani [7]	Purchase	√	Constant	√	√
Liao et al. [8]	Purchase		Constant		√
Banu et al. [21]	Production		Display stock and credit period		√
Min et al. [22]	Purchase		Display stock		√
Panda et al. [23]	Purchase		Display stock	√	√
Jaggi et al. [41]	Purchase		Selling price		√
Mohanty et al. [42]	Purchase		Time		√
Aliabadi et al. [43]	Purchase		Credit period and selling price	√	√
Shaikh et al. [45]	Production		Selling price		√
Das et al. [46]	Purchase		Selling price	√	
Chakraborty et al. [47]	Purchase		Display stock	√	
Manna et al. [48]	Purchase		Time	√	
Barman et al. [49]	Supply chain		Green level and selling price		
Li and Teng [50]	Purchase		Display shelf space, stock dependent (power function), and expiration rate		
This paper	Purchase	√	Display shelf space, stock dependent (power function), and expiration rate	Partial backlogging	Two-level trade credit policy approach

Shelf space is one of the effective strategies of the inventory systems. The main aim of this strategy is to display to customers the products in the shelf space in order that they can check the product's features by themselves. In this context, Bai and Kendall [51] proposed an inventory model under a shelf space and a freshness-dependent demand. Piramuthu and Zhou [52] studied an inventory model with shelf space and freshness-dependent demand for a deteriorating item. Chen et al. [53] explored an inventory model with freshness and stock-dependent demand for fresh products. Wu et al. [54] took into consideration an inventory model for a fresh product whose demand is dependent on time-varying freshness. Dobson et al. [55] proposed an inventory model for deteriorating items whose demand is dependent on the age of the product. Banerjee and Agrawal [56] examined the freshness and price-dependent demand in an inventory model. Soni and Suthar [57] optimized the shelf space for fresh products with an expiration date. Agi et al. and Soni [58] proposed a joint pricing inventory model for deteriorating items with price- and age-dependent demand. Recently, Macías-López et al. [59] studied an inventory model with a shelf space and nonlinear holding cost for a perishable item. Sebatjane and Adetunji [60] suggested a three-echelons supply chain inventory model for age-dependent deterioration with lot-sizing and shipment decisions.

This research work formulates an inventory model with expiration date, shelf-space, and stock-dependent demand under two-level trade credit financing and partial backlogging. In the two-level trade credit system, the supplier offers a trade credit facility to his/her retailer, and then the retailer also provides a trade credit to his/her customers. The shortage is allowed and this is partially backlogged with a constant backlogging rate. Basically, the following cases are discussed: (i) without zero ending inventory; (ii) with zero ending inventory; (iii) shortages. It is important to remark that the cases (i) and (iii) belong to case (ii) when the shortage level is zero or the inventory level is zero.

The rest of this paper is organized as follows: Section 2 defines the notation and the assumptions. Section 3 formulates the inventory model with the expiration date, shelf-space, and stock-dependent demand under two-level trade credit financing and partial backlogging. Section 4 develops a solution procedure and the Algorithm 1 to determine the optimal solution. Section 5 presents and solves some numerical examples. Section 6 performs a sensitivity analysis. Finally, Section 7 provides some conclusions and directions for future research.

2. Notation and Assumptions

The notation used in the development of the inventory model is given in Notation in back matter.

Assumptions

The assumptions on which the inventory model is based are as follows:

(i) The lot size (Q) in a single batch is delivered.
(ii) The inventory planning horizon is infinite.
(iii) Replenishments are instantaneous.
(iv) The freshness of the product depends on several factors such as the temperature of the environment, conditions of the stocking place, and humidity, among others. For these causes the product deteriorates over time continuously and it has an expiration date. In the beginning, when the lot size arrives, the freshness of the product is assumed to equal 1 when the item is in stock, in the storeroom, and in its displayed place. After that, the freshness of the product is decreased, and finally it reaches the expiration date. Thereofore, the freshness function is mathematically expressed as follows:

$$f(t) = \frac{(m-t)}{m} 0 \leq t \leq m$$

At the start, it is assumed that the retailer receives Q units, and from these Q units, W units are stored in the display shelf. The rest of the units are kept in the storeroom. When the product is sold from the display room, then the product moves to a display place from the storeroom. It is also considered that the demand in the time interval $[0, t_1]$ is according to the following function:

$$D(t) = \frac{\alpha W^{\beta}(m-t)}{m} 0 < t \leq t_1$$

where $\alpha > 0$ and $0 \leq \beta < 1$

(i) It is also considered that when the cycle length finishes before the end of the stock (i.e., without ending zero inventory) on that time the demand depends on the power function of the inventory level, as well as the expiration rate. Therefore, the demand is given by

$$D(t) = \alpha[I(t)]^{\beta}\frac{m-t}{m},\ t_1 < t \le T.$$

where $\alpha > 0$ and $0 \le \beta < 1$. When shortages appear, the demand is according to the following function:

$$D(t) = \begin{cases} \alpha[I(t)]^{\beta}\frac{m-t}{m} & t_1 < t \le t_2 \\ \alpha & t_2 < t \le T \end{cases}$$

where $\alpha > 0$ *and* $0 \le \beta < 1$.

(ii) The deteriorated products are neither repaired nor refunded.

(iii) Shortages are allowed with partial backlogging with a rate of δ.

(iv) The supplier gives the credit time (M) facility to his/her retailer. The retailer offers credit time (N) facility to his/her customers.

(v) The retailer uses the sales revenue to obtain the interest with a rate I_e according to the terms and conditions given by the agreement. When the cycle length finishes, the credit is settled, and then interest charges start to be paid by the retailer on the items in stock with a rate I_p.

3. Mathematical Formulation of the Inventory Model with Expiration Date, Shelf Space, and Stock-Dependent Demand under Two-Level Trade Credit Financing and Partial Backlogging

3.1. Mathematical Derivation When B > 0

Initially, the retailer purchases a lot size of Q units of the product. Moreover, from these Q units, W units are immediately displayed at the showroom in order to attract customers. These Q units are continuously decreasing due to customers' demand. At the time $t = T$, some products are remaining in the inventory, and this stock level is named B. Afterwards, the cycle length T repeats itself again.

Consider that the inventory level is represented by $I(t)$ at any time $t \ge 0$. Then, the $I(t)$ satisfies the following differential equations at any time t:

$$\frac{dI(t)}{dt} = -\alpha W^{\beta}\frac{m-t}{m} \quad 0 \le t \le t_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -\alpha[I(t)]^{\beta}\frac{m-t}{m} \quad t_1 < t \le T \tag{2}$$

with the boundary conditions:

$$I(t) = Q \text{ at } t = 0,\ I(t) = W \text{ at } t = t_1 \text{ and } I(t) = B \text{ at } t = T \tag{3}$$

Furthermore, $I(t)$ is continuous at the time $t = t_1$.

By using the conditions (3), the differential Equations (1) and (2) are solved and their solutions are as follows:

$$I(t) = \frac{1}{2m}\alpha W^{\beta}t^2 - \alpha W^{\beta}t + Q \qquad 0 < t \le t_1 \tag{4}$$

$$I(t) == \left[\frac{\alpha(1-\beta)}{2m}\left\{t^2 + 2m(T-t) - T^2\right\} + B^{1-\beta}\right]^{\frac{1}{1-\beta}} \qquad t_1 < t \le T \tag{5}$$

From (4), $I(t) = W$ at $t = t_1$. Then,

$$Q = W + \frac{\alpha W^{\beta}}{2m}\left(2mt_1 - t_1^2\right) \ge W \tag{6}$$

Using continuity of condition of Equations (4) and (5), hence

$$\frac{1}{2m}\alpha W^{\beta}t_1^2-\alpha W^{\beta}t_1+Q=\left[\frac{\alpha(1-\beta)}{2m}\left\{t_1^2+2m(T-t_1)-T^2\right\}+B^{1-\beta}\right]^{\frac{1}{1-\beta}} \tag{7}$$

From Equation (7), it is obtained that

$$t_1=m-\sqrt{(m-T)^2+\frac{2m\left(W^{1-\beta}-B^{1-\beta}\right)}{\alpha(1-\beta)}}\geq 0 \tag{8}$$

It is informed to the reader that the detailed calculations are shown in the Supplementary material.

Using Equations (6) and (8) the following result is determined:

$$Q=W+\frac{\alpha W^{\beta}}{2m}\left[m^2-(m-T)^2-\frac{2m\left(W^{1-\beta}-B^{1-\beta}\right)}{\alpha(1-\beta)}\right] \tag{9}$$

Now, the costs of the inventory system are calculated below:

(i) Ordering cost = c_o

(ii) Holding cost = $chol=h\left\{\int\limits_0^{t_1}I_1(t)dt+\int\limits_{t_1}^{t_2}I_2(t)dt\right\}=h(chol_1+chol_2)$, where $chol_1=\left[\frac{\alpha W^{\beta}}{6m}t_1^3-\frac{\alpha W^{\beta}}{2}t_1^2+Qt_1\right]$ and $chol_2=\frac{(W+B)(T-t_1)}{2}$.

According to the credit period facility (M), five cases occur and these are described as follows: Case-1: $N<M\leq t_1<T$, Case-2: $N<t_1\leq M<T$, Case-3: $t_1\leq N<M<T$, Case-4: $t_1<N<T\leq M$ and Case-5: $t_1<T\leq N<M$.

Now, all these cases are discussed in detail below.

Case-1: When $N<M\leq t_1<T$

In this case, the credit period facility (M) is given by $N<M\leq t_1<T$ then the retailer earns interest during the period $[N,M]$. Hence, the interest earned is obtained as follows.

$$\text{Interest earned (IE)}=pI_e\frac{\alpha W^{\beta}}{m}\int\limits_N^M\int\limits_0^t(m-u)dudt=pI_e\frac{\alpha W^{\beta}}{m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{\left(M^3-N^3\right)}{6}\right]$$

Here, the credit time facility M is always less than or equal to t_1. The retailer makes the required payment during the period $[M,T]$, and thus the interest paid is computed as follows.

$$\text{Interest paid (IP)}=cI_p\left[\int\limits_M^{t_1}I_1(t)dt+\int\limits_{t_1}^{T}I_2(t)dt\right]$$

$$\text{Interest paid (IP)}=cI_p\left[\frac{\alpha W^{\beta}}{6m}\left(t_1^3-M^3\right)-\frac{\alpha W^{\beta}}{2}\left(t_1^2-M^2\right)+Q(t_1-M)+\frac{(W+B)(T-t_1)}{2}\right]$$

The total profit for Case-1 is calculated with $X_1=p(Q-B)+SB+IE-cQ-c_o-chol-IP-uW$. Thus, the total profit per unit of time of Case-1 is expressed below.

$$Z_1(W,B,T)=\frac{p(Q-B)+SB+IE-cQ-c_o-chol-IP-uW}{T} \tag{10}$$

$$\left.\begin{array}{ll}\text{Problem-1} & \text{Max}\,TP_1(W,B,T)=Z_1(W,B,T)\\ \text{Subject to} & N<M\leq t_1<T\end{array}\right\} \tag{11}$$

Case-2: When $N<t_1\leq M<T$

In this case, the credit period facility (M) is between $N<t_1\leq M<T$. Therefore, the retailer gets interest during the period $[N,M]$ and the retailer must cover the payment

during the period $[M, T]$. Hence, the interest earned and the interest paid are determined in the following manner:

$$\text{Interest earned (IE)} = pI_e\left[\frac{\alpha W^{\beta}}{m}\int_{N}^{t_1}\int_{0}^{t}(m-u)dudt + \frac{\alpha}{m}\int_{t_1}^{M}\int_{0}^{t}I(u)(m-u)dudt\right]$$

$$\text{Interest earned (IE)} = pI_e\left[\frac{\alpha W^{\beta}}{m}\left\{\frac{m}{2}\left(t_1^2 - N^2\right) - \frac{1}{6}\left(t_1^3 - N^3\right)\right\} + \frac{\alpha(W+B)^{\beta}}{2m}\left\{\frac{m}{2}\left(M^2 - t_1^2\right) - \frac{1}{6}\left(M^3 - t_1^3\right)\right\}\right]$$

$$\text{Interest paid (IP)} = cI_p\left[\int_{M}^{T}I(t)dt\right] = cI_p\frac{(W+B)(T-M)}{2}$$

The total profit for Case-2 is expressed as $X_2 = p(Q-B) + SB + IE - cQ - c_o - chol - IP - uW$. Now, the total profit per unit of time of Case-2 is

$$Z_2(W,B,T) = \frac{p(Q-B) + SB + IE - cQ - c_o - chol - IP - uW}{T} \tag{12}$$

$$\left.\begin{array}{ll}\text{Problem-2} & \text{Max}\, TP_2(W,B,T) = Z_2(W,B,T) \\ \text{Subject to} & N < t_1 \le M < T\end{array}\right\} \tag{13}$$

Case-3: When $t_1 \le N < M < T$

Here, the credit period facility (M) belongs to $t_1 \le N < M < T$. In this interval, the retailer wins interest during the period $[N, M]$. However, in this situation, the retailer needs to pay interest during the period payment $[M, T]$. For this case the interest earned and the interest paid are calculated as follows.

$$\text{Interest earned (IE)} = \frac{\alpha}{m}\int_{N}^{M}\int_{0}^{t}I(u)(m-u)dudt = pI_e\frac{\alpha(W+B)^{\beta}}{2m}\left[\frac{m}{2}\left(M^2 - N^2\right) - \frac{1}{6}\left(M^3 - N^3\right)\right]$$

$$\text{Interest paid (IP)} = cI_p\frac{(W+B)(T-M)}{2}$$

The total profit for Case-3 is given by $X_3 = p(Q-B) + SB + IE - cQ - c_o - chol - IP - uW$. Here, the total profit per unit of time of Case-3 is

$$Z_3(W,B,T) = \frac{p(Q-B) + SB + IE - cQ - c_o - chol - IP - uW}{T} \tag{14}$$

$$\left.\begin{array}{ll}\text{Problem-3} & \text{Max}\, TP_3(W,B,T) = Z_3(W,B,T) \\ \text{Subject to} & t_1 \le N < M < T\end{array}\right\} \tag{15}$$

Case-4: $t_1 < N < T \le M$

Here, the retailer obtains interest during the time interval $[N, M]$.

$$\text{Interest earned (IE)} = \frac{\alpha}{m}\int_{N}^{T}\int_{0}^{t}I(u)(m-u)dudt[M-T] = pI_e\frac{\alpha(W+B)^{\beta}}{2m}\left[\frac{m}{2}\left(T^2 - N^2\right) - \frac{1}{6}\left(T^3 - N^3\right)\right][M-T]$$

Due to the fact that the cycle length T is less than or equal to credit period M, the interest paid (IP) = 0.

The total profit for Case-4 is $X_4 = p(Q-B) + SB + IE - cQ - c_o - chol - uW$. Then the total profit per unit time of Case-4 is

$$Z_4(W,B,T) = \frac{p(Q-B) + SB + IE - cQ - c_o - chol - uW}{T} \tag{16}$$

$$\left.\begin{array}{ll}\text{Problem-4} & \text{Max}\, TP_4(W,B,T) = Z_4(W,B,T) \\ \text{Subject to} & t_1 < N < T \leq M\end{array}\right\} \tag{17}$$

Case-5: $t_1 < T \leq N < M$

In this case, the retailer makes interest during the time interval $[N, M]$. Therefore, the interest earned (*IE*) = $pI_e(Q-B)(M-N)$ and the interest paid (*IP*) = 0.

The total profit for Case-5 is given by $X_5 = p(Q-B) + SB + IE - cQ - c_o - chol - uW$. The total profit per unit of time of Case-5 is

$$Z_5(W,B,T) = \frac{p(Q-B) + SB + IE - cQ - c_o - chol - uW}{T} \tag{18}$$

$$\left.\begin{array}{ll}\text{Problem-5} & \text{Max}\, TP_5(W,B,T) = Z_5(W,B,T) \\ \text{Subject to} & t_1 < T \leq N < M\end{array}\right\} \tag{19}$$

3.2. Mathematical Derivation When B < 0

In the beginning, the retailer purchases $Q + B$ units of the product. This shows that the initial stock level is Q at $t = 0$ after satisfying the backordering quantity of the previous cycle. Out of these Q units, W units are displayed in the shelf space to motivate the customers to buy the product. These Q units are decreasing due to consumer demand. After some time, the inventory level attains to a zero level at the time $t = t_2$. Then, at time $t = t_2$, shortages start to appear with the backlogging rate δ until the time $t = T$; where $\delta > 0$. Then, the inventory level reaches the maximum shortage level B at time $t = T$. This whole inventory cycle repeats itself after the cycle length T.

Consider that the inventory level is represented by $I(t)$ at any time $t \geq 0$. Then, the inventory level $I(t)$ satisfies the following differential equations at any time t:

$$\frac{dI(t)}{dt} = -\alpha W^{\beta} \frac{m-t}{m} \quad 0 \leq t \leq t_1 \tag{20}$$

$$\frac{dI(t)}{dt} = -\alpha [I(t)]^{\beta} \frac{m-t}{m} \quad t_1 < t \leq t_2 \tag{21}$$

$$\frac{dI(t)}{dt} = -\alpha\delta \quad t_2 < t \leq T \tag{22}$$

With the following boundary conditions,

$$I(t) = Q \text{ at } t = 0,\ I(t) = W \text{ at } t = t_1 \text{ and } I(t) = -B \text{ at } t = T \tag{23}$$

notice that the $I(t)$ is continuous at the time $t = t_1, t_2$.

Using the conditions (23), the solutions to the differential Equations (20)–(22) are:

$$I(t) = \frac{1}{2m}\alpha W^{\beta} t^2 - \alpha W^{\beta} t + Q \quad 0 < t \leq t_1 \tag{24}$$

$$I(t) = \left[\frac{\alpha(1-\beta)}{2m}\left\{t^2 + 2m(t_2 - t) - {t_2}^2\right\} + B^{1-\beta}\right]^{\frac{1}{1-\beta}} \quad t_1 < t \leq t_2 \tag{25}$$

$$I(t) = \alpha\delta(T - t_2) \quad t_2 < t \leq T \tag{26}$$

From (24), $I(t) = W$ at $t = t_1$. Then, the lot size is obtained with

$$Q = W + \frac{\alpha W^{\beta}}{2m}\left(2mt_1 - t_1^2\right) \geq W \tag{27}$$

Using continuity of condition of Equations (24) and (25), the following result is found

$$\frac{1}{2m}\alpha W^{\beta} t_1^2 - \alpha W^{\beta} t_1 + Q = \left[\frac{\alpha(1-\beta)}{2m}\left\{t_1^2 + 2m(t_2 - t_1) - t_2^2\right\}\right]^{\frac{1}{1-\beta}} \tag{28}$$

From Equation (28), it is obtained

$$t_1 = m - \sqrt{(m - t_2)^2 + \frac{2mW^{1-\beta}}{\alpha(1-\beta)}} \geq 0 \tag{29}$$

The detailed calculations are shown in Supplementary A in the Supplementary material. Using conditions given in (23), from Equation (26), it is determined

$$t_2 = T - \frac{B}{\alpha\delta} \tag{30}$$

By substituting Equations (29) and (30) into Equation (27), then the lot size is calculated as follows

$$Q = W + \frac{\alpha W^\beta}{2m}\left[m^2 - \left(m - T + \frac{B}{\alpha\delta}\right)^2 - \frac{2mW^{1-\beta}}{\alpha(1-\beta)}\right] \tag{31}$$

Here, the costs of the total inventory system are determined in the following way:

(i) Ordering cost = c_o

(ii) Holding cost = $chol = h\left\{\int_0^{t_1} I_1(t)dt + \int_{t_1}^{t_2} I_2(t)dt\right\} = h(chol_1 + chol_2)$, where $chol_1 = \left[\frac{\alpha W^\beta}{6m}t_1^3 - \frac{\alpha W^\beta}{2}t_1^2 + Qt_1\right]$ and $chol_2 = \frac{W(t_2-t_1)}{2}$

(iii) Shortage cost = $csho = c_b\int_{t_2}^{T}[-I(t)]dt = c_b\int_{t_2}^{T}\alpha\delta(t-t_2)dt = \frac{c_bB^2}{2\delta\alpha}$

(iv) Cost of lost sale = $ocls = c_l(1-\delta)\alpha\int_{t_2}^{T}dt = \frac{c_l(1-\delta)B}{\delta}$

According to the credit period facility (M), there are five cases that occur. These are described and named as follows: Case-6: $N < M \leq t_1 < T$, Case-7: $N < t_1 \leq M < T$, Case-8: $t_1 \leq N < M < T$, Case-9: $t_1 < N < T \leq M$ and Case-10: $t_1 < T \leq N < M$.

Case-6: When $N < M \leq t_1 < T$

In this case, the credit period facility is given by $N < M \leq t_1 < T$ then retailer generates interest during the interval time $[N, M]$. Thus,

$$\text{Interest earned (IE)} = pI_e\frac{\alpha W^\beta}{m}\int_N^M\int_0^t(m-u)dudt = pI_e\frac{\alpha W^\beta}{m}\left[\frac{m}{2}\left(M^2-N^2\right) - \frac{\left(M^3-N^3\right)}{6}\right]$$

Note that the credit period facility M is always less than or equal to t_1. The retailer has to pay the required interest in the period $[M, T]$. Hence, the interest paid is:

$$\text{Interest paid (IP)} = cI_p\left[\int_M^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\right][T - t_2]$$

$$\text{Interest paid (IP)} = cI_p\left[\frac{\alpha W^\beta}{6m}\left(t_1^3 - M^3\right) - \frac{\alpha W^\beta}{2}\left(t_1^2 - M^2\right) + Q(t_1 - M) + \frac{W(t_2-t_1)}{2}\right][T - t_2]$$

The total profit for Case-6 is $X_6 = (p-c)(Q+B) + IE - c_o - chol - csho - ocls - IP - uW$. The total profit per unit of time of Case-6 is

$$Z_6(W,B,T) = \frac{(p-c)(Q+B) + IE - c_o - chol - csho - ocls - IP - uW}{T} \tag{32}$$

$$\left.\begin{array}{ll}\text{Problem-6} & \text{Max}\,TP_6(W,B,T) = Z_6(W,B,T) \\ \text{Subject to} & N < M \leq t_1 < T\end{array}\right\} \tag{33}$$

Case-7: When $N < t_1 \leq M < T$

The credit period facility (M) lies between $N < t_1 \leq M < T$. The retailer produces interest during the period $[N, M]$. Therefore, the interest earned is

$$\text{Interest earned (IE)} = pI_e\left[\frac{\alpha W^\beta}{m}\int_N^{t_1}\int_0^t (m-u)dudt + \frac{\alpha}{m}\int_{t_1}^M\int_0^t I(u)(m-u)dudt\right]$$

$$\text{Interest earned (IE)} = pI_e\left[\frac{\alpha W^\beta}{m}\left\{\frac{m}{2}\left(t_1^2 - N^2\right) - \frac{1}{6}\left(t_1^3 - N^3\right)\right\} + \frac{\alpha W^\beta}{2m}\left\{\frac{m}{2}\left(M^2 - t_1^2\right) - \frac{1}{6}\left(M^3 - t_1^3\right)\right\}\right].$$

The retailer must give the payment of interest paid of the period $[M, T]$, which is determined in the following way.

$$\text{Interest paid (IP)} = cI_p\left[\int_M^{t_2} I(t)dt\right][T - t_2] = cI_p\frac{W(t_2 - M)(T - t_2)}{2}$$

Total net profit for Case-7 is $X_7 = (p-c)(Q+B) + IE - c_o - chol - csho - ocls - IP - uW$. The total profit per unit of time of Case-7 is

$$Z_7(W, B, T) = \frac{(p-c)(Q+B) + IE - c_o - chol - csho - ocls - IP - uW}{T} \tag{34}$$

$$\left.\begin{array}{ll}\text{Problem-7} & \text{Max}\, TP_7(W, B, T) = Z_7(W, B, T) \\ \text{Subject to} & N < t_1 \leq M < T\end{array}\right\} \tag{35}$$

Case-8: When $t_1 \leq N < M < T$

The credit period facility (M) has its place in $t_1 \leq N < M < T$. In this interval, retailers have earned during the period $[N, M]$.

Therefore,

$$\text{Interest earned (IE)} = \frac{\alpha}{m}\int_N^M\int_0^t I(u)(m-u)dudt = pI_e\frac{\alpha W^\beta}{2m}\left[\frac{m}{2}\left(M^2 - N^2\right) - \frac{1}{6}\left(M^3 - N^3\right)\right]$$

The retailer needs to pay the interest paid for the time interval payment $[M, T]$. Thus,

$$\text{The interest paid (IP)} = cI_p\frac{W(t_2 - M)(T - t_2)}{2}$$

The total profit for Case-8 is $X_8 = (p-c)(Q+B) + IE - c_o - chol - csho - ocls - IP - uW$. The total profit per unit time of Case-8 is

$$Z_8(W, B, T) = \frac{(p-c)(Q+B) + IE - c_o - chol - csho - ocls - IP - uW}{T} \tag{36}$$

$$\left.\begin{array}{ll}\text{Problem-8} & \text{Max}\ \ TP_8(W, B, T) = Z_8(W, B, T) \\ \text{Subject to} & t_1 \leq N < M < T\end{array}\right\} \tag{37}$$

Case-9: $t_1 < T \leq N < M$

Here, the retailer gains interest in the period $[N, M]$. Therefore, the interest earned is

$$\text{Interest earned (IE)} = \frac{\alpha}{m}\int_N^T\int_0^t I(u)(m-u)dudt[M - T] = pI_e\frac{\alpha W^\beta}{2m}\left[\frac{m}{2}\left(t_2^2 - N^2\right) - \frac{1}{6}\left(t_2^3 - N^3\right)\right][M - t_2]$$

In this case, the interest paid (IP) = 0. The total profit for Case-9 is $X_9 = (p-c)(Q+B) + IE - c_o - chol - csho - ocls - uW$. The total profit per unit of time of Case-9 is

$$Z_9(W,B,T) = \frac{(p-c)(Q+B) + IE - c_o - chol - csho - ocls - uW}{T} \tag{38}$$

$$\left.\begin{array}{ll} \text{Problem-9} & \text{Max}\, TP_9(W,B,T) = Z_9(W,B,T) \\ \text{Subject to} & t_1 < N < T \leq M \end{array}\right\} \tag{39}$$

Case-10: $t_1 < T \leq N < M$

In this situation, the retailer earns interest during the time interval $[N, M]$. Therefore, the interest earned is obtained as follows: Interest earned (*IE*) = $pI_eQ(M-N)$ and the interest paid (*IP*) = 0. The total profit for Case-10 is $X_{10} = (p-c)(Q+B) + IE - c_o - chol - csho - ocls - uW$. The total profit per unit of time of Case-10 is

$$Z_{10}(W,B,T) = \frac{(p-c)(Q+B) + IE - c_o - chol - csho - ocls - uW}{T} \tag{40}$$

$$\left.\begin{array}{ll} \text{Problem-10} & \text{Max}\, TP_{10}(W,B,T) = Z_{10}(W,B,T) \\ \text{Subject to} & t_1 < T \leq N < M \end{array}\right\} \tag{41}$$

4. Solution Procedure

This section presents the solution procedure for solving the above-defined optimization problems related to the inventory model. The detailed mathematical derivations are shown in Supplementary A in the Supplementary material. First, the inventory model without ending zero inventory is discussed. For, without the ending zero inventory case, there are five optimization problems under the two-level trade credit policy. The concave fractional programming is applied in order to solve the optimization problems 1 to 5. According to Cambini and Martein (2009), it is well-know that for any real-valued function,

$$V_i(x) = \frac{y_i(x)}{g(x)} \tag{42}$$

Equation (42) is strictly pseudo-concave if $y_i(x)$ is differentiable, and is strictly concave, non-negative, and $g(x)$ is differentiable, positive, and convex. Let J_i be

$$J_i = (p-c)\frac{\partial^2 Q}{\partial T^2} + \frac{\partial^2 chol}{\partial T^2} + \frac{\partial^2 IP}{\partial T^2} - \frac{\partial^2 IE}{\partial T^2} \tag{43}$$

For any given values of W and B, by applying (42), if $J_i > 0$ then it is proved that the total profit functions $TP_i(W,B,T)$ in Equations (11), (13), (15), (17) and (19) are strictly pseudo concave with respect to T. Therefore, it is possible to determine a unique global optimal solution of T^* which maximizes the total profit function. Then, the following theorem is proposed.

Theorem 1. *If $J_i > 0$ for any given values of W and B then the total profit functions $TP_i(W,B,T)$ are strictly pseudo-concave with respect to T and there exists a unique optimal solution of T^* (for $i = 1, \ldots, 5$).*

Proof. Please see Supplementary B in the Supplementary material. □

For any given values of W and B, taking the first-order derivative of the profit function $TP_i(W,B,T)$ with respect to T and set it equal to zero, $\frac{\partial TP_i(W,B,T)}{\partial T} = 0$, then the necessary and sufficient conditions for the optimal cycle time T^* are obtained. These are shown below:

$$T\frac{\partial y_i(T)}{\partial T} = y_i(T) \tag{44}$$

Detailed calculations are supplied in Supplementary C in the Supplementary material. Hence, the following corollary is established.

Corollary 1. *For any given values of W and B both t_1 and Q are increasing and concave downward in T.*

Proof. Please see Supplementary D in the Supplementary material. □

It is easy to show that $L_i < 0$ and that $M_i < 0$. Then, the following theorem is stated below.

Theorem 2. *For any given value of T, if $L_i < 0$, $M_i < 0$ and $L_i M_i - K_i^2 > 0$, then equation (42) is strictly concave function with respect to both W and B and there exists a unique optimal solution for W^* and B^*.*

Proof. Please see Supplementary E in the Supplementary material. □

For any given value of T, and the first-order derivative of the profit function $TP_i(W, B, T)$ with respect to W and B and set these equal to zero, $\frac{\partial TP_i(W,B,T)}{\partial W} = 0$ and $\frac{\partial TP_i(W,B,T)}{\partial B} = 0$, then the necessary and sufficient conditions are obtained. These are given below.

$$(p-c)\frac{\partial Q}{\partial W} + \frac{\partial IE}{\partial W} - \frac{\partial chol}{\partial W} - \frac{\partial IP}{\partial W} - u = 0 \tag{45}$$

$$(p-c)\frac{\partial Q}{\partial B} + \frac{\partial IE}{\partial B} - \frac{\partial chol}{\partial B} - \frac{\partial IP}{\partial B} = 0 \tag{46}$$

For a detailed calculation see Supplementary A in the Supplementary material. Thus, the following corollary is proposed.

Corollary 2. *For any given value of T, (i) the t_1 and t_2 are increasing with respect to B and decreasing respect to W and (ii) Q is increasing and concave with respect to both W and B.*

Proof. This is clear due to the expressions of (D.1), (D.2), (D.6), (D.7), (D.12), and (D.13). Please see Supplementary D in the Supplementary material. □

For the shortage case, it is applied that the same technique is used to find the optimal solution of each decision variable. Let J_i set

$$J_i = (p-c)\frac{\partial^2 Q}{\partial T^2} + \frac{\partial^2 chol}{\partial T^2} + \frac{\partial^2 csho}{\partial T^2} + \frac{\partial^2 ocls}{\partial T^2} + \frac{\partial^2 IP}{\partial T^2} - \frac{\partial^2 IE}{\partial T^2} \tag{47}$$

For any given values of W and B and by applying (42), if $J_i > 0$ then it is shown that the profit function $TP_i(W, B, T)$ in Equations (33), (35), (37), (39) and (41) are strictly pseudo-concave with respect to T. As a result, there is a unique global optimal solution of T^* which maximizes the total profit function.

For any given values of W and B, taking the first-order derivative of the profit function $TP_i(W, B, T)$ with respect to T and set it equal to zero, $\frac{\partial TP_i(W,B,T)}{\partial T} = 0$, then the necessary and sufficient condition for the optimal cycle time T^* is determined and this is given below:

$$T\frac{\partial y_i(T)}{\partial T} = y_i(T) \tag{48}$$

The detailed calculations are supplied in Supplementary C in the Supplementary material.

For any given value of T, taking the first-order derivatives of the profit function $TP_i(W, B, T)$ with respect to W and B and set these equal to zero, $\frac{\partial TP_i(W,B,T)}{\partial W} = 0$ and $\frac{\partial TP_i(W,B,T)}{\partial B} = 0$, thus the necessary and sufficient conditions are determined. These are as follows:

$$(p-c)\frac{\partial Q}{\partial W} + \frac{\partial IE}{\partial W} - \frac{\partial chol}{\partial W} - \frac{\partial csho}{\partial W} - \frac{\partial ocls}{\partial W} - \frac{\partial IP}{\partial W} - u = 0 \quad (49)$$

$$(p-c)\frac{\partial Q}{\partial B} + \frac{\partial IE}{\partial B} - \frac{\partial chol}{\partial B} - \frac{\partial csho}{\partial B} - \frac{\partial ocls}{\partial B} - \frac{\partial IP}{\partial B} = 0 \quad (50)$$

By considering the theoretical results obtained before the following Algorithm 1 is developed.

Algorithm 1. Steps to determine the solution to the inventory model.

Step 1. Input all inventory parameters values.
Step 2. Select without ending zero inventory situation.
Step 3. Set the following initial values for $W = 1$, $B = 1$, and $j = 0$ into Equation (44) and state the accuracy as $\varepsilon = 10^{-5}$.
Step 4. Solve the Equation (44) to compute the value of T_j^*.
Step 5. Set $T = T_j^*$ and $B = 1$ into Equation (45) to calculate the value of W_j^*.
Step 6. Set $T = T_j^*$ and $W = W_j^*$ into Equation (46) to obtain the value of B_j^*.
Step 7. Iterate Equations (44), (45), and (46) until the accuracy is satisfied $\left|T_{j+1}^* - T_j\right| < \varepsilon$, $\left|W_{j+1}^* - W_j\right| < \varepsilon$, $\left|B_{j+1}^* - B_j\right| < \varepsilon$ and $\left|TP_{i,j+1}^*(.) - TP_{i,j}(.)\right| < \varepsilon$. If all the conditions are satisfied, then store $T_j = T_{j+1}^*$, $W_j = W_{j+1}^*$, $B_j = B_{j+1}^*$ and $TP_{i,j}(.) = TP_{i,j+1}^*(.)$. Otherwise, put $j = 1$ and repeat Step 4 to Step 6.
Step 8. Compare the value of the objective function to select the best profit $TP(W^*, B^*, T^*) = \underset{i=1,\ldots,5}{Max}\{TP_i(W, B, T)\}$.
Step 9. Select partial backlogged shortages situation.
Step 10. Set the initial values for $W = 1$, $B = 1$, $j = 0$ into Equation (48) and state the accuracy as $\varepsilon = 10^{-5}$.
Step 11. Solve the Equation (48) to calculate the value of T_j^*.
Step 12. Set $T = T_j^*$ and $B = 1$ into Equation (49) to obtain the value of W_j^*.
Step 13. Set $T = T_j^*$ and $W = W_j^*$ into Equation (50) to get the value of B_j^*.
Step 14. Iterate Equations (48), (49), and (50) until the accuracy is satisfied $\left|T_{j+1}^* - T_j\right| < \varepsilon$, $\left|W_{j+1}^* - W_j\right| < \varepsilon$, $\left|B_{j+1}^* - B_j\right| < \varepsilon$ and $\left|TP_{i,j+1}^*(.) - TP_{i,j}(.)\right| < \varepsilon$. If all the conditions are satisfied, then store $T_j = T_{j+1}^*$, $W_j = W_{j+1}^*$, $B_j = B_{j+1}^*$ and $TP_{i,j}(.) = TP_{i,j+1}^*(.)$. Otherwise, put $j = 1$ and repeat Step 11 to Step 13.
Step 15. Compare the value of the objective function to select the best profit $TP(W^*, B^*, T^*) = \underset{i=6,\ldots,10}{Max}\{TP_i(W, B, T)\}$.
Step 16. Report the solution and stop.

5. Numerical Examples

This section solves two numerical examples in order to illustrate and validate the proposed inventory model.

Example 1. *Without ending zero inventory model. The data are as follows:*

$$p = \$40/\text{ unit}, S = \$10/\text{unit}, c_o = \$10/\text{order}, \ c = \$20/\text{unit}, h = \$4/\text{unit/year}, \ u = \$5/\text{unit},$$
$$m = 0.4\,\text{years}, M = 30/365\,\text{years}, N = 15/365\,\text{years}, I_e = 7\%/\text{year}, I_p = 12\%/\text{year}, \alpha = 50, \ \beta = 0.7$$

Table 2 presents the optimal solution to the Example 1 for the situation of without ending zero inventory ($B > 0$). Table 3 shows the optimal solution to the Example 1 when the ending inventory level is zero ($B = 0$). The bold denotes the best solution.

Table 2. The optimal solution for the Example 1 without ending zero inventory ($B > 0$).

Q	B	W	t_1	T	Total Profit	Case
2590.934	**964.5861**	**1873.553**	**0.08180**	**0.3057132**	**35357.71**	**1**
2591.885	965.3788	1871.911	0.08219178	0.3057701	35357.59	2
2414.413	850.9161	2013.839	0.04109	0.3011034	33605.92	3
145.3784	58.75093	106.8986	0.030395	0.08219	6911.664	4
18.23070	7.478258	13.44068	0.015854	0.041095	1514.526	5

Table 3. The optimal solution for the Example 1 when the ending inventory level is zero ($B = 0$).

Q	B	W	t_1	T	Total Profit	Case
30.73261	0.0000	16.72789	0.041095	0.3318969	1521.353	1
33.28520	0.0000	23.52837	0.021995	0.3269761	1576.334	2
33.29935	**0.0000**	**23.58922**	**0.021846**	**0.3269347**	**1577.467**	**3**
1.351514	0.0000	0.9899749	0.0073494	0.082191	144.7568	4
0.1615969	0.0000	0.1189071	0.0038087	0.041095	−178.9718	5

Example 2. *The shortage case ($B < 0$). The input values for the parameters are given below.*

$$p = \$30/\text{ unit}, \; c_o = \$10/\text{order}, c = \$20/\text{unit}, h = \$4/\text{unit/year}, c_b = \$22/\text{unit/year},$$
$$c_l = \$25/\text{unit/year}, u = \$5/\text{unit}, \delta = 0.8, m = 0.2\,\text{years}, \; M = 30/365\,\text{years}, N = 15/365\,\text{years},$$
$$I_e = 7\%/\text{year}, I_p = 12\%/\text{year}, \alpha = 150, \; \beta = 0.9$$

Table 4 displays the optimal solution for the Example 2 for the situation of shortages ($B < 0$).

Table 4. The optimal solution for the Example 2 when shortages are allowed ($B < 0$).

Q	B	W	t_1	t_2	T	Total Profit	Case
3.924188	**0.5730634**	**0.5730634**	**0.04109**	**0.1947118**	**0.1994873**	**141.2968**	**6**
3.924255	0.5730742	0.5730742	0.041095	0.1947162	0.1994919	140.8073	7
1.563487	0.6658734	0.6658734	0.0312564	0.0901456	0.0942564	−5.830897	8
0.8069315	0.7086727	0.7086727	0.0008951	0.082191	0.088097	−33.83189	9
0.002679	0.0026796	0.0026796	0.000006	0.041095	0.0411182	−223.2291	10

6. Sensitivity Analysis

This section performs a sensitivity analysis taking into consideration Example 1. The sensitivity analysis is used to investigate the impacts of a variation in the values of the input parameters on the optimal values for the time of the displayed units (*t1*), the cycle length (T), ending inventory level (B), the initial stock level (Q), shelf space (W), and the maximum total profit of the inventory system. The percentage changes if the optimal values are considered as measures of sensitivity. The parameters are changed (increased and decreased) by −20% to +20%. The results of the sensitivity analysis for Example 1 are presented in Table 5.

Table 5. Sensitivity analysis for Example 1.

Parameter	% Change of Parameter	% Change in					
		Total Profit	Q	B	W	t_1	T
α	−20	−52.52	−52.46	−52.46	−52.46	0.01	0.03
	−10	−29.64	−29.61	−29.61	−29.61	0.01	0.01
	10	37.43	37.39	37.39	37.39	0.00	−0.01
	20	83.70	83.61	83.61	83.61	−0.01	−0.01

Table 5. *Cont.*

Parameter	% Change of Parameter	% Change in					
		Total Profit	Q	B	W	t_1	T
β	−20	−89.24	−94.15	−95.29	−94.50	−1.60	−12.35
	−10	−74.25	−81.29	−83.02	−81.80	−0.66	−6.01
	10	175.79	185.74	195.19	136.66	0.38	5.62
	20	–	–	–	–	–	–
c	−20	179.27	211.44	329.40	227.76	19.80	−1.61
	−10	64.41	71.22	102.69	75.99	7.84	−0.66
	10	−38.14	−39.06	−49.55	−41.01	−4.98	0.46
	20	−61.61	−61.76	−74.46	−64.41	−7.75	0.77
h	−20	3.80	5.43	6.94	5.69	0.88	0.87
	−10	1.87	2.66	3.39	2.79	0.43	0.43
	10	−1.82	−2.57	−3.25	−2.68	−0.43	−0.43
	20	−3.59	−5.04	−6.36	−5.27	−0.85	−0.85
m	−20	−36.98	−48.24	−47.05	−48.04	−18.85	−18.85
	−10	−19.47	−26.58	−25.74	−26.44	−9.36	−9.36
	10	21.34	31.82	30.35	31.57	9.24	9.24
	20	44.46	69.14	65.42	68.50	18.36	18.36
c_o	−20	0.02	0.00	0.00	0.00	0.00	−0.01
	−10	0.01	0.00	0.00	0.00	0.00	0.00
	10	−0.01	0.00	0.00	0.00	0.00	0.00
	20	−0.02	0.00	0.00	0.00	0.00	0.01
p	−20	−78.15	−71.21	−82.64	−75.78	11.28	−0.26
	−10	−48.75	−42.83	−53.21	−46.59	4.43	−0.09
	10	73.64	60.69	85.17	68.56	−3.11	0.04
	20	178.04	142.81	211.04	163.91	−5.42	0.07
S	−20	−15.18	−20.4	−27.23	−20.49	−7.01	0.75
	−10	−8.21	−11.19	−15.14	−11.24	−3.72	0.40
	10	9.77	13.76	19.25	13.83	4.23	−0.45
	20	21.56	30.97	44.14	31.14	9.09	−0.97
u	−20	19.72	21.94	28.90	28.55	−13.51	−0.51
	−10	9.23	10.21	13.26	13.11	−6.57	−0.25
	10	−8.16	−8.92	−11.32	−11.21	6.22	0.23
	20	−15.4	−16.76	−21.03	−20.84	12.14	0.44

From Table 5 the following observations are obtained:

1. The total profit is highly sensitive to all parameters except to the ordering cost parameter c_o and it is infeasible when the value of β is +20%.
2. The initial stock level (Q) is highly sensitive to all parameters except to the ordering cost parameter c_o and it is infeasible when the value of β is +20%.
3. The ending inventory level (B) is highly sensitive to all parameters except to the ordering cost parameter c_o and it is infeasible when the value of β is +20%.
4. The shelf-space (W) is highly sensitive to all parameters except the ordering cost parameter c_o and it is infeasible the value of β is +20%.
5. The time of the displayed units (t_1) is insensible to the parameters α and c_o; and it is moderately sensitive to the parameter h, and highly sensitive to the rest parameters, and it is infeasible when the value of β is +20%.
6. The cycle length (T) is insensible to the parameters α and c_o. It is moderately sensitive to the parameters h, p, s, and u. It is highly sensible to the rest parameters, and it is infeasible the value of β is +20%.

7. Conclusions

This research work develops a two-level trade credit policy inventory model considering expiration rate and stock-dependent demand under a non-zero inventory and partial backlogged shortages. In fact, this research work extends Wu et al. [55]'s inventory model by considering partial backlogged shortage and a two-level trade credit policy.

For future research, academicians and researchers can explore an extention of the proposed inventory model by taking into account some characteristics such as inflation, deterioration, price and stock dependent demand and nonlinear holding cost, among other things. The inventory model may also be extended to a two-warehouse inventory enviroment. Additionally, it can be also considered that the inventory parameters are fuzzy valued and interval-valued. Another line to investigate is to propose an inventory model with a partial trade credit policy, as well as a credit risk customer. These are some interesting and challenging research ideas that academicians and researchers can perform in the near future.

Supplementary Materials: The following are available online at https://www.mdpi.com/article/10.3390/su132313493/s1, Supplementary A: Detailed calculations, Supplementary B: Proof of Theorem 1, Supplementary C: Finding the optimal solution of the cycle length, Supplementary D: Proof of Corollary 1, Supplementary E: Proof of Theorem 2.

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Notation

Parameter	**Description**
p	Selling price per unit ($/unit)
S	Salvage price per unit ($/unit)
c_o	Ordering cost per order ($/order)
c	Purchase cost per unit ($/unit)
h	Holding cost per unit per time unit ($/unit/time unit)
c_b	Shortage cost per unit per time unit ($/unit/time unit)
c_l	Unit opportunity cost due to lost sale per unit per time unit ($/unit/time unit)
u	Shelf space cost ($/unit)
δ	Backlogging parameter
m	The expiration time of the product (time unit)
M	Credit period offered by the supplier to his/her retailer (time unit)
N	Credit period offered by the retailer to the customer (time unit)
I_e	Interest earned by the retailer (%/time unit)
I_p	Interest charged by the suppliers to the retailers (%/time unit)
$I(t)$	Inventory level at time t (units)
$TP_i^{(.)}$	The total profit ($/time unit)
Dependent decision variables	
t_1	Time of the displayed units (time unit)
t_2	Time when inventory level reaches zero (time unit)
Q	The stock level at the beginning of the period (units)
Decision variables	
T	Cycle length (time unit)
W	The number of units displayed in shelf space (units).
B	Ending inventory level at time T (units)

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