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Third-Order Hankel Determinant for Certain Class of Analytic Functions Related with Exponential Function

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Abstract: Let S_l^* denote the class of analytic functions f in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$, which is subordinate to exponential function, $\frac{zf'(z)}{f(z)} \prec e^z$ $(z \in \mathbb{D})$. In this paper, we aim to investigate the third-order Hankel determinant $H_3(1)$ for this function class S_l^* associated with exponential function and obtain the upper bound of the determinant $H_3(1)$. Meanwhile, we give two examples to illustrate the results obtained.

Keywords: analytic function; Hankel determinant; exponential function; upper bound

MSC: 30C45; 30C50; 30C80

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1. Introduction

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1}$$

Assume that $\mathcal{P}$ denote the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

and satisfying the condition Re $p(z) > 0$ $(z \in \mathbb{D})$.

It is easy to see that, if $p(z) \in \mathcal{P}$, then exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that (see [1])

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \quad (z \in \mathbb{D}).$$

Now, we start with recalling the definition of subordination.

Suppose that f and g are two analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) \quad (z \in \mathbb{D}),$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that (see [2])

$$g(z) = f(\omega(z)) \quad (z \in \mathbb{D}).$$

Recently, Mendiratta et al. in [3] introduced the following subclass S_l^* of analytic functions associated with exponential function.

Definition 1. *(see [3]). A function $f \in \mathcal{S}$ is said to be in the class S_l^*, if it satisfies the following condition:*

$$\frac{zf'(z)}{f(z)} \prec e^z \ (z \in \mathbb{D}). \tag{2}$$

We easily observe that, $f \in S_l^*$, if and only if

$$\left|\log \frac{zf'(z)}{f(z)}\right| < 1 \ \ (z \in \mathbb{D}). \tag{3}$$

In fact, if we choose $f(z) = z + \frac{1}{4}z^2$, then, from Equation (3), we can sketch the figure of the function class S_l^* (see Figure 1).

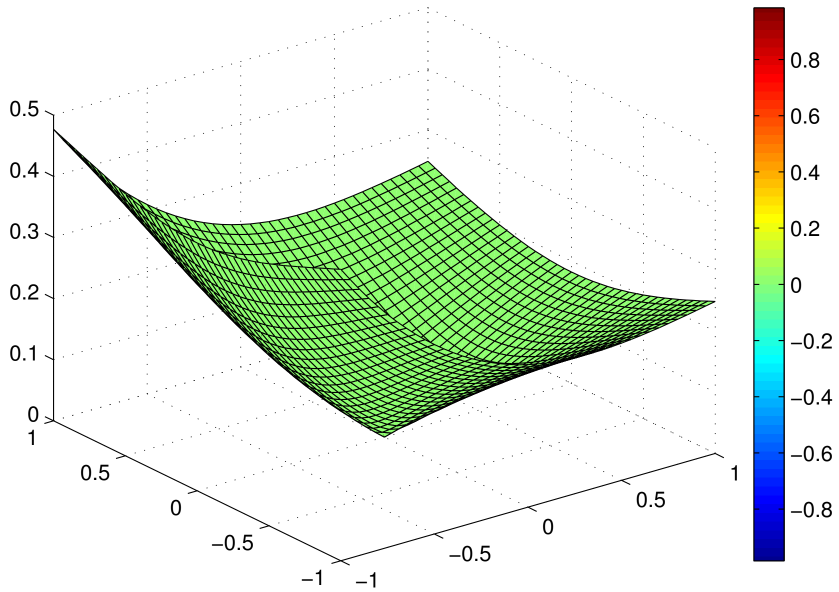

Figure 1. the figure of the function class S_l^* for $f(z) = z + \frac{1}{4}z^2$.

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ is stated by Noonan and Thomas [4] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

This determinant has been considered by several authors, for example, Noor [5] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f(z)$ given by Equation (1) with bounded boundary and Ehrenborg [6] studied the Hankel determinant of exponential polynomials.

In particular, we have

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \ \ (a_1 = 1,\ n = 1,\ q = 2),$$

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \quad (n=2,\ q=2),$$

and

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \quad (n=1,\ q=3).$$

Since $f \in \mathcal{S},\ a_1 = 1$, thus

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

We note that $H_2(1)$ is the well-known Fekete-Szego functional (see, for instance, [7–12]).

In recent years, many authors studied the second-order Hankel determinant $H_2(2)$ and the third-order Hankel determinant $H_3(1)$ for various classes of functions, the interested readers can see, for example, [13–22]. We note that, they discussed the determinants $H_2(2)$ and $H_3(1)$ based on the function classes, which are all subordinate to a certain function $\frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1;\ z \in \mathbb{D})$. Until now, very few researchers have studied the above determinants for the function class, subordinated to e^z $(z \in \mathbb{D})$. So, in this paper, we aim to investigate the third-order Hankel determinant $H_3(1)$ for the function class S_l^*, which is associated with exponential function, and obtain the upper bound of the above determinant.

2. Main Results

In order to prove our desired results, we shall require the following lemmas.

Lemma 1. *(see [23]). If $p(z) \in \mathcal{P}$, then exists some $x,\ z$ with $|x| \le 1,\ |z| \le 1$, such that*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - (4 - c_1^2)c_1x^2 + 2(4 - c_1^2)(1 - |x|^2)z.$$

Lemma 2. *(see [24]). Let $p(z) \in \mathcal{P}$, then*

$$|c_n| \le 2,\ n = 1, 2, \cdots.$$

Lemma 3. *(see [3]). If the function $f(z) \in S_l^*$ and of the form Equation (1), then*

$$|a_2| \le 1,\quad |a_3| \le \frac{3}{4},\quad |a_4| \le \frac{17}{36},\quad |a_5| \le 1. \tag{4}$$

We now state and prove the main results of our present investigation.

Theorem 1. *If the function $f(z) \in S_l^*$ and of the form Equation (1), then we have*

$$|a_3 - a_2^2| \le \frac{1}{2}. \tag{5}$$

Proof. Since $f(z) \in S_l^*$, according to the definition of subordination, then there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\frac{zf'(z)}{f(z)} = e^{\omega(z)}.$$

Now

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{z+\sum_{n=2}^{\infty} na_n z^n}{z+\sum_{n=2}^{\infty} a_n z^n} \\ &= (1+\sum_{n=2}^{\infty} na_n z^{n-1})[1 - a_2 z + (a_2^2 - a_3)z^2 - (a_2^3 - 2a_2a_3 + a_4)z^3 + \cdots] \\ &= 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 + \cdots . \end{aligned} \tag{6}$$

Define a function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots .$$

Then, we notice that $p(z) \in \mathcal{P}$ and

$$\omega(z) = \frac{p(z)-1}{1+p(z)} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

On the other hand,

$$\begin{aligned} & e^{\omega(z)} = 1 + \omega(z) + \tfrac{\omega(z)^2}{2!} + \tfrac{\omega(z)^3}{3!} + \cdots \\ &= 1 + \tfrac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots} + \tfrac{1}{2}(\tfrac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots})^2 + \tfrac{1}{6}(\tfrac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots})^3 + \cdots \\ &= 1 + \tfrac{1}{2}(c_1 z + c_2 z^2 + c_3 z^3 + \cdots)[1 - \tfrac{c_1 z}{2} + (\tfrac{c_1^2}{4} - \tfrac{c_2}{2})z^2 - (\tfrac{c_1^3}{8} - \tfrac{c_1 c_2}{2} + \tfrac{c_3}{2})z^3 + \cdots] \\ &+ \tfrac{1}{8}(c_1 z + c_2 z^2 + c_3 z^3 + \cdots)^2[1 - \tfrac{c_1 z}{2} + (\tfrac{c_1^2}{4} - \tfrac{c_2}{2})z^2 - (\tfrac{c_1^3}{8} - \tfrac{c_1 c_2}{2} + \tfrac{c_3}{2})z^3 + \cdots]^2 \\ &+ \tfrac{1}{48}(c_1 z + c_2 z^2 + c_3 z^3 + \cdots)^3[1 - \tfrac{c_1 z}{2} + (\tfrac{c_1^2}{4} - \tfrac{c_2}{2})z^2 - (\tfrac{c_1^3}{8} - \tfrac{c_1 c_2}{2} + \tfrac{c_3}{2})z^3 + \cdots]^3 + \cdots \\ &= 1 + \tfrac{1}{2}c_1 z + (\tfrac{c_2}{2} - \tfrac{c_1^2}{8})z^2 + (\tfrac{c_1^3}{48} - \tfrac{c_1 c_2}{4} + \tfrac{c_3}{2})z^3 + \cdots . \end{aligned} \tag{7}$$

On comparing the coefficients of $z,\ z^2,\ z^3$ between the Equations (6) and (7), we obtain

$$a_2 = \frac{c_1}{2},\ a_3 = \frac{c_2}{4} + \frac{c_1^2}{16},\ a_4 = \frac{c_3}{6} + \frac{c_1 c_2}{24} - \frac{c_1^3}{288}. \tag{8}$$

So,

$$|a_3 - a_2^2| = \left| \frac{c_2}{4} + \frac{c_1^2}{16} - \frac{c_1^2}{4} \right| = \left| \frac{c_2}{4} - \frac{3c_1^2}{16} \right|.$$

Using Lemma 1, we thus know that

$$|a_3 - a_2^2| = \left| \frac{x(4 - c_1^2)}{8} - \frac{c_1^2}{16} \right|.$$

Letting $|x| = t \in [0,1],\ c_1 = c \in [0,2]$ and applying the triangle inequality, the above equation reduces to

$$|a_3 - a_2^2| \le \frac{t(4-c^2)}{8} + \frac{c^2}{16}.$$

Suppose that

$$F(c,t) := \frac{t(4-c^2)}{8} + \frac{c^2}{16},$$

then we get

$$\frac{\partial F}{\partial t} = \frac{4-c^2}{8} \geq 0,$$

which shows that $F(c,t)$ is an increasing function on the closed interval [0,1] about t. Therefore, the function $F(c,t)$ can get the maximum value at $t=1$, that is

$$\max F(c,t) = F(c,1) = \frac{(4-c^2)}{8} + \frac{c^2}{16}.$$

Next, let

$$G(c) := \frac{(4-c^2)}{8} + \frac{c^2}{16} = \frac{1}{2} - \frac{c^2}{16}.$$

Then, we easily find the function $G(c)$ have a maximum value at $c=0$, also which is

$$|a_3 - a_2^2| \leq G(0) = \frac{1}{2}.$$

The proof of Theorem 1 is thus completed. □

Theorem 2. *If the function $f(z) \in S_l^*$ and of the form Equation (1), then we have*

$$|a_2a_3 - a_4| \leq \frac{896\sqrt{2}+385}{3087}. \tag{9}$$

Proof. From the Equation (8), we have

$$\begin{aligned} |a_2a_3 - a_4| &= |\tfrac{c_1c_2}{8} + \tfrac{c_1^3}{32} - \tfrac{c_3}{6} - \tfrac{c_1c_2}{24} + \tfrac{c_1^3}{288}| \\ &= |\tfrac{c_1c_2}{12} - \tfrac{c_3}{6} + \tfrac{5c_1^3}{144}|. \end{aligned}$$

Again, by applying Lemma 1, we get

$$|a_2a_3 - a_4| = \left| \frac{(4-c_1^2)c_1x^2}{24} - \frac{(4-c_1^2)c_1x}{24} - \frac{(4-c_1^2)(1-|x|^2)z}{12} + \frac{5c_1^3}{144} \right|.$$

Assume that $|x| = t \in [0,1],\ c_1 = c \in [0,2]$. Then, using the triangle inequality, we deduce that

$$|a_2a_3 - a_4| \leq \frac{(4-c^2)ct^2}{24} + \frac{(4-c^2)ct}{24} + \frac{(4-c^2)}{12} + \frac{5c^3}{144}.$$

Setting

$$F(c,t) := \frac{(4-c^2)ct^2}{24} + \frac{(4-c^2)ct}{24} + \frac{(4-c^2)}{12} + \frac{5c^3}{144}.$$

Hence, we have

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)ct}{12} + \frac{(4-c^2)c}{24} \geq 0,$$

namely, that $F(c,t)$ is an increasing function on the closed interval [0,1] about t. This implies that the maximum value of $F(c,t)$ occurs at $t=1$, which is

$$\max F(c,t) = F(c,1) = \frac{(4-c^2)c}{24} + \frac{(4-c^2)c}{24} + \frac{(4-c^2)}{12} + \frac{5c^3}{144}.$$

Now define

$$G(c) := \frac{(4-c^2)c}{24} + \frac{(4-c^2)c}{24} + \frac{(4-c^2)}{12} + \frac{5c^3}{144},$$

then

$$G'(c) = \frac{(4-c^2)}{12} - \frac{c^2}{6} - \frac{c}{6} + \frac{15c^2}{144}.$$

Let $G'(c) = 0$, then the root is $c = r = \frac{-4+8\sqrt{2}}{7}$. And so the function $G(c)$ have a maximum value attained at $c = r = \frac{-4+8\sqrt{2}}{7}$, also which is

$$|a_2a_3 - a_4| \leq G(r) = \frac{896\sqrt{2}+385}{3087}.$$

The proof of Theorem 2 is completed. □

Theorem 3. *If the function $f(z) \in S_l^*$ and of the form Equation (1), then we have*

$$|a_2a_4 - a_3^2| \leq \frac{7}{12}. \tag{10}$$

Proof. Suppose that $f(z) \in S_l^*$, then from Equation (8), we have

$$\begin{aligned}|a_2a_4 - a_3^2| &= |\tfrac{c_1c_3}{12} + \tfrac{c_1^2c_2}{48} - \tfrac{c_1^4}{576} - (\tfrac{c_2}{4} + \tfrac{c_1^2}{16})^2| \\ &= |\tfrac{c_1c_3}{12} - \tfrac{c_1^2c_2}{96} - \tfrac{c_1^4}{576} - \tfrac{c_2^2}{16} - \tfrac{c_1^4}{256}|.\end{aligned}$$

In view of Lemma 1, we thus obtain

$$\begin{aligned}|a_2a_4 - a_3^2| &= \left|\tfrac{c_1c_3}{12} + \tfrac{c_1^2c_2}{48} - \tfrac{c_1^4}{576} - (\tfrac{c_2}{4} + \tfrac{c_1^2}{16})^2\right| \\ &= \left|\tfrac{xc_1^2(4-c_1^2)}{192} - \tfrac{x^2c_1^2(4-c_1^2)}{48} - \tfrac{x^2(4-c_1^2)^2}{64} - \tfrac{c_1(4-c_1^2)(1-|x|^2)z}{24} - \tfrac{c_1^4}{256}\right|.\end{aligned}$$

Also, let $|x| = t \in [0,1]$, $c_1 = c \in [0,2]$. Then, using the triangle inequality, we get

$$|a_2a_4 - a_3^2| \leq \frac{tc^2(4-c^2)}{192} + \frac{t^2c^2(4-c^2)}{48} + \frac{t^2(4-c^2)^2}{64} + \frac{(4-c^2)}{12} + \frac{c^4}{256}.$$

Assume that

$$F(c,t) := \frac{tc^2(4-c^2)}{192} + \frac{t^2c^2(4-c^2)}{48} + \frac{t^2(4-c^2)^2}{64} + \frac{(4-c^2)}{12} + \frac{c^4}{256},$$

thus, we have

$$\frac{\partial F}{\partial t} = \frac{c^2(4-c^2)}{192} + \frac{tc^2(4-c^2)}{24} + \frac{t(4-c^2)^2}{32} \geq 0,$$

which implies that $F(c,t)$ increases on the closed interval [0,1] about t. That is, that $F(c,t)$ have a maximum value at $t = 1$, which is

$$\max F(c,t) = F(c,1) = \frac{5c^2(4-c^2)}{192} + \frac{(4-c^2)^2}{64} + \frac{(4-c^2)}{12} + \frac{c^4}{256}.$$

Taking

$$G(c) := \frac{5c^2(4-c^2)}{192} + \frac{(4-c^2)^2}{64} + \frac{(4-c^2)}{12} + \frac{c^4}{256},$$

then we have

$$G'(c) = \frac{5c(4-c^2)}{96} - \frac{c(4-c^2)}{16} - \frac{c}{6} - \frac{5c^3}{96} + \frac{c^3}{64}.$$

If $G'(c) = 0$, then the root is $c = 0$. After a simple calculation, we can deduce that $G''(0) < 0$, which means that the function $G(c)$ can take the maximum value at $c = 0$, also which is

$$|a_2a_4 - a_3^2| \le G(0) = \frac{7}{12},$$

and so we complete the proof of Theorem 3. □

Theorem 4. *If the function $f(z) \in S_l^*$ and of the form Equation (1), then we have*

$$|H_3(1)| \le \frac{165,095 + 60,928\sqrt{2}}{444,528} \approx 0.565. \tag{11}$$

Proof. Because

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

so, by applying the triangle inequality, we obtain

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{12}$$

Next, substituting Equations (4), (5), (8) and (10) into (12), we easily get the desired assertion Equation (11).

Finally, we give two examples to illustrate the results obtained. □

Example 1. *If we choose the function $f(z) = e^z - 1 = z + \sum_{n=2}^{\infty} \frac{z^n}{n!} \in S_l^*$, then we have*

$$\begin{aligned}|H_3(1)| &\le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \\ &= \frac{1}{3!} \times |\frac{1}{2!} \times \frac{1}{4!} - \frac{1}{3!} \times \frac{1}{3!}| + \frac{1}{4!} \times |\frac{1}{4!} - \frac{1}{2!} \times \frac{1}{3!}| + \frac{1}{5!} \times |\frac{1}{3!} - \frac{1}{2!} \times \frac{1}{2!}| \\ &\approx 0.004 < 0.565.\end{aligned}$$

Example 2. *If we put the function $f(z) = -\log(1 - z) = z + \sum_{n=2}^{\infty} \frac{z^n}{n} \in S_l^*$, then we get*

$$\begin{aligned}|H_3(1)| &\le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \\ &= \frac{1}{3} \times |\frac{1}{2} \times \frac{1}{4} - \frac{1}{3} \times \frac{1}{3}| + \frac{1}{4} \times |\frac{1}{4} - \frac{1}{2} \times \frac{1}{3}| + \frac{1}{5} \times |\frac{1}{3} - \frac{1}{2} \times \frac{1}{2}| \\ &\approx 0.042 < 0.565.\end{aligned}$$

3. Conclusions

In this paper, we mainly investigate the third-order Hankel determinant $H_3(1)$ for the function class S_l^*, which is subordinate to exponential function, and obtain the upper bound of the above determinant. The results obtained generalize and unify the theories of Hankel determinants in geometric function theory.

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