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A Note on the Minimum Size of a Point Set Containing Three Nonintersecting Empty Convex Polygons

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Abstract: Let P be a planar point set with no three points collinear, k points of P be a k-*hole* of P if the k points are the vertices of a convex polygon without points of P. This article proves 13 is the smallest integer such that any planar points set containing at least 13 points with no three points collinear, contains a 3-hole, a 4-hole and a 5-hole which are pairwise disjoint.

Keywords: planar point set; convex polygon; disjoint holes

1. Introduction

In this paper, we deal with the finite planar point set P in *general position*, that is to say, no three points in P are collinear. In 1935, Erdős and Szekeres [1], posed a famous combinational geometry question: Whether for every positive integer $m \geq 3$, there exists a smallest integer $ES(m)$, such that any set of n points ($n \geq ES(m)$), contains a subset of m points which are the vertices of a convex polygon. It is a long standing open problem to evaluate the exact value of $ES(m)$. Erdős and Szekeres [2] showed that $ES(m) \geq 2^{m-2} + 1$, which is also conjectured to be sharp. We have known that $ES(4) = 5$ and $ES(5) = 9$. Then by using computer, Szekeres and Peters [3] proved that $ES(6) = 17$. The value of $ES(m)$ for all $m > 6$ is unknown.

For a planar point set P, let k points of P be a k-*hole* of P if the k points are the vertices of a convex polygon whose interior contains no points of P. Erdős posed another famous question in 1978. He asked whether for every positive integer k, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, contains a k-*hole*. It is obvious that $H(3) = 3$. Esther Klein showed $H(4) = 5$. Harborth [4] determined $H(5) = 10$, and also gave the configuration of nine points with no empty convex pentagons. Horton [5] showed that it was possible to construct arbitrarily large set of points without a 7-hole, That is to say $H(k)$ does not exist for $k \geq 7$. The existence of $H(6)$ had been proved by Gerken [6] and Nicolás [7], independently. In [8], Urabe first studied the disjoint holes problems when hewas considering the question about partitioning of planar point sets. Let $Ch(P)$ stand for the *convex hull* of a point set P. A family of holes $\{H_i\}_{i \in I}$ is called pairwise disjoint if $Ch(H_i) \cap Ch(H_j) = \emptyset, i \neq j; i \in I, j \in I$. These holes are disjoint with each other. Determine the smallest integer $n(k_1, ..., k_l)$, $k_1 \leq k_2 \leq ... \leq k_l$, such that any set of at least $n(k_1, ..., k_l)$ points of the plane, contains a k_i-*hole* for every i, $1 \leq i \leq l$, where the holes are disjoint. From [9], we know $n(2,4) = 6$, $n(3,3) = 6$. Urabe [8] showed that $n(3,4) = 7$, while Hosono and Urabe [10] showed that $n(4,4) = 9$. In [11], Hosono and Urabe also gave $n(3,5) = 10$, $12 \leq n(4,5) \leq 14$ and $16 \leq n(5,5) \leq 20$. The result $n(3,4) = 7$ and $n(4,5) \leq 14$ were re-authentication by Wu and Ding [12]. Hosono and

Urabe [9] proved $n(4,5) \leq 13$. $n(4,5) = 12$ by Bhattacharya and Das was published in [13], who also discussed the convex polygons and pseudo-triangles [14]. Hosono and Urabe also changed the lower bound on $H(5,5)$ to 17 [9], and Bhattacharya and Das showed the upper bound on $n(5,5)$ to 19 [15]. Recently, more detailed discussions about two holes are published in [16]. Hosono and Urabe in [9] showed $n(2,3,4) = 9$, $n(2,3,5) = 11$, $n(4,4,4) = 16$. We showed $n(3,3,5) = 12$ in [17]. We have proved that $n(3,3,5) = 12$ [17], $n(4,4,5) \leq 16$ [18] and also discuss a disjoint holes problem in preference [19]. In this paper, we will continue discussing this problem and prove that $n(3,4,5) = 13$.

2. Definitions

The *vertices* are on convex hull of the given points,from the remaining *interior points*. Let $V(P)$ denote a set of the vertices and $I(P)$ be a set of the interior points of P. $|P|$ stands for the number of points contained in P. Let $p_1, p_2, ..., p_k$ be k points of P, we know that $p_1, p_2, ..., p_k$ be a k-hole H when the k points are the vertices of a convex polygon whose interior does not contain any point of P. And we simply say $H = (p_1 p_2 ... p_k)_k$. As in [9], let $l(a,b)$ be the line passing points a and b. Determine the closed half-plane with $l(a,b)$, who contains c or does not contain c by $H(c;ab)$ or $H(\bar{c};ab)$, respectively. R is a region in the plane. An interior point of R is an element of a given point set P in its interior, and we say R is empty when R contains no interior points, and simply $R = \emptyset$. The interior region of the angular domain determined by the points a, b and c is a convex cone. It is denoted by $\gamma(a;b,c)$. a is the apex. b and c are on the boundary of the angular domain. If $\gamma(a;b,c)$ is not empty, we define an interior point of $\gamma(a;b,c)$ be attack point $\alpha(a;b,c)$, such that $\gamma(a;b,\alpha(a;b,c))$ is empty, as shown in Figure 1.

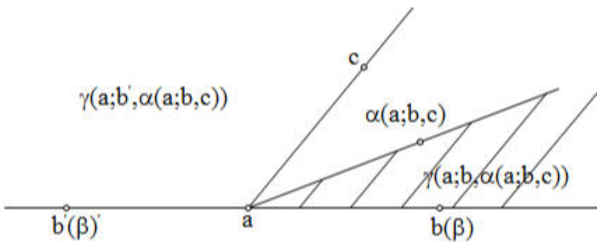

Figure 1. Figure of attack point.

For $\beta = b$ or $\beta = c$ of $\gamma(a;b,c)$, let β' be a point such that a is on the line segment $\overline{\beta\beta'}$. $\gamma(a;b',c)$ means that a lies on the segment $\overline{bb'}$. Let $v_1, v_2, v_3, v_4 \in P$ and $(v_1 v_2 v_3 v_4)_4$ be a 4-hole, as shown in Figure 2. We name $l(v_3, v_4)$ a separating line, denoted by $SL(v_3, v_4)$ or SL_4 for simple, when all of the remaining points of P locate in $H(\overline{v_1}; v_3 v_4)$.

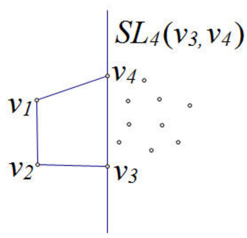

Figure 2. Figure of separating line.

We identify indices modulo t, when indexing a set of t points.

3. Main Result and Proof

Theorem 1. *[9] For any planar point set with at least 13 points in general position, if there exists a separating line SL_4, which separates a 4-hole from all of the remaining points, we always can find a 3-hole, a 4-hole and a 5-hole which are pairwise disjoint.*

From [20], we know that $13 \leq n(3,4,5) \leq 14$. In this note we will give the exact value of $n(3,4,5)$, that is the following theorem.

Theorem 2. *$n(3,4,5) = 13$, that is to say, 13 is the smallest integer such that any planar point set with at least 13 points in general position, we always can find a 3-hole, a 4-hole and a 5-hole which are pairwise disjoint.*

Proof. Let P be a 13 points set. $CH(P) = \{v_1, v_2, ..., v_l\}$. If we can find a 5-hole and a disjoint convex region with at least 7 points remained, we are done by $n(3,4) = 7$ [8]. That is to say, if we find a straight line which separates a 5-hole from at least 7 points remained, the result is correct. We call such a line a cutting line through two points u and v in P, denoted by $L_5(u,v)$. If we can find a 4-hole and the vertices number of the remaining points is more than 4, we are done by Theorem 1, where the two parts are disjoint. That is to say, if we can find such a cutting line through two points m and n in P, denoted by $L_4(m,n)$, our conclusion is correct. Therefore, in the following proof, if we can find a cutting line $L_5(u,v)$ or $L_4(m,n)$, our conclusion must be true.

In the following, we will assume there does not exist a separating line SL_4. Then there must exist a point p_i, such that $\gamma(p_i; v_i, v'_{i-1})$ and $\gamma(p_i; v_{i-1}, v'_i)$ are empty, as shown in Figure 3. Considering the 13 points, it is easy to know the conclusion is obvious right when $|V(P)| \geq 7$. Next, we discuss the considerations that $3 \leq |V(P)| \leq 6$.

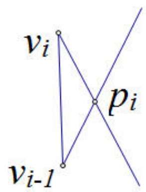

Figure 3. Figure of point determined by two separating lines.

Case 1 $|V(P)| = 6$.

Let $v_i \in V(P)$ for $i = 1,2,...6$. As shown in Figure 4, we have the points p_i for $i = 1,2,...6$, such that the shaded region is empty and we have 1 point p_7 remained.

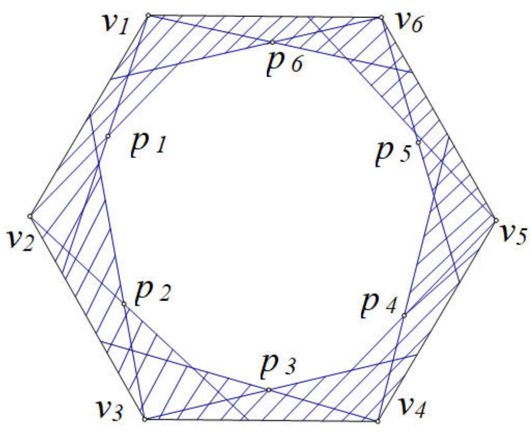

Figure 4. Figure of $|\mathrm{V(P)}| = \mathbf{6}$

As the isomorphism of geometry from Figure 4, we only discuss one case. And the rest could be obtained in the same way.

Assume $\gamma(v_1; p_1, v_3) \cap \gamma(v_3; v_1, p_2) = \emptyset$. We have a cutting line $L_5(v_1, \alpha(v_1; v_3, v_6))$.

Assume $\gamma(v_1; p_1, v_3) \cap \gamma(v_3; v_1, p_2) \neq \emptyset$. We have a cutting line $L_5(v_1, p_4)$.

Case 2 $|V(P)| = 5$.

Let $v_i \in V(P)$ for $i = 1, 2, 3, 4, 5$. We have 5 friend points p_i for $i = 1, 2, 3, 4, 5$ as shown in Figure 5. Then we have 3 points r_1, r_2, r_3 remained.

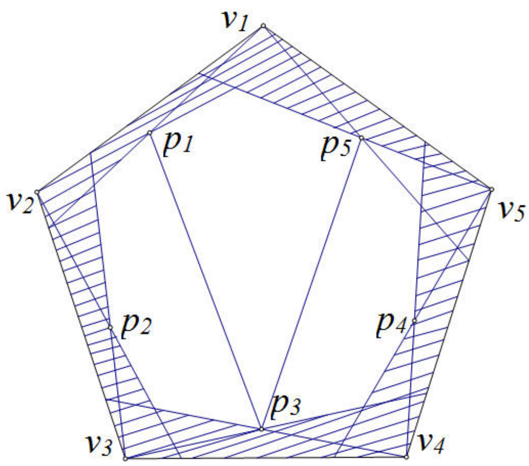

Figure 5. Figure of $|\mathrm{V(P)}| = 5$.

Assume $\gamma(p_1; v_1', p_3) \cap \gamma(p_2; v_3', v_2') = \emptyset$. We have a cutting line $L_5(p_1, \alpha(p_1; p_3, v_2'))$.

Assume $\gamma(p_3; v_3', p_5) \cap \gamma(p_4; v_4', v_5') = \emptyset$. We have a cutting line $L_5(p_3, \alpha(p_3; p_5, p_1))$.

Assume $\gamma(p_1; v_1', p_3) \cap \gamma(p_2; v_3', v_2') \neq \emptyset$ and $\gamma(p_3; v_3', p_5) \cap \gamma(p_4; v_4', v_5') \neq \emptyset$. Suppose $\gamma(p_1; v_2', p_3) \cap \gamma(p_5; v_5', p_3) = \emptyset$. If $\gamma(p_1; v_1', p_3) \cap \gamma(p_2; v_3', v_2')$ has two of the remaining points say r_1, r_2, $r_3 \in \gamma(p_5; p_3, v_5)$, let $r_1 = \alpha(p_3; p_1, v_4')$: and if $r_2 \in \gamma(r_1; p_2, p_3') \neq \emptyset$, we have a cutting line $L_5(r_1, p_3)$; and if $r_2 \in \gamma(r_1; p_1', p_3)$, we have $(v_2 v_3 p_2)_3$, $(p_1 r_1 r_2 p_3 v_1)_5$ and a 4-hole from the remaining points; and if $r_2 \in \gamma(r_1; p_2, p_1')$, we have a cutting line $L_5(p_1, r_1)$. If $\gamma(p_5; p_3, v_5) \cap \gamma(v_4; p_3, p_4)$ has two of the remaining points, symmetrically, the conclusion is also right. Suppose $\gamma(p_1; v_2', p_3) \cap \gamma(p_5; v_5', p_3) \neq \emptyset$. We may suppose $r_1 \in \gamma(p_1; v_1', p_3) \cap \gamma(p_2; v_3', v_2')$, $r_2 \in \gamma(p_1; v_2', p_3) \cap \gamma(p_5; v_5', p_3)$, $r_3 \in \gamma(p_3; v_4, p_5) \cap \gamma(p_4; v_4', v_5')$. If $\gamma(r_2; p_1, p_3') \neq \emptyset$, we have $(v_2 v_3 p_2)_3$, $(p_1 r_1 p_3 r_2 v_1)_5$ and a 4-hole from the remaining points. If $\gamma(r_2; p_3, p_1') \neq \emptyset$, we have $(v_2 v_3 p_2)_3$, $(r_2 p_1 r_1 p_3 \alpha(r_2; p_3, p_1'))_5$ and a 4-hole from the remaining points. If $\gamma(r_2; p_1, p_3') = \emptyset$ and $\gamma(r_2; p_3, p_1') = \emptyset$, we have $(v_4 v_5 p_4)_3$, $(r_3 p_5 v_1 r_2 p_3)_5$ and a 4-hole from the remaining points.

Case 3 $|V(P)| = 4$.

Let $v_i \in V(P)$ for $i = 1, 2, 3, 4$. We have 4 friend points p_i for $i = 1, 2, 3, 4$. Then we have 5 points r_1, r_2, r_3, r_4, r_5 remained as shown in Figure 6.

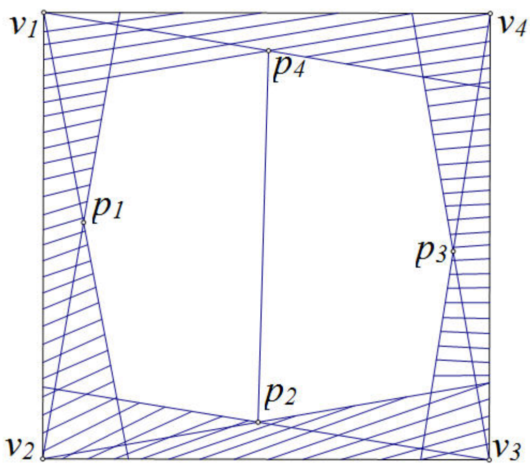

Figure 6. Figure of $|\mathrm{V}(\mathrm{P})| = 4$.

If $\gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4) = \emptyset$ or $\gamma(p_3; v_3', v_4') \cap H(p_3; p_2p_4) = \emptyset$, we have a cutting line $L_5(p_4, \alpha(p_4; p_2, v_1'))$ or $L_5(p_4, \alpha(p_4; p_2, v_4'))$. Then we will consider that $\gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4) \neq \emptyset$ and $\gamma(p_3; v_3', v_4') \cap H(p_3; p_2p_4) \neq \emptyset$.

Assume one of the five points say $r_1 \in \gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4)$ and the remaining four say $r_i \in \gamma(p_3; v_3', v_4') \cap H(p_3; p_2p_4)$, $i = 2, 3, 4, 5$. (If $\gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4)$ has four points and $\gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4)$ has one point, symmetrically, the conclusion is also right). Let $r_2 = \alpha(p_4; p_2, v_1')$.

Suppose $r_1 \in \gamma(p_1; v_1', p_2)$ or $r_1 \in \gamma(p_1; v_2', p_4)$. We always have a cutting line $L_5(p_2, p_4)$. Suppose $r_1 \in \gamma(p_1; p_4, r_2) \cap H(p_1; p_2p_4))$. We have $(v_1v_4p_4)_3$, $(p_1v_2p_2r_2r_1)_5$ and a 4-hole from the remaining points. Suppose $r_1 \in \gamma(p_1; p_2, r_2) \cap H(p_1; p_2p_4)$. We have $(v_2v_3p_2)_3$, $(p_1v_1p_4r_2r_1)_5$ and a 4-hole from the remaining points.

Assume two of the five points, say $r_1, r_2 \in \gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4)$ and the remaining three say $r_i \in \gamma(p_3; v_3', v_4') \cap H(p_3; p_2p_4)$, $i = 3, 4, 5$. (If $\gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4)$ has three points and $\gamma(p_1; v_1', v_2') \cap H(p_1; p_2p_4)$ has two points, symmetrically, our conclusion is also right.)

Suppose $\gamma(p_2; v_1, p_4) = \emptyset$. If $\gamma(p_2; v_1, p_1) \neq \emptyset$, let $r_1 = \alpha(p_2; v_1, p_1)$, we have $(r_2p_1v_2)_3$, $(p_4v_1r_1p_2\alpha(p_2; p_4, v_2'))_5$ and a 4-hole from the remaining points. If $\gamma(p_2; v_1, p_1) = \emptyset$, we have $(r_1r_2v_2)_3$, $(p_4v_1p_1p_2\alpha(p_2; p_4, v_2'))_5$ and a 4-hole from the remaining points. Suppose $\gamma(p_2; v_1, p_4) \neq \emptyset$. Let $r_1 = \alpha(p_2; p_4, v_1)$. If $r_2 \in \gamma(r_1; p_1, p_2')$, we have $(v_1v_4p_4)_3$, $(r_1r_2p_1v_2p_2)_5$ and a 4-hole from the remaining points. If $r_2 \in \gamma(r_1; p_1, p_4')$, we have $(v_2p_2v_3)_3$, $(v_1p_1r_2r_1p_4)_5$ and a 4-hole from the remaining points. If $r_2 \in \gamma(r_1; p_2, p_4')$, we have $(v_1v_2p_1)_3$, $(p_4r_1r_2p_2\alpha(p_2; p_4, v_2'))_5$ and a 4-hole from the remaining points.

Case 4 $|V(P)| = 3$.

Let $v_1, v_2, v_3 \in V(P)$. We have 3 friend points p_1, p_2, p_3 and 7 points remained. As shown in Figure 7, denote $\gamma(p_1; v_2', p_3) \cap \gamma(p_3; v_3', p_1) = T_1$, $\gamma(p_1; v_1', p_2) \cap \gamma(p_2; v_3', p_1) = T_2$, $\gamma(p_2; v_2', p_3) \cap \gamma(p_3; v_1', p_1) = T_3$.

Without loss of generality, we assume $|T_3| \geq |T_1| \geq |T_2|$.

(1) $|T_3| = 7$.

We have a cutting line $L_5(p_2, \alpha(p_2; p_3, v_2')))$.

(2) $|T_3| = 6$.

Name the remaining one r_1. If $r_1 \in \gamma(p_3; v_3', p_1)$ or $r_1 \in \gamma(p_2; v_3', p_1)$, we have a cutting line $L_5(p_2, p_3)$. If $r_1 \in \gamma(p_3; p_1, p_2) \cap \gamma(p_1; p_2, p_3)$: and if $\gamma(r_1; p_3, p_1') \neq \emptyset$, we have a cutting line

$L_5(r_1, \alpha(r_1; p_3, p_1^{'}))$; and if $\gamma(r_1; p_3, p_1^{'}) = \emptyset$, we have $(v_1v_3p_3)_3$, $(r_1p_1v_2p_2\alpha(r_1; p_2, p_1^{'}))_5$ and a 4-hole from the remaining points.

(3) $|T_3| = 5$.

Name the remaining two points r_1, r_2. Then we will discuss the region $\gamma(p_3; v_1, p_1)$, as shown in Figure 8.

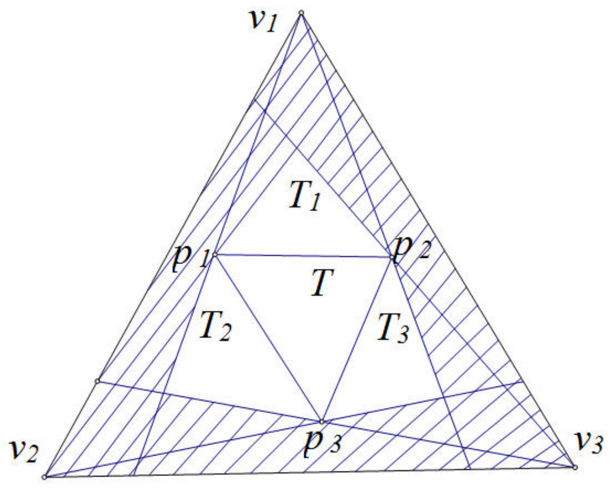

Figure 7. Figure of $|V(P)| = 5$.

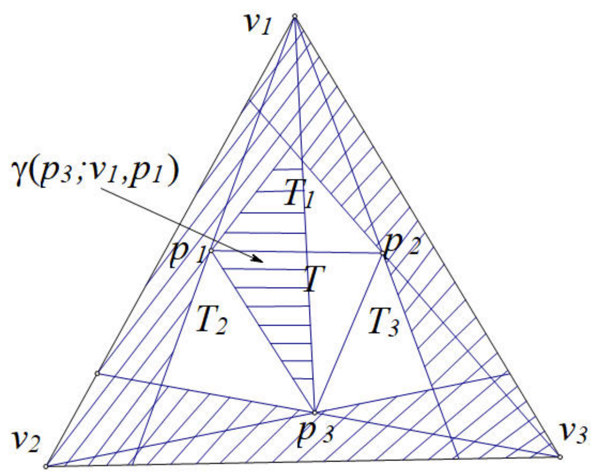

Figure 8. Figure of $|T_3| = 5$

Assume $\gamma(p_3; v_1, p_1) = \emptyset$. (If $\gamma(p_1; p_2, v_2) = \emptyset$, by the similar reason our conclusion is also right.) Let $r_1 = \alpha(p_3; p_1, p_2)$. Suppose $r_1 \in \gamma(p_2; p_1, p_3)$.

If $r_2 \in \gamma(r_1; p_3, p_1^{'})$, we have a cutting line $L_5(p_3, r_2)$. If $r_2 \in \gamma(p_2; r_1, p_1)$: and if $\gamma(r_1; p_3, p_1^{'}) \neq \emptyset$, we have $(r_2p_2v_2)_3$, $(p_3v_1p_1r_1\alpha(r_1; p_3, p_1^{'}))_5$ and a 4-hole from the remaining points; and if $\gamma(r_1; p_3, p_1^{'}) = \emptyset$, we have $(v_1v_2p_1)_3$, $(p_3r_1r_2p_2\alpha(p_2; p_3, v_3))_5$ and a 4-hole from the remaining points. Suppose $r_1 \in \gamma(p_2; p_1, v_3^{'})$. If $r_2 \in \gamma(r_1; p_3, p_1^{'})$, we have a cutting line $L_5(p_3, r_2)$. If $r_2 \in \gamma(r_1; p_1^{'}, p_3^{'})$, we have $(r_1v_2p_2)_3$, $(p_3v_1p_1r_1\alpha(r_1; p_3, p_2))_5$ and a 4-hole from the remaining points.

Assume $\gamma(p_3; v_1, p_1) \neq \emptyset$ and $\gamma(p_1; p_2, v_2) \neq \emptyset$. Then we suppose $\gamma(p_3; v_1, p_1)$ has one point say r_1 and $\gamma(p_1; p_2, v_2)$ has one point say r_2. If $\gamma(r_1; p_1, p_2^{'}) \neq \emptyset$, we have a cutting line $L_5(p_2, r_1)$. If $\gamma(r_1; p_1, p_2^{'}) = \emptyset$, we have a cutting line $L_5(r_1, \alpha(r_1; p_2, p_3))$.

(4) $|T_3| = 4$.

Name the remaining three points r_1, r_2, r_3. Then we will discuss the region $\gamma(p_3; p_1, v_3^{'})$, as shown in Figure 9.

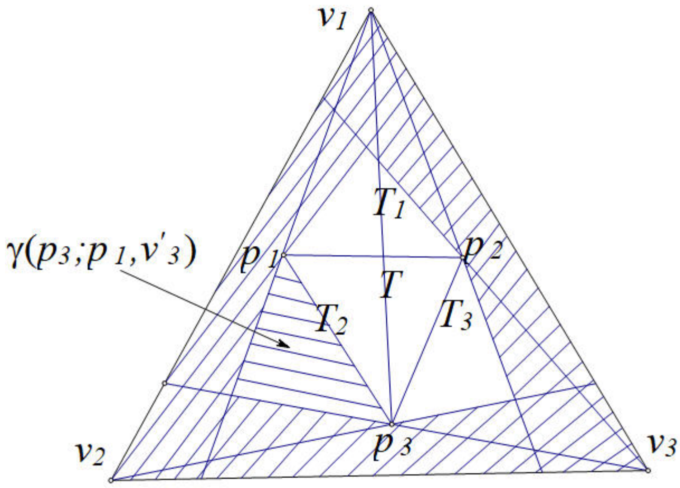

Figure 9. Figure of $|\mathrm{T}_3| = \mathbf{4}$

(a) Assume $r_1, r_2, r_3 \in \gamma(p_3; p_1, v_3^{'})$. Let $r_1 = \alpha(p_3; p_1, v_3^{'})$. We have $(v_1p_2p_3)_3$, $(r_1p_1v_2p_2p_3)_5$ and a 4-hole from the remaining points.

(b) Assume two of $r_i, i = 1, 2, 3$, say $r_1, r_2 \in \gamma(p_3; p_1, v_3^{'})$. Suppose $r_3 \in \gamma(p_2; p_1, p_3) \cap \gamma(p_3; p_1, p_2)$. If $\gamma(r_3; p_1, p_2^{'}) \neq \emptyset$: we have a 4-hole from $\{r_4, r_5, r_6, r_7, v_3\}$, $(p_1v_2p_2r_3\alpha(r_3; p_1, p_2^{'}))_5$ and a 3-hole from the remaining points. If $\gamma(r_3; p_1, p_2^{'}) \neq \emptyset$, we have$(r_3p_1v_2p_2\alpha(r_3; p_2, p_1^{'}))_5$, $(r_1r_2v_1)_3$ and a 4-hole from the remaining points. If $\gamma(r_3; p_1, p_2^{'}) = \emptyset$ and $\gamma(r_3; p_1, p_2^{'}) = \emptyset$, we have a cutting line $L_4(p_2, r_3)$.

(c) Assume one of $r_i, i = 1, 2, 3$, say $r_1 \in \gamma(p_3; p_1, v_3^{'})$.

Suppose $\gamma(r_3; p_1, v_2) = \emptyset$. We have a cutting line $L_5(p_3, r_2)$.

Suppose $\gamma(p_3; p_1, v_2) \neq \emptyset$. Let $r_2 = \alpha(p_3; p_1, v_2)$. If $r_2 \in \gamma(p_1; v_1^{'}, p_2)$, we have a cutting line $L_5(r_2, \alpha(r_2; p_3, p_2))$. Then we suppose $r_2 \in \gamma(p_1; p_2, p_3)$. If $r_1 \in \gamma(r_2; p_2^{'}, p_1)$: and if $r_3 \in \gamma(r_2; p_3, p_1^{'})$, we have a cutting line $L_5(r_2, r_3)$; and if $r_3 \in \gamma(r_2; p_2, p_1^{'})$, we have $(v_1r_1p_3)_3$, $(p_1v_2p_2r_3r_2)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; p_2, v_2)$, we have $(v_1v_3p_3)_3$, $(r_1p_1v_2r_3r_2)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; v_2, p_3^{'})$, we have a cutting line $L_5(v_2, p_3)$. If $r_1 \in \gamma(r_2; p_2^{'}, p_3)$: and if $r_3 \in \gamma(r_2; p_3, p_1^{'})$, we have a cutting line $L_5(r_2, \alpha(r_2; p_3, p_1^{'}))$; and if $r_3 \in \gamma(r_2; p_2, p_1^{'})$, we have $(v_1r_1p_3)_3$, $(r_2p_1v_2p_2r_3)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; r_1^{'}, p_2)$, we have $(v_1v_2p_1)_3$, $(p_3r_1r_2r_3p_2)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; v_2, r_1^{'})$, we have $(v_1v_3p_3)_3$, $(r_1p_1v_2r_3r_2)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; p_3^{'}, v_2)$, we have $(v_1r_1p_1)_3$, $(p_3r_2r_3v_2p_2)_5$ and a 4-hole from the remaining points.

(d) Assume $\gamma(p_3; p_1, v_3^{'}) = \emptyset$. By the same reason, we also assume $\gamma(p_1; p_2, v_1^{'}) = \emptyset$. Then we will discuss the region $\gamma(v_1; p_1, p_2)$, as shown in Figure 10.

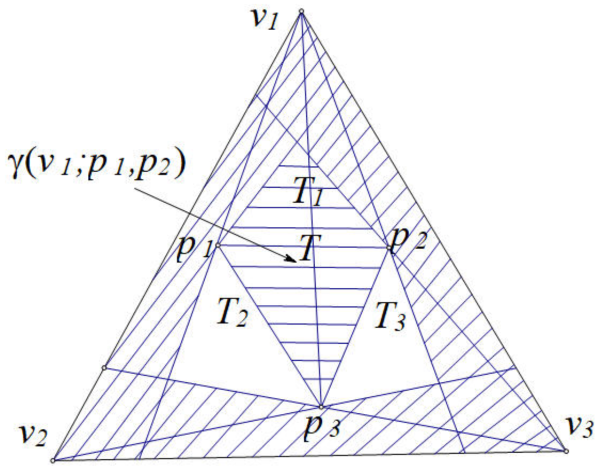

Figure 10. Figure of $|T_3| = 4$ with shaded region nonempty.

(d1) Suppose $\gamma(v_1; p_1, p_2) = \emptyset$. Let $r_1 = \alpha(p_1; p_3, p_2)$ within $(p_1p_2p_3)$.

If $\gamma(r_1; p_1, p_3') \neq \emptyset$, we have $(v_2v_3p_2)_3$, $(r_1p_3v_1p_1\alpha(r_1; p_1, p_3'))_5$ and a 4-hole from the remaining points.

If $\gamma(r_1; p_1, p_3') = \emptyset$: and if $\gamma(r_1; p_3, p_1') \neq \emptyset$, we have $(v_2v_3p_2)_3$, $(p_3v_1p_1r_1\alpha(r_1; p_3, p_1))_5$ and a 4-hole from the remaining points; and if $\gamma(r_1; p_3, p_1') = \emptyset$, let $r_2 = \alpha(r_1; p_3', p_1')$ within $(p_1p_2p_3)$, we have $(v_1v_3p_3)_3$, $(r_1p_1v_2r_2r_3)_5$ and a 4-hole from the remaining points when $r_3 \in \gamma(r_2; r_1, v_2') \cap \gamma(r_1; r_2, p_1')$, we have $(v_1p_1r_1p_3)_4$, $(r_3r_2v_2p_2\alpha(r_3; p_2, r_2))_5$ and a 3-hole from the remaining points when $r_3 \in \gamma(r_2; p_1', v_2')$ and $\gamma(p_3; p_2, v_1') \cap \gamma(r_3; p_2, r_2') \neq \emptyset$, we have $(v_1v_2p_1)_3$, $(p_3r_1r_2r_3\alpha(r_3; p_3, r_2'))_5$ and a 4-hole from the remaining points when $r_3 \in \gamma(r_2; p_1', v_2')$ and $\gamma(p_3; p_2, v_1') \cap \gamma(r_3; p_2, r_2') = \emptyset$, we have $(v_1r_1p_3)_3$, $(p_1v_2p_2r_3r_2)_5$ and a 4-hole from the remaining points when $r_3 \in \gamma(r_2; p_1', p_2)$, we have $(v_1v_2p_1)_3$, $(p_3r_1r_2r_3p_2)_5$ and a 4-hole from the remaining points when $r_3 \in \gamma(r_2; r_1', p_2)$.

(d2) Suppose $\gamma(v_1; p_1, p_2)$ has one of the r_1, r_2, r_3, say $r_1 \in \gamma(r_1; p_1, p_2)$. Let $r_2 = \alpha(p_2; p_1, p_3)$.

If $r_2 \in \gamma(r_1; p_2, p_3)$, we have $(v_2v_3p_2)_3$, $(r_1p_1v_1p_3r_2)_5$ and a 4-hole from the remaining points.

If $r_2 \in \gamma(r_1; p_1, p_3)$: and if $r_3 \in \gamma(r_2; r_1, p_3)$, we have $(v_1v_2p_1)_3$, $(r_3r_2r_1p_2p_3)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; p_3, p_1')$, we have $(v_2v_3p_2)_3$, $(v_1p_1r_2r_3p_3)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; p_1', v_1')$, we have a cutting line $L_5(r_1, \alpha(r_1; p_2, p_1'))$ when $\gamma(r_1; p_2, p_1') \neq \emptyset$, we have $(v_2v_3p_2)_3$, $(r_3r_2p_1r_1\alpha(r_3; r_1, r_2'))_5$ and a 4-hole from the remaining points when $\gamma(r_1; p_2, p_1') = \emptyset$ and $\gamma(r_3; r_1, r_2') \neq \emptyset$, we have $(r_1p_1v_2p_2)_4$, $(p_3v_1r_2r_3\alpha(r_3; p_3, r_2'))_5$ and a 3-hole from the remaining points when $\gamma(r_1; p_2, p_1') = \emptyset$ and $\gamma(r_3; r_1, r_2') = \emptyset$.

If $r_2 \in \gamma(r_1; p_2, p_3)$, we have $(v_2v_3p_2)_3$, $(p_3v_1p_1r_1r_2)_5$ and a 4-hole from the remaining points.

(d3) Suppose $\gamma(v_1; p_1, p_2)$ has two of the points r_1, r_2, r_3, say $r_1, r_2 \in \gamma(r_1; p_1, p_2)$. Let $r_1 = \alpha(p_2; p_1, p_3)$.

If $\gamma(r_1; p_2, p_1') \neq \emptyset$, we have a cutting line $L_5(r_1, \alpha(r_1; p_2, p_1'))$.

If $\gamma(r_1; p_2, p_1') = \emptyset$, let $r_2 = \alpha(p_1; p_2, p_2)$: and if $r_2 \in \gamma(v_1; p_2, p_3)$, we have a cutting line $L_5(r_2, r_3)$ when $r_3 \in \gamma(r_2; p_1, p_3)$, we have $(v_1p_1v_2)_3$, $(p_3r_2r_3r_1p_2)_5$ and a 4-hole from the remaining points when $r_3 \in \gamma(r_2; p_3, r_1)$, we have $(v_2v_3p_2)_3$, $(r_2v_1p_1r_1p_3)_5$

and a 4-hole from the remaining points when $r_3 \in \gamma(r_1; r_2, p_1^{'})$, we have $(v_1p_3v_3)_3$, $(r_1p_1v_2p_2r_3)_5$ and a 4-hole from the remaining points when $r_3 \in \gamma(r_1; p_2, p_1^{'})$; and if $r_2 \in \gamma(v_1; p_1, p_2)$, we have $(p_1r_1p_2v_2)_4$, $(p_1v_1r_1r_3\alpha(r_3; p_3, r_2^{'}))_5$ and a 3-hole from the remaining points when $\gamma(r_2; r_3, p_1^{'}) \cap \gamma(p_2; p_3, v_2^{'}) \neq \emptyset$, we have a cutting line $L_5(r_1, \alpha(r_1; p_2, p_1^{'}))$, when $\gamma(r_1; p_2, p_1^{'}) \neq \emptyset$, we have $(v_1v_3p_3)_3$, $(r_3r_2p_1r_1\alpha(r_1; r_3, p_1^{'}))_5$ and a 4-hole from the remaining points when $\gamma(r_2; r_3, p_1^{'}) \cap \gamma(p_2; p_3, v_2^{'}) = \emptyset$ and $\gamma(r_1; p_2, p_1^{'}) = \emptyset$.

(d4) Suppose $\gamma(v_1; p_1, p_2)$ has all of the three points r_1, r_2, r_3. Let $r_1 = \alpha(p_1; p_3, p_2)$, $r_2 = \alpha(p_1; p_2, p_3)$.

If $\gamma(r_1; p_3, p_1^{'}) \neq \emptyset$ or $\gamma(r_2; p_2, p_1^{'}) \neq \emptyset$, we always have a cutting line L_5.

If $\gamma(r_1; p_3, p_1^{'}) = \emptyset$ and $\gamma(r_2; p_2, p_1^{'}) = \emptyset$: and if $r_3 \in \gamma(r_1; p_1, p_3^{'})$, we have a cutting line $L_5(p_3, r_1)$; and if $r_3 \in \gamma(r_1; p_3^{'}, p_2) \cap \gamma(r_2; p_2^{'}, r_1)$, we have $(v_1v_2p_1)_3$, $(p_3r_1r_3r_2p_2)_5$ and a 4-hole from the remaining points; and if $r_3 \in \gamma(r_2; p_1, p_2^{'})$, we have a cutting line $L_5(p_2, r_2)$; and if $r_3 \in \gamma(r_2; p_1^{'}, v_2^{'})$, we have $(v_1p_1r_1p_3)_4$, $(r_3r_2v_2p_2\alpha(r_3; p_2, r_2^{'}))_5$ and a 3-hole from the remaining points when $\gamma(r_3; p_2, r_2^{'}) \cap \gamma(v_1; p_2, p_3) \neq \emptyset$, we have $(v_2v_3p_2)_3$, $(r_1p_1r_2r_3\alpha(r_3; p_3, r_2^{'}))_5$ and a 4-hole from the remaining points when $\gamma(r_3; p_2, r_2^{'}) \cap \gamma(v_1; p_2, p_3) \neq \emptyset$.

(5) $|T_3| = 3$. Let $r_1, r_2, r_3 \in T_3$.

(a) $|T_1| = 3$.

Let $r_4, r_5, r_6 \in T_1$. Name the remaining one point r_7. Assume $r_7 \in \gamma(v_2; p_3, p_2)$, as shown in Figure 11.

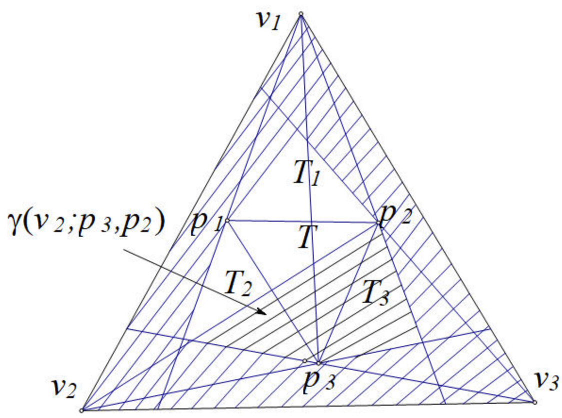

Figure 11. Figure of $|T_1| = 3$.

Symmetrically, our conclusion is also right when $r_7 \in \gamma(v_2; p_3, p_1)$. Let $r_4 = \alpha(p_3; p_1, v_3^{'})$. We have $(r_5r_6v_1)_3$, $(r_4p_1v_2r_7p_3)_5$ and a 4-hole from the remaining points.

(b) $|T_1| = 2$.

Let $r_4, r_5 \in T_1$. Name the remaining two points r_6, r_7.

(b1) $|T_2| = 2$. Let $r_6, r_7 \in T_2$.

Assume $\gamma(v_1; p_1, p_2) = \emptyset$. Let $r_4 = \alpha(p_2; v_1, p_3)$. Suppose $r_5 \in \gamma(r_4; p_2^{'}, p_3)$. We have a cutting line $L_5(p_1, p_3)$. Suppose $r_5 \in \gamma(r_4; v_1^{'}, p_3)$. If $\gamma(r_5; p_3, r_4^{'}) \neq \emptyset$, we have a cutting line $L_5(r_5, \alpha(r_5; p_3, v_4^{'}))$. If $\gamma(r_5; p_3, r_4^{'}) = \emptyset$, we have a cutting line $L_5(r_1, \alpha(r_1; p_1, p_2)$ where $r_1 = \alpha(p_1; p_3, p_2)$. Suppose $r_5 \in \gamma(r_4; p_2^{'}, v_1^{'})$. We have $(r_6r_7v_2)_3$, $(r_4v_1p_1p_2r_5)_5$ and a 4-hole from the remaining points.

Assume $\gamma(v_1; p_1, p_2)$ has one of r_4, r_5. Let $r_4 \in \alpha(v_1; p_1, p_2)$. Suppose $r_5 \in \alpha(r_4; p_1', v_1)$. If $\gamma(r_4; p_2, v_1') = \emptyset$, we have $(r_2r_3v_3)_3$, $(r_5r_4p_2r_1p_3)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_2; p_3, v_3)$. If $\gamma(r_4; p_2, v_1') \neq \emptyset$, we have $(p_1v_2r_7)_3$, $(v_1r_4r_6p_2r_5)_5$ and a 4-hole from the remaining points where $r_6 = \alpha(r_4; p_2, v_1')$.

Assume $\gamma(v_1; p_1, p_2)$ has r_4, r_5. Let $r_4 \in \alpha(p_2; v_1, p_1)$, $r_1 = \alpha(p_2; p_3, v_3)$. we have $(r_2r_3v_3)_3$, $(p_2r_1p_3v_1r_4)_5$ and a 4-hole from the remaining points.

(b2) $|T_2| = 1$.

Let $r_6 \in T_2$ and $r_7 \in (p_1p_2p_3)$, as shown in Figure 12.

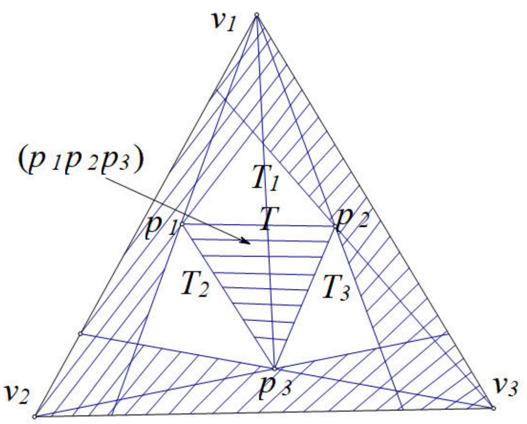

Figure 12. Figure of $|T_2| = 1$.

Assume $r_6 \in \gamma(r_7; p_3', p_2)$. We have $(r_2r_3v_3)_3$, $(p_3r_7r_6p_2r_1)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_2; p_3, v_3)$. Assume $r_6 \in \gamma(r_7; p_3', v_2)$. We have $(r_4r_5v_1)_3$, $(p_1v_2r_6r_7p_3)_5$ and a 4-hole from the remaining points. Assume $r_6 \in \gamma(r_7; p_1, v_2)$. If $\gamma(r_7; r_6', p_2) \neq \emptyset$, we have $(r_2r_3v_3)_3$, $(p_6r_7r_1p_2v_2)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(r_7; p_2, r_6')$. If $\gamma(r_7; r_6', p_2) = \emptyset$, we have $(v_2v_3p_2)_3$, $(p_1r_6r_7r_1p_3)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(r_7; p_3, p_1')$.

(b3) $|T_2| = 0$.

Let $r_6, r_7 \in (p_1p_2p_3)$. Then we will discuss the region $\gamma(p_3; p_1, v_1)$, as shown in Figure 13.

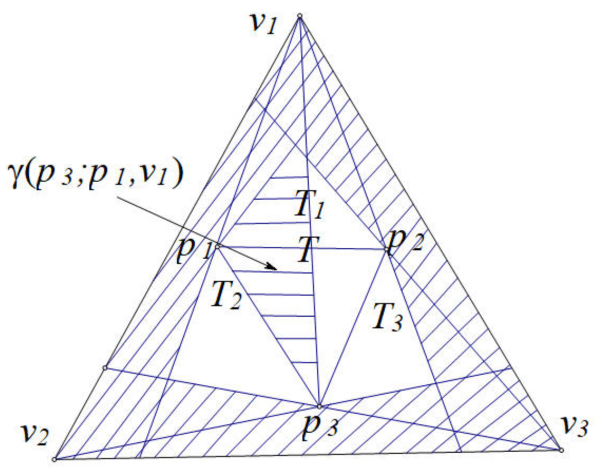

Figure 13. Figure of $|T_2| = 0$.

Assume $\gamma(p_3; p_1, v_1) = \emptyset$. Suppose $\gamma(r_6; p_3, v_2') \neq \emptyset$. We have $(p_3 p_1 v_2 r_6 \alpha(r_6; p_3, v_2'))_5$, $(r_4 r_5 v_1)_3$ and a 4-hole from the remaining points. Suppose $\gamma(r_6; p_3, v_2') = \emptyset$. If $\gamma(r_7; r_6', p_2) \cap \gamma(p_2; r_7, v_2') \neq \emptyset$, we have $(r_2 r_3 v_3)_3$, $(r_6 v_2 p_2 r_1 r_7)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(r_7; p_2, r_6')$. If $\gamma(r_7; r_6', p_2) \cap \gamma(p_2; r_7, v_2') = \emptyset$, we have $(v_2 v_3 p_2)_3$, $(r_1 r_7 r_6 p_1 \alpha(r_1; p_1, p_3))_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_1; r_7, p_3)$ within $\gamma(p_3; p_2, v_3)$.

Assume $\gamma(p_3; p_2, v_1) = \emptyset$. We have $(r_2 r_3 v_3)_3$, $(r_1 p_3 r_6 v_2 p_2)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_3; p_2, v_3)$ and $r_6 = \alpha(p_3; v_2, p_1)$.

Assume $\gamma(p_3; p_1, v_2) \neq \emptyset$ and $\gamma(p_3; p_1, v_2) \neq \emptyset$. We may assume $r_6 \in \gamma(p_3; p_1, v_2)$ and $r_7 \in \gamma(p_3; p_1, v_2)$. Suppose $r_7 \in \gamma(r_6; p_2, p_1')$. We have $(r_4 r_5 v_1)_3$, $(r_6 r_7 p_2 v_2 p_1)_5$ and a 4-hole from the remaining points. Suppose $r_7 \in \gamma(r_6; p_3, p_1')$. If $\gamma(r_7; r_6', p_2) \neq \emptyset$, we have $(r_2 r_3 v_3)_3$, $(r_7 r_6 v_2 p_2 r_1)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_2; r_7, v_2')$. If $\gamma(r_7; r_6', p_2) = \emptyset$: and if $\gamma(r_7; p_1, v_1) = \emptyset$, we have $(r_2 v_3 p_2)_3$, $(v_1 p_1 r_6 r_7 r_4)_5$ and a 4-hole from the remaining points where $r_4 = \alpha(r_7; p_1, p_3)$ within $\gamma(p_3; p_1, v_1)$; and if $\gamma(r_7; p_1, v_1) \neq \emptyset$, we have $(r_2 v_3 p_2)_3$, $(r_4 p_1 r_6 r_7 r_1)_5$ and a 4-hole from the remaining points where $r_4 = \alpha(r_7; p_1, v_1)$.

(c) $|T_1| = \mathbf{1}$. Let $r_4 \in T_1$.

(c1) $|T_2| = \mathbf{1}$. Let $r_5 \in T_2$ and $r_6, r_7 \in (p_1 p_2 p_3)$.

Firstly, consider $r_4 \in \gamma(v_1; p_1, p_2)$, then we will discuss the region $\gamma(v_1; p_1, p_2) \cap (p_1 p_2 p_3) = \emptyset$, as shown in Figure 14.

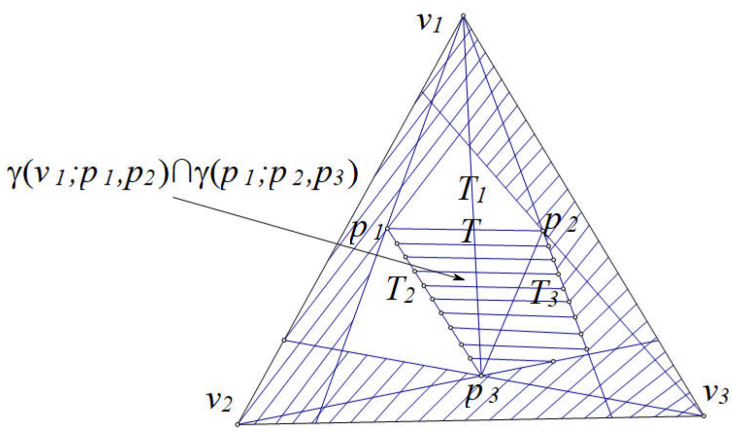

Figure 14. Figure of $|\mathrm{T}_1| = 1$ and $|\mathrm{T}_2| = \mathbf{1}$.

Assume $\gamma(v_1; p_1, p_2) \cap (p_1 p_2 p_3) = \emptyset$. We have a cutting line $L_5(r_4, \alpha(r_4; p_2, p_1'))$. Assume $\gamma(v_1; p_1, p_2) \cap (p_1 p_2 p_3) \neq \emptyset$. Let $r_6 = \alpha(p_2; p_1, v_1)$. If $\gamma(r_6; p_2, p_1') \neq \emptyset$, we have a cutting line $L_5(r_4, \alpha(r_6; p_2, p_1'))$. Then we may assume $\gamma(r_6; p_2, p_1') = \emptyset$.

Suppose $r_5 \in \gamma(r_6; v_2, r_4')$. If $r_7 \in \gamma(r_6; p_2', r_4)$, we have $(p_1 v_2 r_5)_3$, $(r_4 r_7 r_6 p_2 \alpha(r_4; p_2, p_3))_5$ and a 4-hole from the remaining points. If $r_7 \in \gamma(r_6; p_1', r_4)$, we have a cutting line $L_5(r_4, r_6)$.

Suppose $r_5 \in \gamma(r_6; v_2, p_1)$. If $\gamma(r_6; r_5', p_1') \neq \emptyset$, we have $(v_1 r_4 p_1)_3$, $(r_6 r_5 v_2 p_2 \alpha(r_6; p_2, r_5'))_5$ and a 4-hole from the remaining points. If $\gamma(r_6; r_5', p_1') = \emptyset$: and if $r_7 \in \gamma(r_6; r_4, r_5')$, we have $(v_2 v_3 p_2)_3$, $(r_4 p_1 r_5 r_6 r_7)_5$ and a 4-hole from the remaining points; and if $r_7 \in \gamma(r_6; r_4, p_2')$, we have $(p_1 v_2 r_5)_3$, $(r_4 r_7 r_6 p_2 \alpha(r_4; p_2, p_3))_5$ and a 4-hole from the remaining points. Suppose $r_5 \in \gamma(r_6; p_2, r_4')$. If $r_7 \in \gamma(r_6; p_2', r_4)$, we have $(p_1v_2r_5)_3$, $(p_2r_6r_7r_4\alpha(r_4;p_2,p_3))_5$ and a 4-hole from the remaining points. If $r_7 \in \gamma(r_6;r_4,p_2) \cap H(r_6;r_4p_2)$, we have $(p_1v_2r_6r_5)_3$, $(p_3r_4r_7p_2\alpha(p_3;p_2,v_1'))_5$ and a 3-hole from the remaining points. If $r_7 \in \gamma(p_2;r_4,v_1)$, we have $(v_1v_2p_1)_3$, $(r_1r_6r_5p_2r_7)_5$ and a 4-hole from the remaining points. If $r_7 \in \gamma(p_2;v_1,p_3)$, we have $(p_1v_2r_5)_3$, $(v_1r_4r_6p_2r_7)_5$ and a 4-hole from the remaining points.

Secondly, consider $r_4 \in \gamma(v_1;p_2,p_3)$, then we will discuss the region $\gamma(r_4;p_2,p_3) \cap (p_1p_2p_3) = \emptyset$, as shown in Figure 15.

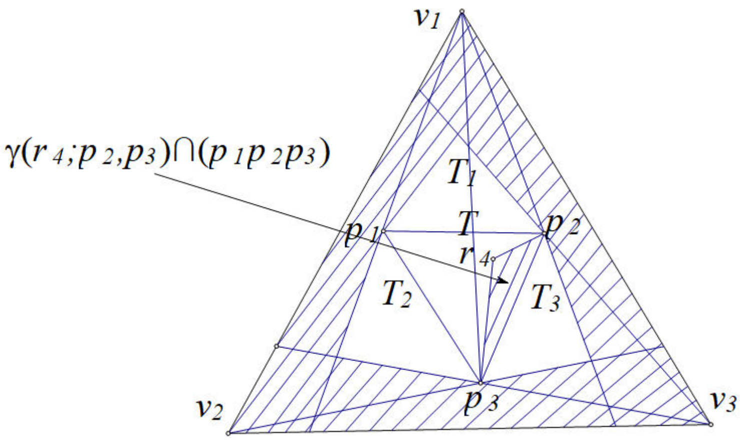

Figure 15. Figure of $|\mathrm{T}_1| = \mathbf{1}$ and $|\mathrm{T}_2| = \mathbf{1}$ with shaded region nonempty.

Assume $\gamma(r_4;p_2,p_3) \cap (p_1p_2p_3) = \emptyset$. We have $(r_2r_3v_3)_3$, $(r_1p_3r_4r_6p_2)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_3;p_2,v_3)$, $r_6 = \alpha(r_4;p_2,p_1)$. Assume $\gamma(r_4;p_1,p_2) \cap (p_1p_2p_3) = \emptyset$. We have $L_5(p_2,r_4)$. Assume $\gamma(r_4;p_2,p_3) \cap (p_1p_2p_3) \neq \emptyset$ and $\gamma(r_4;p_1,p_2) \cap (p_1p_2p_3) \neq \emptyset$. Then we may assume $r_6 \in \gamma(r_4;p_2,p_3)$, $r_7 \in \gamma(r_4;p_1,p_2)$. Suppose $r_6 \in \gamma(r_4;v_1',p_3) \cap (p_1p_2p_3)$. If $\gamma(r_6;r_4',p_3) \neq \emptyset$, we have $(p_1r_5p_2r_7)_4$, $(v_1r_4r_6r_1p_3)_5$ and $(r_2r_3v_3)_3$ where $r_1 = \alpha(r_6;p_3,r_4')$. If $\gamma(r_6;r_4',p_3) = \emptyset$: and if $r_7 \in \gamma(r_4;v_1,p_2) \cap \gamma(v_1;p_2,r_4)$, we have $L_5(p_2;r_4)$; and if $r_7 \in \gamma(r_4;r_5,p_2) \cap \gamma(p_2;p_1,v_1)$, we have $L_5(r_4;r_7)$; and if $r_7 \in \gamma(r_4;p_1,r_5)$, we have $(v_1v_2p_1)_3$, $(r_4r_7r_5p_2r_6)_5$ and a 4-hole from the remaining points. Suppose $r_6 \in \gamma(r_4;v_1',p_2) \cap (p_1p_2p_3)$. If $r_7 \in \gamma(v_1;p_1,p_2) \cap (p_1p_2p_3)$, we have $(r_5v_2p_1)_3$, $(v_1r_7p_2r_6r_4)_5$ and a 4-hole from the remaining points. If $r_7 \in \gamma(v_1;p_2,r_4) \cap \gamma(r_4;p_1,p_2)$: and if $r_7 \in \gamma(r_7;r_4',p_1)$, we have $L_5(r_4,r_7)$; and if $r_5 \in \gamma(r_7;r_4',p_2)$, we have $(v_1v_2p_1)_3$, $(r_4r_7r_5p_2r_6)_5$ and a 4-hole from the remaining points.

(c2) $|T_2| = \mathbf{0}$.

Denote $r_1,r_2,r_3 \in T_3$, $r_4 \in T_2$, $r_5,r_6,r_7 \in (p_1p_2p_3)$. Let $r_5 = \alpha(p_3;p_1,p_2)$ within $(p_1p_2p_3)$. If $\gamma(r_5;p_1',p_3) \neq \emptyset$, we have $L_5(r_5;\alpha(r_5;p_3,p_1'))$. Then we assume $\gamma(r_5;p_1',p_3) = \emptyset$, and we will discuss the region $\gamma(r_5;p_1,p_2) \cap (p_1p_2p_3) = \emptyset$, as hown in Figure 16.

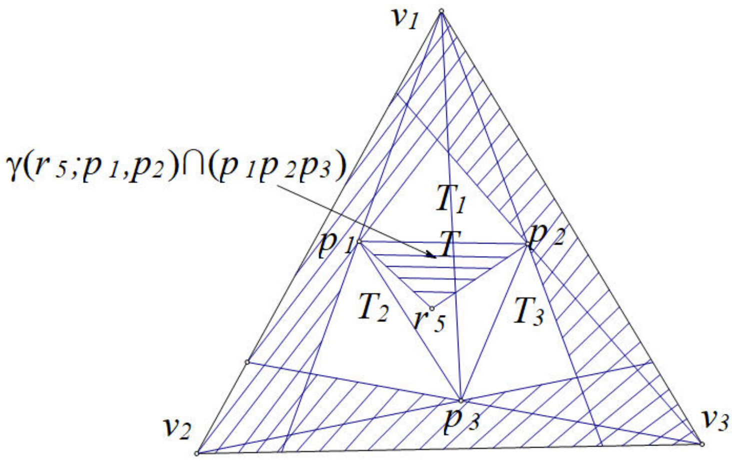

Figure 16. Figure of $|\mathrm{T}_1| = \mathbf{1}$ and $|\mathrm{T}_2| = \mathbf{0}$.

Assume $\gamma(r_5; p_1, p_2) \cap (p_1p_2p_3) = \emptyset$, we have $(v_1r_4p_3)_3$, $(r_5p_1v_2p_2\alpha(p_2; r_5, p_3))_5$ and a 4-hole from the remaining points.

Assume $\gamma(r_5; p_2, p_3) \cap (p_1p_2p_3) = \emptyset$. Let $p_6 = \alpha(r_5; p_2, p_3^{'})$. Suppose $r_4 \in \gamma(r_5; p_3, r_6^{'})$. We have $(r_2r_3v_3)_3$, $(p_2r_1p_3r_4\alpha(r_4; p_2, p_3^{'}))_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_2; p_3, v_2^{'})$. Suppose $r_4 \in \gamma(r_5; p_1, r_6^{'})$. We have $(r_2r_3r_4)_3$, $(p_2r_1p_3r_5r_6)_5$ and a 4-hole from the remaining points where $r_1 = \alpha(p_2; p_3, v_2^{'})$.

Assume $\gamma(r_5; p_1, p_2) \cap (p_1p_2p_3) \neq \emptyset$ and $\gamma(r_5; p_2, p_3) \cap (p_1p_2p_3) \neq \emptyset$. Without loss of generality, we suppose $r_6 \in \gamma(r_5; p_1, p_2)$, $r_7 \in \gamma(r_5; p_2, p_3)$.

Firstly, we may assume $r_6 \in \gamma(r_5; v_2, p_2)$. Suppose $r_4 \in \gamma(r_6; p_7, p_2^{'})$. We have $L_5(p_2, r_6)$. Suppose $r_4 \in \gamma(r_5; r_6^{'}, p_1) \cap H(r_5; r_6p_2)$. We have a cutting line $L_5(r_5, r_6)$. Suppose $r_4 \in \gamma(r_6; p_6^{'}, p_1^{'})$. If $r_7 \in \gamma(r_4; p_2, p_3)$, we have $(v_1v_2p_1)_3$, $(v_4r_5r_6p_2r_7)_5$ and a 4-hole from the remaining points. If $r_7 \in \gamma(p_2; r_4, p_5)$, we have $(v_3r_2r_3)_3$, $(v_1p_1r_6r_5)_4$ and $(p_3r_4r_7p_2r_1)_5$ where $r_1 = \alpha(p_2; p_3, v_2^{'})$.

Secondly, we have may assume $r_6 \in \gamma(r_5; v_2, p_3^{'})$, we have $(v_1p_1r_4)_3$, $(r_5r_6v_2p_2r_7)_5$ and a 4-hole from the remaining points.

(d) $|T_1| = \mathbf{0}$. $|T_2| = \mathbf{0}$.

Let $r_4, r_5, r_6, r_7 \in (p_1p_2p_3)$. And $r_1 = \alpha(p_3; p_2, v_3)$, $r_4 = \alpha(p_3; p_2, p_1)$, $r_5 = \alpha(p_2; p_1, r_4)$. If $\gamma(r_4; p_2, p_3^{'}) \neq \emptyset$, we have $(r_2r_3v_3)_3$, $(p_2r_1p_3r_4\alpha(r_4; p_2, p_3^{'}))_5$ and a 4-hole from the remaining points. Assume $r_5 \in \gamma(r_4; p_1, p_3)$. If $\gamma(r_5; p_2, p_1^{'}) \neq \emptyset$, we have a cutting line $L_5(r_5; \alpha(r_5; p_2, p_1^{'}))$. Then we will discuss the region $\gamma(v_4; p_1, p_2^{'}) \cap (p_1p_2p_3)$ and $\gamma(r_4; p_1, p_3^{'}) \cap \gamma(p_1; p_5, r_4)$, as shown in Figure 17.

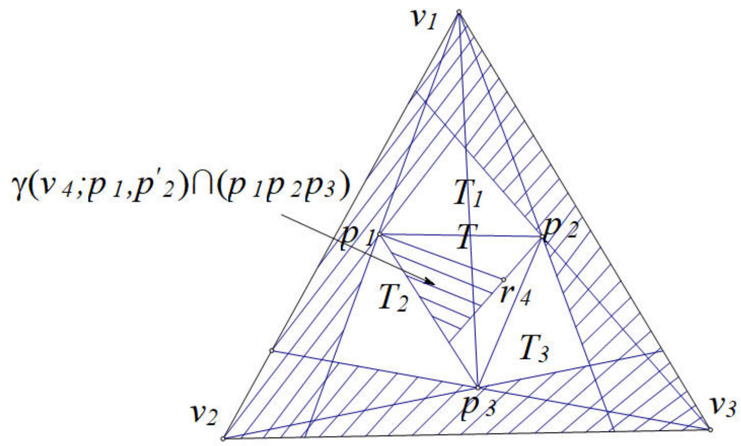

Figure 17. Figure of $|\mathrm{T}_1| = \mathbf{1}$ and $|\mathrm{T}_2| = \mathbf{0}$ with shaded region nonempty.

Assume $\gamma(v_4; p_1, p_2') \cap (p_1p_2p_3) = \emptyset$. We have $(r_7r_5v_2)_3$, $(r_4p_3v_1p_1r_6)_5$ and a 4-hole from the remaining points where $r_6 = \alpha(r_4; p_1, p_3')$.

Assume $\gamma(r_4; p_1, p_3') \cap \gamma(p_1; p_5, r_4) = \emptyset$. Let $r_6 = \alpha(p_1; p_3, r_4)$. Suppose $r_7 \in \gamma(r_6; r_4, p_1')$. We have $(v_1v_2p_1)_3$, $(p_2r_4r_7r_6r_5)_5$ and a 4-hole from the remaining points. Suppose $r_7 \in \gamma(r_6; r_4, v_1') \cap \gamma(r_4; p_1, p_2')$. We have $(r_1r_2r_3)_3$, $(p_1v_2p_2r_5)_4$ and $(r_4p_3v_1r_6r_7)_5$. Suppose $r_7 \in \gamma(r_6; r_5, v_1') \cap \gamma(r_4; p_1, p_2')$. We have $(v_3r_2r_3)_3$, $(r_4p_2r_1p_3)_4$ and $(r_6v_1p_2r_5r_7)_5$. Suppose $r_7 \in \gamma(r_6; r_5, p_2')$. We have $(v_1v_2p_2)_3$, $(r_4p_3r_6r_7r_5)_5$ and a 4-hole from the remaining points. Suppose $r_7 \in \gamma(r_6; p_1, r_3')$. We have a cutting line $L_5(p_3, r_6)$.

Assume $\gamma(v_4; p_1, p_2') \cap (p_1p_2p_3) \neq \emptyset$ and $\gamma(r_4; p_1, p_3') \cap \gamma(p_1; p_5, r_4) \neq \emptyset$. Without loss of generality, assume $r_6 \in \gamma(r_4; p_1, p_2') \cap (p_1p_2p_3)$, $r_7 \in \gamma(r_4; p_1, p_3') \cap \gamma(p_1; p_5, r_4')$.

Suppose $r_6 \in \gamma(r_5; p_3, p_1')$. We have a cutting line $L_5(r_6, \alpha(r_6; p_1, p_3')$.

Suppose $r_6 \in \gamma(r_5; p_3, p_1) \cap \gamma(v_1; r_4, p_3)$. If $r_7 \in \gamma(r_5; p_3, p_2') \cap \gamma(p_1; r_5, r_4)$, we have $(v_2p_2r_5)_3$, $(v_1p_1r_7r_4r_6)_5$ and a 4-hole from the remaining points. If $r_7 \in \gamma(r_5; p_3', p_1') \cap \gamma(r_4; p_1, p_3')$, we have $(v_2p_2p_3)_3$, $(v_1p_1r_5r_7r_6)_5$ and a 4-hole from the remaining points.

Suppose $r_6 \in \gamma(r_5; p_3, p_1) \cap \gamma(v_1; r_1, p_1)$. If $r_7 \in \gamma(r_6; r_4, v_1')$, we have $(v_1r_6r_7r_4p_3)_5$, $(p_1v_2p_2r_5)_4$ and $(r_1r_2r_3)_3$. If $r_7 \in \gamma(r_6; r_5, v_1')$, we have $(v_2v_3p_2)_3$, $(v_1p_1r_5r_7r_6)_5$ and a 4-hole from the remaining points.

(6) $|T_3| = \mathbf{2}$.

Let $r_1, r_2 \in T_3$ and $r_1 = \alpha(p_2; p_3, v_1')$. Assume $r_2 \in \gamma(r_1; p_2, v_3)$. We have $(p_2r_1r_2v_3)_4$ and the remaining 9 points are in $H(\overline{v_3}; p_2p_3)$, as shown in Figure 18.

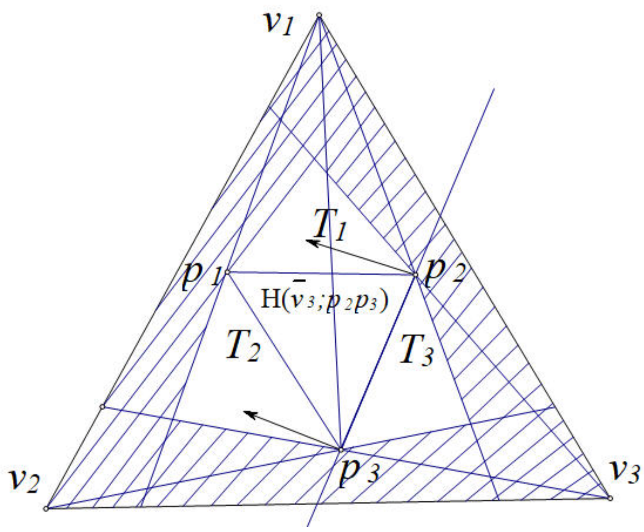

Figure 18. Figure of $|T_3| = \mathbf{2}$.

By the discussion of Part One, we know our conclusion is right. Assume $r_2 \in \gamma(r_1; p_3', v_2)$. We have $(p_3 r_1 r_2 v_2)_4$. By the discussion of Part One, we know our conclusion is also right. Assume $r_2 \in \gamma(r_1; p_3', p_2)$. We have a cutting line $L_5(p_2, \alpha(p_2; p_3, p_1))$.

(7) $|T_3| = \mathbf{1}$.

Let $r_1 \in T_3$, $r_2 \in T_1$, $r_3 \in T_2$ and $r_4, r_5, r_6, r_7 \in (p_1 p_2 p_3)$. Let $r_4 = \alpha(p_3; p_2, p_1)$ within $(p_1 p_2 p_3)$. Assume $r_4 \in \gamma(p_3; p_1, v_1)$, as shown in Figure 19.

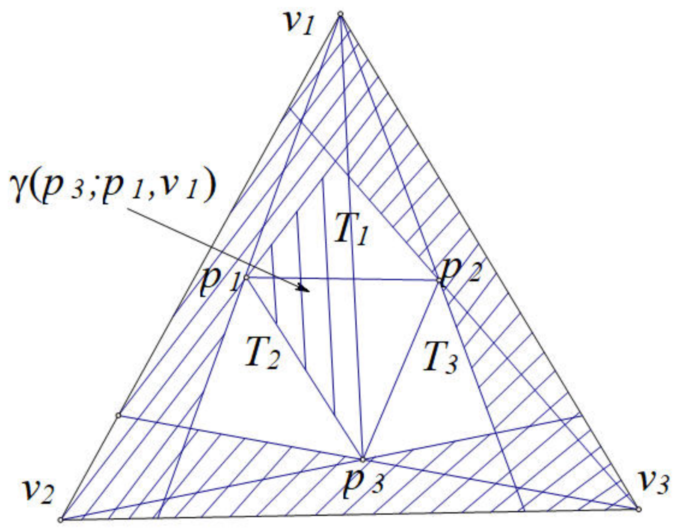

Figure 19. Figure of $|T_3| = \mathbf{1}$.

If $r_2 \in \gamma(v_1; p_2, p_3)$, we have a cutting line $L_5(r_2, \alpha(r_2; p_2, p_1))$. If $r_2 \in \gamma(v_1; p_2, p_1)$, we have a cutting line $L_5(v_1, \alpha(v_1; p_2, p_1))$. Assume $r_4 \in \gamma(p_3; p_2, v_1)$. If $\gamma(r_4; p_3, p_2') \neq \emptyset$, we have a cutting line $L_5(r_4, \alpha(r_4; p_3, p_2'))$. If $\gamma(r_4; p_2, p_3') \neq \emptyset$, we have a cutting line $L_5(r_4; \alpha(r_4; p_2, p_3'))$. If $\gamma(r_4; p_3, p_2') = \emptyset$ and $\gamma(r_4; p_2, p_3') = \emptyset$: and if $r_1 \in \gamma(r_4; p_2, v_3)$, we have $(r_4 p_3 v_3 r_1)_4$; and if $r_1 \in \gamma(r_4; p_3, v_3)$, we have $(p_2 r_4 r_1 v_3)_4$. Then the remaining 9 points are all in $H(\overline{v_3}; p_2 p_3)$. By the discussion of Part One, our conclusion is right.

(8) $|T_3| = \mathbf{0}$.

Then $|T_2| = 0$, $|T_1| = 0$ and $r_i \in (p_1p_2p_3)$ for $i = 1, ..., 7$. Let $r_1 = \alpha(p_1; p_3, p_2)$.
If $r_1 \in \gamma(p_1; p_3, v_3)$, as shown in Figure 20.

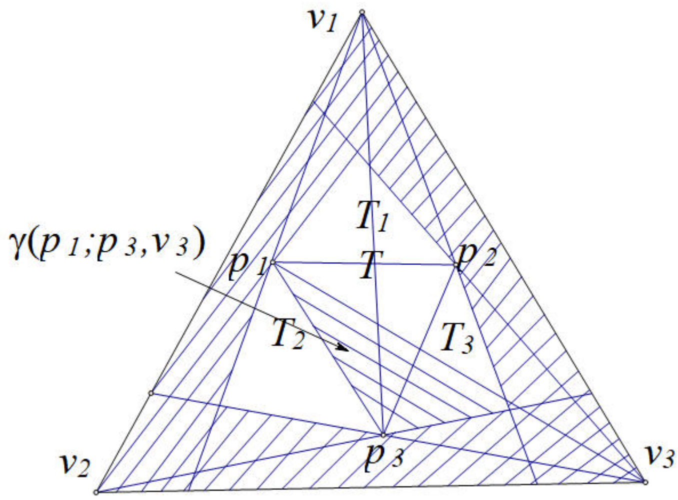

Figure 20. Figure of $|\mathrm{T}_1| = 0$ and $|\mathrm{T}_2| = 0$.

We have $(v_1p_1r_1p_3)_4$ and the remaining 9 points are all in $H(\overline{p_3}; p_1r_1)$. By the discussion of Part One, our conclusion is right. If $r_1 \in \gamma(p_1; p_3, v_3)$: and if $\gamma(r_1; p_1, p_3') = \emptyset$, we have $(v_1p_3r_1p_1)_4$ and the remaining 9 points are all in $H(\overline{v_1}; p_3r_1)$; and if $\gamma(r_1; p_1, p_3') \neq \emptyset$, we have a cutting line $L_5(r_1, \alpha(r_1; p_1, p_3'))$. $\square$

4. Conclusions

In this paper, we discuss a classical discrete geometry problem. After detailed proof, conclusion shows that a general planar point set contains a 3-hole, a 4-hole and a 5-hole, with at least 13 points. As $30 \leq n(6) \leq 463$ [16,21] and $n(7)$ does not exist, the proposed theorem will contribute to the theoretical research to some degree. Discrete geometry is a meaningful tool to study social networks. Therefore, our conclusion could be used to deal with some complex network problems. For example, under the environment of competition social structure, the structural holes which have been studied by many economists, are part of an important research branch of discrete geometry.

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