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Analysis of $M/M/1/N$ Stochastic Queueing—Inventory System with Discretionary Priority Service and Retrial Facility

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Abstract: In this paper, we analyze a queueing–inventory system with two classes of customers, high priority (HP) and low priority (LP), under the discretionary priority discipline. The LP customers are served in two stages: preliminary service in stage-I and main service in stage-II. In contrast, HP customers require only the main service. Whenever the inventory level is less than the threshold level during the stage-I service of an LP customer, an arriving HP customer is allowed to interrupt the service of an LP customer by adopting the mixed-priority discipline. Otherwise, non-preemptive priority discipline is used in both stages. The interrupted LP customer moves to orbit and retries for the service whenever the server is free. The waiting hall of finite capacity is afforded for the HP customer only. The orbital search is provided for LP customers in orbit. The inventory is replenished following the (s, Q) ordering policy, with the lifetimes of the items being exponentially distributed. An expression for the stability condition is determined explicitly, and system performance measures are evaluated. Numerical examples are formulated for different sets of input values of the parameters.

Keywords: discretionary priority; mixed priority; preemptive priority; non-preemptive priority; infinite orbit

MSC: 60K25

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1. Introduction

The model developed in this paper was motivated by the author's personal experience at a fertilizer company located in Thiruvannamalai, a town in the state of Tamilnadu. The customers of the company were farmers who came to buy fertilizers for their agricultural land. To do so, the company made it mandatory to register their cultivation land details beforehand, to buy the required amounts of fertilizer using cash. The farmers who had registered earlier were considered registered farmers, and those who were new to this process were considered unregistered farmers. Therefore, the registered farmer directly went to buy the product, whereas an unregistered farmer had to go through the registration process first and then buy the product. Consider the registration of land with the survey number to be a stage-I service, and buying the product to be a stage-II service. When an unregistered farmer is in stage-I, and in the meantime, a registered farmer arrives, the seller pauses the service for the unregistered farmer to assist the registered farmer. The farmer whose service has been paused goes around the town to make other purchases or waits nearby until the registered farmer's order is completed. This situation motivated the author to focus on such stochastic modelling.

The service rule applied in the above real-life situation illustrates the discretionary priority service process. It describes an intermediate priority between preemptive and non-preemptive priorities. This type of service system is widely used in telecommunication systems, hospitals, industrial administration, railway network services, flight landings in an airport, signal processing, military fields, etc. When observing a railway network, the administration gives a green signal to super-fast trains while pausing local trains in cases of insufficient track facilities at a particular junction. Once a local train has crossed the junction, the super-fast train cannot be allowed to interrupt the service of the local train. Therefore, providing preemptive priority in one stage and non-preemptive priority in the next stage of the service process is called discretionary service discipline.

Literature Review

In a queueing–inventory model, every customer is provided with an item from the inventory at the end of the service. Melikov and Molchanov [1] discussed the optimal policies for reordering stocks in an inventory system with queueing of order requests. Sigman and Simichi [2] introduced a service facility in an $M/G/1$ queueing–inventory model with limited inventory under light traffic. Berman and Kim [3] formulated two cases of a queueing–inventory model with a service facility. The queueing capacity of one was infinite, and the other was finite. Further, Berman and Sapna [4] developed the aforementioned models with arbitrarily distributed service time. Recently, Sangeetha and Sivakumar [5] discussed a perishable inventory system with Markovian arrivals and an exponentially distributed service rate. Jeganathan et al. [6] modeled an $M/M/1/N$ queueing–inventory system with a retrial facility. The reader can refer to the recent papers on retrial systems in [7–9]. For more details, the references [10–22] are essential for understanding queueing–inventory systems with service facilities.

With a single server, customers of multiple classes can be served following various service disciplines in many real-life situations. For instance, there have been many COVID-19 vaccination clinics opened in the country recently; in all such clinics, two types of individuals arrive and are categorized: individuals who booked their slots beforehand and those who come without bookings. Both individuals are administered by a single server. The server makes sure that all individuals who booked ahead are served completely before moving serving individuals without bookings. Additionally, suppose a "booked" individual arrives during the provision of the service to an "unbooked" one. In that case, the former has to wait until the service's completion, and their vaccination will start immediately after the latter's departure. This way of providing services is known as non-preemptive priority discipline, a service carried out without any interruption.

Falin et al. [23] studied in-depth a more general model with two classes of customers, using different service distributions for both types of customers, to examine the stochastic decomposition and asymptotic behavior of the stationary characteristics. Jeganathan et al. [24] developed a queueing–inventory model with two types of customers based on their priority levels and rendered services through non-preemptive priority discipline. Chakravarthy [25] studied a non-preemptive priority queue with two classes of customers and introduced a new dynamic rule to provide services to lower-priority customers in the presence of higher-priority customers. Korenevskaya et al. [26] considered a retrial queueing system with Poisson arrival following non-preemptive priority. Recently, Krishnamoorthy and Divya [27] analyzed a queue with non-preemptive priority under a marked Markovian arrival process with a distinct phase-type distribution of services.

Non-preemptive priority discipline cannot be applied in all priority queues, because some situations need preemptive priority discipline. For instance, in a retail shop, two types of customers approach: those of one type have used buy online and pick up in-store (BOPIS), and those of the other come for in-store purchases. The services to in-store purchasers are interrupted on the arrival of BOPIS customers, as the service times required for BOPIS customers are low compared with those for shop-in-store customers. A service that is carried out with interruption of a high-priority (HP) customer in favor of a low-priority (LP) customer is known as preemptive priority discipline. Krishna Kumar et al. [28] discussed the $M/G/1$ queue under preemptive resume priority with a service in two phases. Tarabia [29] discussed a queueing system with two priority classes under preemptive discipline, where an interrupted service of a low priority is resumed when there is no high priority need in the system. Jeganathan et al. [30] discussed a single-server inventory system with an interruption and a finite waiting room. Gao and Wang [31] analyzed a single-server retrial queue with Poisson arrival. The customers are served through preemptive priority with orbital search.

The preemptive and non-preemptive service disciplines are both labored in some situations and are known as mixed-priority discipline. Like in situations as explained for preemptive priority, a BOPIS customer may opt to wait until the service completion of an ongoing in-store purchase. Thus, they adopt a non-preemptive priority. However, not all BOPIS customers would opt to wait, and some may interrupt and avail service following preemptive priority. Hence, a service discipline that focuses on both preemptive and non-preemptive priority customer under the Bernoulli's schedule is known as mixed-priority discipline.

Adiri and Domb [32] used this service discipline in their model that serves m priority classes of customers. If the positive value d is fixed, and if the variation between the class indices is outreach d, then preemptive priority discipline is allowed. Otherwise, non-preemptive priority is followed. Cho and Un [33] analyzed the $M/G/1$ queue combining preemptive and non-preemptive discipline and considered three schemes: the elapsed service time, the ratio of elapsed to total service time, and the remaining service time as the discretion for preemption, where each is based on the parameters of the low-priority job. Fajardo and Drekic [34] categorized two distinct types of priority classes: (i) urgent class and (ii) non-urgent class among N priority classes. Within each category, non-preemptive priority is adopted, and between the categories, preemptive priority is followed.

A priority mechanism is a method of scheduling that allows customers of different classes to receive different levels of service from a single server. It is used in communication systems, healthcare systems, and inventory systems. The priority discipline in queueing systems can be preemptive or non-preemptive. However, both disciplines have flaws. Under preemptive discipline, a low-priority (LP) job whose service is almost complete may be preempted. Even if a low-priority (LP) job with a long service time has just entered service, a high-priority (HP) job may wait under non-preemptive discipline. Allowing the server to decide whether or not to continue or stop the LP customer's service can help avoid these unpleasant situations. In reality, especially in the areas of telecommunication, communication networks, and the chemical industry, among the entire service procedure, some service stages can be interrupted while some service stages cannot be interrupted.

From all these priorities, a new priority rule known as discretionary priority was formulated and was first studied by Avi-itzhak et al. [35]. This priority discipline is widely utilized in telecommunication and communication networks. Melkonian and Kaiser [36] examined discretionary priority with general service time distribution. Lian and Zhao [37] modeled an $M/G/1$ queue with service in two stages, following discretionary priority with preemptive priority in stage-I and non-preemptive priority in stage-II. Zhao and Lian [38] formulated a two-stage $M/G/1$ queue with discretionary priority and constructed an embedded Markov model for specific time points on the time axis.

Lian and Zhao [39] investigated the two-stage service process for two classes of customers. They are called LP and HP customers. They assumed that each class of customer would have a two-stage service. However, the crucial assumption of their model is that the HP customer can be allowed to interrupt the service of an LP customer while they are being serviced in stage-I only. In stage-II, interruption is not allowed. Ning Zhao and Yaya Guo [40] investigated the same two-stage service for two classes of customers whose arrival occurs according to the Markovian arrival process and whose service process follows the phase-type distribution. Jeganathan et al. [11] analyzed the threshold-based priority services in the queueing inventory system.

To the best of our knowledge, so far, the models following discretionary priority discipline are discussed only in queueing systems. There is no queueing system attached to an inventory, which is considered under discretionary priority discipline, which creates a research gap in the queueing–inventory system. Of course, the discretionary priority discipline applied in [37–40] mostly assumed that the interruption of HP customers would happen only at stage-I service but not at stage-II without any other restriction. Even though the service discipline involving discretionary priority rules is not applied in the QIS, the concept of providing discretionary priority service methodology based on threshold level is not yet discussed even in queueing theory. As per the literature review analysis on the QIS, the discretionary priority service discipline is not applied. This is to be considered a research gap in the QIS. This model is specifically designed to fill such a gap. However, the novelty of this proposed model is that it introduces the discretionary priority service methodology with the threshold-based inventory level. Additionally, the research work is an attempt to weigh between the priority disciplines and find the optimal priority discipline that helps improve a business setup. Together with discretionary priority, we combined the concepts of orbital search, retrial facility, and interruption, based on Bernoulli's schedule.

The paper is organized as follows: In Section 2, the description of the model and notations are defined. Section 3 deals with analysis, the steady-state solution of the model, the system performance measures, and cost analysis. In Section 4, some numerical examples pertaining to the real-time usage of the model are explained. Finally, Section 5 holds the conclusion of the paper.

2. Model Description

2.1. Definitions

2.1.1. Non-Preemptive Priority (Head-of-The-Line Priority Discipline)

During the service of an LP customer, the interruption of an HP customer is strictly not allowed, and this service discipline is called non-preemptive priority.

2.1.2. Preemptive Priority Discipline

When the server is busy with an LP customer, the interruption of an HP customer is always allowed, and this service discipline is called preemptive priority.

2.1.3. Mixed-Priority Discipline

In mixed-priority service, both preemptive and non-preemptive priority discipline can be adopted. If an HP customer arrives while an LP customer is being served, the arriving HP customer may or may not interrupt the LP customer's service based on the Bernoulli's schedule.

2.1.4. Discretionary Priority Discipline

Discretionary priority service discipline is defined as in which the server allows preemption of HP customer when the server is busy with LP customer at stage-I and preemption of HP customer is not allowed if the server is busy with LP customer in stage-II (see [38]).

2.2. Notations

The following standard notations are used in this paper.

A_{ij}	: The matrix A as a element at (i, j)th position
$\mathbf{0}$	: Zero matrix
I	: Identitiy matrix
δ_{ij}	: $\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
$H(y)$	: $\begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}$
$\mathbf{e}$	: Column vector with 1 in each entry
W	:The Set of all Whole Numbers
V_1	:$\{1, 2, \ldots, S\}$
V_2	:$\{0, 1, 2, \ldots, N\}$

2.3. Explanation of the Model

This model explores the discretionary priority service discipline in a single-server, perishable queueing–inventory system (SSPQIS) with two classes of customers. The inventory system is made up of a single commodity with the size S, a finite waiting hall with the size N, and an infinite orbit. This system provides the service facility for two types of customers, say HP and LP, served in two distinct stages. Every customer of the system has to go through two stages of service, namely, preliminary service and main service. In preliminary service, a customer undergoes registration with the server, whereas the customer purchases the product in the main service. Every customer in the system goes through preliminary service only once. Once the registration process has been completed successfully, the customer can directly advance to stage-II service in the current and upcoming purchases. Depending on this service scheme, the customer who has already completed the preliminary service and approaches the system only to avail of the main service is initialized as an HP customer, and the customer who is new to the system and accordingly has to go through the preliminary service before moving to obtain the main service is termed as an LP customer.

The service discipline utilized in the system depends upon the threshold level. The threshold, L, is a predetermined level on a certain amount of inventory. Suppose the inventory level is greater than this threshold level. In that case, the server follows non-preemptive priority discipline in stage-I service. That is, if an LP customer is being served with preliminary service, then the arriving HP customer cannot be allowed to interrupt the service. On the other hand, if the quantity of inventory is less than or equal to the threshold, then discretionary priority service discipline is adopted, allowing the interruption of HP customers during the service of LP customers in stage-I under Bernoulli's schedule. The interrupted LP customer moves into orbit. During the stage-II service, the server always follows non-preemptive priority discipline.

While the server follows mixed-priority service discipline, the arriving HP customer may interrupt the service or decide to wait until the service of the LP customer is completed in stage-I. When inventory is less than or equal to L, the arriving HP customer may either interrupt the service of an LP customer with probability r or move to the waiting hall with probability $(1 - r)$. If an HP customer decides to wait, then the upcoming HP customers also have to wait in the queue. However, if the HP customer interrupts, then the LP customer moves to orbit with probability, p, and tries for service when the system is void of any HP customers. If $r = 0$, then there is no HP customer interrupts an LP customer. Thus, the server always follows non-discretionary priority discipline. If $r \in (0, 1]$, then the discussion completely comes under the discretionary priority discipline. When $r = 1$, the preemption of HP customers towards the LP customers' service will certainly occur. Thus, the server adopts a preemptive priority discipline. Additionally, if $r \in (0, 1)$, then the priority discipline is called mixed-priority discipline.

The parameters λ_H and λ_L denote the arrival rate of HP and LP customers, and both independently follow the Poisson process. The waiting hall is used only for HP customers. An arriving HP customer moves to the waiting hall in any one of the following circumstances:

- The HP customer is at service;
- The LP customer is receiving stage-II service;
- The LP customer is at stage-I service, when the inventory is greater than L;
- The HP customer decides to wait with probability $(1-r)$;
- With zero inventory level.

Similarly, an LP customer will move to an orbit with probability p in any one of the following circumstances:

- When the inventory is zero;
- On interruption from an HP customer at stage-I;
- If the server is busy with HP or LP customers.

Otherwise, they are considered lost with probability $(1-p)$. The retrial customer approaches the system under the classical retrial policy, and the subsequent retrial process follows an exponential distribution. The service rate of HP customer in stage-II is μ_H, and LP customer in stage-I is μ_{L_1} and stage-II is μ_{L_1}. All the service rates follow the exponential distribution independently. Here, we assume $\mu_{L_1} < \mu_{L_2}$. Inventory decreases one by one after each service is completed. Whenever the server is free with positive inventory, they search for an LP customer from the orbit with probability q, or does not search with probability $(1-q)$. The items in the inventory are subject to perishables except for the items in the service process. The perishable rate is defined by γ. If the inventory reaches s, $Q(=S-s)$ items are ordered using the (s,Q) ordering policy. The parameter β indicates the replenishment rate of the reorder process. The lifetime and replenishment time follow the exponential distribution.

3. Analysis

From the suppositions made in the system, we see that the stochastic process $\{A(t), t \geq 0\} = \{(A_1(t), A_2(t), A_3(t), A_4(t)), t \geq 0\}$ is a continuous time Markov chain with state space E, where

$$\begin{aligned}
A_1(t) &: \text{Number of Customers in the Orbit at time } t\\
A_2(t) &: \text{Inventory level at time } t\\
A_3(t) &: \text{The Status of Server at time } t\\
A_4(t) &: \text{Number of HP in Waiting Hall at time } t\\
A_3(t) &: \begin{cases} 0 & \text{the server is free.}\\ 1 & \text{the server is busy with LP customer in stage-I.}\\ 2 & \text{the server is busy with LP customer in stage-II.}\\ 3 & \text{the server is busy with HP customer in stage-II.}\end{cases}\\
E_1 &: \{(i, j=0, k=0, l) : i \in W, l \in V_2\}\\
E_2 &: \{(i, j, k=0, l=0) : i \in W, j \in V_1\}\\
E_3 &: \{(i, j, k, l) : i \in W, j \in V_1, k \in \{1,2,3\}, l \in V_2\}\\
E &: E_1 \cup E_2 \cup E_3.
\end{aligned}$$

The transition of infinitesimal generator matrix is of the following structure:

$$K = \begin{array}{cc} & \begin{array}{cccccccc} (0) & (1) & (2) & \cdots & (M-1) & M & (M+1) & \cdots \end{array} \\ \begin{array}{c} (0)\\ (1)\\ (2)\\ \vdots\\ (M-1)\\ (M)\\ (M+1)\\ \vdots \end{array} & \begin{pmatrix} \mathbb{K}_{00} & \mathbb{K}_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\\ \mathbb{K}_{10} & \mathbb{K}_{11} & \mathbb{K}_{01} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\\ \mathbf{0} & \mathbb{K}_{21} & \mathbb{K}_{22} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{K}_{M-1,M-1} & \mathbb{K}_{01} & \mathbf{0} & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{K}_{M,M-1} & \mathbb{K}_{M,M} & \mathbb{K}_{01} & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbb{K}_{M+1,M} & \mathbb{K}_{M+1,M+1} & \cdots\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} \tag{1}$$

Here, we assume that i_1, i_2 represent the number of customer in the orbit, j_1, j_2 represent the present stock level of the system, k_1, k_2 denote the server status and l_1, l_2 indicate the number of HP customer in the waiting hall to explore the transition of states of the system, where $K_{i_1 i_2}, \quad i_2 = i_1 + 1, \quad i_1 = 0, 1, 2, \ldots$

$$
\begin{aligned}
K_{i_1 i_2} &= \begin{cases} B_0, & j_2 = j_1, \quad j_1 = 0 \\ B_1, & j_2 = j_1, \quad j_1 = 1, 2, \ldots, L \\ B_2, & j_2 = j_1, \quad j_1 = L+1, L+2, \ldots, S \\ \mathbf{0} & otherwise. \end{cases} \\
B_0 &= \begin{cases} p\lambda_L, & k_2 = k_1, \quad k_1 = 0 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N \\ 0, & otherwise. \end{cases} \\
B_1 &= \begin{cases} p\lambda_L, & k_2 = k_1, \quad k_1 = 1, 2, 3. \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N \\ r\lambda_H, & k_2 = 3, \quad k_1 = 1 \\ & l_2 = l_1, \quad l_1 = 0 \\ 0, & otherwise. \end{cases} \\
B_2 &= \begin{cases} p\lambda_L, & k_2 = k_1, \quad k_1 = 1, 2, 3. \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N \\ 0, & otherwise. \end{cases}
\end{aligned}
\tag{2}
$$

$K_{i_1 i_2}, \quad i_2 = i_1 - 1, \quad i_1 = 1, 2, \ldots$

$$
K_{i_1 i_2} = \begin{cases} G_1 & j_2 = j_1, \quad j_1 = 1, 2, \ldots, S \\ H & j_2 = j_1 - 1, \quad j_1 = 2, 3, \ldots, s \\ \mathbf{0}, & otherwise. \end{cases}
$$

$$
G_1 = \begin{cases} i_1\lambda_r, & k_2 = 1, \quad k_1 = 0 \\ & l_2 = 0, \quad l_1 = 0 \\ 0, & otherwise. \end{cases}
\tag{3}
$$

$$
H = \begin{cases} q\mu_{L_2}, & k_2 = 1, \quad k_1 = 2 \\ & l_2 = 0, \quad l_1 = 0 \\ q\mu_H, & k_2 = 1, \quad k_1 = 3 \\ & l_2 = 0, \quad l_1 = 0 \\ 0, & otherwise. \end{cases}
$$

$K_{i_1 i_2}, \quad i_2 = i_1$ and $i_1 = 0, 1, 2, \ldots$

$$
K_{00} = \begin{cases} F_{j_1} & j_2 = j_1, \quad j_1 = 0, 1, \ldots, S \\ C_{j_1} & j_2 = j_1 - 1, \quad j_1 = 1, 2, \ldots, S \\ D_0 & j_2 = Q, \quad j_1 = 0 \\ D_1 & j_2 = Q + j_1, \quad j_1 = 1, \ldots, s \\ \mathbf{0}, & otherwise. \end{cases}
\tag{4}
$$

and

$$K_{i_1 i_2} = \begin{cases} F_{j_1} & j_2 = j_1, \quad j_1 = 0, 1, \ldots, S \\ C_{j_1} & j_2 = j_1 - 1, \quad j_1 = 1, 2, \ldots, S \\ D_0 & j_2 = Q, \quad j_1 = 0 \\ D_1 & j_2 = Q + j_1, \quad j_1 = 1, \ldots, s \\ \mathbf{0}, & otherwise. \end{cases} \tag{5}$$

where

$$F_0 = \begin{cases} \lambda_H + \beta & k_2 = k_1, \quad k_1 = 0 \\ & l_2 = l_1 + 1, \quad l_1 = 0, 1, \ldots, N \\ -(\lambda_H + \beta + p\lambda_L) & k_2 = k_1, \quad k_1 = 0 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N-1 \\ -(\beta + p\lambda_L) & k_2 = k_1, \quad k_1 = 0 \\ & l_2 = l_1, \quad l_1 = N \\ 0, & otherwise. \end{cases}$$

$$F_{j_1} = \begin{cases} \lambda_H & k_2 = 3, \quad k_1 = 1 \\ & l_2 = l_1 = 0 \\ \mu_{L_1} & k_2 = k_1 + 1, \quad k_1 = 1 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N \\ \lambda_H & k_2 = k_1, \quad k_1 = 2, 3 \\ & l_2 = l_1 + 1, \quad l_1 = 0, 1, \ldots, N \\ (1-r)\lambda_H & \mathrm{k}_2 = k_1, \quad k_1 = 1 \\ & l_2 = l_1 + 1, \quad l_1 = 0 \\ \lambda_H & k_2 = k_1, \quad k_1 = 1 \\ & l_2 = l_1 + 1, \quad l_1 = 1, 2, \ldots, N-1 \\ -((j_1 - 1)\gamma + \lambda_H + H(s - j_1)\beta + \bar{\delta}_{0 i_1} i_1 \lambda_r) & k_2 = k_1, \quad k_1 = 0 \\ & l_2 = l_1, \quad l_1 = 0 \\ -((j_1 - 1)\gamma + \lambda_H + \mu_{L_1} + p\lambda_L + H(s - j_1)\beta) & k_2 = k_1, \quad k_1 = 1 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N-1 \\ -((j_1 - 1)\gamma + \mu_{L_1} + p\lambda_L + H(s - j_1)\beta) & k_2 = k_1, \quad k_1 = 1 \\ & l_2 = l_1, \quad l_1 = N \\ -((j_1 - 1)\gamma + \mu_{L_2} + \lambda_H + p\lambda_L + H(s - j_1)\beta) & k_2 = k_1, \quad k_1 = 2 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N-1 \\ -((j_1 - 1)\gamma + \mu_{L_2} + p\lambda_L + H(s - j_1)\beta) & k_2 = k_1, \quad k_1 = 2 \\ & l_2 = l_1, \quad l_1 = N \\ -((j_1 - 1)\gamma + \mu_H + \lambda_H + p\lambda_L + H(s - j_1)\beta) & k_2 = k_1, \quad k_1 = 3 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N-1 \\ -((j_1 - 1)\gamma + \mu_H + p\lambda_L + H(s - j_1)\beta) & k_2 = k_1, \quad k_1 = 3 \\ & l_2 = l_1, \quad l_1 = N \\ 0, & otherwise. \end{cases}$$

$$C_1 = \begin{cases} \mu_{L_2}, & k_2 = 0, \quad k_1 = 2 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N \\ \mu_H, & k_2 = 0, \quad k_1 = 3 \\ & l_2 = l_1, \quad l_1 = 0, 1, \ldots, N \\ 0, & otherwise. \end{cases}$$

$$C_{j_1} = \begin{cases} (j-1)\gamma, & k_2 = k_1 \quad k_1 = 0,1,2,3 \\ & l_2 = l_1, \quad l_1 = 0,1,\ldots,N \\ \mu_{L_2}, & k_2 = 0 \quad k_1 = 2 \\ & l_2 = 0 \quad l_1 = 0 \\ \mu_{L_2}, & k_2 = 3, \quad k_1 = 2 \\ & l_2 = l_1 - 1, \quad l_1 = 1,2,\ldots,N \\ \mu_H, & k_2 = 0, \quad k_1 = 3 \\ & l_2 = 0, \quad l_1 = 0 \\ \mu_H, & k_2 = 3 \quad k_1 = 3 \\ & l_2 = l_1 - 1 \quad l_1 = 1,2,\ldots,N \\ 0, & otherwise. \end{cases}$$

$$D_0 = \begin{cases} \beta & k_2 = 0, \quad k_1 = 0 \\ & l_2 = l_1, \quad l_1 = 0,1,\ldots,N \\ 0, & otherwise. \end{cases}$$

$$D_1 = \begin{cases} \beta, & k_2 = k_1, \quad k_1 = 0,1,2,3 \\ & l_2 = l_1, \quad l_1 = 0,1,2,\ldots,N \\ 0, & otherwise. \end{cases}$$

$$C_1 = \begin{cases} (1-q)\mu_{L_2}, & k_2 = 0, \quad k_1 = 2 \\ & l_2 = l_1, \quad l_1 = 0,1,\ldots,N \\ (1-q)\mu_H, & k_2 = 0, \quad k_1 = 3 \\ & l_2 = l_1, \quad l_1 = 0,1,\ldots,N \\ 0, & otherwise. \end{cases}$$

$$C_{j_1} = \begin{cases} (j-1)\gamma, & k_2 = k_1 \quad k_1 = 0,1,2,3 \\ & l_2 = l_1, \quad l_1 = 0,1,\ldots,N \\ (1-q)\mu_{L_2}, & k_2 = 0 \quad k_1 = 2 \\ & l_2 = 0 \quad l_1 = 0 \\ (1-q)\mu_{L_2}, & k_2 = 3, \quad k_1 = 2 \\ & l_2 = l_1 - 1, \quad l_1 = 1,2,\ldots,N \\ (1-q)\mu_H, & k_2 = 0, \quad k_1 = 3 \\ & l_2 = 0, \quad l_1 = 0 \\ (1-q)\mu_H, & k_2 = 3 \quad k_1 = 3 \\ & l_2 = l_1 - 1 \quad l_1 = 1,2,\ldots,N \\ 0, & otherwise. \end{cases}$$

3.1. Steady-State Analysis

According to the Neuts [41] matrix geometric method, the level-dependent quasi-birth-and-death process is truncated at the point $M \geq 1$, where M denotes the number of customers in orbit. After this truncation point, the LDQBD process changed into a level-independent QBD process. That is, the retrial customers follow the constant retrial policy to approach the system. To proceed further, we need to find the stability condition of the system at this truncation point. First, we require the stationary probability vector to the rate matrix K_M is formed by $K_M = K_{M(M-1)} + K_{MM} + K_{01}$, which is

$$K_M = \begin{array}{cc} & \begin{array}{cccccccccccc} (0) & (1) & (2) & \cdots & (s-1) & (s) & \cdots & (Q) & (Q+1) & \cdots & (S-1) & (S) \end{array} \\ \begin{array}{c} (0) \\ (1) \\ (2) \\ \vdots \\ (s-1) \\ (s) \\ \vdots \\ (Q) \\ (Q+1) \\ \vdots \\ (S-1) \\ (S) \end{array} & \begin{pmatrix} F_0 & & & & & & & D_0 & & & & \\ C_1 & F_1 & & & & & & & D_1 & & & \\ & C_2 & F_2 & & & & & & & & & \\ & & \ddots & \ddots & & & & & & \ddots & & \\ & & & & F_{s-1} & & & & & & D_1 & \\ & & & & C_s & F_s & & & & & & D_1 \\ & & & & & \ddots & \ddots & & & & & \\ & & & & & & & F_Q & & & & \\ & & & & & & & C_Q & F_{Q+1} & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & & F_{S-1} & \\ & & & & & & & & & & C_S & F_S \end{pmatrix} \end{array}$$

Theorem 1. *The steady-state probability vector $\pi = (\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(S)})$, related to the rate matrix K_M, is given by*

$$\pi^{(l)} = \pi^{(Q)}\Omega_l, \quad l = 0, 1, 2, \ldots S. \tag{6}$$

where

$$\Omega_l = \begin{cases} (-1)^{Q-l} C_Q F_{Q-1}^{-1} C_{Q-1} F_{Q-2}^{-1} \ldots C_{l+1} F_l^{-1}, & l = 0, 1, 2, \ldots, Q-1, \\ I, & l = Q, \\ (-1)^{2Q+1-l} \sum_{i=0}^{S-l} [(C_Q F_{Q-1}^{-1} C_{Q-1} F_{Q-2}^{-1} \ldots C_{s-i+1} F_{s-i}^{-1}) & \\ \qquad (D_1 F_{S-i})(C_{S-i} F_{S-i-1}^{-1} \ldots C_{l+1} F_l^{-1})] & l = Q+1, \ldots, S \end{cases}$$

and $\pi^{(Q)}$ can be obtained by solving

$$\pi^{(Q)} \Big[(-1)^Q C_Q F_{Q-1}^{-1} C_{Q-1} F_{Q-2}^{-1} \ldots C_1 F_0^{-1} D_0 + F_Q + (-1)^Q \Big[\sum_{i=0}^{s-1} (C_Q F_{Q-1}^{-1} C_{Q-1} F_{Q-2}^{-1} \ldots \\ C_{s-i+1} F_{s-i}^{-1})(D_1 F_{S-i})(C_{S-i} F_{S-i-1}^{-1} \ldots C_{Q+2} F_{Q+1}^{-1}) \Big] C_{Q+1} \Big] = \mathbf{0} \tag{7}$$

and

$$\pi^{(Q)} \Big[\sum_{i=0}^{Q-1} ((-1)^{Q-l} C_Q F_{Q-1}^{-1} C_{Q-1} F_{Q-2}^{-1} \ldots C_{l+1} F_l^{-1}) + I + \sum_{i=0}^{2Q+1-l} ((-1)^{2Q+1-l} (C_Q F_{Q-1}^{-1} \\ C_{Q-1} F_{Q-2}^{-1} \ldots C_{s-i+1} F_{s-i}^{-1})(D_1 F_{S-i})(C_{S-i} F_{S-i-1}^{-1} \ldots C_{l+1} F_l^{-1})) \Big] \boldsymbol{e} = 1 \tag{8}$$

Proof. We know that $\pi K_M = \mathbf{0}$ and $\pi \mathbf{e} = \mathbf{0}$. The first equality of the above produces the set of equations as follows:

$$\pi^{(l)} F_l + \pi^{(l+1)} C_{l+1} = \mathbf{0} \quad l = 0, 1, 2, \ldots, Q-1 \tag{9}$$

$$\pi^{(l-Q)} D_0 + \pi^{(l)} F_l + \pi^{(l+1)} C_{l+1} = \mathbf{0} \quad l = Q \tag{10}$$

$$\pi^{(l-Q)} D_1 + \pi^{(l)} F_l + \pi^{(l+1)} C_{l+1} = \mathbf{0} \quad l = Q+1, Q+2, \ldots, S-1 \tag{11}$$

$$\pi^{(l-Q)} D_1 + \pi^{(l)} F_l = \mathbf{0} \quad l = S \tag{12}$$

On solving the above equations recursively, we obtain Equation (6). Additionally, on substituting the value of Ω_l in Equation (6) and in the normalizing condition, we yield $\pi^{(Q)}$. □

After finding the stationary probability vector, π, to the generator matrix, K_M, the stability condition of the system can be derived easily. To derive such condition, we require the Neuts [41] inequality, which is formed at the truncation point, M. So using Theorem 1, we derive the stability condition of the system.

Theorem 2. *The stability condition at the truncation point, M, of the proposed system is given by*

$$\sum_{j=0}^{N} \pi^{(0,0,j)} p\lambda_L + \sum_{i=1}^{L} \sum_{K=1}^{3} \sum_{j=0}^{N} \pi^{(i,k,j)} p\lambda_L + \sum_{i=L+1}^{S} \sum_{K=1}^{3} \sum_{j=0}^{N} \pi^{(i,k,j)} r\lambda_H < \sum_{i=1}^{S} \pi^{(i,0,0)} M\lambda_r + \sum_{i=2}^{S} \pi^{(i,2,0)} q\mu_{L_2} + \sum_{i=2}^{S} \pi^{(i,3,0)} q\mu_H \tag{13}$$

Proof. From the Neuts [41] matrix-geometric approach, we obtain the following constrain to derive a stability condition:

$$\pi K_{01}\mathbf{e} < \pi K_{M(M-1)}\mathbf{e}$$

$$[\pi^{(0)}B_0, \pi^{(1)}B_1, \ldots, \pi^{(L)}B_1, \pi^{(L+1)}B_2, \ldots, \pi^{(S)}B_2]\mathbf{e} < [\pi^{(1)}G_1 + \pi^{(2)}H, \ldots, \pi^{(S-1)}G_1 + \pi^{(S)}H, \pi^{(S)}G_1]\mathbf{e}.$$

On computing the above inequality explicitly, we obtain
LHS:

$$\pi K_{01}\mathbf{e} = [\theta_0 + (\theta_1 + \Psi_1) + (\theta_2 + \Psi_2) + \ldots + (\theta_L + \Psi_L) + \theta_{(L+1)} + \ldots + \theta_S]$$

where
$\theta_0 = \sum_{j=0}^{N} \pi^{(0,0,j)} p\lambda_L,$
$\theta_i = \sum_{k=1}^{3} \sum_{j=0}^{N} \pi^{(i,k,j)} p\lambda_L \quad i = 1, 2, \ldots, S \quad and \quad \Psi_i = \pi^{(i,0,0)} r\lambda_H \quad i = 1, 2, \ldots, S$
simplifying further, we obtain

$$\sum_{j=0}^{N} \pi^{(0,0,j)} p\lambda_L + \sum_{i=1}^{L} \sum_{k=1}^{3} \sum_{j=0}^{N} \pi^{(i,k,j)} p\lambda_L + \sum_{i=L+1}^{S} \sum_{k=1}^{3} \sum_{j=0}^{N} \pi^{(i,k,j)} r\lambda_H \tag{14}$$

RHS:

$$\begin{aligned} \pi K_{M(M-1)}\mathbf{e} &= \pi^{(1,0,0)} M\lambda_r + \pi^{(2,2,0)} q\mu_{L_2} + \pi^{(2,3,0)} q\mu_H + pi^{(2,0,0)} M\lambda_r + \\ &\pi^{(3,2,0)} q\mu_{L_2} + \pi^{(3,3,0)} q\mu_H + \ldots + \pi^{(S-1,0,0)} M\lambda_r + \pi^{(S,2,0)} q\mu_{L_2} + \pi^{(S,3,0)} q\mu_H + \pi^{(S,0,0)} M\lambda_r \\ &= \sum_{i=1}^{S} \pi^{(i,0,0)} M\lambda_r + \sum_{i=2}^{S} \pi^{(i,2,0)} q\mu_{L_2} + \sum_{i=2}^{S} \pi^{(i,3,0)} q\mu_H. \end{aligned} \tag{15}$$

Therefore, from (14) and (15), the result holds. □

3.2. Computation of R-Matrix

After the truncation point M, the stationary probability vector u to the infinitesimal generator matrix K is dependent on the matrix R, where R is the minimal non-negative solution of the matrix quadratic equation $R^2K_{M,M-1} + RK_{MM} + K_{01} = \mathbf{0}$.

Theorem 3. *Due to the specific structure of K, R can be determined by*

$$R^2K_{M,M-1} + RK_{MM} + K_{01} = \boldsymbol{0} \tag{16}$$

where R is the minimal non-negative solution of the quadratic Equation (16). The square matrix R is of dimension $S[(3N+4)+(N+1)]$ *and is defined by*

$$R = \begin{pmatrix} R_{(0,0)} & R_{(0,1)} & R_{(0,2)} & \cdots & R_{(0,S)} \\ R_{(1,0)} & R_{(1,1)} & R_{(1,2)} & \cdots & R_{(1,S)} \\ R_{(2,0)} & R_{(2,1)} & R_{(2,2)} & \cdots & R_{(2,S)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{(S,S)} & R_{(S,1)} & R_{(S,2)} & \cdots & R_{(S,S)} \end{pmatrix}. \quad (17)$$

where the block $R_{(0,0)}$ *has dimension (N+1)(N+1) and of the form*

$$R_{(0,0)} = \begin{pmatrix} r_{00} & r_{01} & r_{02} & \cdots & r_{0N} \\ r_{10} & r_{11} & r_{12} & \cdots & r_{1N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N0} & r_{N1} & r_{N2} & \cdots & r_{NN} \end{pmatrix}.$$

The consecutive blocks in the first row with dimension (N+1)(3N+4) will be

$$R_{(0,j)} = \begin{bmatrix} R_{0j}^{(0,0)} & R_{0j}^{(0,1)} & R_{0j}^{(0,2)} & R_{0j}^{(0,3)} \end{bmatrix}, j = 1,2,\ldots,S.$$

where

$$R_{0j}^{(0,0)} = \begin{bmatrix} r_{00}^{(00,j0)} \\ r_{10}^{(00,j0)} \\ \vdots \\ r_{N0}^{(00,j0)} \end{bmatrix} \text{ and}$$

$$R_{0j}^{(0,k)} = \begin{bmatrix} r_{00}^{(00,jk)} & r_{01}^{(00,jk)} & r_{02}^{(00,jk)} & \cdots & r_{0N}^{(00,jk)} \\ r_{10}^{(00,jk)} & r_{11}^{(00,jk)} & r_{12}^{(00,jk)} & \cdots & r_{1N}^{(00,jk)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N0}^{(00,jk)} & r_{N1}^{(00,jk)} & r_{N2}^{(00,jk)} & \cdots & r_{NN}^{(00,jk)} \end{bmatrix}, j = 1,2,\ldots,S., \quad k = 1,2,3.$$

similarly, for the blocks in the first column, $R_{(i,0)}, i = 1,2,\ldots,S.$

$$R_{(i,0)} = \begin{bmatrix} R_{i0}^{(0,0)} \\ R_{i0}^{(1,0)} \\ R_{i0}^{(2,0)} \\ R_{i0}^{(3,0)} \end{bmatrix}, i = 1,2,\ldots,S.$$

where

$$R_{0}^{(0,0)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and}$$

$$R_{i0}^{(k,0)} = \begin{bmatrix} r_{00}^{(ik,00)} & r_{01}^{(ik,00)} & r_{02}^{(ik,00)} & \cdots & r_{0N}^{(ik,00)} \\ r_{10}^{(ik,00)} & r_{11}^{(ik,00)} & r_{12}^{(ik,00)} & \cdots & r_{1N}^{(ik,00)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N0}^{(ik,00)} & r_{N1}^{(ik,00)} & r_{N2}^{(ik,00)} & \cdots & r_{NN}^{(ik,00)} \end{bmatrix}, i = 1,2,\ldots,S, \quad k = 1,2,3.$$

Considering the remaining blocks $R_{(i,j)}, i = 1,2,\ldots,S$ *and* $j = 1,2,\ldots,S$

$$R_{(i,j)} = \begin{bmatrix} R_{ij}^{(0,0)} & R_{ij}^{(0,1)} & R_{ij}^{(0,2)} & R_{ij}^{(0,3)} \\ R_{ij}^{(1,0)} & R_{ij}^{(1,1)} & R_{ij}^{(1,2)} & R_{ij}^{(1,3)} \\ R_{ij}^{(2,0)} & R_{ij}^{(2,1)} & R_{ij}^{(2,2)} & R_{ij}^{(2,3)} \\ R_{ij}^{(3,0)} & R_{ij}^{(3,1)} & R_{ij}^{(3,2)} & R_{ij}^{(3,3)} \end{bmatrix}$$

where

$$R_{ij}^{(k,0)} = \begin{bmatrix} r_{00}^{(ik,j0)} \\ r_{01}^{(ik,j0)} \\ r_{02}^{(ik,j0)} \\ \vdots \\ r_{0N}^{(ik,j0)} \end{bmatrix}$$

$$R_{ij}^{(k,l)} = \begin{bmatrix} r_{00}^{(ik,jl)} & r_{01}^{(ik,jl)} & r_{02}^{(ik,jl)} & \cdots & r_{0N}^{(ik,jl)} \\ r_{10}^{(ik,jl)} & r_{11}^{(ik,jl)} & r_{12}^{(ik,jl)} & \cdots & r_{1N}^{(ik,jl)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N0}^{(ik,jl)} & r_{N1}^{(ik,jl)} & r_{N2}^{(ik,jl)} & \cdots & r_{NN}^{(ik,jl)} \end{bmatrix}$$

$$R_{ij}^{(0,l)} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \textit{ and}$$

$$R_{ij}^{(0,0)} = \begin{bmatrix} 0 \end{bmatrix},\ i = 1,2,\ldots,S.,\ j = 1,2,\ldots,S.,\ k = 1,2,3. \textit{ and } l = 1,2,3.$$

Proof. The structure of R matrix has $S[(3N+3)+(N+1)]$ non-zero rows and can be formed depending on the special structure of K_{01} matrix .

$$K_{01} = \begin{array}{c} \\ (0) \\ (1) \\ (2) \\ \vdots \\ (L) \\ (L+1) \\ \vdots \\ (S) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0) & (1) & (2) & \cdots & (L) & (L+1) & \cdots & (S) \end{array} \\ \begin{pmatrix} B_0 & & & & & & & \\ & B_1 & & & & & & \\ & & B_1 & & & & & \\ & & & \ddots & & & & \\ & & & & B_1 & & & \\ & & & & & B_2 & & \\ & & & & & & \ddots & \\ & & & & & & & B_2 \end{pmatrix} \end{array}$$

where the blocks B_0 has all positive rows and B_1, B_2 has one zero row in each. Therefore, the R-matrix has $S[(3N+3)+(N+1)]$ non-zero rows. Let us assume the R matrix is as in (17). With this assumption, exploiting the structure of the matrices $K_{M,M-1}, K_{MM}$ and R in Equation (16), we obtain the following set of nonlinear equations:

$$R_{i0}B_0 + R_{i1}C_1 + \delta_{i0}B_0 = \mathbf{0}, \quad i = 0,1,2,\ldots,S. \tag{18}$$

$$G_1\left\{\sum_{j=0}^{S} R_{ij}R_{jk}\right\} + H\left\{\sum_{j=0}^{S} R_{ij}R_{j(k+1)}\right\} + R_{ik}F_k + R_{i(k+1)}C_{k+1} + \delta_{ik}B_1 = \mathbf{0} \quad i = 0,1,2,\ldots,S, \quad k = 1,2,\ldots,L. \tag{19}$$

$$G_1\left\{\sum_{j=0}^{S} R_{ij}R_{jk}\right\} + H\left\{\sum_{j=0}^{S} R_{ij}R_{j(k+1)}\right\} + R_{ik}F_k + R_{i(k+1)}C_{k+1} + \delta_{ik}B_2 = \mathbf{0} \quad i = 0,1,2,\ldots,S, \quad k = L+1,L+2,\ldots,Q-1. \tag{20}$$

$$G_1\left\{\sum_{j=0}^{S} R_{ij}R_{jQ}\right\} + H\left\{\sum_{j=0}^{S} R_{ij}R_{j(Q+1)}\right\} + R_{iQ}F_Q + R_{i0}D_0 + R_{i(Q+1)}C_{Q+1} + B_2 = \mathbf{0} \quad i = Q. \tag{21}$$

$$G_1\left\{\sum_{j=0}^{S} R_{ij}R_{jk}\right\} + H\left\{\sum_{j=0}^{S} R_{ij}R_{j(k+1)}\right\} + R_{ik}F_k + R_{i0}D_0 + R_{i(k+1)}C_{k+1} + \delta_{ik}B_2 = \mathbf{0} \quad i = 0,1,2,\ldots,S, \quad k = Q+1,Q+2,\ldots,S-1. \tag{22}$$

$$G_1\left\{\sum_{j=0}^{S} R_{ij}R_{jS}\right\} + R_{i(S-Q)}D_1 + R_{iS}F_S + \delta_{iS}B_2 = \mathbf{0} \quad i = 0,1,2,\ldots,S. \tag{23}$$

After solving all Equations (18)–(23) by the Gauss–Seidel iterative method, we obtain the entries of the R-matrix. □

Now, we derive the stationary probability vector, $u = (u(0), u(1), u(2), \ldots)$, to the infinitesimal generator matrix, K, as given below.

Theorem 4. *Due to the special structure of K, the stationary probability vector $u = (u(0), u(1), u(2), \ldots)$ can be expressed as*

$$u(i+M-1) = u(M-1)R^i; \qquad i \geq 0 \tag{24}$$

where the matrix R is the unique non-negative solution with spectral radius less than 1, of the following equation:

$$R^2K_{M,M-1} + RK_{MM} + K_{01} = \mathbf{0} \tag{25}$$

yields the solution to the vectors $(u(0), u(1), \ldots)$ as

$$u(i) = \begin{cases} \delta v(0)\prod_{j=i}^{M} K_{j,j-1}(-K_{j-1})^{-1} & 0 \leq i \leq M-1 \\ \delta v(0)R^{i-M} & i \geq M \end{cases} \tag{26}$$

where $v(0)$ is the unique solution of the system

$$\begin{aligned} v(0)(K_{MM} + RK_{M+1,M}) &= \mathbf{0} \\ v(0)(I-R)^{-1}\boldsymbol{e} &= 1 \end{aligned} \tag{27}$$

and

$$\delta = [1 + v(0)\sum_{i=0}^{M-1}\prod_{j=i}^{M} K_{j,j-1}(-K_{j-1})^{-1}\mathbf{e}]^{-1}. \tag{28}$$

Proof. The sub-vector, $(u(0), \ldots, u(M-1))$, can be obtained by solving

$$u(0)K_{00} + u(1)K_{21} = \mathbf{0}, \tag{29}$$

$$u(i-1)K_{01} + u(i)K_{ii} + u(i+1)K_{i+1,i} = \mathbf{0}, \quad 1 \le i \le M-1. \tag{30}$$

Now, simplifying (29) and (30) over the range $1 \le i \le M-1$, we obtain

$$u(i) = u(i+1)K_{i+1,i}(-K_i)^{-1} \quad 1 \le i \le M-1$$

where

$$K'_i = \begin{cases} K_{00} & i = 0 \\ K_{ii} + K_{i,i-1}(-K'_{i-1})^{-1} & 1 \le i \le M. \end{cases}$$

Now, by applying block Gaussian elimination, the partitioned sub-vector (u(M), u(M+1), $\cdots$) corresponding to non-boundary states, satisfies the relation.

$$\begin{bmatrix} u(M) & u(M+1) & \cdots \end{bmatrix} \begin{bmatrix} K'_M & K_{01} & 0 & 0 & \cdots \\ K_{M+1,M} & K_{M+1,M+1} & K_0 & 0 & \cdots \\ 0 & K_{M+2,M+1} & K_{M+2,M+2} & K_{01} & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0} \tag{31}$$

Let

$$\delta = \sum_{i=M}^{\infty} u(i)\mathbf{e} \tag{32}$$

$$\text{and} \quad v(i) = \delta^{-1}u(M+i), \quad i \ge 0 \tag{33}$$

From (31), we obtain

$$u(M)K'_M + u(M+1)K_{M+1,M} = \mathbf{0}$$
$$u(M+i) = u(M+i-1)R, \quad i \ge 1$$

Using (25), we obtain

$$u(M)K'_M + u(M)RK_{M+1,M} = \mathbf{0}$$
$$v(0)K'_M + v(1)K_{M+1,M} = \mathbf{0} \tag{34}$$

Since $\sum_{i=0}^{\infty} v(i)\mathbf{e} = 1$, we obtain

$$v(0)(I-R)^{-1}\mathbf{e} = 1 \tag{35}$$

Additionally,

$$u(i) = \delta v(0)R^{i-M}, \quad i \ge M. \tag{36}$$

From (32), we obtain

$$u(i) = \delta v(0) \prod_{j=i}^{M} K_{j,j-1}(-K'_{j-1})^{-1}, \quad 0 \leq i \leq M-1 \tag{37}$$

From Equations (34) and (35), we obtain (27)

where $v(0)$ is the unique solution of the system (28). Additionally, from (36) and (37), we obtain (28)

$$\delta = [1 + v(0) \sum_{i=0}^{M-1} \prod_{j=i}^{M} K_{j,j-1}(-K'_{j-1})^{-1}\mathbf{e}]^{-1}$$

□

3.3. System Performance Measures

In this section, we put forward some important performance measures in order to construct the mean total cost of the proposed model.

1. Expected inventory level:

$$E_I = \sum_{i_1=0}^{\infty} \sum_{j_1=1}^{S} j_1 \left\{ u(i_1, j_1, 0, 0) + \sum_{k_1=1}^{3} \sum_{l_1=0}^{N} u(i_1, j_1, k_1, l_1) \right\}$$

2. Expected perishable rate:

$$E_P = \sum_{i_1=0}^{\infty} \left\{ \sum_{j_1=1}^{S} j_1 \gamma u(i_1, j_1, 0, 0) + \sum_{j_1=1}^{S} \sum_{k_1=1}^{3} \sum_{l_1=0}^{N} (j_1 - 1)\gamma u(i_1, j_1, k_1, l_1) \right\}$$

3. Expected reorder rate:

$$E_R = \sum_{i_1=0}^{\infty} (s+1)\gamma u(i_1, (s+1), 0, 0) + \sum_{i_1=0}^{\infty} \sum_{k_1=1}^{3} \sum_{l_1=0}^{N} s\gamma u(i_1, (s+1), k_1, l_1) + \sum_{i_1=0}^{\infty} \mu_{L_2} u(i_1, (s+1), 2, 0) + \sum_{i_1=0}^{\infty} \sum_{l_1=0}^{N} \mu_H \, u(i_1, (s+1), 3, l_1).$$

4. Expected number of HP customers in the waiting hall:

$$E_{WH} = \sum_{i_1=0}^{\infty} \sum_{l_1=0}^{N} l_1 \left\{ u(i_1, 0, 0, l_1) + \sum_{j_1=1}^{S} \sum_{k_1=1}^{3} u(i_1, j_1, k_1, l_1) \right\}$$

5. Expectation of HP customer entering into the waiting hall:

$$E_{EWH} = \sum_{i_1=0}^{\infty} \left\{ \sum_{l_1=0}^{N-1} \lambda_H \, u(i_1, 0, 0, l_1) + \sum_{j_1=1}^{S} \lambda_H u(i_1, j_1, 0, 0) + \sum_{j_1=1}^{S} \sum_{k_1=1}^{3} \sum_{l_1=0}^{N-1} \lambda_H \, u(i_1, j_1, k_1, l_1) \right\}$$

6. Expected waiting time of an HP:

$$E_{WHP} = E_{WH}/E_{EWH}$$

7. Expected number of LP customers in the orbit:

$$E_O = \sum_{i_1=0}^{\infty} i_1 \, u(i_1)\mathbf{e}$$

8. Expected number of LP customers lost:
$$E_{LPL} = \sum_{i_1=0}^{\infty} \sum_{l_1=0}^{N} (1-p)\lambda_L \, u(i_1, 0, 0, l_1) + \sum_{i_1=0}^{\infty} \sum_{1_1=1}^{S} \sum_{k_1=1}^{3} \sum_{l_1=0}^{N} (1-p)\lambda_L \, u(i_1, j_1, k_1, l_1)$$
$$+ \sum_{i_1=0}^{\infty} \sum_{j_1=1}^{L} \sum_{l_1=0}^{N} r\lambda_H \, u(i_1, j_1, 0, l_1)$$

9. Expected number of HP customers lost:

$$E_{HPL} = \sum_{i_1=0}^{\infty} \left\{ \lambda_H \, u(i_1, 0, 0, N) + \sum_{j_1=1}^{S} \sum_{k_1=1}^{3} \lambda_H \, u(i_1, j_1, k_1, N) \right\}$$

10. Expected number of times an HP customer interrupts an LP customer:

$$E_{HPIL} = \sum_{i_1=0}^{\infty} \sum_{j_1=1}^{L} r\lambda_H \, u(i_1, j_1, 1, 0)$$

11. Probability of server being idle:

$$E_{SI} = \sum_{i_1=0}^{\infty} \left\{ \sum_{l_1=0}^{N} u(i_1, 0, 0, l_1) + \sum_{j_1=1}^{S} u(i_1, j_1, 0, 0) \right\}$$

12. Probability of server being busy:

$$E_{SB} = \sum_{i_1=0}^{\infty} \sum_{j_1=1}^{S} \sum_{k_1=1}^{3} \sum_{l_1=0}^{N} u(i_1, j_1, k_1, l_1)$$

13. Expected number of times a server carries out the orbital search:

$$E_{OS} = \sum_{i_1=1}^{\infty} \sum_{j_1=1}^{S} \{ q\mu_{L_2} \, u(i_1, j_1, 2, 0) + q\mu_H \, u(i_1, j_1, 3, 0) \}$$

14. Overall rate of retrial:

$$E_{ORR} = \sum_{i_1=1}^{\infty} i_1 \lambda_r \, u(i_1)\mathbf{e}$$

15. Successful rate of retrial:

$$E_{SRR} = \sum_{i_1=1}^{\infty} \sum_{j_1=1}^{S} i_1 \lambda_r \, u(i_1, j_1, 0, 0)$$

16. Fraction of successful rate of retrial:

$$E_{FSRR} = E_{SRR} / E_{ORR}$$

3.4. Cost Analysis

The different costs we used are as follows:
c_h = cost of carrying the inventory per unit item per unit time;
c_r = cost of placing an order;
c_p = perishable cost of an item per unit time;
c_{wh} = cost due to waiting time of a HP customer per unit time;
c_{wo} = cost due to waiting time of a LP customer per unit time;
c_{ll} = cost due to loss of an LP customer per unit time;
c_{hl} = cost due to loss of an HP customer per unit time;
The expected total cost (TC) is given by:

$$TC(S,s) = c_h * E_I + c_p * E_P + c_r * E_R + c_{wh} * E_{WHP} + c_{wo} * E_O + c_{ll} * E_{LPL} + c_{hl} * E_{HPL}$$

4. Numerical Discussions

In this section, we discuss significant examples that illustrate the effectiveness of the proposed queueing–inventory model more practically. Through numerical discussion, we are provided with a broader view of the model in a more practical situation. Additionally, to enhance understanding and to find solutions to real-life problems, the results discussed would be more useful. To do so, we first fix the values for parameters. The optimal set of values for the parameters and costs are $L = 6$; $N = 8$; $Q = S - s$; $\lambda_H = 0.90$; $\lambda_L = 0.47$; $\lambda_r = 0.64$; $\mu_{L_1} = 2.45$; $\mu_{L_2} = 2.63$; $\mu_H = 2.79$; $p = 0.6$; $q = 0.67$; $c_h = 0.006$; $c_r = 2.4$; $c_p = 0.1$; $c_{wh} = 1.4$; $c_{wo} = 0.5$; $c_{ll} = 2$; $c_{hl} = 1$.

Example 1. *Optimal Analysis: To determine the optimal value for S, s, and r, we vary S and s under a finite set of values and r from 0 to 1 as it is the probability of interruption. Therefore, the total cost as a function of S and s for different values of r is tabulated in Tables 1–3.*

Table 1. Convex with $r = 0$.

S | s	1	2	3	4	5	6	7
28	3.187610	3.132378	3.114067	**3.102593**	3.110016	3.132837	3.164871
29	3.170042	3.116397	3.108295	**3.093581**	3.099043	3.120513	3.151056
30	3.157677	3.103865	3.097418	**3.088793**	3.092117	3.111897	3.140980
31	3.153721	3.101571	3.095904	**3.089001**	3.095888	3.114345	3.142301
32	3.162313	3.113600	3.111644	**3.106273**	3.114821	3.135832	3.163509

Table 2. Convex with $r = 0.23$.

S | s	1	2	3	4	5	6	7
28	3.156901	3.107505	3.094919	**3.087682**	3.098520	3.119306	3.155629
29	3.140263	3.092402	3.089834	**3.079195**	3.088026	3.107578	3.142287
30	3.128573	3.080375	3.079441	**3.074810**	3.081410	3.099400	3.132570
31	3.125040	3.078352	3.078033	**3.075131**	3.085328	3.102140	3.134130
32	3.133964	3.090639	3.094001	**3.092575**	3.104508	3.124117	3.155706

In Tables 1–3, the values shown in bold are the least in the respective rows and the values underlined are the least in that respective column. Therefore, the value that is bold and underlined is the optimal value of total cost.

Case (i): Non-discretionary priority discipline.

From Table 1, we see that for $r = 0$ the optimum value of total cost is obtained at $S^* = 30$, $s^* = 4$ and $TC^* = 3.088793$. This provides the optimum result when the interruption of an HP customer is not allowed.

Table 3. Convex with $r = 1$.

S | s	1	2	3	4	5	6	7
28	3.027256	**3.005787**	3.018169	3.027169	3.050287	3.068396	3.120758
29	3.014473	**2.994246**	3.015881	3.020835	3.041772	3.058894	3.109163
30	3.005563	**2.984233**	3.007418	3.018146	3.036500	3.052381	3.100798
31	3.003744	**2.983274**	3.006425	3.018997	3.041181	3.056248	3.103275
32	3.013988	**2.996584**	3.023343	3.037225	3.061479	3.080069	3.126179
33	3.022344	**3.005740**	3.032852	3.047970	3.072497	3.092072	3.138950

Case (ii): Discretionary priority discipline.

- *Mixed priority: At $r = 0.23$, the optimum value is obtained similarly at $S^* = 30$, $s^* = 4$ and $TC^* = 3.074810$. as shown in Table 2. This TC^* shows the impact of interruption, based on the discretionary priority. On comparing TC^* with case (i), as we expected, the discretionary priority service process gives the minimum optimal total cost rather than the usual priority service patterns.*
- *Preemptive priority discipline: At $r = 1$, the optimum total cost is obtained at $S^* = 31$, $s^* = 2$, and $TC^* = 2.983274$, as shown in Table 3. Even though the TC^* of this case is minimum than that of the case (i), we conclude that case (ii) provides the best service discipline. This is because, when a company allows preemptive priority, it may lose its LP customers. The growth of the company obviously depends upon all types of customers. So, the company must satisfy them by providing their best service. In such a way, discretionary priority discipline is considered the best one.*

Example 2. *In this example we study the influence of β, γ, λ_H, λ_r, and μ_H under various interruption rates, r, on the total cost. The results are tabulated in Table 4 and observations made are the following: with the increase in perishable rate as expected, the total cost increases and, on increased reorder rate, the total cost decreases. Additionally, the TC^* increases when the arrival rate increases and service rate decreases.*

Case (i): Non-discretionary priority discipline.

For $r = 0$, the optimum total cost $TC^* = 2.925533$ is obtained at $\gamma = 0.05$ and $\beta = 1.8$. For higher values of either arrival rate or perishable rate, the TC^* increases.

Case (ii): Discretionary priority discipline.

- *Mixed priority: At r = 0.5, the total cost TC^* follows the same pattern of increment and decrement as in case (i), for all the parameters considered in Table 4, respectively. The optimum total cost obtained is $TC^* = 2.890126$ at $\gamma = 0.05$ and $\beta = 1.8$. Thus, the optimum value obtained in discretionary priority is less than that of non-preemptive priority.*
- *Preemptive priority discipline: At $r = 1$, for the same value of the parameter as in case (i), we obtain the optimum total cost to be $TC^* = 2.48149$, which is less than those of non-preemptive and discretionary priority disciplines. However, depending on this TC^*, we cannot make the inference that preemptive priority discipline is economical. Although it seems like preemptive priority discipline is more profitable than the other priority discipline, it leads to the loss of new patrons for the business. A new LP customer who is ignored or made to wait for a long duration may leave the system before trying the product even once. Hence, discretionary priority is best suited to increase the customer base.*

Table 4. The effect of the parameters γ, β, λ_H, λ_r, μ_H, and r.

γ	λ_H	λ_r	μ_H	$r=0$		$r=0.5$		$r=1$	
				$\beta=1.7$	$\beta=1.8$	$\beta=1.7$	$\beta=1.8$	$\beta=1.7$	$\beta=1.8$
0.05	0.85	0.60	2.75	2.979013	2.973860	2.942602	2.938585	2.899332	2.896864
			2.80	2.954551	2.948898	2.918055	2.913559	2.874628	2.871711
			2.85	2.931692	2.925533	2.895107	2.890126	2.851522	2.848149
		0.70	2.75	2.990586	2.985467	2.954113	2.950135	2.910731	2.908310
			2.80	2.966412	2.960791	2.929858	2.925397	2.886322	2.883448
			2.85	2.943827	2.937697	2.907188	2.902239	2.863496	2.860163
	0.95	0.60	2.75	2.991737	2.987089	2.957554	2.953994	2.916601	2.914532
			2.80	2.966064	2.960974	2.931800	2.927819	2.890698	2.888237
			2.85	2.941959	2.936423	2.907617	2.903210	2.866367	2.863511
		0.70	2.75	2.991737	2.996258	2.957554	2.963084	2.916601	2.923494
			2.80	2.966064	2.970501	2.931800	2.937274	2.890698	2.897572
			2.85	2.941959	2.946289	2.907617	2.913009	2.866367	2.873197
0.1	0.85	0.60	2.75	3.106973	3.099336	3.064676	3.058380	3.014288	3.009778
			2.80	3.083714	3.075479	3.041333	3.034460	2.990770	2.985716
			2.85	3.062043	3.053205	3.019577	3.012126	2.968839	2.963238
		0.70	2.75	3.118530	3.110927	3.076163	3.069904	3.025633	3.021167
			2.80	3.095568	3.087364	3.053121	3.046281	3.002421	2.997407
			2.85	3.074181	3.065369	3.031652	3.024231	2.980781	2.975217
	0.95	0.60	2.75	3.118081	3.111224	3.078620	3.073032	3.031223	3.027343
			2.80	3.093557	3.086174	3.054006	3.047916	3.006432	3.002082
			2.85	3.070585	3.062675	3.030949	3.024355	2.983203	2.978382
		0.70	2.75	3.127136	3.120334	3.087575	3.082047	3.040010	3.036197
			2.80	3.102982	3.095650	3.063340	3.057305	3.015606	3.011320
			2.85	3.080360	3.072498	3.040641	3.034099	2.992743	2.987982
0.15	0.85	0.60	2.75	3.227871	3.124522	3.180782	3.172613	3.124522	3.118335
			2.80	3.205585	3.101958	3.158412	3.149586	3.101958	3.095151
			2.85	3.184858	3.080966	3.137602	3.128126	3.080966	3.073538
		0.70	2.75	3.239407	3.229767	3.239407	3.184112	3.135819	3.129678
			2.80	3.217425	3.207099	3.217425	3.161390	3.113568	3.106804
			2.85	3.196987	3.185980	3.196987	3.140222	3.092876	3.085485
	0.95	0.60	2.75	3.237823	3.229153	3.194032	3.186784	3.141275	3.135916
			2.80	3.214209	3.204947	3.170322	3.162508	3.117363	3.111471
			2.85	3.192134	3.182281	3.148160	3.139778	3.095006	3.088582
		0.70	2.75	3.246812	3.238204	3.202911	3.195730	3.149959	3.144675
			2.80	3.223575	3.214372	3.179589	3.171837	3.126443	3.120623
			2.85	3.201857	3.192058	3.157792	3.149469	3.104462	3.098105

Example 3. *Waiting Time Analysis: In this example, we investigate the waiting times of HP customers and LP customers for changes in arrival rate and service rate. The data are presented in graphs in Figures 1–6, with the following observations.*

Case (i): Non-discretionary priority discipline.

At $r=0$, the measure E_{WHP} increases with an increase in arrival rate and decreases with an increase in service rate. Similarly, the waiting time of an LP customer, E_{WLP}, is directly proportional to the arrival rate and inversely proportional to the service rates, μ_{L_1} and μ_{L_2}. Additionally, with an increase in retrial rate, E_{WLP} decreases.

Case (ii): Discretionary priority discipline.

- *Mixed priority: At $r = 0.5$, the measures E_{WHP} and E_{WLP} show the same variations as in case (i) for the respective parameters, but the difference is that the waiting time of HP customers is less, and the waiting time of LP customers is higher, compared with those in case (i).*
- *Preemptive Priority discipline: At $r = 1$, similar to case (i), the waiting times remain with the same variation with respect to the parameters considered in Figures 1–6, and the measure E_{WHP} is considerably less and E_{WLP} is much higher compared with those in case (i).*

With these observations, we can understand that the waiting time of HP customers will reach the peak with non-preemptive priority and that the waiting time of LP customers will reach the peak with preemptive priority. However, discretionary priority discipline is preferable to the other disciplines for maintaining an optimal waiting time for each type of customer.

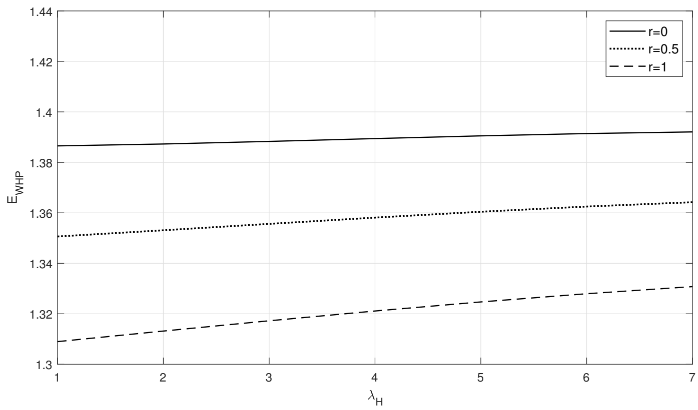

Figure 1. r vs. λ_H on E_{WHP}.

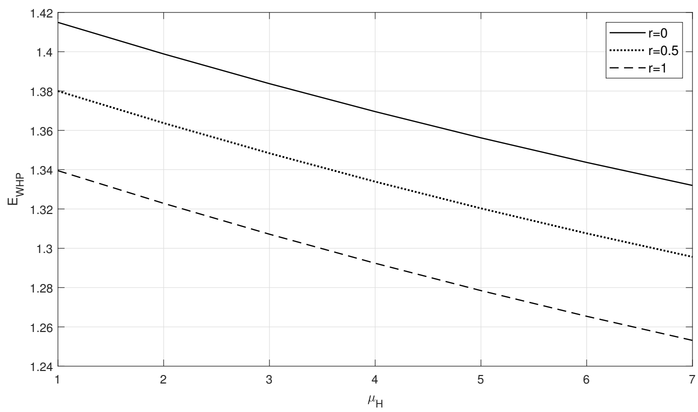

Figure 2. r vs. μ_H on E_{WHP}.

Example 4. *This example examines the expected number of HP customers and LP customers lost. The results of E_{HPL} and E_{LPL} are tabulated in Tables 5 and 6, and the observations are as follows:*

Case (i): Non-discretionary Priority discipline.

In terms of $r = 0$, E_{HPL}, and E_{LPL}, they show an increase in the arrival rate of HP customers. However, with the HP customers, a gradual decline in the loss of customers can be noticed with an increase in waiting for hall capacity and reorder rate. For E_{LPL}, the loss slightly reduces with an increase in service rates.

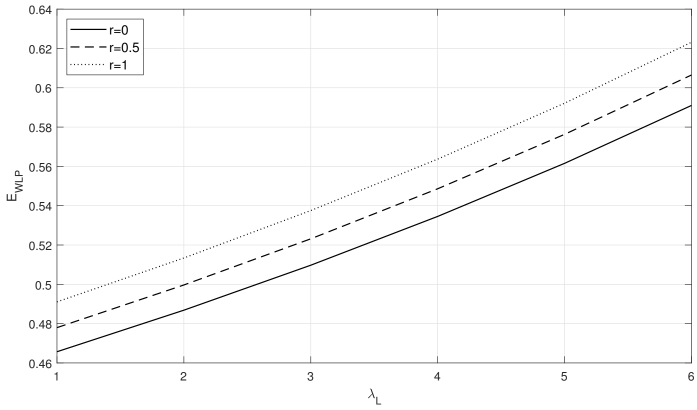

Figure 3. r vs. λ_L on E_{WLP}.

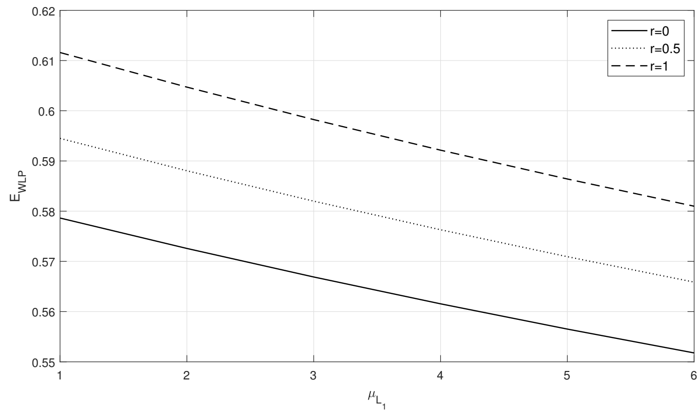

Figure 4. r vs. μ_{L_1} on E_{WLP}.

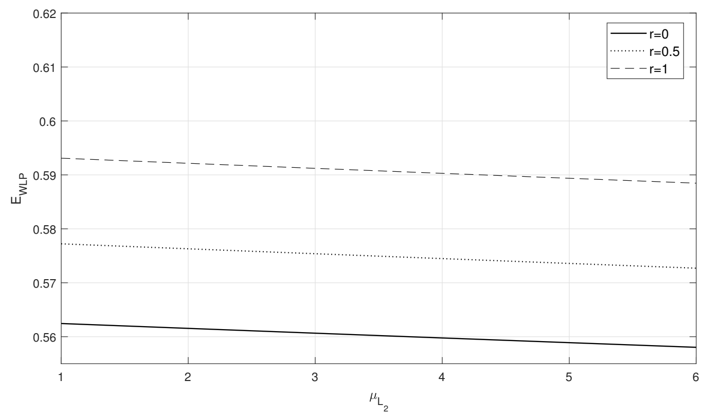

Figure 5. r vs. μ_{L_2} on E_{WLP}.

Case (ii): Discretionary priority discipline.

- *Mixed priority: At* $r = 0.5$*, the loss of HP customers encountered by following discretionary priority is less compared with non-preemptive priority. In the same way, the loss of LP customers by following discretionary priority is greater compared with non-preemptive priority.*
- *Preemptive Priority: As seen previously, the measure* E_{HPL} *is a little less compared with case (i), and* E_{LPL} *is slightly greater than mixed priority. Thus, the loss of an HP customer heightens with non-discretionary priority, and an LP customer rises with preemptive priority. However, the intensified loss of any one type of customer will affect the reputation of the organization, which in turn decreases the growth of the business. Hence, to maintain an admissible loss of any type of customer, the following discretionary priority is advisable.*

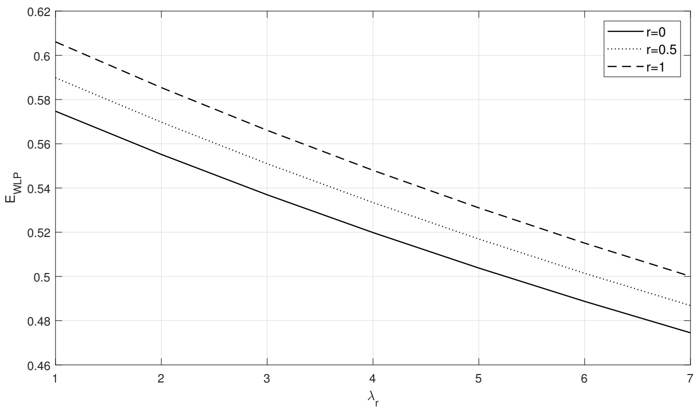

Figure 6. r vs. λ_r on E_{WLP}.

Table 5. Expected number of HP customers lost.

N	β	μ_H	$r = 0$		$r = 0.5$		$r = 1$	
			$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$
7	1.6	2.75	0.003336	0.005413	0.003179	0.005173	0.002990	0.004882
		2.80	0.003239	0.005245	0.003083	0.005006	0.002894	0.004717
		2.85	0.003156	0.005097	0.002999	0.004859	0.002811	0.004570
	1.7	2.75	0.003173	0.005191	0.003024	0.004963	0.002845	0.004687
		2.80	0.003068	0.005012	0.002920	0.004786	0.002742	0.004511
		2.85	0.002977	0.004854	0.002829	0.004628	0.002652	0.004355
	1.8	2.75	0.003054	0.005025	0.002911	0.004806	0.002741	0.004542
		2.80	0.002943	0.004839	0.002802	0.004622	0.002633	0.004360
		2.85	0.002846	0.004673	0.002706	0.004457	0.002537	0.004197
8	1.6	2.75	0.001648	0.002828	0.001570	0.002702	0.001476	0.002549
		2.80	0.001591	0.002723	0.001514	0.002598	0.001421	0.002447
		2.85	0.001543	0.002631	0.001466	0.002507	0.001373	0.002358
	1.7	2.75	0.001547	0.002683	0.001474	0.002565	0.001387	0.002422
		2.80	0.001486	0.002572	0.001414	0.002456	0.001327	0.002314
		2.85	0.001432	0.002474	0.001361	0.002359	0.001275	0.002219
	1.8	2.75	0.001474	0.002576	0.001405	0.002464	0.001323	0.002328
		2.80	0.001409	0.002461	0.001341	0.002350	0.001260	0.002217
		2.85	0.001353	0.002358	0.001286	0.002249	0.001205	0.002117
9	1.6	2.75	0.000821	0.001491	0.000782	0.001424	0.000735	0.001344
		2.80	0.000789	0.001426	0.000750	0.001361	0.000704	0.001282
		2.85	0.000761	0.001371	0.000723	0.001306	0.000677	0.001228
	1.7	2.75	0.000760	0.001399	0.000724	0.001337	0.000681	0.001262
		2.80	0.000725	0.001331	0.000690	0.001271	0.000647	0.001197
		2.85	0.000695	0.001272	0.000660	0.001212	0.000618	0.001140
	1.8	2.75	0.000716	0.001331	0.000683	0.001273	0.000642	0.001203
		2.80	0.000679	0.001261	0.000646	0.001205	0.000607	0.001136
		2.85	0.000647	0.001199	0.000615	0.001144	0.000576	0.001076

Table 6. Expected number of LP customers lost.

μ_H	μ_{L_2}	μ_{L_1}	$r = 0$		$r = 0.5$		$r = 1$	
			$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$
2.75	2.60	2.40	0.223480	0.223851	0.223730	0.224161	0.224073	0.224585
		2.45	0.223453	0.223827	0.223700	0.224133	0.224038	0.224551
		2.50	0.223427	0.223803	0.223671	0.224107	0.224004	0.224519
	2.65	2.40	0.223458	0.223831	0.223708	0.224141	0.224051	0.224566
		2.45	0.223431	0.223807	0.223678	0.224113	0.224016	0.224532
		2.50	0.223405	0.223783	0.223649	0.224087	0.223983	0.224500
	2.70	2.40	0.223436	0.223812	0.223686	0.224122	0.224030	0.224547
		2.45	0.223409	0.223787	0.223656	0.224094	0.223995	0.224513
		2.50	0.223383	0.223764	0.223628	0.224068	0.223962	0.224481
2.80	2.60	2.40	0.223403	0.223775	0.223653	0.224086	0.223998	0.224514
		2.45	0.223374	0.223750	0.223622	0.224057	0.223962	0.224479
		2.50	0.223347	0.223726	0.223592	0.224030	0.223927	0.224445
	2.65	2.40	0.223379	0.223754	0.223630	0.224065	0.223975	0.224493
		2.45	0.223351	0.223729	0.223598	0.224037	0.223939	0.224458
		2.50	0.223324	0.223705	0.223569	0.224009	0.223904	0.224425
	2.70	2.40	0.223357	0.223734	0.223608	0.224046	0.223953	0.224474
		2.45	0.223329	0.223709	0.223576	0.224017	0.223917	0.224439
		2.50	0.223302	0.223685	0.223547	0.223989	0.223883	0.224406
2.85	2.60	2.40	0.223327	0.223701	0.223577	0.224013	0.223924	0.224443
		2.45	0.223297	0.223674	0.223545	0.223983	0.223887	0.224407
		2.50	0.223269	0.223649	0.223514	0.223954	0.223851	0.224373
	2.65	2.40	0.223302	0.223679	0.223553	0.223991	0.223900	0.224422
		2.45	0.223273	0.223653	0.223521	0.223961	0.223863	0.224386
		2.50	0.223245	0.223628	0.223490	0.223933	0.223827	0.224351
	2.70	2.40	0.223279	0.223658	0.223530	0.223970	0.223878	0.224402
		2.45	0.223250	0.223632	0.223498	0.223941	0.223840	0.224366
		2.50	0.223222	0.223607	0.223467	0.223912	0.223804	0.224331

Example 5. *System performances with respect to r, μ_H, μ_{L_1}, and λ_H: In this example, we discuss the other system performances with respect to r, μ_H, μ_{L_1}, and λ_H. The corresponding values are tabulated in Table 7 and the observations are as follows:*

1. *The E_{HPIL} increases with μ_H despite the increment in μ_{L_1} on positive interruption. Since an increase in service rates allows the server to serve all customers in less time, the measure E_{SB} decreases with an increase in service rate and increases with arrival rate.*
2. *It is clear from the table that, as the service rate, μ_H, increases, the server may complete the service to all HP customers, and the LP customers obtain a successful chance of retrial.*

Example 6. *In this example, we deal with the expected number of HP customers in the waiting hall (E_{WH}) and of LP customers in the orbit (E_O). The results are tabulated in Tables 8 and 9. The observations are as follows.*

Case (i): Non-preemptive priority discipline.

At $r = 0$, E_{WH} increase with N, λ_H and γ and decreases with increase in μ_H and r. Additionally, E_O increases with r and λ_H, whereas it decreases with an increase in service rates μ_H, μ_{L_1}, and μ_{L_2}.

Table 7. System performances with respect to r, μ_H, μ_{L_1}, and λ_H.

r	μ_H	μ_{L_1}	E_{HPIL}		E_{SB}		E_{SRR}	
			$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$
0.5	2.75	2.40	0.001919	0.002247	0.971408	0.972786	0.015359	0.014580
		2.45	0.001921	0.002249	0.971198	0.972595	0.015423	0.014632
		2.50	0.001923	0.002251	0.970997	0.972412	0.015484	0.014682
	2.80	2.40	0.001921	0.002251	0.970795	0.972169	0.015546	0.014750
		2.45	0.001923	0.002253	0.970575	0.971969	0.015615	0.014806
		2.50	0.001925	0.002255	0.970364	0.971778	0.015679	0.014859
	2.85	2.40	0.001922	0.002255	0.970186	0.971555	0.015732	0.015732
		2.45	0.001923	0.002257	0.969956	0.971346	0.015804	0.015804
		2.50	0.001925	0.002259	0.969736	0.971146	0.015873	0.015873
1	2.75	2.40	0.003858	0.004527	0.970324	0.971718	0.015939	0.015181
		2.45	0.003863	0.004532	0.970121	0.971534	0.015995	0.015224
		2.50	0.003868	0.004536	0.969926	0.971358	0.016048	0.015266
	2.80	2.40	0.003860	0.004536	0.969673	0.971062	0.016135	0.015358
		2.45	0.003865	0.004540	0.969460	0.970869	0.016196	0.015406
		2.50	0.003870	0.004544	0.969255	0.970684	0.016252	0.015451
	2.85	2.40	0.003860	0.004543	0.969027	0.970408	0.016329	0.015536
		2.45	0.003865	0.004547	0.968804	0.970206	0.016394	0.015588
		2.50	0.003870	0.004551	0.968589	0.970012	0.016454	0.015636

Table 8. Expected number of HP customers in the waiting hall.

N	γ	μ_H	$r = 0$		$r = 0.5$		$r = 1$	
			$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$
7	0.05	2.75	1.166812	1.294542	1.133508	1.259561	1.092839	1.216619
		2.80	1.155714	1.281839	1.122121	1.246556	1.081005	1.203148
		2.85	1.145370	1.269916	1.111490	1.234334	1.069927	1.190461
	0.1	2.75	1.176179	1.304286	1.138123	1.264415	1.092326	1.216121
		2.80	1.165538	1.292048	1.127166	1.251849	1.080899	1.203065
		2.85	1.155644	1.280587	1.116958	1.240061	1.070223	1.190789
	0.15	2.75	1.185531	1.314007	1.143894	1.270412	1.094334	1.218132
		2.80	1.175302	1.302194	1.133339	1.258259	1.083308	1.205488
		2.85	1.165815	1.291153	1.123528	1.246881	1.073028	1.193620
8	0.05	2.75	1.196611	1.333320	1.162571	1.297436	1.120983	1.253362
		2.80	1.184390	1.319239	1.150062	1.283055	1.108028	1.238515
		2.85	1.173006	1.306032	1.138391	1.269550	1.095910	1.224544
	0.1	2.75	1.207098	1.344284	1.168216	1.303401	1.121370	1.253814
		2.80	1.195381	1.330726	1.156191	1.289522	1.108882	1.239453
		2.85	1.184499	1.318037	1.145001	1.276513	1.097230	1.225963
	0.15	2.75	1.217523	1.355182	1.174969	1.310463	1.124237	1.256745
		2.80	1.206273	1.342103	1.163403	1.297056	1.112209	1.242855
		2.85	1.195845	1.329887	1.152660	1.284515	1.101009	1.229833

Table 8. *Cont.*

N	γ	μ_H	$r = 0$		$r = 0.5$		$r = 1$	
			$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$
9	0.05	2.75	1.217496	1.361326	1.182931	1.324789	1.140678	1.279882
		2.80	1.204370	1.346108	1.169522	1.309276	1.126827	1.263907
		2.85	1.192158	1.331848	1.157027	1.294722	1.113888	1.248891
	0.1	2.75	1.228956	1.373376	1.189474	1.331744	1.141857	1.281196
		2.80	1.216383	1.358733	1.176600	1.316790	1.128530	1.265767
		2.85	1.204714	1.345043	1.164633	1.302789	1.116108	1.251293
	0.15	2.75	1.240316	1.385311	1.197085	1.339751	1.145488	1.284957
		2.80	1.228252	1.371198	1.184714	1.325320	1.132663	1.270051
		2.85	1.217078	1.358028	1.173237	1.311835	1.120736	1.256093

Table 9. Expected number of LP Customers in the orbit.

μ_H	μ_{L_2}	μ_{L_1}	$r = 0$		$r = 0.5$		$r = 1$	
			$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$	$\lambda_H = 0.85$	$\lambda_H = 0.95$
2.75	2.60	2.40	0.168374	0.205167	0.172715	0.210847	0.177406	0.217027
		2.45	0.166878	0.203129	0.171119	0.208675	0.175696	0.214703
		2.50	0.165467	0.201211	0.169613	0.206630	0.174082	0.212513
	2.65	2.40	0.167100	0.203432	0.171412	0.209068	0.176071	0.215202
		2.45	0.165626	0.201427	0.169838	0.206931	0.174385	0.212914
		2.50	0.164236	0.199539	0.168354	0.204918	0.172793	0.210758
	2.70	2.40	0.165892	0.201789	0.170176	0.207385	0.174805	0.213474
		2.45	0.164439	0.199815	0.168624	0.205280	0.173141	0.211219
		2.50	0.163068	0.197957	0.167160	0.203297	0.171571	0.209095
2.80	2.60	2.40	0.162540	0.197251	0.166733	0.202716	0.171263	0.208663
		2.45	0.161115	0.195324	0.165212	0.200662	0.169633	0.206463
		2.50	0.159772	0.193510	0.163777	0.198726	0.168094	0.204388
	2.65	2.40	0.161327	0.195611	0.165492	0.201035	0.169992	0.206938
		2.45	0.159923	0.193714	0.163992	0.199012	0.168384	0.204770
		2.50	0.158599	0.191929	0.162577	0.197107	0.166866	0.202727
	2.70	2.40	0.160177	0.194057	0.164315	0.199443	0.168786	0.205303
		2.45	0.158792	0.192190	0.162836	0.197450	0.167199	0.203167
		2.50	0.157487	0.190431	0.161440	0.195573	0.165701	0.201154
2.85	2.60	2.40	0.157094	0.189915	0.161148	0.195181	0.165528	0.200911
		2.45	0.155734	0.188089	0.159696	0.193232	0.163970	0.198822
		2.50	0.154452	0.186369	0.158325	0.191396	0.162500	0.196853
	2.65	2.40	0.155937	0.188360	0.159964	0.193587	0.164315	0.199275
		2.45	0.154597	0.186563	0.158532	0.191669	0.162779	0.197218
		2.50	0.153333	0.184870	0.157181	0.189861	0.161329	0.195278
	2.70	2.40	0.154839	0.186888	0.158841	0.192078	0.163164	0.197726
		2.45	0.153518	0.185118	0.157428	0.190188	0.161648	0.195698
		2.50	0.152272	0.183450	0.156095	0.188406	0.160217	0.193786

Case (ii): Discretionary priority discipline.

- *Mixed priority: In contrast to case (i), we obtain the same increment and reduction in the case $r = 0.5$ for the parameters, respectively. However, with discretionary priority, E_{WH} decreases and E_O increases.*
- *Preemptive priority discipline: As discussed in case (i), the changes with respect to other parameters at $r = 1$ are in the same order, but E_{WH} decreases, and E_O increases slightly higher compared with case (i). With this, we conclude that the waiting hall accumulates non-preemptive priority, and the customers in orbit increase with preemptive priority. This also affects the business environment and once again shows the importance of discretionary priority discipline.*

5. Conclusions

The queueing–inventory model studied here considered two classes of customers who require two stages of service for LP customers and a single stage of service for HP customers. In this model, the case $r = 0$ provides the result of a non-discretionary priority service process. In contrast, the case $r \in (0, 1]$ provides the result of a discretionary priority service process. In this paper, a comparison between non-discretionary and discretionary priority service discipline is carried out. We conclude that each of them shows a significant result for discretionary priority rather than non-discretionary priority from the numerical illustrations. A priority discipline, either preemptive or non-preemptive, adopted individually might reflect negatively on the growth of the customer base, the organization's reputation, and the company's profit. Therefore, to overcome such economic issues, the organization has to plan wisely when it comes to service provision. Our model recommends that through discretionary priority discipline, the above mentioned issues can be overcome to a certain extent. This work is an attempt to utilize discretionary priority in the queueing–inventory system. The model can be extended by changing the arrival rate from the Poisson process to the Markovian process. This model deals with the discretion rule based on the predetermined inventory level. Instead, a discretion rule based on service time can be formulated, which is another extension.

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References

1. Melikov, A.Z.; Molchanov, A.A. Stock optimization in transport/storage. *Cybern. Syst. Anal.* **1992**, *28*, 484–487. [CrossRef]
2. Sigman, K.; Simichi-Levi, D. Light traffic heuristic for an $M/G/1$ queue with limited inventory. *Ann. Oper. Res.* **1992**, *40*, 371–380. [CrossRef]
3. Berman,O.; Kim, E. Stochastic models for inventory management at service facility. *Commun. Stat. Stoch. Model.* **1999**, *15*, 695–718. [CrossRef]

4. Berman, O.; Sapna, K.P. Inventory Management for service facility for systems with arbitrarily distributed service time. *Commun. Stat. Stoch. Model.* **2000**, *16*, 343–360. [CrossRef]
5. Sangeetha, N.; Sivakumar, B. Optimal service rates of a perishable inventory system with service facility. *Int. J. Math. Oper. Res.* **2020**, *16*, 515–550. [CrossRef]
6. Jeganathan, K.; Selvakumar, S.; Anbazhagan, N.; Amutha, S.; Porpattama, H. Stochastic modeling on $M/M/1/N$ inventory system with queue-dependent service rate and retrial facility. *Aims Math.* **2021**, *6*, 7386–7420. [CrossRef]
7. Jacob, J.; Shajin, D.; Krishnamoorthy, A.; Vishnevsky, V.; Kozyrev, D. Queueing-Inventory with One Essential and m Optional Items with Environment Change Process Forming Correlated Renewal Process (MEP). *Mathematics* **2022**, *10*, 104. [CrossRef]
8. Sugapriya, C.; Nithya, M.; Jeganathan, K.; Anbazhagan, N.; Joshi, G.P.; Yang, E.; Seo, S. Analysis of Stock-Dependent Arrival Process in a Retrial Stochastic Inventory System with Server Vacation. *Processes* **2022**, *10*, 176. [CrossRef]
9. Klimenok, V.I.; Dudin, A.N.; Vishnevsky, V.M.; Semenova, O.V. Retrial $BMAP/PH/N$ Queueing System with a Threshold-Dependent Inter-Retrial Time Distribution. *Mathematics* **2022**, *10*, 269. [CrossRef]
10. Amirthakodi, M.; Sivakumar, B. An inventory system with service facility and feedback customers. *Int. J. Ind. Syst. Eng.* **2019**, *33*, 374–411.
11. Jeganathan, K.; Anbazhagan, N.; Amutha, S.; Vinitha, V. A queueing inventory system with threshold based priority service and a flexible server. *J. Stat. Manag. Syst.* **2020**, *24*, 663–684. [CrossRef]
12. Jeganathan, K.; Selvakumar, S.; Saravanan, S.; Anbazhagan, N.; Amutha, S.; Cho, W.; Joshi, G.P.; Ryoo, J. Performance of Stochastic Inventory System with a Fresh Item, Returned Item, Refurbished Item, and Multi-Class Customers. *Mathematics* **2022**, *10*, 1137. [CrossRef]
13. Jeganathan, K.; Reiyas, M.A.; Selvakumar, S.; Anbazhagan, N. Analysis of Retrial Queueing-Inventory System with Stock Dependent Demand Rate: (s,S) Versus (s,Q) Ordering Policies. *Int. J. Appl. Comput. Math.* **2020**, *5*, 1–27. [CrossRef]
14. Wang, J.; Qiu, Q.; Wang, H. Joint optimization of condition-based and age-based replacement policy and inventory policy for a two-unit series system. *Reliab. Eng. Syst. Saf.* **2021**, *205*, 107251. [CrossRef]
15. Zhang, J.; Zhao, X.; Song, Y.; Qiu, Q. Joint optimization of condition-based maintenance and spares inventory for a series–parallel system with two failure modes, *Comput. Ind. Eng.* **2022**, *168*, 108094. [CrossRef]
16. Krishnamoorthy, A.; Joshua, A.N.; Kozyrev, D. Analysis of a Batch Arrival, Batch Service Queuing-Inventory System with Processing of Inventory While on Vacation. *Mathematics* **2021**, *9*, 419. [CrossRef]
17. Maqbali, K.A.K.A.; Joshua, V.C., Krishnamoorthy A. On a Single Server Queueing Inventory System with Common Life Time for Inventoried Items. *Commun. Comput. Inf. Sci.* **2021**, *1391*, 186–197.
18. Reshmi, P.S.; Jose, K.P. A queueing–inventory system with perishable items and retrial of customers. *Malaya J. Math.* **2019**, *7*, 165–170. [CrossRef]
19. Sivakumar, B.; Arivarignan, A. A modified lost sales inventory system with two types of customers. *Qual. Technol. Quant. Manag.* **2008**, *5*, 339–349. [CrossRef]
20. Uzunoglu Kocer, U.; Yalçin, B. Continuous review (s, Q) inventory system with random lifetime and two demand classes. *Opsearch* **2019**, *57*, 104–118. [CrossRef]
21. Yadavalli, V.S.S.; Anbazhagan, N.; Jeganathan, K. A retrial inventory system with impatient customers. *Int. J. Appl. Math. Inf. Sci.* **2015**, *9*, 637–650.
22. Zhang, J.; Zhao, X.; Song, Y.; Qiu, Q. Joint optimization of maintenance and spares ordering policy for a use-oriented product-service system with multiple failure modes. *Appl. Stoch. Model. Bus Ind.* **2021**, *37*, 1123– 1142. [CrossRef]
23. Falin, G.I.; Artalejo, J.R.; Martin, M. On the single server retrial queue with priority customers. *Queueing Syst.* **1993**, *14*, 439–455. [CrossRef]
24. Jeganathan, K.; Anbazhagan, N.; Kathiresan, J. A Retrial Inventory System with Non-preemptive Priority Service. *Int. J. Inf. Manag. Sci.* **2013**, *24*, 57–77.
25. Chakravarthy, S.R. A Dynamic Non-preemptive Priority Queueing Model with Two Types of Customers. In Proceedings of the Mathematics and Computing, ICMC, Varanasi, India, 9–11 January 2018; Springer: Singapore, 2018; Volume 253, pp. 23–42
26. Korenevskaya, M.; Zayats, O.; Ilyashenko, A.; Muliukha, V. Retrial Queuing System with Randomized Push-Out Mechanism and Non-Preemptive Priority. *Procedia Comput. Sci.* **2019**, *150*, 716–725. [CrossRef]
27. Krishnamoorthy, A.; Divya, V. (M, MAP)/(PH, PH)/1 Queue with Non-preemptive Priority and Working Vacation under N-Policy. *J. Indian Soc. Probab. Stat.* **2020**, *21*, 69–122. [CrossRef]
28. Krishna, Kumar, B.; Pavai, Madheswari, S.; Arivudainambi, D. 2005. On the Busy Period of an $M/G/1$ Retrial Queueing System with Two-Phase Service and Preemptive Resume. *Stoch. Model. Appl.* **2005**, *8*, 18–34.
29. Tarabia, A.M.K. Analysis of $M/M/1$ Queueing System with two Priority Classes. *OPSEARCH* **2007**, *44*, 346–365. [CrossRef]
30. Jeganathan, K.; Kathiresan, J.; Anbazhagan, N. A retrial inventory system with priority customers and second optional service. *Opsearch* **2016**, *53*, 808–834 [CrossRef]
31. Gao, S.; Wang, J. Stochastic analysis of a preemptive retrial queue with orbital search and multiple vacations. *Rairo-Oper. Res.* **2018**, *54*, 231–249. [CrossRef]
32. Adiri, I.; Domb, I. A Single Server Queueing System Working under Mixed Priority Discipline. *Oper. Res.* **1982**, *30*, 97–115. [CrossRef]

33. Cho, Y.; Un, C. Analysis of the M/G/1 Queue under a combined Preemptive/Nonpreemptive Priority Discipline. *IEEE Trans. Commun.* **1993**, *41*, 132–141. [CrossRef]
34. Fajardo, V. A.; Drekic, S. On a General Mixed Priority Queue with Server Discretion. *Stoch. Model.* **2016**, *32*, 1–31. [CrossRef]
35. Avi-Itzhak, B.; Brosh, I.; Naor, P. On discretionary priority queueing. *Zamm-J. Appl. Math. Mech. Fiir Angew. Math. Mech.* **1964**, *44*, 235–242. [CrossRef]
36. Melkonian, V.; Kaiser, M. Discretionary Priority Discipline: A Reasonable Compromise between Preemptive and Nonpreemtive Disciplines. *Appl. Math. Lett.* **1996**, *9*, 91–94. [CrossRef]
37. Lian, Z.; Zhao, N. A two-stage M/G/1 queue with discretionary priority. In Proceedings of the 2011 IEEE International Conference on Industrial Engineering and Engineering Management, Singapore, 6–9 December 2011; pp. 1402–1406.
38. Zhao, N.; Guo, Y.; Lian, Z.; Wang, M. Analysis of a $MAP/PH/1$ queue with discretionary priority. In Proceedings of the 2015 Winter Simulation Conference (WSC), Huntington Beach, CA, USA, 6–9 December 2015; pp. 2872–2883.
39. Lian, Z.; Zhao, N. A Two-Stage Discretionary Priority Service System with Markovian Arrival Inputs. In Proceedings of the 2010 IEEE International Conference on Industrial Engineering and Engineering Management, Macao, China, 7–10 December 2010; pp. 438–442.
40. Zhao, N.; Lian, Z.; Wu, K. Analysis of a $MAP/PH/1$ Queue with Discretionary Priority Based on Service Stages. *Asia-Pac. J. Oper. Res.* **2015**, *32*, 1550042. [CrossRef]
41. Neuts, M. *Matrix Geometric Solutions in Stochastic Models: An Algorithmic Approach*; John Hopkins University Press: Baltimore, MD, USA, 1981.