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Supplementary material for the paper:

Two Level Trade Credit Policy Approach in Inventory Model with Expiration Rate and Stock Dependent Demand under Nonzero Inventory and Partial Backlogged Shortages

By Ali Akbar Shaikh, Leopoldo Eduardo Cárdenas-Barrón, Amalesh Kumar Manna, Armando Céspedes-Mota and Gerardo Treviño-Garza

Supplementary A Detailed calculations

Equation (7) is

$$\frac{1}{2m}\alpha W^{\beta} t_1^2 - \alpha W^{\beta} t_1 + Q = \left[\frac{\alpha(1-\beta)}{2m}\left\{t_1^2 + 2m(T-t_1) - T^2\right\} + B^{1-\beta}\right]^{\frac{1}{1-\beta}} \tag{A.1}$$

By using equation (6), it is obtained

$$W = \left[\frac{\alpha(1-\beta)}{2m}\left\{t_1^2 + 2m(T-t_1) - T^2\right\} + B^{1-\beta}\right]^{\frac{1}{1-\beta}} \tag{A.2}$$

From equation (A.2), it is determined the value of t_1 is given by

$$t_1 = m \pm \sqrt{(m-T)^2 + \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}} \geq 0 \tag{A.3}$$

Here, it is considered that the value of t_1 is given in the following form

$$t_1 = m - \sqrt{(m-T)^2 + \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}} \geq 0 \tag{A.4}$$

The initial stock Q is

$$Q = W + \frac{\alpha W^{\beta}}{2m}\left[m^2 - (m-T)^2 - \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}\right] \tag{A.5}$$

The first partial derivative of equation (A.5) with respect to W is

$$\frac{\partial Q}{\partial W} = \frac{\alpha\beta W^{\beta-1}}{2m}\left[m^2 - (m-T)^2 - \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}\right]$$

$$\frac{\partial Q}{\partial W} = \frac{\beta}{W}\frac{\alpha W^{\beta}}{2m}\left[m^2 - (m-T)^2 - \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}\right]$$

$$\frac{\partial Q}{\partial W} = \frac{\beta(Q-W)}{W} > 0$$

(A.6)

The first partial derivative of equation (A.5) with respect to B is

$$\frac{\partial Q}{\partial B}=\left(\frac{W}{B}\right)^{\beta}>0 \tag{A.7}$$

The second partial derivative of equation (A.5) with respect to W is

$$\frac{\partial^2 Q}{\partial W^2}=\frac{-\beta W\left(\frac{\beta(Q-W)}{W}-1\right)-\beta(Q-W)}{W^2}<0 \tag{A.8}$$

The second partial derivative of equation (A.5) with respect to B is

$$\frac{\partial^2 Q}{\partial B^2}=\frac{-\beta W^{\beta}}{B^{1+\beta}}<0 \tag{A.9}$$

The cross partial derivative of equation (A.7) with respect to W is

$$\frac{\partial^2 Q}{\partial B \partial W}=\frac{\beta W^{\beta-1}}{B^{\beta}}>0$$

(A.10)

The first partial derivative of the holding cost with respect to B is

$$\frac{\partial chol}{\partial B}=h\left(\frac{\partial chol_1}{\partial B}+\frac{\partial chol_2}{\partial B}\right) \tag{A.11}$$

The second partial derivative of the holding cost with respect to B *is*

$$\frac{\partial^2 chol}{\partial B^2}=h\left(\frac{\partial^2 chol_1}{\partial B^2}+\frac{\partial^2 chol_2}{\partial B^2}\right) \tag{A.12}$$

The expressions of the first and second derivatives for $chol_1$ and $chol_2$ with respect to B are given below.

$$\frac{\partial chol_1}{\partial B}=\frac{\partial Q}{\partial B}t_1+Q\frac{\partial t_1}{\partial B}+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial t_1}{\partial B}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial B}$$

$$\frac{\partial chol_2}{\partial B}=\frac{(T-t_1)}{2}-\frac{(W+B)}{2}\frac{\partial t_1}{\partial B}$$

$$\frac{\partial^2 chol_1}{\partial B^2}=\frac{\partial^2 Q}{\partial B^2}t_1+2\frac{\partial Q}{\partial B}\frac{\partial t_1}{\partial B}+Q\frac{\partial^2 t_1}{\partial B^2}+\frac{\alpha W^{\beta}t_1}{m}\left(\frac{\partial t_1}{\partial B}\right)^2+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial^2 t_1}{\partial B^2}-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial B}\right)^2-\alpha W^{\beta}\frac{\partial^2 t_1}{\partial B^2}$$

$$\frac{\partial^2 chol_2}{\partial B^2}=-\frac{1}{2}\frac{\partial t_1}{\partial B}-\frac{1}{2}\frac{\partial t_1}{\partial B}-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial B^2}$$

$$\frac{\partial^2 chol_2}{\partial B^2}=-\frac{\partial t_1}{\partial B}-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial B^2}$$

The first partial derivative of the holding cost with respect to W is

$$\frac{\partial chol}{\partial W}=h\left(\frac{\partial chol_1}{\partial W}+\frac{\partial chol_2}{\partial W}\right) \tag{A.13}$$

The second partial derivative of the holding cost with respect to W is

$$\frac{\partial^2 chol}{\partial W^2}=h\left(\frac{\partial^2 chol_1}{\partial W^2}+\frac{\partial^2 chol_2}{\partial W^2}\right) \tag{A.14}$$

The expressions of the first and second derivatives for $chol_1$ and $chol_2$ with respect to W are given bellow

$$\frac{\partial chol_1}{\partial W}=\frac{\partial Q}{\partial W}t_1+Q\frac{\partial t_1}{\partial W}+\frac{\alpha\beta W^{\beta-1}t_1^3}{6m}+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial t_1}{\partial W}-\frac{\alpha\beta W^{\beta-1}t_1^2}{2}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial W}$$

$$\frac{\partial chol_2}{\partial W}=\frac{(T-t_1)}{2}-\frac{(W+B)}{2}\frac{\partial t_1}{\partial W}$$

$$\begin{aligned}\frac{\partial^2 chol_1}{\partial W^2}&=\frac{\partial^2 Q}{\partial W^2}t_1+2\frac{\partial Q}{\partial W}\frac{\partial t_1}{\partial W}+Q\frac{\partial^2 t_1}{\partial W^2}+\frac{\alpha\beta(\beta-1)W^{\beta-2}t_1^3}{6m}+\frac{\alpha\beta W^{\beta-1}t_1^2}{2m}\frac{\partial t_1}{\partial W}+\frac{\alpha\beta W^{\beta-1}t_1^2}{2m}\frac{\partial t_1}{\partial W}\\&+\frac{\alpha W^{\beta}t_1}{m}\left(\frac{\partial t_1}{\partial W}\right)^2+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial^2 t_1}{\partial W^2}-\frac{\alpha\beta(\beta-1)W^{\beta-2}t_1^2}{2}-\alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W}-\alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W}\\&-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial W}\right)^2-\alpha W^{\beta}t_1\frac{\partial^2 t_1}{\partial W^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 chol_2}{\partial W^2}&=-\frac{1}{2}\frac{\partial t_1}{\partial W}-\frac{1}{2}\frac{\partial t_1}{\partial W}-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial W^2}\\&=-\frac{\partial t_1}{\partial W}-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial W^2}\end{aligned}$$

The first partial derivative of the holding cost with respect to T is

$$\frac{\partial chol}{\partial T}=h\left(\frac{\partial chol_1}{\partial T}+\frac{\partial chol_2}{\partial T}\right) \tag{A.15}$$

The second partial derivative of the holding cost respect to T is

$$\frac{\partial^2 chol}{\partial T^2}=h\left(\frac{\partial^2 chol_1}{\partial T^2}+\frac{\partial^2 chol_2}{\partial T^2}\right) \tag{A.16}$$

The expressions of the first and second derivatives for $chol_1$ and $chol_2$ with respect to T are given bellow

$$\frac{\partial chol_1}{\partial T}=\frac{\partial Q}{\partial T}t_1+Q\frac{\partial t_1}{\partial T}+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial t_1}{\partial T}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial T}$$

$$\frac{\partial chol_2}{\partial T}=\frac{(W+B)}{2}\left\{1-\frac{\partial t_1}{\partial T}\right\}$$

$$\frac{\partial^2 chol_1}{\partial T^2}=\frac{\partial^2 Q}{\partial T^2}t_1+2\frac{\partial Q}{\partial T}\frac{\partial t_1}{\partial T}+Q\frac{\partial^2 t_1}{\partial T^2}+\frac{\alpha W^{\beta}t_1}{m}\left(\frac{\partial t_1}{\partial T}\right)^2+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial^2 t_1}{\partial T^2}-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial T}\right)^2-\alpha W^{\beta}\frac{\partial^2 t_1}{\partial T^2}$$

$$\frac{\partial^2 chol_2}{\partial T^2}=-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial T^2}$$

Interest earned for Case 1

The first partial derivative of the interest earned for Case 1 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e\frac{\alpha\beta W^{\beta-1}}{m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{\left(M^3-N^3\right)}{6}\right]$$

(A.17)

The second partial derivative of interest earned for Case 1 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e\frac{\alpha\beta(\beta-1)W^{\beta-2}}{m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{\left(M^3-N^3\right)}{6}\right] \tag{A.18}$$

The first partial derivative of the interest earned for Case 1 with respect to B is

$$\frac{\partial IE}{\partial B}=0 \tag{A.19}$$

So, the second partial derivative is also zero.
The first partial derivative of the interest earned for Case 1 with respect T is

$$\frac{\partial IE}{\partial T} = 0 \tag{A.20}$$

So, the second partial derivative is also zero.

Interest earned for Case 2
The first partial derivative of the interest earne for Case 2 with respect to W is

$$\frac{\partial IE}{\partial W} = pI_e \left[\begin{array}{l} \frac{\alpha\beta W^{\beta-1}}{m}\left\{\frac{m}{2}\left(t_1^2 - N^2\right) - \frac{1}{6}\left(t_1^3 - N^3\right)\right\} + \frac{\alpha W^{\beta}}{m}\left\{mt_1 \frac{\partial t_1}{\partial W} - \frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\} \\ + \frac{\alpha\beta(W+B)^{\beta-1}}{2m}\left\{\frac{m}{2}\left(M^2 - t_1^2\right) - \frac{1}{6}\left(M^3 - t_1^3\right)\right\} + \frac{\alpha(W+B)^{\beta}}{2m}\left\{-mt_1\frac{\partial t_1}{\partial W} + \frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\} \end{array} \right] \tag{A.21}$$

The second partial derivative of interest earned for Case 2 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2} = pI_e \left[\begin{array}{l} \frac{\alpha\beta(\beta-1)W^{\beta-2}}{m}\left\{\frac{m}{2}\left(t_1^2 - N^2\right) - \frac{1}{6}\left(t_1^3 - N^3\right)\right\} + 2\frac{\alpha\beta W^{\beta-1}}{m}\left\{mt_1\frac{\partial t_1}{\partial W} - \frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\} \\ + \frac{\alpha W^{\beta}}{m}\left\{m\left(\frac{\partial t_1}{\partial W}\right)^2 + mt_1\frac{\partial^2 t_1}{\partial W^2} - t_1\left(\frac{\partial t_1}{\partial W}\right)^2 - \frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial W^2}\right\} \\ + \frac{\alpha\beta(\beta-1)(W+B)^{\beta-2}}{2m}\left\{\frac{m}{2}\left(M^2 - t_1^2\right) - \frac{1}{6}\left(M^3 - t_1^3\right)\right\} + \frac{\alpha\beta(W+B)^{\beta-1}}{m}\left\{-mt_1\frac{\partial t_1}{\partial W} + \frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\} \\ \frac{\alpha(W+B)^{\beta}}{2m}\left\{-m\left(\frac{\partial t_1}{\partial W}\right)^2 - mt_1\frac{\partial^2 t_1}{\partial W^2} + t_1\left(\frac{\partial t_1}{\partial W}\right)^2 + \frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial W^2}\right\} \end{array} \right] \tag{A.22}$$

The first partial derivative of the interest earned for Case 2 with respect to B is

$$\frac{\partial IE}{\partial B} = pI_e \left[\begin{array}{l} \frac{\alpha W^{\beta}}{m}\left\{mt_1\frac{\partial t_1}{\partial B} - \frac{t_1^2}{2}\frac{\partial t_1}{\partial B}\right\} + \frac{\alpha\beta(W+B)^{\beta-1}}{2m}\left\{\frac{m}{2}\left(M^2 - t_1^2\right) - \frac{1}{6}\left(M^3 - t_1^3\right)\right\} \\ + \frac{\alpha(W+B)^{\beta}}{2m}\left\{-mt_1\frac{\partial t_1}{\partial B} + \frac{t_1^2}{2}\frac{\partial t_1}{\partial B}\right\} \end{array} \right] \tag{A.23}$$

The second partial derivative of the interest earned for Case 2 with of respect to B is

$$\frac{\partial^2 IE}{\partial B^2} = pI_e \left[\begin{array}{l} \frac{\alpha W^{\beta}}{m}\left\{m\left(\frac{\partial t_1}{\partial B}\right)^2 + mt_1\frac{\partial^2 t_1}{\partial B^2} - t_1\left(\frac{\partial t_1}{\partial B}\right)^2 - \frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial B^2}\right\} + \frac{\alpha\beta(\beta-1)(W+B)^{\beta-2}}{2m}\left\{\frac{m}{2}\left(M^2 - t_1^2\right) - \frac{1}{6}\left(M^3 - t_1^3\right)\right\} \\ + \frac{\alpha\beta(W+B)^{\beta-1}}{m}\left\{-mt_1\frac{\partial t_1}{\partial B} + \frac{t_1^2}{2}\frac{\partial t_1}{\partial B}\right\} + \frac{\alpha(W+B)^{\beta}}{2m}\left\{-m\left(\frac{\partial t_1}{\partial B}\right)^2 - mt_1\frac{\partial^2 t_1}{\partial B^2} + t_1\left(\frac{\partial t_1}{\partial B}\right)^2 + \frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial B^2}\right\} \end{array} \right] \tag{A.24}$$

The first partial derivative of the interest earned for Case 2 with respect to T is

$$\frac{\partial IE}{\partial T} = pI_e \left[\frac{\alpha W^{\beta}}{m}\left\{mt_1\frac{\partial t_1}{\partial T} - \frac{t_1^2}{2}\frac{\partial t_1}{\partial T}\right\} + \frac{\alpha(W+B)^{\beta}}{2m}\left\{-mt_1\frac{\partial t_1}{\partial T} + \frac{t_1^2}{2}\frac{\partial t_1}{\partial T}\right\} \right]$$

(A.25)

The partial derivative of equation (A.25) with respect to T is

$$\frac{\partial^2 IE}{\partial T^2} = pI_e \left[\begin{array}{l} \frac{\alpha W^{\beta}}{m}\left\{m\left(\frac{\partial t_1}{\partial T}\right)^2 + mt_1\frac{\partial^2 t_1}{\partial T^2} - t_1\left(\frac{\partial t_1}{\partial T}\right)^2 - \frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial T^2}\right\} \\ + \frac{\alpha(W+B)^{\beta}}{2m}\left\{-m\left(\frac{\partial t_1}{\partial T}\right)^2 - mt_1\frac{\partial^2 t_1}{\partial T^2} + t_1\left(\frac{\partial t_1}{\partial T}\right)^2 + \frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial T^2}\right\} \end{array} \right] \tag{A.26}$$

Interest earned for Case 3

The first partial derivative of the interest earned for Case 3 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e\frac{\alpha\beta(W+B)^{\beta-1}}{2m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{1}{6}\left(M^3-N^3\right)\right] \quad (A.27)$$

The second partial derivative of interest earned for Case 3 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e\frac{\alpha\beta(\beta-1)(W+B)^{\beta-2}}{2m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{1}{6}\left(M^3-N^3\right)\right] \quad (A.28)$$

The first partial derivative of the interest earned for Case 3 with respect to B is

$$\frac{\partial IE}{\partial B}=pI_e\frac{\alpha\beta(W+B)^{\beta-1}}{2m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{1}{6}\left(M^3-N^3\right)\right] \quad (A.29)$$

The second partial derivative of interest earned for Case 3 with respect to B is

$$\frac{\partial^2 IE}{\partial B^2}=pI_e\frac{\alpha\beta(\beta-1)(W+B)^{\beta-2}}{2m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{1}{6}\left(M^3-N^3\right)\right] \quad (A.30)$$

The first partial derivative of the interest earned for Case 3 with respect to T is

$$\frac{\partial IE}{\partial T}=0 \quad (A.31)$$

So, the second partial derivative of interest earned is zero.

Interest earned of Case 4

The first partial derivative of the interest earned for Case 4 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e\frac{\alpha\beta(W+B)^{\beta-1}}{2m}\left[\frac{m}{2}\left(T^2-N^2\right)-\frac{1}{6}\left(T^3-N^3\right)\right][M-T]$$

(A.32)

The second partial derivative of the interest earned for Case 4 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e\frac{\alpha\beta(\beta-1)(W+B)^{\beta-2}}{2m}\left[\frac{m}{2}\left(T^2-N^2\right)-\frac{1}{6}\left(T^3-N^3\right)\right][M-T] \quad (A.33)$$

The first partial derivative of the interest earned for Case 4 with respect to B is

$$\frac{\partial IE}{\partial B}=pI_e\frac{\alpha\beta(W+B)^{\beta-1}}{2m}\left[\frac{m}{2}\left(T^2-N^2\right)-\frac{1}{6}\left(T^3-N^3\right)\right][M-T] \quad (A.34)$$

The second partial derivative of the interest earned for Case 4 with respect to B is

$$\frac{\partial^2 IE}{\partial B^2}=pI_e\frac{\alpha\beta(\beta-1)(W+B)^{\beta-2}}{2m}\left[\frac{m}{2}\left(T^2-N^2\right)-\frac{1}{6}\left(T^3-N^3\right)\right][M-T] \quad (A.35)$$

The first partial derivative of the interest earned for Case 4 with respect to T is

$$\frac{\partial IE}{\partial T}=pI_e\frac{\alpha(W+B)^{\beta}}{2m}\left\{mT-\frac{T^2}{2}\right\}[M-T]-pI_e\frac{\alpha(W+B)^{\beta}}{2m}\left[\frac{m}{2}\left(T^2-N^2\right)-\frac{1}{6}\left(T^3-N^3\right)\right] \quad (A.36)$$

The second partial derivative of the interest earned for Case 4 with respect to B is

$$\frac{\partial^2 IE}{\partial T^2}=pI_e\frac{\alpha(W+B)^{\beta}(m-T)}{2m}[M-T]-2pI_e\frac{\alpha(W+B)^{\beta}}{2m}\left\{mT-\frac{T^2}{2}\right\} \quad (A.37)$$

Interest earned of Case 5

The first partial derivative of the interest earned for Case 5 with respect to *W* is

$$\frac{\partial IE}{\partial W} = pI_e(M-N)\frac{\partial Q}{\partial W} \tag{A.38}$$

The second partial derivative of the interest earned for Case 5 with respect to *W* is

$$\frac{\partial^2 IE}{\partial W^2} = pI_e(M-N)\frac{\partial^2 Q}{\partial W^2}$$

(A.39)

The first partial derivative of the interest earned for Case 5 with respect to *B* is

$$\frac{\partial IE}{\partial B} = pI_e(M-N)\left\{\frac{\partial Q}{\partial B} - 1\right\} \tag{A.40}$$

The second partial derivative of the interest earned for Case 5 with respect to *B* is

$$\frac{\partial^2 IE}{\partial B^2} = pI_e(M-N)\frac{\partial^2 Q}{\partial B^2} \tag{A.41}$$

The first partial derivative of the interest earned for Case 5 with respect to *T* is

$$\frac{\partial IE}{\partial T} = pI_e(M-N)\frac{\partial Q}{\partial T} \tag{A.42}$$

The partial derivative of the interest earned for Case 5 with respect to *B* is

$$\frac{\partial^2 IE}{\partial T^2} = pI_e(M-N)\frac{\partial^2 Q}{\partial T^2} \tag{A.43}$$

Interest paid of Case 1

The first partial derivative of the interest paid for Case 1 with respect to *W* is

$$\frac{\partial IP}{\partial W} = cI_p\left[\begin{array}{l}\frac{\alpha\beta W^{\beta-1}}{6m}\left(t_1^3 - M^3\right) + \frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial t_1}{\partial W} - \frac{\alpha\beta W^{\beta-1}}{2}\left(t_1^2 - M^2\right) - \alpha W^{\beta}t_1\frac{\partial t_1}{\partial W} \\ + \frac{\partial Q}{\partial W}(t_1 - M) + Q\frac{\partial t_1}{\partial W} + \frac{(T-t_1)}{2} - \frac{(W+B)}{2}\frac{\partial t_1}{\partial W}\end{array}\right] \tag{A.44}$$

The second partial derivative of interest paid for Case 1 with respect to *W* is

$$\frac{\partial^2 IP}{\partial W^2} = cI_p\left[\begin{array}{l}\frac{\alpha\beta(\beta-1)W^{\beta-2}}{6m}\left(t_1^3 - M^3\right) + \frac{\alpha\beta W^{\beta-1}}{6m}t_1^2\frac{\partial t_1}{\partial W} + \frac{\alpha\beta W^{\beta-1}}{2m}t_1^2\frac{\partial t_1}{\partial W} + \frac{\alpha W^{\beta}}{m}t_1\left(\frac{\partial t_1}{\partial W}\right)^2 \\ + \frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial^2 t_1}{\partial W^2} - \frac{\alpha\beta(\beta-1)W^{\beta-2}}{2}\left(t_1^2 - M^2\right) - \alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W} - \alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W} \\ -\alpha W^{\beta}\left(\frac{\partial t_1}{\partial W}\right)^2 - \alpha W^{\beta}t_1\frac{\partial^2 t_1}{\partial W^2} + \frac{\partial^2 Q}{\partial W^2}(t_1 - M) + 2\frac{\partial Q}{\partial W}\frac{\partial t_1}{\partial W} + Q\frac{\partial^2 t_1}{\partial W^2} - \frac{\partial t_1}{\partial W} - \frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial W^2}\end{array}\right] \tag{A.45}$$

The first partial derivative of the interest paid for Case 1 with respect to *B* is

$$\frac{\partial IP}{\partial B} = cI_p\left[\frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial t_1}{\partial B} - \alpha W^{\beta}t_1\frac{\partial t_1}{\partial B} + \frac{\partial Q}{\partial B}(t_1 - M) + Q\frac{\partial t_1}{\partial B} + \frac{(T-t_1)}{2} - \frac{(W+B)}{2}\frac{\partial t_1}{\partial B}\right] \tag{A.46}$$

The second partial derivative of interest paid for Case 1 with respect to *B* is

$$\frac{\partial^2 IP}{\partial B^2}=cI_p\left[\begin{array}{l}\frac{\alpha W^{\beta}}{m}t_1\frac{\partial t_1}{\partial B}+\frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial^2 t_1}{\partial B^2}-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial B}\right)^2-\alpha W^{\beta}t_1\frac{\partial^2 t_1}{\partial B^2}+\frac{\partial^2 Q}{\partial B^2}(t_1-M)\\+2\frac{\partial Q}{\partial B}\frac{\partial t_1}{\partial B}+Q\frac{\partial^2 t_1}{\partial B^2}-\frac{\partial t_1}{\partial B}-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial B^2}\end{array}\right] \tag{A.47}$$

The first partial derivative of the interest paid for Case 1 with respect to T is

$$\frac{\partial IP}{\partial T}=cI_p\left[\frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial t_1}{\partial T}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial T}+\frac{\partial Q}{\partial T}(t_1-M)+Q\frac{\partial t_1}{\partial T}+\frac{(W+B)}{2}\left\{1-\frac{\partial t_1}{\partial T}\right\}\right] \tag{A.48}$$

The second partial derivative of interest paid for Case 1 with respect to T is

$$\frac{\partial^2 IP}{\partial T^2}=cI_p\left[\begin{array}{l}\frac{\alpha W^{\beta}}{m}t_1\frac{\partial t_1}{\partial T}+\frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial^2 t_1}{\partial T^2}-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial T}\right)^2-\alpha W^{\beta}t_1\frac{\partial^2 t_1}{\partial T^2}+\frac{\partial^2 Q}{\partial T^2}(t_1-M)\\+2\frac{\partial Q}{\partial T}\frac{\partial t_1}{\partial T}+Q\frac{\partial^2 t_1}{\partial T^2}-\frac{(W+B)}{2}\frac{\partial^2 t_1}{\partial T^2}\end{array}\right] \tag{A.49}$$

Interest paid for Case 2

The first partial derivative of the interest paid for Case 2 with respect to W is

$$\frac{\partial IP}{\partial W}=cI_p\frac{(T-M)}{2} \tag{A.50}$$

The second partial derivative of the interest paid for Case 2 with respect to W is

$$\frac{\partial^2 IP}{\partial W^2}=0 \tag{A.51}$$

The first partial derivative of the interest paid for Case 2 with respect to B is

$$\frac{\partial IP}{\partial B}=cI_p\frac{(T-M)}{2} \tag{A.52}$$

The second partial derivative of the interest paid for Case 2 with respect to B is

$$\frac{\partial^2 IP}{\partial B^2}=0 \tag{A.53}$$

The first partial derivative of the interest paid for Case 2 with respect to T is

$$\frac{\partial IP}{\partial T}=cI_p\frac{(W+B)}{2} \tag{A.54}$$

The partial derivative of the interest paid for Case 2 with respect to T is

$$\frac{\partial^2 IP}{\partial T^2}=0 \tag{A.55}$$

Interest paid of Case 3

For the Case 3, all the expressions are similar to Case 2.

Interest paid of Case 4 and Case 5.

There is not interest paid**.**

For shortage ($B<0$)

$$Q = W + \frac{\alpha W^{\beta}}{2m}\left(2mt_1 - t_1^2\right) \geq W$$

Using the continuity of equations (24) and (25), it is determined

$$\frac{1}{2m}\alpha W^{\beta} t_1^2 - \alpha W^{\beta} t_1 + Q = \left[\frac{\alpha(1-\beta)}{2m}\left\{t_1^2 + 2m(t_2 - t_1) - t_2^2\right\}\right]^{\frac{1}{1-\beta}} \tag{A.56}$$

By using equation (27), it is obtained

$$W = \left[\frac{\alpha(1-\beta)}{2m}\left\{t_1^2 + 2m(t_2 - t_1) - t_2^2\right\}\right]^{\frac{1}{1-\beta}} \tag{A.57}$$

Solving equation (A.57) for t_1,

$$\frac{\alpha(1-\beta)}{2m}\left\{t_1^2 + 2m(t_2 - t_1) - t_2^2\right\} = W^{1-\beta}$$

$$\left\{t_1^2 + 2m(t_2 - t_1) - t_2^2\right\} = \frac{2mW^{1-\beta}}{\alpha(1-\beta)}$$

$$\left\{t_1^2 - 2mt_1 + m^2 - m^2 + 2mt_2 - t_2^2\right\} = \frac{2mW^{1-\beta}}{\alpha(1-\beta)}$$

$$\left\{(t_1 - m)^2 - (t_2 - m)^2\right\} = \frac{2mW^{1-\beta}}{\alpha(1-\beta)}$$

$$(t_1 - m)^2 = (t_2 - m)^2 + \frac{2mW^{1-\beta}}{\alpha(1-\beta)}$$

$$t_1 = m \pm \sqrt{(m - t_2)^2 + \frac{2mW^{1-\beta}}{\alpha(1-\beta)}} \geq 0$$

It is only considered the following value of t_1,

$$t_1 = m - \sqrt{(m - t_2)^2 + \frac{2mW^{1-\beta}}{\alpha(1-\beta)}} \geq 0 \tag{A.58}$$

Substitute the value of t_2 into equation (A.58), hence

$$t_1 = m - \sqrt{\left(m - T + \frac{B}{\alpha\delta}\right)^2 + \frac{2mW^{1-\beta}}{\alpha(1-\beta)}} \geq 0 \tag{A.59}$$

The initial stock Q is

$$Q = W + \frac{\alpha W^{\beta}}{2m}\left[m^2 - \left(m - T + \frac{B}{\alpha\delta}\right)^2 - \frac{2mW^{1-\beta}}{\alpha(1-\beta)}\right] \tag{A.60}$$

The first partial derivative of equation (A.60) with respect to B is

$$\frac{\partial Q}{\partial B} = -\frac{W^{\beta}}{\delta m}\left(m - T + \frac{B}{\alpha\delta}\right) < 0 \tag{A.61}$$

The second partial derivative of equation (A.60) with respect to B *is*

$$\frac{\partial^2 Q}{\partial B^2} = -\frac{W^{\beta}}{\alpha \delta^2 m} < 0 \tag{A.62}$$

The cross partial derivative of equation (A.61) with respect to *W* is

$$\frac{\partial^2 Q}{\partial B \partial W} = -\frac{\beta W^{\beta-1}}{\delta m}\left(m - T + \frac{B}{\alpha\delta}\right) < 0 \tag{A.63}$$

The first partial derivative of (A.60) with respect to *W* is

$$\frac{\partial Q}{\partial W} = \frac{\alpha\beta W^{\beta-1}}{2m}\left[m^2 - \left(m - T + \frac{B}{\alpha\delta}\right)^2 - \frac{2mW^{1-\beta}}{\alpha(1-\beta)}\right]$$

$$\frac{\partial Q}{\partial W} = \frac{\beta(Q-W)}{W} > 0$$

(A.64)

The second partial derivative of equation (A.60) with respect to *W* is

$$\frac{\partial^2 Q}{\partial W^2} = \frac{-\beta W\left(\frac{\beta(Q-W)}{W} - 1\right) - \beta(Q-W)}{W^2} < 0 \tag{A.65}$$

The first partial derivative of the holding cost with respect to *B* is

$$\frac{\partial chol}{\partial B} = h\left(\frac{\partial chol_1}{\partial B} + \frac{\partial chol_2}{\partial B}\right) \tag{A.66}$$

The secodn partial derivative of the holding cost with respect to *B is*

$$\frac{\partial^2 chol}{\partial B^2} = h\left(\frac{\partial^2 chol_1}{\partial B^2} + \frac{\partial^2 chol_2}{\partial B^2}\right) \tag{A.67}$$

The expressions of the first and second derivatives for $chol_1$ and $chol_2$ with respect to *B* are given below.

$$\frac{\partial chol_1}{\partial B} = \frac{\partial Q}{\partial B} t_1 + Q\frac{\partial t_1}{\partial B} + \frac{\alpha W^{\beta} t_1^2}{2m}\frac{\partial t_1}{\partial B} - \alpha W^{\beta} t_1 \frac{\partial t_1}{\partial B}$$

$$\frac{\partial chol_2}{\partial B} = \frac{W\left(\frac{\partial t_2}{\partial B} - \frac{\partial t_1}{\partial B}\right)}{2}$$

$$\frac{\partial^2 chol_1}{\partial B^2} = \frac{\partial^2 Q}{\partial B^2} t_1 + 2\frac{\partial Q}{\partial B}\frac{\partial t_1}{\partial B} + Q\frac{\partial^2 t_1}{\partial B^2} + \frac{\alpha W^{\beta} t_1}{m}\left(\frac{\partial t_1}{\partial B}\right)^2 + \frac{\alpha W^{\beta} t_1^2}{2m}\frac{\partial^2 t_1}{\partial B^2} - \alpha W^{\beta}\left(\frac{\partial t_1}{\partial B}\right)^2 - \alpha W^{\beta}\frac{\partial^2 t_1}{\partial B^2}$$

$$\frac{\partial^2 chol_2}{\partial B^2} = \frac{W\left(\frac{\partial^2 t_2}{\partial B^2} - \frac{\partial^2 t_1}{\partial B^2}\right)}{2}$$

The first partial derivative of the holding cost with respect to *W* is

$$\frac{\partial chol}{\partial W} = h\left(\frac{\partial chol_1}{\partial W} + \frac{\partial chol_2}{\partial W}\right) \tag{A.68}$$

The second partial derivative of the holding cost with respect to *W* is

$$\frac{\partial^2 chol}{\partial W^2} = h\left(\frac{\partial^2 chol_1}{\partial W^2} + \frac{\partial^2 chol_2}{\partial W^2}\right) \tag{A.69}$$

The expressions of the first and second derivatives for $chol_1$ and $chol_2$ with respect to W are given bellow

$$\frac{\partial chol_1}{\partial W}=\frac{\partial Q}{\partial W}t_1+Q\frac{\partial t_1}{\partial W}+\frac{\alpha\beta W^{\beta-1}t_1^3}{6m}+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial t_1}{\partial W}-\frac{\alpha\beta W^{\beta-1}t_1^2}{2}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial W}$$

$$\frac{\partial chol_2}{\partial W}=\frac{(t_2-t_1)}{2}-\frac{W}{2}\left(\frac{\partial t_2}{\partial W}-\frac{\partial t_1}{\partial W}\right)$$

$$\begin{aligned}\frac{\partial^2 chol_1}{\partial W^2}=&\frac{\partial^2 Q}{\partial W^2}t_1+2\frac{\partial Q}{\partial W}\frac{\partial t_1}{\partial W}+Q\frac{\partial^2 t_1}{\partial W^2}+\frac{\alpha\beta(\beta-1)W^{\beta-2}t_1^3}{6m}+\frac{\alpha\beta W^{\beta-1}t_1^2}{2m}\frac{\partial t_1}{\partial W}+\frac{\alpha\beta W^{\beta-1}t_1^2}{2m}\frac{\partial t_1}{\partial W}\\&+\frac{\alpha W^{\beta}t_1}{m}\left(\frac{\partial t_1}{\partial W}\right)^2+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial^2 t_1}{\partial W^2}-\frac{\alpha\beta(\beta-1)W^{\beta-2}t_1^2}{2}-\alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W}-\alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W}\\&-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial W}\right)^2-\alpha W^{\beta}t_1\frac{\partial^2 t_1}{\partial W^2}\end{aligned}$$

$$\frac{\partial^2 chol_2}{\partial W^2}=\frac{1}{2}\left(\frac{\partial t_2}{\partial W}-\frac{\partial t_1}{\partial W}\right)-\frac{1}{2}\left(\frac{\partial t_2}{\partial W}-\frac{\partial t_1}{\partial W}\right)-\frac{W}{2}\left(\frac{\partial^2 t_2}{\partial W^2}-\frac{\partial^2 t_1}{\partial W^2}\right)$$

$$\frac{\partial^2 chol_2}{\partial W^2}=-\frac{W}{2}\left(\frac{\partial^2 t_2}{\partial W^2}-\frac{\partial^2 t_1}{\partial W^2}\right)$$

The first partial derivative of the holding cost with respect to T is

$$\frac{\partial chol}{\partial T}=h\left(\frac{\partial chol_1}{\partial T}+\frac{\partial chol_2}{\partial T}\right) \tag{A.70}$$

The second partial derivative of the holding cost with respect to T is

$$\frac{\partial^2 chol}{\partial T^2}=h\left(\frac{\partial^2 chol_1}{\partial T^2}+\frac{\partial^2 chol_2}{\partial T^2}\right) \tag{A.71}$$

The expressions of the first and second derivatives for $chol_1$ and $chol_2$ with respect to T are given bellow

$$\frac{\partial chol_1}{\partial T}=\frac{\partial Q}{\partial T}t_1+Q\frac{\partial t_1}{\partial T}+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial t_1}{\partial T}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial T}$$

$$\frac{\partial chol_2}{\partial T}=\frac{W}{2}\left\{\frac{\partial t_2}{\partial T}-\frac{\partial t_1}{\partial T}\right\}$$

$$\frac{\partial^2 chol_1}{\partial T^2}=\frac{\partial^2 Q}{\partial T^2}t_1+2\frac{\partial Q}{\partial T}\frac{\partial t_1}{\partial T}+Q\frac{\partial^2 t_1}{\partial T^2}+\frac{\alpha W^{\beta}t_1}{m}\left(\frac{\partial t_1}{\partial T}\right)^2+\frac{\alpha W^{\beta}t_1^2}{2m}\frac{\partial^2 t_1}{\partial T^2}-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial T}\right)^2-\alpha W^{\beta}\frac{\partial^2 t_1}{\partial T^2}$$

$$\frac{\partial^2 chol_2}{\partial T^2}=\frac{W}{2}\left\{\frac{\partial^2 t_2}{\partial T^2}-\frac{\partial^2 t_1}{\partial T^2}\right\}$$

The first partial derivatives of the shortage cost with respect to W and T are

$$\frac{\partial csho}{\partial W}=0 \text{ and } \frac{\partial csho}{\partial T}=0 \tag{A.72}$$

So, the second partial derivatives with respect to W and T are also zero.

The first partial derivative of the shortage cost with respect to B is

$$\frac{\partial csho}{\partial B}=\frac{c_b B}{\delta\alpha} \tag{A.73}$$

The second partial derivative of the shortage cost with respect to B is

$$\frac{\partial^2 csho}{\partial B^2}=\frac{c_b}{\delta\alpha} \quad (A.74)$$

The first partial derivatives of the lost sale cost with respect to W and T are

$$\frac{\partial ocls}{\partial W}=0 \text{ and } \frac{\partial ocls}{\partial T}=0 \quad (A.75)$$

So, the second partial derivatives with respect to W and T are also zero.

The first partial derivative of the lost sale cost with respect to B is

$$\frac{\partial ocls}{\partial B}=\frac{c_l(1-\delta)}{\delta} \quad (A.76)$$

The second partial derivative of the lost sale cost with respect to B is

$$\frac{\partial^2 ocls}{\partial B^2}=0 \quad (A.77)$$

Interest earned for Case 6

The details derivation of interest earned for Case 6 is similar as without shortage case.

Interest earned for Case 7

The first partial derivative of the interest earned for Case 7 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e\left[\begin{array}{l}\frac{\alpha\beta W^{\beta-1}}{m}\left\{\frac{m}{2}\left(t_1^2-N^2\right)-\frac{1}{6}\left(t_1^3-N^3\right)\right\}+\frac{\alpha W^{\beta}}{m}\left\{mt_1\frac{\partial t_1}{\partial W}-\frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\}\\ +\frac{\alpha\beta(W)^{\beta-1}}{2m}\left\{\frac{m}{2}\left(M^2-t_1^2\right)-\frac{1}{6}\left(M^3-t_1^3\right)\right\}+\frac{\alpha(W)^{\beta}}{2m}\left\{-mt_1\frac{\partial t_1}{\partial W}+\frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\}\end{array}\right] \quad (A.78)$$

The second partial derivative of interest earned for Case 7 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e\left[\begin{array}{l}\frac{\alpha\beta(\beta-1)W^{\beta-2}}{m}\left\{\frac{m}{2}\left(t_1^2-N^2\right)-\frac{1}{6}\left(t_1^3-N^3\right)\right\}+2\frac{\alpha\beta W^{\beta-1}}{m}\left\{mt_1\frac{\partial t_1}{\partial W}-\frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\}\\ +\frac{\alpha W^{\beta}}{m}\left\{m\left(\frac{\partial t_1}{\partial W}\right)^2+mt_1\frac{\partial^2 t_1}{\partial W^2}-t_1\left(\frac{\partial t_1}{\partial W}\right)^2-\frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial W^2}\right\}\\ +\frac{\alpha\beta(\beta-1)(W)^{\beta-2}}{2m}\left\{\frac{m}{2}\left(M^2-t_1^2\right)-\frac{1}{6}\left(M^3-t_1^3\right)\right\}+\frac{\alpha\beta(W)^{\beta-1}}{m}\left\{-mt_1\frac{\partial t_1}{\partial W}+\frac{t_1^2}{2}\frac{\partial t_1}{\partial W}\right\}\\ \frac{\alpha(W)^{\beta}}{2m}\left\{-m\left(\frac{\partial t_1}{\partial W}\right)^2-mt_1\frac{\partial^2 t_1}{\partial W^2}+t_1\left(\frac{\partial t_1}{\partial W}\right)^2+\frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial W^2}\right\}\end{array}\right] \quad (A.79)$$

The first partial derivative of the interest earned for Case 7 with respect to B is

$$\frac{\partial IE}{\partial B}=pI_e\left[\frac{\alpha W^{\beta}}{m}\left\{mt_1\frac{\partial t_1}{\partial B}-\frac{t_1^2}{2}\frac{\partial t_1}{\partial B}\right\}+\frac{\alpha(W)^{\beta}}{2m}\left\{-mt_1\frac{\partial t_1}{\partial B}+\frac{t_1^2}{2}\frac{\partial t_1}{\partial B}\right\}\right] \quad (A.80)$$

The second partial derivative of the interest earned for Case 7 with respect to B is

$$\frac{\partial^2 IE}{\partial B^2}=pI_e\left[\begin{array}{l}\frac{\alpha W^{\beta}}{m}\left\{m\left(\frac{\partial t_1}{\partial B}\right)^2+mt_1\frac{\partial^2 t_1}{\partial B^2}-t_1\left(\frac{\partial t_1}{\partial B}\right)^2-\frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial B^2}\right\}\\+\frac{\alpha(W+B)^{\beta}}{2m}\left\{-m\left(\frac{\partial t_1}{\partial B}\right)^2-mt_1\frac{\partial^2 t_1}{\partial B^2}+t_1\left(\frac{\partial t_1}{\partial B}\right)^2+\frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial B^2}\right\}\end{array}\right]$$

(A.81)

The first partial derivative of the interest earned for Case 7 with respect to T is

$$\frac{\partial IE}{\partial T}=pI_e\left[\frac{\alpha W^{\beta}}{m}\left\{mt_1\frac{\partial t_1}{\partial T}-\frac{t_1^2}{2}\frac{\partial t_1}{\partial T}\right\}+\frac{\alpha(W)^{\beta}}{2m}\left\{-mt_1\frac{\partial t_1}{\partial T}+\frac{t_1^2}{2}\frac{\partial t_1}{\partial T}\right\}\right] \tag{A.82}$$

The second partial derivative of the interest earned for Case 7 with respect to B is

$$\frac{\partial^2 IE}{\partial T^2}=pI_e\left[\begin{array}{l}\frac{\alpha W^{\beta}}{m}\left\{m\left(\frac{\partial t_1}{\partial T}\right)^2+mt_1\frac{\partial^2 t_1}{\partial T^2}-t_1\left(\frac{\partial t_1}{\partial T}\right)^2-\frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial T^2}\right\}\\+\frac{\alpha(W)^{\beta}}{2m}\left\{-m\left(\frac{\partial t_1}{\partial T}\right)^2-mt_1\frac{\partial^2 t_1}{\partial T^2}+t_1\left(\frac{\partial t_1}{\partial T}\right)^2+\frac{t_1^2}{2}\frac{\partial^2 t_1}{\partial T^2}\right\}\end{array}\right] \tag{A.83}$$

Interest earned for Case 8

The first partial derivative of the interest earn for case-3 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e\frac{\alpha\beta(W)^{\beta-1}}{2m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{1}{6}\left(M^3-N^3\right)\right] \tag{A.84}$$

The second partial derivative of the interest earned for Case 8 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e\frac{\alpha\beta(\beta-1)(W)^{\beta-2}}{2m}\left[\frac{m}{2}\left(M^2-N^2\right)-\frac{1}{6}\left(M^3-N^3\right)\right] \tag{A.85}$$

The first partial derivative of the interest earned for Case 8 with respect to B is

$$\frac{\partial IE}{\partial B}=0 \tag{A.86}$$

So, the second partial derivative is zero.

The first partial derivative of the interest earned for Case 8 with respect to T is

$$\frac{\partial IE}{\partial T}=0 \tag{A.87}$$

So, the second partial derivative of interest earn is equal to 0.

Interest earned of Case 9

The partial derivative of the interest earned for Case 9 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e\frac{\alpha\beta(W)^{\beta-1}}{2m}\left[\frac{m}{2}\left(t_2^2-N^2\right)-\frac{1}{6}\left(t_2^3-N^3\right)\right][M-t_2] \tag{A.88}$$

The second partial derivative of the interest earned for Case 9 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e\frac{\alpha\beta(\beta-1)(W)^{\beta-2}}{2m}\left[\frac{m}{2}\left(t_2^2-N^2\right)-\frac{1}{6}\left(t_2^3-N^3\right)\right][M-t_2] \tag{A.89}$$

The first partial derivative of the interest earned for Case 9 with respect to B is

$$\frac{\partial IE}{\partial B}=\left[pI_e\frac{\alpha W^{\beta}}{2m}\left\{mt_2\frac{\partial t_2}{\partial B}-\frac{t_2^2}{2}\frac{\partial t_2}{\partial B}\right\}[M-t_2]-pI_e\frac{\alpha W^{\beta}}{2m}\left\{\frac{m}{2}\left(t_2^2-N^2\right)-\frac{1}{6}\left(t_2^3-N^3\right)\right\}\frac{\partial t_2}{\partial B}\right] \tag{A.90}$$

The second partial derivative of the the interest earned for Case 9 with respect to B is

$$\frac{\partial^2 IE}{\partial B^2}=\left[\begin{array}{l}pI_e\frac{\alpha W^{\beta}}{2m}\left\{m\left(\frac{\partial t_2}{\partial B}\right)^2+mt_2\frac{\partial^2 t_2}{\partial B^2}-t_2\left(\frac{\partial t_2}{\partial B}\right)^2\frac{t_2^2}{2}\frac{\partial^2 t_2}{\partial B^2}\right\}[M-t_2]\\ -2pI_e\frac{\alpha W^{\beta}}{2m}\left\{mt_2\frac{\partial t_2}{\partial B}-\frac{t_2^2}{2}\frac{\partial t_2}{\partial B}\right\}\frac{\partial t_2}{\partial B}-pI_e\frac{\alpha W^{\beta}}{2m}\left\{\frac{m}{2}\left(t_2^2-N^2\right)-\frac{1}{6}\left(t_2^3-N^3\right)\right\}\frac{\partial^2 t_2}{\partial B^2}\end{array}\right] \tag{A.91}$$

The partial derivative of the interest earn for Case 9 with respect to T is

$$\frac{\partial IE}{\partial T}=\left[pI_e\frac{\alpha W^{\beta}}{2m}\left\{mt_2\frac{\partial t_2}{\partial T}-\frac{t_2^2}{2}\frac{\partial t_2}{\partial T}\right\}[M-t_2]-pI_e\frac{\alpha W^{\beta}}{2m}\left\{\frac{m}{2}\left(t_2^2-N^2\right)-\frac{1}{6}\left(t_2^3-N^3\right)\right\}\frac{\partial t_2}{\partial T}\right] \tag{A.92}$$

The partial derivative of the equation (A.92) with respect to B is

$$\frac{\partial^2 IE}{\partial T^2}=\left[\begin{array}{l}pI_e\frac{\alpha W^{\beta}}{2m}\left\{m\left(\frac{\partial t_2}{\partial T}\right)^2+mt_2\frac{\partial^2 t_2}{\partial T^2}-t_2\left(\frac{\partial t_2}{\partial T}\right)^2\frac{t_2^2}{2}\frac{\partial^2 t_2}{\partial T^2}\right\}[M-t_2]\\ -2pI_e\frac{\alpha W^{\beta}}{2m}\left\{mt_2\frac{\partial t_2}{\partial T}-\frac{t_2^2}{2}\frac{\partial t_2}{\partial T}\right\}\frac{\partial t_2}{\partial T}-pI_e\frac{\alpha W^{\beta}}{2m}\left\{\frac{m}{2}\left(t_2^2-N^2\right)-\frac{1}{6}\left(t_2^3-N^3\right)\right\}\frac{\partial^2 t_2}{\partial T^2}\end{array}\right] \tag{A.93}$$

Interest earned of Case 10

The first partial derivative of the interest earned for Case 10 with respect to W is

$$\frac{\partial IE}{\partial W}=pI_e(M-N)\frac{\partial Q}{\partial W} \tag{A.94}$$

The second partial derivative of the interest earned for Case 10 with respect to W is

$$\frac{\partial^2 IE}{\partial W^2}=pI_e(M-N)\frac{\partial^2 Q}{\partial W^2} \tag{A.95}$$

The first partial derivative of the interest earned for Case 10 with respect to B is

$$\frac{\partial IE}{\partial B}=pI_e(M-N)\frac{\partial Q}{\partial B} \tag{A.96}$$

The second partial derivative of the interest earned for Case 10 with respect to B is

$$\frac{\partial^2 IE}{\partial B^2}=pI_e(M-N)\frac{\partial^2 Q}{\partial B^2} \tag{A.97}$$

The first partial derivative of the interest earned for Case 10 with respect to T is

$$\frac{\partial IE}{\partial T}=pI_e(M-N)\frac{\partial Q}{\partial T} \tag{A.98}$$

The partial derivative of the equation (A.98) with respect to B is

$$\frac{\partial^2 IE}{\partial T^2}=pI_e(M-N)\frac{\partial^2 Q}{\partial T^2} \tag{A.99}$$

Interest paid of Case 6

The first partial derivative of the interest paid for Case 6 with respect to W is

$$\frac{\partial IP}{\partial W}=cI_p\left[\begin{array}{l}\frac{\alpha\beta W^{\beta-1}}{6m}\left(t_1^3-M^3\right)+\frac{\alpha W^\beta}{2m}t_1^2\frac{\partial t_1}{\partial W}-\frac{\alpha\beta W^{\beta-1}}{2}\left(t_1^2-M^2\right)-\alpha W^\beta t_1\frac{\partial t_1}{\partial W}\\+\frac{\partial Q}{\partial W}(t_1-M)+Q\frac{\partial t_1}{\partial W}+\frac{(t_2-t_1)}{2}-\frac{W}{2}\left(\frac{\partial t_2}{\partial W}-\frac{\partial t_1}{\partial W}\right)\end{array}\right] \tag{A.100}$$

The second partial derivative of interest paid for Case 6 with respect to W is

$$\frac{\partial^2 IP}{\partial W^2}=cI_p\left[\begin{array}{l}\frac{\alpha\beta(\beta-1)W^{\beta-2}}{6m}\left(t_1^3-M^3\right)+\frac{\alpha\beta W^{\beta-1}}{6m}t_1^2\frac{\partial t_1}{\partial W}+\frac{\alpha\beta W^{\beta-1}}{2m}t_1^2\frac{\partial t_1}{\partial W}+\frac{\alpha W^\beta}{m}t_1\left(\frac{\partial t_1}{\partial W}\right)^2\\+\frac{\alpha W^\beta}{2m}t_1^2\frac{\partial^2 t_1}{\partial W^2}-\frac{\alpha\beta(\beta-1)W^{\beta-2}}{2}\left(t_1^2-M^2\right)-\alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W}-\alpha\beta W^{\beta-1}t_1\frac{\partial t_1}{\partial W}\\-\alpha W^\beta\left(\frac{\partial t_1}{\partial W}\right)^2-\alpha W^\beta t_1\frac{\partial^2 t_1}{\partial W^2}+\frac{\partial^2 Q}{\partial W^2}(t_1-M)+2\frac{\partial Q}{\partial W}\frac{\partial t_1}{\partial W}+Q\frac{\partial^2 t_1}{\partial W^2}-\left(\frac{\partial t_2}{\partial W}-\frac{\partial t_1}{\partial W}\right)\\-\frac{W}{2}\left(\frac{\partial^2 t_2}{\partial W^2}-\frac{\partial^2 t_1}{\partial W^2}\right)\end{array}\right] \tag{A.101}$$

The first partial derivative of the interest paid for Case 6 with respect to B is

$$\frac{\partial IP}{\partial B}=cI_p\left[\begin{array}{l}\left\{\frac{\alpha W^\beta}{2m}t_1^2\frac{\partial t_1}{\partial B}-\alpha W^\beta t_1\frac{\partial t_1}{\partial B}+\frac{\partial Q}{\partial B}(t_1-M)+Q\frac{\partial t_1}{\partial B}-\frac{W}{2}\left(\frac{\partial t_2}{\partial B}-\frac{\partial t_1}{\partial B}\right)\right\}(T-t_2)\\-\left\{\frac{\alpha W^\beta}{6m}\left(t_1^3-M^3\right)-\frac{\alpha W^\beta}{2}\left(t_1^2-M^2\right)+Q(t_1-M)+\frac{W(t_2-t_1)}{2}\right\}\frac{\partial t_2}{\partial B}\end{array}\right] \tag{A.102}$$

The second partial derivative of interest paid for Case 6 with respect to B is

$$\frac{\partial^2 IP}{\partial B^2}=cI_p\left[\begin{array}{l}\left\{\begin{array}{l}\frac{\alpha W^\beta}{m}t_1\frac{\partial t_1}{\partial B}+\frac{\alpha W^\beta}{2m}t_1^2\frac{\partial^2 t_1}{\partial B^2}-\alpha W^\beta\left(\frac{\partial t_1}{\partial B}\right)^2-\alpha W^\beta t_1\frac{\partial^2 t_1}{\partial B^2}+\frac{\partial^2 Q}{\partial B^2}(t_1-M)\\+2\frac{\partial Q}{\partial B}\frac{\partial t_1}{\partial B}+Q\frac{\partial^2 t_1}{\partial B^2}-\frac{W}{2}\left(\frac{\partial^2 t_2}{\partial B^2}-\frac{\partial^2 t_1}{\partial B^2}\right)\end{array}\right\}(T-t_2)\\-2\left\{\frac{\alpha W^\beta}{2m}t_1^2\frac{\partial t_1}{\partial B}-\alpha W^\beta t_1\frac{\partial t_1}{\partial B}+\frac{\partial Q}{\partial B}(t_1-M)+Q\frac{\partial t_1}{\partial B}-\frac{W}{2}\left(\frac{\partial t_2}{\partial B}-\frac{\partial t_1}{\partial B}\right)\right\}\frac{\partial t_2}{\partial B}\\-\left\{\frac{\alpha W^\beta}{6m}\left(t_1^3-M^3\right)-\frac{\alpha W^\beta}{2}\left(t_1^2-M^2\right)+Q(t_1-M)+\frac{W(t_2-t_1)}{2}\right\}\frac{\partial^2 t_2}{\partial B^2}\end{array}\right] \tag{A.103}$$

The first partial derivative of the interest paid for Case 6 with respect to T is

$$\frac{\partial IP}{\partial T}=cI_p\left[\begin{array}{l}\left\{\frac{\alpha W^\beta}{2m}t_1^2\frac{\partial t_1}{\partial T}-\alpha W^\beta t_1\frac{\partial t_1}{\partial T}+\frac{\partial Q}{\partial T}(t_1-M)+Q\frac{\partial t_1}{\partial T}-\frac{W}{2}\left(\frac{\partial t_2}{\partial T}-\frac{\partial t_1}{\partial T}\right)\right\}(T-t_2)\\+\left\{\frac{\alpha W^\beta}{6m}\left(t_1^3-M^3\right)-\frac{\alpha W^\beta}{2}\left(t_1^2-M^2\right)+Q(t_1-M)+\frac{W(t_2-t_1)}{2}\right\}\left(1-\frac{\partial t_2}{\partial T}\right)\end{array}\right] \tag{A.104}$$

The second partial derivative of interest paid for Case 6 with respect to T is

$$\frac{\partial^2 IP}{\partial T^2}=cI_p\left[\begin{array}{l}\left\{\begin{array}{l}\frac{\alpha W^{\beta}}{m}t_1\frac{\partial t_1}{\partial T}+\frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial^2 t_1}{\partial T^2}-\alpha W^{\beta}\left(\frac{\partial t_1}{\partial T}\right)^2-\alpha W^{\beta}t_1\frac{\partial^2 t_1}{\partial T^2}+\frac{\partial^2 Q}{\partial T^2}(t_1-M)\\+2\frac{\partial Q}{\partial T}\frac{\partial t_1}{\partial T}+Q\frac{\partial^2 t_1}{\partial T^2}-\frac{W}{2}\left(\frac{\partial^2 t_2}{\partial T^2}-\frac{\partial^2 t_1}{\partial T^2}\right)\end{array}\right\}(T-t_2)\\+2\left\{\frac{\alpha W^{\beta}}{2m}t_1^2\frac{\partial t_1}{\partial T}-\alpha W^{\beta}t_1\frac{\partial t_1}{\partial T}+\frac{\partial Q}{\partial T}(t_1-M)+Q\frac{\partial t_1}{\partial T}-\frac{W}{2}\left(\frac{\partial t_2}{\partial T}-\frac{\partial t_1}{\partial T}\right)\right\}\left(1-\frac{\partial t_2}{\partial T}\right)\\-\left\{\frac{\alpha W^{\beta}}{6m}\left(t_1^3-M^3\right)-\frac{\alpha W^{\beta}}{2}\left(t_1^2-M^2\right)+Q(t_1-M)+\frac{W(t_2-t_1)}{2}\right\}\frac{\partial^2 t_2}{\partial T^2}\end{array}\right] \quad (A.105)$$

Interest paid for Case 7

The first partial derivative of the interest paid for Case 7 with respect to W is

$$\frac{\partial IP}{\partial W}=cI_p\frac{(t_2-M)(T-t_2)}{2} \quad (A.106)$$

The second partial derivative of interest paid for Case 7 with respect to W is

$$\frac{\partial^2 IP}{\partial W^2}=0 \quad (A.107)$$

The first partial derivative of the interest paid for Case 7 with respect to B is

$$\frac{\partial IP}{\partial B}=cI_p\frac{W}{2}\left\{(T-t_2)\frac{\partial t_2}{\partial B}-(t_2-M)\frac{\partial t_2}{\partial B}\right\}$$
$$\frac{\partial IP}{\partial B}=cI_p\frac{W}{2}\{T-2t_2-M\}\frac{\partial t_2}{\partial B} \quad (A.108)$$

The second partial derivative of interest paid for Case 7 with respect to B is

$$\frac{\partial^2 IP}{\partial B^2}=cI_p\frac{W}{2}\left[-2\left(\frac{\partial t_2}{\partial B}\right)^2+\{T-2t_2-M\}\frac{\partial^2 t_2}{\partial B^2}\right]$$

(A.109)

The first partial derivative of the interest paid for Case 7 with respect to T is

$$\frac{\partial IP}{\partial T}=cI_p\frac{W}{2}\left\{(T-t_2)\frac{\partial t_2}{\partial T}-(t_2-M)\left(1-\frac{\partial t_2}{\partial T}\right)\right\} \quad (A.110)$$

The second partial derivative of interest paid for Case 7 with respect to T is

$$\frac{\partial^2 IP}{\partial T^2}=cI_p\frac{W}{2}\left[\left(1-\frac{\partial t_2}{\partial T}\right)\frac{\partial t_2}{\partial T}+(T-t_2)\frac{\partial^2 t_2}{\partial T^2}-\frac{\partial t_2}{\partial T}\left(1-\frac{\partial t_2}{\partial T}\right)+(t_2-M)\frac{\partial^2 t_2}{\partial T^2}\right] \quad (A.111)$$

Interest paid of Case 8

For the interst paid of Case 8 all the expressions are similar to Case 7.

Interest paid of Case 9 and Case 10

There is not interest paid.

Supplementary B. Proof of Theorem 1

For any given value of W and B, the first and second partial derivatives of equation (A.4) with respect to T are as follows:

$$\frac{\partial t_1}{\partial T}=\frac{m-T}{\sqrt{\left[(m-T)^2+\frac{2m\left(W^{1-\beta}-B^{1-\beta}\right)}{\alpha(1-\beta)}\right]}} \tag{B.1}$$

$$\frac{\partial t_1}{\partial T}=\frac{m-T}{m-t_1}>0 \tag{B.2}$$

$$\frac{\partial^2 t_1}{\partial T^2}=\frac{-(m-t_1)-(m-T)\frac{\partial t_1}{\partial T}}{(m-t_1)^2}$$
$$\frac{\partial^2 t_1}{\partial T^2}=\frac{-(m-t_1)-(m-T)\frac{(m-T)}{(m-t_1)}}{(m-t_1)^2}<0 \tag{B.3}$$

The first partial derivative of equation (A.5) with respect to T is

$$\frac{\partial Q}{\partial T}=\frac{\alpha W^{\beta}}{m}(m-T)>0 \tag{B.4}$$

The second partial derivative of equation (A.5) with respect to T is

$$\frac{\partial^2 Q}{\partial T^2}=-\frac{\alpha W^{\beta}}{m}<0 \tag{B.5}$$

$$y_i(T)=p(Q-B)+SB+IE-cQ-c_o-chol-IP-uW \tag{B.6}$$

and

$$f(T)=T>0 \tag{B.7}$$

therefore,

$$q(T)=\frac{y_i(T)}{f(T)}=Z_i(p,B,T) \quad \text{for } i=1,\ldots 5 \tag{B.8}$$

Differentiate of equation (B.6) with respect to T,

$$\frac{\partial y_i(T)}{\partial T}=(p-c)\frac{\partial Q}{\partial T}+\frac{\partial IE}{\partial T}-\frac{\partial chol}{\partial T}-\frac{\partial IP}{\partial T} \tag{B.9}$$

Differentiate the equation (B.9) with respect to T,

$$\frac{\partial^2 y_i(T)}{\partial T^2}=(p-c)\frac{\partial^2 Q}{\partial T^2}+\frac{\partial^2 IE}{\partial T^2}-\frac{\partial^2 chol}{\partial T^2}-\frac{\partial^2 IP}{\partial T^2}$$
$$\frac{\partial^2 y_i(T)}{\partial T^2}=-\left[(p-c)\frac{\alpha W^{\beta}}{m}-\frac{\partial^2 IE}{\partial T^2}+\frac{\partial^2 chol}{\partial T^2}+\frac{\partial^2 IP}{\partial T^2}\right] \tag{B.10}$$
$$\frac{\partial^2 y_i(T)}{\partial T^2}=-J_i$$

Therefore, if $J_i > 0$ then $\frac{\partial^2 y_i(T)}{\partial T^2}<0$. Hence, $y_i(T)$ is a strictly concave for (i=1,..,5), differentiable and nonnegative function. As a result, if $J_i>0$ then $TP_i(W,B,T)$ is a strictly pseudo-concave function of T. Therefore, there exists a unique optimal solution.

Supplementary C. Finding the optimal solution of the cycle length T^*.

From equation (B.6) to (B.7),

$$\frac{y_i(T)}{f(T)} = TP_i(W,B,T) \tag{C.1}$$

Differentiate equation (C.1) with respect to T is

$$\frac{\partial TP_i(W,B,T)}{\partial T} = \frac{\frac{\partial y_i(T)}{\partial T}}{T} - \frac{y_i(T)}{T^2} \tag{C.2}$$

From equation (C.2) the necessary and sufficient condition to obtain the value of T^* is $\frac{\partial TP_i(W,B,T)}{\partial T} = 0$. Thus,

$$T\frac{\partial y_i(T)}{\partial T} - y_i(T) = 0 \tag{C.3}$$

If $J_i > 0$ then

$$T\frac{\partial y_i(T)}{\partial T} = y_i(T) \tag{C.4}$$

Supplementary D. Proof of Corollary 1

Differentiate equation (A.4) with respect to B and W,

$$\frac{\partial t_1}{\partial B} = \frac{-1}{2\sqrt{(m-T)^2 + \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}}}\left[-\frac{2m(1-\beta)}{\alpha(1-\beta)}B^{-\beta}\right] \tag{D.1}$$

$$\frac{\partial t_1}{\partial B} = \frac{m}{\alpha(m-t_1)B^{\beta}} > 0$$

$$\frac{\partial t_2}{\partial W} = \frac{-1}{2\sqrt{(m-T)^2 + \frac{2m\left(W^{1-\beta} - B^{1-\beta}\right)}{\alpha(1-\beta)}}}\left[\frac{2m(1-\beta)W^{-\beta}}{\alpha(1-\beta)}\right] \tag{D.2}$$

$$= \frac{-m}{\alpha(m-t_1)W^{\beta}} < 0$$

Differentiate equation (D.1) with respect to B,

$$\frac{\partial^2 t_1}{\partial B^2} = \frac{m}{\alpha}\left[\frac{-(m-t_1)B^{-\beta-1} + B^{-\beta}\frac{\partial t_1}{\partial B}}{(m-t_1)^2}\right] \tag{D.3}$$

Differentiate equation (D.1) with respect to W,

$$\frac{\partial^2 t_1}{\partial B \partial W} = \frac{-m^2}{\alpha(m-t_1)^3(BW)^{\beta}} \tag{D.4}$$

Differentiate equation (D.2) with respect to W,

$$\frac{\partial^2 t_1}{\partial W^2} = \frac{-m}{\alpha}\left[\frac{-\beta(m-t_1)W^{-\beta-1} + W^{-\beta}\frac{\partial t_1}{\partial W}}{(m-t_1)^2}\right] \tag{D.5}$$

From equations (A.7), (A.8), (A.9), and (A.10) is

$$\frac{\partial Q}{\partial B} = \left(\frac{W}{B}\right)^{\beta} > 0 \tag{D.6}$$

$$\frac{\partial^2 Q}{\partial W^2} = \frac{-\beta W\left(\frac{\beta(Q-W)}{W}-1\right)-\beta(Q-W)}{W^2} < 0 \tag{D.7}$$

$$\frac{\partial^2 Q}{\partial B^2} = \frac{-\beta W^{\beta}}{B^{1+\beta}} < 0 \tag{D.8}$$

$$\frac{\partial^2 Q}{\partial W \partial B} = \frac{\beta W^{\beta-1}}{B^{\beta}} > 0 \tag{D.9}$$

For any given value of *T*,

$$TP_i(W, B, T) = \frac{1}{T} X_i(p, B) \tag{D.10}$$

where

$$X_i(W,B) = p(Q-B) + SB + IE - cQ - c_o - chol - IP - uW \tag{D.11}$$

Differentiate the equation (D.11) with respect to *B* is

$$\frac{\partial X_i(W,B)}{\partial B} = (p-c)\frac{\partial Q}{\partial B} - p + S + \frac{\partial IE}{\partial B} - \frac{\partial chol}{\partial B} - \frac{\partial IP}{\partial B} \tag{D.12}$$

Differentiate equation (D.12) with respect to *B* is

$$L_i = \frac{\partial^2 X_i(W,B)}{\partial B^2} = (p-c)\frac{\partial^2 Q}{\partial B^2} + \frac{\partial^2 IE}{\partial B^2} - \frac{\partial^2 chol}{\partial B^2} - \frac{\partial^2 IP}{\partial B^2} \tag{D.13}$$

Differentiate equation (D.12) with respect to *W* is

$$K_i = \frac{\partial^2 U_i(W,B)}{\partial B \partial W} = (p-c)\frac{\partial^2 Q}{\partial B \partial W} + \frac{\partial^2 IE}{\partial B \partial W} - \frac{\partial^2 chol}{\partial B \partial W} - \frac{\partial^2 IP}{\partial B \partial W} \tag{D.14}$$

Differentiate equation (D.11) with respect to *W* is

$$\frac{\partial X_i(W,B)}{\partial W} = (p-c)\frac{\partial Q}{\partial W} + \frac{\partial IE}{\partial W} - \frac{\partial chol}{\partial W} - \frac{\partial IP}{\partial W} - u \tag{D.15}$$

Differentiate equation (D.15) with respect to *W* is

$$M_i = \frac{\partial^2 X_i(W,B)}{\partial W^2} = (p-c)\frac{\partial^2 Q}{\partial W^2} + \frac{\partial^2 IE}{\partial W^2} - \frac{\partial^2 chol}{\partial W^2} - \frac{\partial^2 IP}{\partial W^2} \tag{D.16}$$

Supplementary E. Proof of Theorem 2.

Using the equations (D.13), (D.14), and (D.16), we can say $L_i < 0, M_i < 0$ and $L_i M_i - K_i^2 > 0$. For a given value of *T*, the Hessian matrix of the profit function is negative definite.

$$H = \begin{bmatrix} \frac{\partial^2 X_i(W,B)}{\partial B^2} & \frac{\partial^2 X_i(W,B)}{\partial W \partial B} \\ \frac{\partial^2 X_i(W,B)}{\partial W \partial B} & \frac{\partial^2 X_i(W,B)}{\partial W^2} \end{bmatrix} = \begin{bmatrix} L_i & K_i \\ K_i & M_i \end{bmatrix} \tag{E.1}$$

Then the profit functions $TP_i(W,B,T)$ are strictly concave in *E* and *W*. Therefore, there exists a unique optimal solution.