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Some Properties of Interval Shapley Values: An Axiomatic Analysis

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Abstract: Interval games are an extension of cooperative coalitional games, in which players are assumed to face payoff uncertainty. Characteristic functions thus assign a closed interval instead of a real number. This study revisits two interval game versions of Shapley values (i.e., the interval Shapley value and the interval Shapley-like value) and characterizes them using an axiomatic approach. For the interval Shapley value, we show that the existing axiomatization can be generalized to a wider subclass of interval games called size monotonic games. For the interval Shapley-like value, we show that a standard axiomatization using Young's strong monotonicity holds on the whole class of interval games.

Keywords: cooperative interval games; interval uncertainty; Shapley value; axiomatization

1. Introduction

This paper examines cooperative interval games in which players face payoff uncertainty. One of the most familiar representations of cooperative game theory without uncertainty is coalitional form games with transferable utility (so-called coalition form games or TU games). A coalition form game consists of the set N of players and a characteristic function v that gives a real number $v(S)$ (the worth of S) to every subset S of N (coalitions). $v(S)$ is the total payoff that S can obtain by itself without uncertainty. In reality, however, the payoffs a coalition can obtain may entail uncertainty. Therefore, introducing uncertainty into classical coalition form games is a natural extension. Interval games, initially proposed and studied by Branzei et al. [1] and Alparaslan Gök et al. [2], consider interval uncertainty in that the uncertainty regarding coalition payoff is represented by an interval. More specifically, an interval game consists of the set N of players and a characteristic function w that gives a closed interval $w(S)$, rather than a real number, to every coalition S. It should be noted that interval games can be regarded as a generalization of TU games.

In the existing literature on interval games, various solution concepts have been proposed, and their properties have been investigated. Alparslan Gök et al. [2,3] proposed the notion of interval solution concepts. The interval core, the interval stable set, and the interval Shapley value are the solutions included in this category. Fei et al. [4] and Liang and Li [5] investigated the notions of the discount Shapley value and the Banzhaf values, respectively, in the context of interval games. Meng et al. [6] studied the Shapley value in interval fuzzy cooperative games based on Hukuhara's difference operator. Mallozzi and Vidal-Puga [7] examined interval games where players have different attitudes towards uncertainty represented by Hurwicz coefficients. Shino et al. [8] examined the notion of the solution mapping as an alternative to the interval solution concept, proposed Shapley mapping as a specific form of the solution mapping, and showed its axiomatizations. As for applications of interval games to actual economic and social situations, for instance, Palanci et al. [9] introduced uncertainty into transportation games and formalized them as

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interval games. Alparslan Gök et al. [10] examined the Shapley value and Baker–Thompson rule in the interval game version of the airport game. For more details on the literature, see Alparslan Gök [11], Branzei et al. [12] and Ishihara and Shino [13].

In this paper, we revisit existing solution concepts for interval games and investigate their properties using an axiomatic approach. More particularly, we focus on the interval Shapley value (ISV) and the interval Shapley-like value (ISLV), both of which are solution concepts based on the Shapley value [14] for classical coalitional games. First, regarding the ISV, the existing axiomatizations for the ISV were implemented under a specific subclass of interval games called KIG games. That is, Alparslan Gök et al. [3] showed that, within KIG games, the ISV is the unique solution that satisfies the axioms of efficiency, symmetry, the dummy player property, and additivity. Similarly, Alparslan Gök [15] showed that, within KIG games, the ISV is the unique solution that satisfies efficiency, symmetry, and strong monotonicity with respect to the partial operator[1]. In this study, we show that those existing axiomatizations can be generalized to a substantially wider subclass of interval games called size-monotonic games. It should be noted that because the ISV is defined on size monotonic games, this result means that the axiomatization is implemented on the largest possible domain of interval games. Next, as for the ISLV, after Han et al. [16] initially proposed it, Gallardo and Jimenez-Losada [17] completed an axiomatization showing that, in any interval games, the ISLV is the unique solution that satisfies the axioms of indifference efficiency, symmetry, indifference null player property, and additivity[2]. In this study, we show that a standard axiomatization using the strong monotonicity by Young [18] also holds for all interval games.

The remainder of the paper is as organized as follows. Section 2 briefly reviews the models and solution concepts. The main results are presented in Section 3. Section 4 concludes.

2. Models and Solution Concepts

2.1. Coalitional Games and Interval Games

An n-person coalitional game or a transferable utility game consists of a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of players and v: $2^N \to \mathbb{R}$ is a characteristic function that associates a real number $v(S) \in \mathbb{R}$ with each set $S \subset N$, with the condition that $v(\emptyset) = 0$. For a coalition S, let $|S|$ be the number of players in S. A number $v(S)$ is called the worth of S. We refer to S and N as a coalition and grand coalition, respectively. Let CG be the set of all coalitional games with player set N.

Similar to an n-person coalitional game (N, v), an n-person interval game is defined as a pair (N, w), where N is a set of players and w is a characteristic function of type $2^N \to I(\mathbb{R})$ with $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all closed and bounded intervals in $\mathbb{R}$. Therefore, an interval game differs from a coalitional form game in that w assigns a closed interval to each coalition (instead of a real number). Interval $w(S)$ is called the worth set of S and the minimum and the maximum of $w(S)$ are denoted by $\underline{w}(S)$ and $\overline{w}(S)$, respectively; that is, $w(S) = [\underline{w}(S), \overline{w}(S)]$. An interval game (N, w) considers a situation in which the players face "interval uncertainty," in that they know that a coalition S could have $\underline{w}(S)$ as the minimal reward and $\overline{w}(S)$ as the maximal reward, but they do not know which of these will be realized. Let IG be the set of all interval games with the player set N. For simplicity, we denote n-person interval games (N, w) by w.

We provide some interval calculus notations. For a positive number a and a closed interval $I = [\underline{I}, \ \overline{I}]$, we define $aI = [a\underline{I}, \ a\overline{I}]$. Let $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ be two closed intervals. First, when $(\underline{I} + \overline{I})/2 = (\underline{J} + \overline{J})/2$, which means that the medians of the two intervals are identical, we denote this by $I \sim J$. Second, if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$, we denote it by $I \geq J$. Third, if $(\underline{I} + \overline{I})/2 \geq \underline{J} + \overline{J}/2$, then we denote it by $I \succsim J$. The sum of I and J, denoted by $I + J$, is given as $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$. For subtraction between intervals, on the other hand, there are different definitions. First, following Alparslan Gök et al. [19], the partial subtraction operator denoted by "$-$" is defined as $I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. Note that the partial subtraction operator is only defined for an ordered interval pair, i.e., $(I, J) \in I(\mathbb{R}) \times I(\mathbb{R})$ satisfying $\overline{J} - \underline{J} \leq \overline{I} - \underline{I}$. Alternatively, Moore's [20] subtraction operator, which we denote by "$\ominus$" is given by $I \ominus J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$. In contrast to the partial subtraction operator, Moore's operator can be defined for any interval pairs $(I, J) \in I(\mathbb{R}) \times I(\mathbb{R})$.

Players i and j are symmetric if $w(S \cup \{i\}) = w(S \cup \{j\})$ for every $S \subset N \setminus \{i, j\}$. i is a dummy player if $w(S \cup \{i\}) = w(S) + w(\{i\})$ for every $S \in 2^{N \setminus \{i\}}$. For different interval games $w', w'' \in IG$, the sum of the interval games $w' + w'' \in IG$ is also an interval game itself, defined by $(w' + w'')(S) = w'(S) + w''(S)$ for every $S \in 2^N$. $w \in IG$ is called size-monotonic if $\overline{w}(S) - \underline{w}(S) \le \overline{w}(T) - \underline{w}(T)$ for every $S, T \in 2^N$ with $S \subset T$. Let $SMIG$ be the set of all size-monotonic interval games. When an interval game is size-monotonic, the range of the worth set of a coalition, representing the degree of payoff uncertainty, becomes larger as the size of the coalition increases. For $S \in 2^N \backslash \{\emptyset\}$ and $I_S \in I(\mathbb{R})$, the unanimous interval game $I_S u_S$ is defined as

$$I_S u_S(T) = \begin{cases} I_S & \text{if} \quad T \supset S \\ [0, 0] & \text{otherwise.} \end{cases}$$

Let KIG be the set of all interval games that can be expressed as a sum of unanimous interval games. The notion of KIG as a subclass of interval games was initially defined by Alparslan Gök et al. [3]. The idea of focusing on KIG seems based on the fact that, in classical TU game analysis, it is well known that a game can be expressed as a linear combination of unanimity games, and this property has been widely used in existing axiomatizations. Indeed, by focusing on KIG, Alparslan Gök et al. [3] showed that the interval Shapley value can be characterized by a standard axiomatization as that in classical game theory analyses. On the other hand, the following remark and example indicate that KIG covers only a small range of interval games.

Remark 1. *It holds that $KIG \subset SMIG$.*

Proof. [3] For an interval game $w \in KIG$, there exists a set of unanimous interval games $\{I_R u_R\}_{R \in 2^N \backslash \{\emptyset\}}$ that satisfies the following:

$$w(T) = \sum_{R \in 2^N \backslash \{\emptyset\}} I_R u_R(T) \quad \forall T \in 2^N \backslash \{\emptyset\}.$$

For a combination of coalitions $S, T \in 2^N$ with $S \subset T$, it holds that

$$w(S) = \left[\sum_{R \in 2^S \backslash \{\emptyset\}} \underline{I_R}, \sum_{R \in 2^S \backslash \{\emptyset\}} \overline{I_R} \right], \quad w(T) = \left[\sum_{R \in 2^S \backslash \{\emptyset\}} \underline{I_R} + \sum_{R \in 2^T \backslash 2^S} \underline{I_R}, \sum_{R \in 2^S \backslash \{\emptyset\}} \overline{I_R} + \sum_{R \in 2^T \backslash 2^S} \overline{I_R} \right].$$

Therefore,

$$\begin{aligned} (\overline{w}(T) - \underline{w}(T)) - (\overline{w}(S) - \underline{w}(S)) &= \sum_{R \in 2^S \backslash \{\emptyset\}} \overline{I_R} + \sum_{R \in 2^T \backslash 2^S} \overline{I_R} - \sum_{R \in 2^S \backslash \{\emptyset\}} \underline{I_R} \\ &\quad - \sum_{R \in 2^T \backslash 2^S} \underline{I_R} - \sum_{R \in 2^S \backslash \{\emptyset\}} \overline{I_R} + \sum_{R \in 2^S \backslash \{\emptyset\} \underline{I_R}} \sum_{R \in 2^T \backslash 2^S} (\overline{I_R} - \underline{I_R}) \\ &\ge 0, \end{aligned}$$

implying $w \in SMIG$. □

Example 1. *For an arbitrary three-person coalitional game $v \in CG$ and for positive real numbers ϵ and δ, we define the three-person interval game $w_{v,\epsilon,\delta}$ as follows: $w(\emptyset) = [0, 0]$, $w(\{1\}) = [v(\{1\}) - \epsilon, v(\{1\}) + \epsilon]$, $w(\{2\}) = [v(\{2\}) - \epsilon, v(\{2\}) + \epsilon]$, $w(\{3\}) = [v(\{3\}) - \epsilon, v(\{3\}) + \epsilon]$, $w(\{1,2\}) = [v(\{1,2\}) - \delta, v(\{1,2\}) + \delta]$, $w(\{1,3\}) = [v(\{1,3\}) - \delta, v(\{1,3\}) + \delta]$, $w(\{2,3\}) = [v(\{2,3\}) - \delta, v(\{2,3\}) + \delta]$, $w(\{1,2,3\}) = [v(\{1,2,3\}) - 3\epsilon, v(\{1,2,3\}) + 3\epsilon]$. $w_{v,\epsilon,\delta}$ corresponds to the situation in which the degree of uncertainty depends only on the number of coalitions, and the uncertainty regarding the worth of the grand coalition is three times larger* *than that of the singleton coalition. Note that* $w_{v,\epsilon,\delta} \in SMIG$ *when* $\epsilon \leq \delta \leq 3\epsilon$*, but* $w_{v,\epsilon,\delta} \in KIG$ *only when* $\delta = 2\epsilon$*.*

2.2. Solution Concepts

Let a subset of IG be K. A (single-valued) interval solution on K is a function f that associates a single n-dimensional interval vector $f(w) \in I(\mathbb{R})^n$ with each game $w \in K$. This study focuses on two existing interval solutions, i.e., the ISV and ISLV, and investigates their axiomatic characterization. Whereas the ISLV is an interval solution on IG, the ISV is an interval solution on SMIG, i.e., a subclass of IG, because it is defined by using the partial subtraction operator.

For $w \in SMIG$, the ISV, denoted by $\Psi(w) = \big(\Psi_1(w), \ldots, \Psi_n(w)\big)$, is defined as:

$$\text{For } i \in N,\ \ \Psi_i(w) = \sum_{S \in 2^{N\setminus\{i\}}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \{w(S \cup \{i\}) - w(S)\}.$$

For $w \in IG$, the ISLV, denoted by $\Phi(w) = \big(\Phi_1(w), \ldots, \Phi_n(w)\big)$, is defined as

$$\text{For } i \in N,\ \ \Phi_i(w) = \sum_{S \in 2^{N\setminus\{i\}}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \{w(S \cup \{i\}) \ominus w(S)\}.$$

The following Lemma 1 states a relationship between the ISV and the ISLV and Example 2 shows the difference in the ISV and the ISLV in a simple interval game.

Lemma 1. *For any* $w \in SMIG$ *and* $i \in N$*,* $\Psi_i(w) \subset \Phi_i(w)$*.*

Proof. From the definitions of ISV and ISLV, it holds that $\underline{\Psi_i(w)} = \sum_{S \in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!}$ $\{\underline{w}(S \cup \{i\}) - \underline{w}(S)\}$, $\overline{\Psi_i(w)} = \sum_{S \in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!} \{\overline{w}(S \cup \{i\}) - \overline{w}(S)\}$, $\underline{\Phi_i(w)} = \sum_{S \in 2^{N\setminus\{i\}}}$ $\frac{s!(n-s-1)!}{n!} \{\underline{w}(S \cup \{i\}) - \overline{w}(S)\}$ and $\overline{\Phi_i(w)} = \sum_{S \in 2^{N\setminus\{i\}}} \frac{s!(n-s-1)!}{n!} \{\overline{w}(S \cup \{i\}) - \underline{w}(S)\}$. Since this implies $\underline{\Psi_i(w)} - \underline{\Phi_i(w)} \geq 0$ and $\overline{\Phi_i(w)} - \overline{\Psi_i(w)} \geq 0$, it holds that $\underline{\Phi_i(w)} \leq \underline{\Psi_i(w)}$ $\leq \overline{\Psi_i(w)} \leq \overline{\Phi_i(w)}$. □

Example 2. *Consider the following three-person interval game w:* $w(\emptyset) = [0, 0]$, $w(\{1\}) = [6, 12]$, $w(\{2\}) = [12, 24]$, $w(\{3\}) = [18, 36]$, $w(\{1, 2\}) = [24, 48]$, $w(\{1, 3\}) = [30, 60]$, $w(\{2, 3\}) = [36, 72]$, $w(\{1, 2, 3\}) = [90, 180]$. *Since this game is SMIG, both the ISV* Ψ *and the ISLV* Φ *exist:* $\Psi(w) = ([24, 48], [30, 60], [36, 72])$ *and* $\Phi(w) = ([7, 65], [16, 74], [25.83])$*.*

As shown in this example and in line with Lemma 1, the ISLV specifies a wide interval to each player, while that assigned by the ISV is relatively narrow.

3. Main Results

This section reviews existing axiomatizations of the ISV and ISLV and further investigates their properties using a new axiomatic approach. First, following the analysis of strong monotonicity in coalitional games by Peleg and Sudhölter [21], we define the following:

$$\begin{aligned}
\mathcal{D}(w) &= \{S \subset N \mid \text{there exists } T \subset S \text{ with } w(T) \neq [0, 0]\} \\
\mathcal{D}^m(w) &= \{S \in \mathcal{D}(w) \mid \nexists T \in \mathcal{D}(w) \text{ with } T \subsetneq S\} \\
S_0(w) &= \bigcap \{S \mid S \in \mathcal{D}^m(w)\}
\end{aligned}$$

$\mathcal{D}^m(w)$ is the set of minimal coalitions in $\mathcal{D}(w)$. Note that at least one player in $S_0(w)$ is not included in T. Then, $w(T) = [0, 0]$.

3.1. Results for the Interval Shapley Value

First we show the results for the ISV. Letting w be an interval game and f be an interval solution, we consider the following axioms regarding f.

- Axiom 1: Efficiency **(EF)**

$$\sum_{i \in N} f_i(w) = w(N).$$

- Axiom 2: Symmetry **(SYM)**

$$\text{If } w(S \cup \{i\}) = w(S \cup \{j\}) \text{ for every } S \in 2^{N \setminus \{i, j\}}, \text{ then } f_i(w) = f_j(w).$$

- Axiom 3: Dummy Player Property **(DP)**

$$\text{If } w(S \cup \{i\}) = w(S) + w(\{i\}) \text{ for every } S \in 2^{N \setminus \{i\}}, \text{ then } f_i(w) = w(\{i\}).$$

- Axiom 4: Additivity **(AD)**

$$\text{For every } (w', w'') \in IG \times IG,\ f_i(w' + w'') = f_i(w') + f_i(w'') \text{ for every } i \in N.$$

- Axiom 5: Strong Monotonicity w.r.t. the Partial Operator **(SM-P)**

$$\text{If } w(S \cup \{i\}) - w(S) \geq w'(S \cup \{i\}) - w'(S) \text{ for every } 2^{N \setminus \{i\}}, \text{ then } f_i(w) \geq f_i(w').$$

Axiom EF asserts that all $w(N)$ is allocated to players in the game and that no residual exists. Axiom SYM argues that only what a player can obtain on their own in the game should matter, not its specific name. Axiom DP asserts that, if i is a dummy player, f should assign i what i can obtain on its own. Axiom AD means that, when f gives $f_i(w')$ to player i in $w' \in IG$ and $f_i(w'')$ to i in $w'' \in IG$, f should give $f_i(w' + w'')$ to player i in the sum game $(w' + w'') \in IG$. Finally, Axiom SM-P argues that, for two different interval games w and w', if player i's marginal contribution in $w \in IG$ is larger than i's marginal contribution in $w' \in IG$ for every coalition, then what f gives to i in w should be larger than that in w'.

The existing axiomatizations for the ISV have been implemented only for KIG games. That is, Alparslan Gök et al. [3] showed that, within KIG games, the ISV is the unique solution that satisfies EF, SYM, DP, and AD. Similarly, Alparslan Gök [15] showed that, within KIG games, the ISV is the unique solution that satisfies EF, SYM, and SM-P.

For the ISV, our main results are Theorems 1 and 2 below. Note that because the ISV is defined on SMIG, each theorem shows that its associated axiomatization is implemented on the largest possible domain of the interval games.

Theorem 1. *For any $w \in SMIG$, the ISV is the unique solution that satisfies EF, SYM, DP, and AD.*

Theorem 2. *For any $w \in SMIG$, the ISV is the unique solution that satisfies EF, SYM, and SM-P.*

The following Lemmas 2 to 6 are necessary to prove Theorem 1, and all proofs are from Shino et al. [8].

Lemma 2. *For a coalition R, we define a coalitional form game $v_R \in G$ as:*

$$v_R(S) = \begin{cases} 1 & \text{if} \quad R \subset S \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $w \in IG$, there uniquely exists $2(2^n - 1)$ real numbers $(\underline{c_R}, \overline{c_R} : R \subset N)$ that satisfy

$$v_{\underline{w}} = \sum_{R \subset N} \underline{c_R} v_R, \quad v_{\overline{w}} = \sum_{R \subset N} \overline{c_R} v_R \quad \text{where}$$
$$\underline{c_R} = \sum_{T \subset R} (-1)^{|R|-|T|} v_{\underline{w}}(T), \quad \overline{c_R} = \sum_{T \subset R} (-1)^{|R|-|T|} v_{\overline{w}}(T).$$

Lemma 3. *For any coalition $R \subset N$, $w + \sum_{R:\underline{c_R} > \overline{c_R}} [-\underline{c_R}, -\overline{c_R}] v_R = \sum_{R:\underline{c_R} \leq \overline{c_R}} [\underline{c_R}, \overline{c_R}] v_R$.*

Lemma 4. *Suppose that a solution for SMIG f satisfies EF, SYM, and DP. Then, for the interval game $[\underline{c}, \overline{c}]v_R$ (Note : $\underline{c} \leq \overline{c}$),*

$$f_i([\underline{c}, \overline{c}]v_R) = \begin{cases} [\underline{c}, \overline{c}]/|R| & \text{if} \quad i \in R \\ [0, 0] & \text{otherwise.} \end{cases}$$

Proof. See Alparslan Gök et al. [3]. □

Lemma 5. *Let ϕ be the Shapley value for coalitional games. Then, it holds that $\phi_i(v_{\underline{w}}) = \sum_{R \ni i}(\underline{c_R}/|R|)$ and $\phi_i(v_{\overline{w}}) = \sum_{R \ni i}(\overline{c_R}/|R|)$.*

Lemma 6. *Let ϕ be the Shapley value for coalitional games. Then, the ISV for $w \in SMIG$ is $\Psi_i(w) = [\phi_i(v_{\underline{w}}), \phi_i(v_{\overline{w}})]$ and $\phi_i(v_{\underline{w}}) \leq \phi_i(v_{\overline{w}})$.*

Proof of Theorem 1. Alparslan Gök et al. [3] showed that the ISV satisfies EF, SYM, DP, and AD on SMIG in Proposition 3.4, 3.2, 3.3, and 3.1, respectively. Therefore, it suffices to show its uniqueness, i.e., if solution f satisfies EF, SYM, DP, and AD, then $f = \psi$. Suppose that f satisfies EF, SYM, DP, and AD. For an interval game $w \in SMIG$, from Lemmas 2 and 3, and AD, it follows that $f_i(w) + \sum_{R:\underline{c_R} > \overline{c_R}} f_i([-\underline{c_R}, -\overline{c_R}]v_R) = \sum_{R:\underline{c_R} \leq \overline{c_R}} f_i([\underline{c_R}, \overline{c_R}]v_R)$. From Lemma 4, it also holds that $f_i(w) + \sum_{R \ni i:\underline{c_R} > \overline{c_R}}([-\underline{c_R}, -\overline{c_R}]/|R|) = \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}}([\underline{c_R}, \overline{c_R}]/|R|)$. Now, from Lemma 5,

$$\begin{aligned} \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{\overline{c_R}}{|R|} - \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{-\overline{c_R}}{|R|} &= \sum_{R \ni i} \frac{\overline{c_R}}{|R|} = \phi_i(v_{\overline{w}}) \\ \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{\underline{c_R}}{|R|} - \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{-\underline{c_R}}{|R|} &= \sum_{R \ni i} \frac{\underline{c_R}}{|R|} = \phi_i(v_{\overline{w}}). \end{aligned}$$

Therefore, from Lemma 6, we can subtract the interval $\sum_{R \ni i:\underline{c_R} > \overline{c_R}}([\underline{-c_R}, \overline{-c_R}]/|R|)$ from the interval $\sum_{R \ni i:\underline{c_R} \leq \overline{c_R}}([\underline{c_R}, \overline{c_R}]/|R|)$, and it follows that:

$$f_i(w) = \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{[\underline{c_R}, \overline{c_R}]}{|R|} - \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{[-\underline{c_R}, -\overline{c_R}]}{|R|} = [\phi_i(v_{\underline{w}}), \phi_i(v_{\overline{w}})].$$

Therefore, from Lemma 6, $f_i(w) = \Psi_i(w)$. □

Next, we prove Theorem 2.

Proof of Theorem 2. For SMIG, Alparslan Gök et al. [3] showed that the ISV satisfies EF and SYM, and Alparslan Gök. [15] showed that it satisfies SM-P. Therefore, it suffices to show its uniqueness; i.e., that a solution f for $w \in SMIG$ satisfying EF, SYM, and SM-P must be identical to Ψ.

Following Peleg and Sudhölter [21], we use mathematical induction regarding $|\mathcal{D}(w)|$. If $|\mathcal{D}(w)| = 0$, then $f_i(w) = [0, 0]$ for every $i \in N$ by EF and SYM. Because $\Psi_i(w) = [0, 0]$ for every i, $f(w) = \Psi(w)$. Now, assume that $f(w) = \Psi(w)$ for any $w \in SMIG$ satisfying $|\mathcal{D}(w)| \leq k$ and consider any $w \in SMIG$ satisfying $|\mathcal{D}(w)| = k + 1$. For $S \in \mathcal{D}^m(w)$, we define $w_S \in IG$ as $w_S(T) = w(S \cap T)$ for all $T \subset N$ and let $w' \in IG$ be defined by $w' = w - w_S$. Because $S \in \mathcal{D}^m(w)$, the following holds:

$$w_S(T) = \begin{cases} w(S) & \text{if} \quad T \supset S \\ [0,0] & \text{otherwise} \end{cases} \qquad w'(T) = \begin{cases} w(T) - w(S) & \text{if} \quad T \supset S \\ w(T) & \text{otherwise} \end{cases}$$

Note that if $T \supset S$, then $w(T) - w(S)$ is an interval because $w \in SMIG$.

$w'(T \cup \{i\}) - w'(T) = w(T \cup \{i\}) - w(T)$ holds for all $T \subset N$ because for every $i \in N \backslash S$, the following is true:

$$w'(T \cup \{i\}) = \begin{cases} w(T \cup \{i\}) - w(S) & \text{if} \quad T \supset S \\ w(T \cup \{i\}) & \text{otherwise.} \end{cases}$$

First, because f satisfies SM-P, (i) $f_i(w') = f_i(w)$ for all $i \in N \backslash S$. Second, because $|\mathcal{D}(w')| \leq k$, and from the assumptions, (ii) $f_i(w') = \Psi_i(w')$ for all $i \in N \backslash S$. Finally, because Ψ satisfies SM-P, (iii) $\Psi_i(w') = \Psi_i(w)$ for all $i \in N \backslash S$. From (i)–(iii), it holds that $f_i(w) = \Psi_i(w)$ for every $i \in N \backslash S$. As this holds for every $S \in \mathcal{D}^m(w)$, we have

$$f_i(w) = \Psi_i(w) \quad \forall i \in N \backslash S_0(w). \tag{1}$$

As $w(T) = [0,\ 0]$ for every T satisfying $S_0(w) \backslash T \neq \emptyset$, $w(S \cup \{i\}) = w(S \cup \{j\}) = [0,\ 0]$ for every $i,\ j \in S_0(w)$ and all $S \in 2^{N \backslash \{i,\ j\}}$. Furthermore, $f_i(w) = f_j(w)$ and $\Psi_i(w) = \Psi_j(w)$ hold because f and Ψ satisfy SYM. Therefore, from EF and (1), we have:

$$f_i(w) = \Psi_i(w) \quad \forall i \in S_0(w). \tag{2}$$

(1) and (2) imply that $f(w) = \Psi(w)$. $\square$

3.2. Results for the Interval Shapley-like Value

In this subsection, we show the results for the ISLV. Let w be an interval game and f be an interval solution. In addition to SYM and AD, we consider the following axioms.

- Axiom 6: Indifference Efficiency **(IEFF)**

$$\sum_{i \in N} f_i(w) \sim w(N).$$

- Axiom 7: Indifference Null Player Property **(INP)**

  If $w(S \cup \{i\}) = w(S)$ for every $S \in 2^{N \backslash \{i\}}$,
  then there exists $t \in \mathbb{R}$ with $t \geq 0$ such that $f_i(w) = [-t, t]$.

- Axiom 8: Strong Monotonicity with respect to Moore's operator **(SM-M)**

  If $w(S \cup \{i\}) \ominus w(S) \succsim w'(S \cup \{i\}) \ominus w'(S)$ for every $S \in 2^{N \backslash \{i\}}$, then $f_i(w) \succsim f_i(w')$.

Axiom IEFF and INP were initially proposed by Han et al. [16] and then examined by Gallardo and Jiménez-Losada [17], while Axiom SM-M is newly introduced in this study. Axiom IEFF asserts that the median of $w(N)$ should be equal to the median of the sum of all $f_i(w)$. Axiom INP argues that f should assign all null players an interval of which the median is zero. Axiom SM-M is a natural extension of strong monotonicity using Moore's subtraction operator, arguing that for two different interval games w and w', if player i's marginal contribution measured by Moore's subtraction operator in $w \in IG$ is larger than that in $w' \in IG$ for every coalition, then the median of what f gives to i in w should be larger than that in w'.

Gallardo and Jiménez-Losada [17] showed that, in any interval game w, (i) ISLV Φ satisfies IEFF, SYM, INP, and AD, and (ii) if an interval solution f also satisfies IEFF, SYM, INP, and AD, then $f_i(w) \sim \Phi_i(w)$ for every $i \in N$. In other words, they showed Φ's "uniqueness in terms of the medians of allocations".

Our main result for the ISLV is as follows:

Theorem 3. *For any $w \in IG$, the ISLV Φ is the unique solution in terms of the medians of allocations that satisfies IEFF, SYM, and SM-M; that is,*

(i) Φ *satisfies IEFF, SYM, and SM-M.*

(ii) *If an interval solution f also satisfies IEFF, SYM, and SM-M, then $f_i(w) \sim \Phi_i(w)$ for every $i \in N$.*

We prove Theorem 3 by using the following Lemma 7 to Lemma 9.

Lemma 7. *For intervals I_1,I_2 and J, $(I_1 \ominus J) \ominus (I_2 \ominus J) \sim I_1 \ominus I_2$.*

Proof. Since $I_1 \ominus J = [\underline{I_1} - \overline{J}, \overline{I_1} - \underline{J}]$ and $I_2 \ominus J = [\underline{I_2} - \overline{J}, \overline{I_2} - \underline{J}]$, the median of $(I_1 \ominus J) \ominus (I_2 \ominus J)$ $= [\underline{I_1} - \overline{J} - \overline{I_2} + \underline{J}, \overline{I_1} - \underline{J} - \underline{I_2} + \overline{J}]$ is $(\underline{I_1} - \overline{J} - \overline{I_2} + \underline{J} + \overline{I_1} - \underline{J} - \underline{I_2} + \overline{J})/2 = (\underline{I_1} - \overline{I_2} + \overline{I_1} - \underline{I_2})/2$. In addition, the median of $I_1 \ominus I_2 = [\underline{I_1} - \overline{I_2}, \overline{I_1} - \underline{I_2}]$ is $(\underline{I_1} - \overline{I_2} + \overline{I_1} - \underline{I_2})/2$. □

Lemma 8. *For positive numbers a_1,a_2 and intervals I_1,I_2,J_1,J_2, if $I_1 \succsim J_1$ and $I_2 \succsim J_2$, then $a_1 I_1 + a_2 I_2 \succsim a_1 J_1 + a_2 J_2$.*

Proof. If $I_1 \succsim J_1$,$I_2 \succsim J_2$, then $(\underline{I_1} + \overline{I_1})/2 \geq (\underline{J_1} + \overline{J_1})/2$ and $(\underline{I_2} + \overline{I_2})/2 \geq (\underline{J_2} + \overline{J_2})/2$, implying that $(\underline{a_1 I_1 + a_2 I_2} + \overline{a_1 I_1 + a_2 I_2})/2 \geq (\underline{a_1 J_1 + a_2 J_2} + \overline{a_1 J_1 + a_2 J_2})/2$. Therefore $a_1 I_1 + a_2 I_2 \succsim a_1 J_1 + a_2 J_2$. □

Lemma 9. *The ISLV satisfies IEFF, SYM, and SM-M.*

Proof. As Han et al. [16] showed that the ISLV satisfies IEFF and SYM, it suffices to show that the ISLV satisfies SM-M. For $S \in 2^{N \setminus \{i\}}$, if $w(S \cup \{i\}) \ominus w(S) \succsim w'(S \cup \{i\}) \ominus w'(S)$, then

$$\sum_{S \in 2^{N \setminus \{i\}}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \{w(S \cup \{i\}) \ominus w(S)\} \succsim \sum_{S \in 2^{N \setminus \{i\}}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \{w'(S \cup \{i\}) \ominus w'(S)\}$$

is true by Lemma 8. Therefore, $\Phi_i(w) \succsim \Phi_i(w')$. □

Proof of Theorem 3. Suppose a solution f satisfies IEFF, SYM, and SM-M. Then, we show that $f_i(w) \sim \Phi_i(w)$ for every $i \in N$ by mathematical induction regarding $|\mathcal{D}(w)|$. If $|\mathcal{D}(w)| = 0$, then $f_i(w) = [0,\ 0]$ for every $i \in N$ by IEFF and SYM. As $\Phi_i(w) = [0,\ 0]$, $f_i(w) \sim \Phi_i(w)$ for every $i \in N$. Assume that for any $w \in IG$ satisfying $|\mathcal{D}(w)| \leq k$, $f_i(w) \sim \Phi_i(w)$ holds for every $i \in N$ and consider $w \in IG$ satisfying $|\mathcal{D}(w)| = k + 1$. For $S \in \mathcal{D}^m(w)$, we define $w_S \in IG$ as $w_S(T) = w(S \cap T)$ for all $T \subset N$. As $S \in \mathcal{D}^m(w)$,

$$w_S(T) = \begin{cases} w(S) & \text{if} \quad T \supset S \\ [0,0] & \text{otherwise} \end{cases}$$

holds. We define $w' \in IG$ as follows:

$$w'(T) = \begin{cases} w(T) \ominus w(S) & \text{if} \quad T \supsetneq S \\ [0,0] & \text{if} \quad T = S \\ w(T) & \text{otherwise.} \end{cases}$$

Note that for every $i \in N \setminus S$, the following holds:

$$w'(T \cup \{i\}) = \begin{cases} w(T \cup \{i\}) \ominus w(S) & \text{if} \quad T \supsetneq S \\ w(T \cup \{i\}) \ominus w(T) & \text{if} \quad T = S \\ w(T \cup \{i\}) & \text{otherwise.} \end{cases}$$

Therefore, from Lemma 7, $w'(T \cup \{i\}) \ominus w'(T) \sim w(T \cup \{i\}) \ominus w(T)$ for all $T \subset N$.

First, as f satisfies SM-M, (i) $f_i(w') \sim f_i(w)$ for all $i \in N \setminus S$. Second, as $|\mathcal{D}(w')| \leq k$ and from the assumptions, (ii) $f_i(w') \sim \Phi_i(w')$ for all $i \in N \setminus S$. Finally, because Φ satisfies

SM-M from Lemma 9, (iii) $\Phi_i(w') \sim \Phi_i(w)$ for all $i \in N \backslash S$. From (i)–(iii), it holds that $f_i(w) \sim \Phi_i(w)$ for every $i \in N \backslash S$. Because this holds for every $S \in \mathcal{D}^m(w)$,

$$f_i(w) \sim \Phi_i(w) \; \forall i \in N \backslash S_0(w). \tag{3}$$

As $w(T) = [0,\ 0]$ for every T satisfying $S_0(w) \backslash T \neq \emptyset$, $w(S \cup \{i\}) = w(S \cup \{j\}) = [0,\ 0]$ for every $i,\ j \in S_0(w)$ and all $S \in 2^{N \backslash \{i,\ j\}}$. Furthermore, as f satisfies SYM, $f_i(w) = f_j(w)$ and Φ also satisfies SYM by Lemma 9, it holds that $\Phi_i(w) = \Phi_j(w)$. Therefore, from IEFF and (3),

$$f_i(w) \sim \Phi_i(w) \; \forall i \in S_0(w). \tag{4}$$

(3) and (4) implies $f_i(w) \sim \Phi_i(w)$ for every $i \in N$. $\square$

4. Conclusions

In this study, we investigated two interval-game versions of the Shapley value, i.e., the ISV and ISLV, and characterized them with a new axiomatic analysis. For the ISV, we showed that the existing axiomatization can be generalized to a wider subclass of interval games called size-monotonic games. For the ISLV, we showed that a standard axiomatization using Young's strong monotonicity holds on the whole class of interval games.

Here, it should be noted that in the proofs of Theorems 2 and 3, we focused on $|D(w)|$ as Peleg and Sudhölter [21], i.e., the number of coalitions of which at least one subset has a non-zero worth set, and this enabled us to obtain the following intriguing consequences. First, for ISV, by not focusing on games expressed as a sum of unanimous interval games (KIG) but on $|D(w)|$, we succeeded in generalizing the existing axiomatizations to a substantially wider subclass of interval games, namely, size-monotonic games. Second, for ISLV, Jimenez-Losada [17], in their proofs, utilized a linearity of dividends in length games generated by the original interval games. This approach seems particularly powerful when our focus is the medians of worth sets and allocated intervals as well as their relevant axioms, such as those shown in Section 3.2, but this is not necessarily the case in interval game analyses. Since a set of interval games does not have its associated basis in general, our approach, which does not use the notion of linearity, can contribute to future research on interval games. Finally, although the existing axiomatizations employ different approaches, we axiomatized the ISV and ISLV in a unified way in the proofs.

As for topics for further research, first, investigating whether properties of the Shapley value in coalitional games are preserved in interval game analyses is intriguing. For example, it is worth examining whether ISV and ISLV have the consistency property as analyzed by Hart and Mas-Colell's [22] potential approach. Second, Shino et al. [8] proposed a third interval-game version of the Shapley value, called Shapley mapping. It has been characterized by some axiomatizations in [8] but not yet by one that includes strong monotonicity. Investigating this topic would be worthwhile.

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Notes

1 Exact expressions of those axioms will be described in Section 3.

2 More precisely, they showed uniqueness regarding the medians of allocations, as discussed in Section 3.

3 Alparslan Gök. [15] and Alparslan Gök et al. [3] noted this property without proof.

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