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Generalized Liouville–Caputo Fractional Differential Equations and Inclusions with Nonlocal Generalized Fractional Integral and Multipoint Boundary Conditions

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Abstract: We develop the existence criteria for solutions of Liouville–Caputo-type generalized fractional differential equations and inclusions equipped with nonlocal generalized fractional integral and multipoint boundary conditions. Modern techniques of functional analysis are employed to derive the main results. Examples illustrating the main results are also presented. It is imperative to mention that our results correspond to the ones for a symmetric second-order nonlocal multipoint integral boundary value problem under suitable conditions (see the last section).

Keywords: differential equation; differential inclusion; Liouville–Caputo-type fractional derivative; fractional integral; existence; fixed point

1. Introduction

Fractional order differential and integral operators extensively appear in the mathematical modeling of various scientific and engineering phenomena. The main advantage for using these operators is their nonlocal nature, which can describe the past history of processes and material involved in the phenomena. Thus, fractional-order models are more realistic and informative than their corresponding integer-order counterparts. Examples include bio-engineering [1], Chaos and fractional dynamics [2], ecology [3], financial economics [4], etc. Widespread applications of methods of fractional calculus in numerous real world phenomena motivated many researchers to develop this important branch of mathematical analysis—for instance, see the texts [5–8].

Fractional differential equations equipped with a variety of boundary conditions have recently been studied by several researchers. In particular, overwhelming interest has been shown in the study of nonlocal nonlinear fractional-order boundary value problems (FBVPs). The concept of nonlocal conditions dates back to the work of Bitsadze and Samarski [9] and these conditions facilitate describing the physical phenomena taking place inside the boundary of the given domain. In computational fluid dynamics (CFD) studies of blood flow problems, it is hard to justify the assumption of a circular cross-section of a blood vessel due to its changing geometry throughout the vessel. This issue has been addressed by the introduction of integral boundary conditions. In addition, integral boundary conditions are used in regularization of ill-posed parabolic backward problems. Moreover, integral boundary conditions play an important role in mathematical models for bacterial self-regularization [10].

On the other hand, multivalued (inclusions) problems are found to be of special significance in studying dynamical systems and stochastic processes. Examples include granular systems [11,12], control problems [13,14], dynamics of wheeled vehicles [15], etc. For more details, see the text [16], which addresses the pressing issues in stochastic processes, queueing networks, optimization and their application in finance, control, climate control, etc. In previous work [17], synchronization processes involving fractional differential inclusions are studied.

The area of investigation for nonlocal nonlinear fractional boundary value problems includes existence and uniqueness of solutions, stability and oscillatory properties, analytic and numerical methods. The literature on the topic is now much enriched and covers fractional order differential equations and inclusions involving Riemann–Liouville, Liouville–Caputo (Caputo), Hadamard type derivatives, etc. For some recent works on the topic, we refer the reader to a series of papers [18–36] and the references cited therein.

In this paper, we introduce and study a new class of boundary value problems of Liouville–Caputo-type generalized fractional differential equations and inclusions (instead of taking the usual Liouville–Caputo fractional order derivative) supplemented with nonlocal generalized fractional integral and multipoint boundary conditions. Precisely, we consider the problems:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0+}y(t) = f(t, y(t)), \ \ t \in J := [0, T], \\ y(T) = \displaystyle\sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0+} y(\eta_i) + \kappa, \quad \delta y(0) = \sum_{j=1}^{k} \mu_j y(\xi_j), \\ 0 < \eta_1 < \cdots < \eta_i < \cdots < \eta_m < \xi_1 < \cdots < \xi_j < \cdots < \xi_k < T, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0+}y(t) \in F(t, y(t)), \ \ t \in J := [0, T], \\ y(T) = \displaystyle\sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0+} y(\eta_i) + \kappa, \quad \delta y(0) = \sum_{j=1}^{k} \mu_j y(\xi_j), \\ 0 < \eta_1 < \cdots < \eta_i < \cdots < \eta_m < \xi_1 < \cdots < \xi_j < \cdots < \xi_k < T, \end{cases} \tag{2}$$

where ${}^{\rho}_{c}D^{\alpha}_{0+}$ is the Liouville–Caputo-type generalized fractional derivative of order $1 < \alpha \leq 2$, ${}^{\rho}I^{\beta}_{0+}$ is the generalized (Katugampola type) fractional integral of order $\beta > 0, \rho > 0$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\sigma_i, \mu_j, \kappa \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, k$, $\delta = t^{1-\rho}\frac{d}{dt}$, and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

The rest of the paper is arranged as follows: Section 2 contains some preliminary concepts related to our work and a vital lemma associated with the linear variant of the given problem, which is used to convert the given problems into fixed point problems. In Section 3, the existence and uniqueness results for problem (1) are obtained by using a Banach contraction mapping principle, Krasnoselskii's fixed point theorem and Leray–Schauder nonlinear alternative. Existence results for the inclusions problem (2) are studied in Section 4 via Leray–Schauder nonlinear alternative, and Covitz and Nadler fixed point theorem for multi-valued maps. Examples illustrating the obtained results are also included.

2. Preliminaries

Denote by $X^p_c(a, b)$ the space of all complex-valued Lebesgue measurable functions φ on (a, b) equipped with the norm:

$$\|\varphi\|_{X^p_c} = \Big(\int_a^b |x^c \varphi(x)|^p \frac{dx}{x} \Big)^{1/p} < \infty \, c \in \mathbb{R}, 1 \leq p \leq \infty.$$

Let $L^1(a, b)$ represent the space of all Lebesgue measurable functions ψ on (a, b) endowed with the norm:

$$\|\psi\|_{L^1} = \int_a^b |\psi(x)| dx < \infty.$$

We further recall that $AC^n(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x, x', \ldots, x^{(n-1)} \in C(J,\mathbb{R})$ and $x^{(n-1)}$ is absolutely continuous $\}$. For $0 \leq \epsilon < 1$, we define $C_{\epsilon,\rho}(J,\mathbb{R}) = \{f : J \to \mathbb{R} : (t^\rho - a^\rho)^\epsilon f(t) \in C(J,\mathbb{R})\}$ endowed with the norm $\|f\|_{C_{\epsilon,\rho}} = \|(t^\rho - a^\rho)^\epsilon f(t)\|_C$. Moreover, we define the class of functions f that have absolutely continuous δ^{n-1}-derivative, denoted by $AC^n_\delta(J,\mathbb{R})$, as follows: $AC^n_\delta(J,\mathbb{R}) = \Big\{f : J \to \mathbb{R} : \delta^{n-1} f \in AC(J,\mathbb{R}), \delta = t^{1-\rho}\frac{d}{dt}\Big\}$, which is equipped with the norm $\|f\|_{C^n_\delta} = \sum_{k=0}^{n-1} \|\delta^k f\|_C$. More generally, the space of functions endowed with the norm $\|f\|_{C^n_{\delta,\epsilon}} = \sum_{k=0}^{n-1} \|\delta^k f\|_C + \|\delta^n f\|_{C_{\epsilon,\rho}}$ is defined by

$$C^n_{\delta,\epsilon}(J,\mathbb{R}) = \Big\{f : J \to \mathbb{R} : \delta^{n-1} f \in C(J,\mathbb{R}), \delta^n f \in C_{\epsilon,\rho}(J,\mathbb{R}), \delta = t^{1-\rho}\frac{d}{dt}\Big\}.$$

Notice that $C^n_{\delta,0} = C^n_\delta$.

Definition 1 ([37]). *For $-\infty < a < t < b < \infty$, the left-sided and right-sided generalized fractional integrals of $f \in X^p_c(a,b)$ of order $\alpha > 0$ and $\rho > 0$ are respectively defined by*

$$({}^\rho I^\alpha_{a+} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds, \tag{3}$$

$$({}^\rho I^\alpha_{b-} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s)ds. \tag{4}$$

Definition 2 ([38]). *For $0 \leq a < x < b < \infty$, the generalized fractional derivatives, associated with the generalized fractional integrals (3) and (4), are respectively defined by*

$$\begin{aligned}({}^\rho D^\alpha_{a+} f)(t) &= \Big(t^{1-\rho}\frac{d}{dt}\Big)^n ({}^\rho I^{n-\alpha}_{a+} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \Big(t^{1-\rho}\frac{d}{dt}\Big)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} f(s)ds,\end{aligned} \tag{5}$$

$$\begin{aligned}({}^\rho D^\alpha_{b-} f)(t) &= \Big(-t^{1-\rho}\frac{d}{dt}\Big)^n ({}^\rho I^{n-\alpha}_{b-} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \Big(-t^{1-\rho}\frac{d}{dt}\Big)^n \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{\alpha-n+1}} f(s)ds,\end{aligned} \tag{6}$$

if the integrals exist.

Definition 3 ([39]). *The left-sided and right-sided Liouville–Caputo-type generalized fractional derivatives of $f \in AC^n_\delta[a,b]$ of order $\alpha \geq 0$ are respectively defined via the above generalized fractional derivatives as*

$${}^\rho_c D^\alpha_{a+} f(x) = {}^\rho D^\alpha_{a+}\Big[f(t) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^k\Big](x), \quad \delta = x^{1-\rho}\frac{d}{dx}, \tag{7}$$

$${}^\rho_c D^\alpha_{b-} f(x) = {}^\rho D^\alpha_{b-}\Big[f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k f(b)}{k!}\Big(\frac{b^\rho - t^\rho}{\rho}\Big)^k\Big](x), \quad \delta = x^{1-\rho}\frac{d}{dx}, \tag{8}$$

where $n = [\alpha] + 1$.

Lemma 1 ([39]). *Let $\alpha \geq 0, n = [\alpha] + 1$ and $f \in AC^n_\delta[a,b]$, where $0 < a < b < \infty$. Then,*
1. if $\alpha \notin \mathbb{N}$,

$${}^\rho_c D^\alpha_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{n-\alpha-1} \frac{(\delta^n f)(s)ds}{s^{1-\rho}} = {}^\rho I^{n-\alpha}_{a+}(\delta^n f)(t), \tag{9}$$

$$ {}^{\rho}_{c}D^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\frac{s^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} \frac{(-1)^n(\delta^n f)(s)ds}{s^{1-\rho}} = {}^{\rho}I^{n-\alpha}_{b-}(\delta^n f)(t). \tag{10} $$

2. If $\alpha \in \mathbb{N}$,

$$ {}^{\rho}_{c}D^{\alpha}_{a+}f = \delta^n f, \quad {}^{\rho}_{c}D^{\alpha}_{b-}f = (-1)^n \delta^n f. \tag{11} $$

Lemma 2 ([39])**.** *Let $f \in AC^n_\delta[a,b]$ or $C^n_\delta[a,b]$ and $\alpha \in \mathbb{R}$. Then,*

$$ {}^{\rho}I^{\alpha}_{a+}{}^{\rho}_{c}D^{\alpha}_{a+}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\delta^k f)(a)}{k!}\left(\frac{x^\rho - a^\rho}{\rho}\right)^k, $$

$$ {}^{\rho}I^{\alpha}_{b-}{}^{\rho}_{c}D^{\alpha}_{b-}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k(\delta^k f)(a)}{k!}\left(\frac{b^\rho - x^\rho}{\rho}\right)^k. $$

In particular, for $0 < \alpha \le 1$, we have

$$ {}^{\rho}I^{\alpha}_{a+}{}^{\rho}_{c}D^{\alpha}_{a+}f(x) = f(x) - f(a), \ {}^{\rho}I^{\alpha}_{b-}{}^{\rho}_{c}D^{\alpha}_{b-}f(x) = f(x) - f(b). $$

For computational convenience, we introduce the notations:

$$ A_1 = 1 - \sum_{j=1}^{k} \mu_j \frac{\xi_j^\rho}{\rho}, \quad A_2 = \sum_{j=1}^{k} \mu_j, \tag{12} $$

$$ B_1 = \frac{T^\rho}{\rho} - \sum_{i=1}^{m} \sigma_i \frac{\eta_i^{\rho(\beta+1)}}{\rho^{\beta+1}\Gamma(\beta+2)}, \quad B_2 = 1 - \sum_{i=1}^{m} \sigma_i \frac{\eta_i^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}, \tag{13} $$

$$ \Omega = A_1 B_2 + B_1 A_2. \tag{14} $$

The following lemma, related to the linear variant of problem (1), plays a key role in converting the given problem into a fixed point problem.

Lemma 3. *Let $h \in C(0,T) \cap L(0,T)$, $y \in AC^2_\delta(J)$ and $\Omega \neq 0$. Then, the solution of the boundary value problem (BVP):*

$$ \begin{cases} {}^{\rho}_{c}D^{\alpha}_{0+}y(t) = h(t), \ t \in J := [0,T], \\ y(T) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0+}y(\eta_i) + \kappa, \quad \delta y(0) = \sum_{j=1}^{k} \mu_j y(\xi_j), \\ 0 < \eta_1 < \cdots < \eta_i < \cdots < \eta_m < \xi_1 < \cdots < \xi_j < \cdots < \xi_k < T, \end{cases} \tag{15} $$

is given by

$$ \begin{aligned} y(t) = \; & {}^{\rho}I^{\alpha}_{0+}h(t) + \tfrac{1}{\Omega}\Big\{ -B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0+}h(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0+}h(\eta_i) - {}^{\rho}I^{\alpha}_{0+}h(T) + \kappa\Big]\Big\} \\ & + \tfrac{t^\rho}{\rho\Omega}\Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0+}h(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0+}h(\eta_i) - {}^{\rho}I^{\alpha}_{0+}h(T) + \kappa\Big]\Big\}. \end{aligned} \tag{16} $$

Proof. Applying ${}^{\rho}I^{\alpha}_{0+}$ on the fractional differential equation in (15) and using Lemma 2, the solution of fractional differential equation in (15) for $t \in J$ is

$$ y(t) = {}^{\rho}I^{\alpha}_{0+}h(t) + c_1 + c_2\frac{t^\rho}{\rho} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}h(s)ds + c_1 + c_2\frac{t^\rho}{\rho}, \tag{17} $$

for some $c_1, c_2 \in \mathbb{R}$. Taking $\delta - derivative$ of (17), we get

$$\delta y(t) = {}^\rho I_{0+}^{\alpha-1} h(t) + c_2 = \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-2} h(s) ds + c_2. \tag{18}$$

Using the boundary condition $\delta y(0) = \sum_{j=1}^k \mu_j y(\xi_j)$ in (18), we get

$$c_2 = \sum_{j=1}^k \mu_j {}^\rho I_{0+}^\alpha h(\xi_j) + c_1 \sum_{j=1}^k \mu_j + c_2 \sum_{j=1}^k \mu_j \frac{\xi_j^\rho}{\rho},$$

which, on account of (12), takes the form:

$$A_1 c_2 - A_2 c_1 = \sum_{j=1}^k \mu_j {}^\rho I_{0+}^\alpha h(\xi_j). \tag{19}$$

Applying the generalized integral operator ${}^\rho I_{0+}^\beta$ on (17), we get

$${}^\rho I_{0+}^\beta y(t) = {}^\rho I_{0+}^{\alpha+\beta} h(t) + c_1 \frac{t^{\rho\beta}}{\rho^\beta \Gamma(\beta+1)} + c_2 \frac{t^{\rho(\beta+1)}}{\rho^{\beta+1}\Gamma(\beta+2)}, \tag{20}$$

which, together with the boundary condition $y(T) = \sum_{i=1}^m \sigma_i \, {}^\rho I_{0+}^\beta y(\eta_i) + \kappa$, yields

$$\begin{aligned} {}^\rho I_{0+}^\alpha h(T) + c_1 + c_2 \frac{T^\rho}{\rho} &= \sum_{i=1}^m \sigma_i \, {}^\rho I_{0+}^{\alpha+\beta} h(\eta_i) + \sum_{i=1}^m \sigma_i c_1 \frac{\eta_i^{\rho\beta}}{\rho^\beta \Gamma(\beta+1)} \\ &+ \sum_{i=1}^m \sigma_i c_2 \frac{\eta_i^{\rho(\beta+1)}}{\rho^{\beta+1}\Gamma(\beta+2)} + \kappa. \end{aligned} \tag{21}$$

Using the notations (13) in (21), we obtain

$$B_1 c_2 + B_2 c_1 = \sum_{i=1}^m \sigma_i \, {}^\rho I_{0+}^{\alpha+\beta} h(\eta_i) - {}^\rho I_{0+}^\alpha h(T) + \kappa. \tag{22}$$

Solving the system of Equations (19) and (22) for c_1 and c_2, we find that

$$c_1 = \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^k \mu_j {}^\rho I_{0+}^\alpha h(\xi_j) + A_1 \Big[\sum_{i=1}^m \sigma_i \, {}^\rho I_{0+}^{\alpha+\beta} h(\eta_i) - {}^\rho I_{0+}^\alpha h(T) + \kappa \Big] \Big\}. \tag{23}$$

and

$$c_2 = \frac{1}{\Omega}\Big\{ B_2 \sum_{j=1}^k \mu_j {}^\rho I_{0+}^\alpha h(\xi_j) + A_2 \Big[\sum_{i=1}^m \sigma_i \, {}^\rho I_{0+}^{\alpha+\beta} h(\eta_i) - {}^\rho I^\alpha h(T) + \kappa \Big] \Big\}. \tag{24}$$

Substituting the values of c_1 and c_2 in (17), we get Equation (16). The converse follows by direct computation. The proof is completed. □

3. Main Results for the Problem (1)

By Lemma 3, we define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ ($\mathcal{C} = C(J, \mathbb{R})$) associated with problem (1) as

$$\mathcal{G}y(t) = {}^\rho I_{0+}^\alpha f(t, y(t)) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^k \mu_j {}^\rho I_{0+}^\alpha f(\xi_j, y(\xi_j)) + A_1 \Big[\sum_{i=1}^m \sigma_i \, {}^\rho I_{0+}^{\alpha+\beta} f(\eta_i, y(\eta_i))$$

$$-{}^{\rho}I_{0^+}^{\alpha}f(T,y(T))+\kappa\Big]\Big\}+\frac{t^{\rho}}{\rho\Omega}\Big\{B_2\sum_{j=1}^{k}\mu_j{}^{\rho}I_{0^+}^{\alpha}f(\xi_j,y(\xi_j))$$
$$+A_2\Big[\sum_{i=1}^{m}\sigma_i\,{}^{\rho}I_{0^+}^{\alpha+\beta}f(\eta_i,y(\eta_i))-{}^{\rho}I_{0^+}^{\alpha}f(T,y(t))+\kappa\Big]\Big\}. \tag{25}$$

In the following, for brevity, we use the notations:

$$\begin{aligned}\Lambda &= \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+|A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\Big\}\\ &\quad+\frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\Big\}.\end{aligned} \tag{26}$$

In the first result, we establish the existence of solutions for problem (1) via Leray–Schauder nonlinear alternative [40].

Theorem 1. *Suppose that the following conditions hold:*

(H_1) *For a function* $\phi\in L^1([0,T],\mathbb{R}^+)$, *and a nondecreasing function* $\psi:\mathbb{R}^+\to\mathbb{R}^+$ *such that* $|f(t,y)|\le\phi(t)\psi(\|y\|),\ \forall(t,y)\in[0,T]\times\mathbb{R}$;

(H_2) *there exists a positive constant* $\mathcal{M}$ *such that*

$$\frac{\mathcal{M}}{\psi(\mathcal{M})\Lambda_1+\dfrac{|\kappa|(\rho|A_1|+T^{\rho}|A_2|)}{\rho|\Omega|}}>1,$$

where

$$\begin{aligned}\Lambda_1 &= {}^{\rho}I_{0^+}^{\alpha}\phi(T)+\frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^{\rho}I_{0^+}^{\alpha}\phi(\xi_j)+|A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}\phi(\eta_i)+{}^{\rho}I_{0^+}^{\alpha}\phi(T)\Big]\Big\}\\ &\quad+\frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^{\rho}I_{0^+}^{\alpha}\phi(\xi_j)+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}\phi(\eta_i)+{}^{\rho}I_{0^+}^{\alpha}\phi(T)\Big]\Big\}.\end{aligned} \tag{27}$$

Then, there exists at least one solution for problem (1) on $[0,T]$.

Proof. Firstly, we show that the operator $\mathcal{G}:\mathcal{C}\to\mathcal{C}$ defined by (25) is continuous and completely continuous.

Step 1: $\mathcal{G}$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n\to y$ in $\mathcal{C}$. Then,

$$\begin{aligned}|\mathcal{G}(y_n)(t)-\mathcal{G}(y)(t)| &\le {}^{\rho}I_{0^+}^{\alpha}|f(t,y_n(t))-f(t,y(t))|+\frac{1}{|\Omega|}\Bigg\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^{\rho}I_{0^+}^{\alpha}|f(\xi_j,y_n(\xi_j))-f(\xi_j,y(\xi_j))|\\ &\quad+|A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}|f(\eta_i,y_n(\eta_i))-f(\eta_i,y(\eta_i))|+{}^{\rho}I_{0^+}^{\alpha}|f(T,y_n(T))-f(T,y(T))|\Big]\Bigg\}\\ &\quad+\frac{t^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^{\rho}I_{0^+}^{\alpha}|f(\xi_j,y_n(\xi_j))-f(\xi_j,y(\xi_j))|\\ &\quad+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}|f(\eta_i,y_n(\eta_i))-f(\eta_i,y(\eta))|+{}^{\rho}I_{0^+}^{\alpha}|f(T,y_n(T))-f(T,y(T))|\Big]\Big\}\\ &\le \Lambda\|f(\cdot,y_n)-f(\cdot,y)\|.\end{aligned}$$

In view of continuity of f, it follows from the above inequality that

$$\|\mathcal{G}(y_n) - \mathcal{G}(y)\| \leq \Lambda \|f(\cdot, y_n) - f(\cdot, y)\| \to 0, \text{ as } n \to \infty.$$

Step 2: *$\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{C}$.*

For a positive number r, it will be shown that there exists a positive constant ℓ such that $\|\mathcal{G}(y)\| \leq \ell$ for any $y \in B_r = \{y \in \mathcal{C} : \|y\| \leq r\}$. By (H_1), for each $t \in J$, we have

$$\begin{aligned}
|\mathcal{G}(y)(t)| \leq\ & {}^\rho I^\alpha_{0^+}|f(t,y(t))| + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}|f(\xi_j,y(\xi_j))| + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}|f(\eta_i,y(\eta_i))| \\
&+{}^\rho I^\alpha_{0^+}|f(T,y(T))| + |\kappa|\Big]\Big\} + \frac{t^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}|f(\xi_j,y(\xi_j))| \\
&+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}|f(\eta_i,y(\eta_i))| + {}^\rho I^\alpha_{0^+}|f(T,y(T))| + |\kappa|\Big]\Big\} \\
\leq\ & {}^\rho I^\alpha_{0^+}\phi(T)\psi(\|y\|) + \frac{1}{|\Omega|}\left\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j)\psi(\|y\|) + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i)\psi(\|y\|)\right. \\
&+{}^\rho I^\alpha_{0^+}\phi(T)\psi(\|y\|) + |\kappa|\Big]\Big\} + \frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j)\psi(\|y\|) \\
&+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i)\psi(\|y\|) + {}^\rho I^\alpha_{0^+}\phi(T)\psi(\|y\|) + |\kappa|\Big]\Big\} \\
\leq\ & \psi(\|y\|)\Big({}^\rho I^\alpha_{0^+}\phi(T) + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) \\
&+{}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big\} + \frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) + {}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big\}\Big) \\
&+\frac{|\kappa|(\rho|A_1| + T^\rho|A_2|)}{\rho|\Omega|}. \\
\leq\ & \psi(\|r\|)\Big({}^\rho I^\alpha_{0^+}\phi(T) + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) \\
&+{}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big\} + \frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) + {}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big\}\Big) \\
&+\frac{|\kappa|(\rho|A_1| + T^\rho|A_2|)}{\rho|\Omega|} := \ell.
\end{aligned}$$

Step 3: $\mathcal{G}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Let B_r be a bounded set of $\mathcal{C}$ as in Step 2, Then, for $t_1, t_2 \in (0, T]$ with $t_1 < t_2$, and $y \in B_r$, we have

$$\begin{aligned}
&|\mathcal{G}(y)(t_2) - \mathcal{G}(y)(t_1)| \\
\leq\ & \Big|{}^\rho I^\alpha_{0^+}f(t_2,y(t_2)) - {}^\rho I^\alpha_{0^+}f(t_1,y(t_1))\Big| + \frac{|t_2^\rho - t_1^\rho|}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}|f(\xi_j,y(\xi_j))| \\
&+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}|f(\eta_i,y(\eta_i))| + {}^\rho I^\alpha_{0^+}|f(T,y(T))| + |\kappa|\Big]\Big\} \\
\leq\ & \frac{\rho^{1-\alpha}\psi(r)}{\Gamma(\alpha)}\Big|\int_0^{t_1}\Big[\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^\rho - s^\rho)^{1-\alpha}}\Big]\phi(s)ds + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho - s^\rho)^{1-\alpha}}\phi(s)ds\Big| \\
&+\frac{|t_2^\rho - t_1^\rho|}{\rho|\Omega|}\Big\{\psi(r)\Big(|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) + {}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big) + |A_2||\kappa|\Big\}
\end{aligned}$$

$$\to \quad 0 \text{ as } t_2 \to t_1,$$

independently of $y \in B_r$. In view of steps 1–3, it follows by the Arzelá–Ascoli theorem that the operator $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{C}$ is completely continuous.

Step 4: We show that there exists an open set $V \subseteq \mathcal{C}$ with $y \neq \lambda\mathcal{G}(y)$ for $\lambda \in (0,1)$ and $y \in \partial V$.

Let $y \in \mathcal{C}$ be a solution of $y = \lambda\mathcal{G}y$ for $\lambda \in [0,1]$. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
|y(t)| &= |\lambda(\mathcal{G}y)(t)| \\
&\le {}^\rho I^\alpha_{0^+}|f(t,y(t))| + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^k |\mu_j|{}^\rho I^\alpha_{0^+}|f(\xi_j,y(\xi_j))| + |A_1|\Big[\sum_{i=1}^m |\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}|f(\eta_i,y(\eta_i))| \\
&\quad + {}^\rho I^\alpha_{0^+}|f(T,y(T))| + |\kappa|\Big]\Big\} + \frac{t^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^k |\mu_j|{}^\rho I^\alpha_{0^+}|f(\xi_j,y(\xi_j))| \\
&\quad + |A_2|\Big[\sum_{i=1}^m |\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}|f(\eta_i,y(\eta_i))| + {}^\rho I^\alpha_{0^+}|f(T,y(T))| + |\kappa|\Big]\Big\} \\
&\le \psi(\|y\|)\Big({}^\rho I^\alpha_{0^+}|\phi(T) + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^k |\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_1|\Big[\sum_{i=1}^m |\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) \\
&\quad + {}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big\} + \frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^k |\mu_j|{}^\rho I^\alpha_{0^+}\phi(\xi_j) + |A_2|\Big[\sum_{i=1}^m |\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}\phi(\eta_i) + {}^\rho I^\alpha_{0^+}\phi(T)\Big]\Big\}\Big) \\
&\quad + \frac{|\kappa|(\rho|A_1| + T^\rho|A_2|)}{\rho|\Omega|},
\end{aligned}$$

which, on taking the norm for $t \in J$, implies that

$$\frac{\|y\|}{\psi(\|y\|)\Lambda_1 + \dfrac{|\kappa|(\rho|A_1| + T^\rho|A_2|)}{\rho|\Omega|}} \le 1.$$

By the assumption (H_2), we can find a positive number $\mathcal{M}$ such that $\|y\| \neq \mathcal{M}$. Introduce $V = \{y \in \mathcal{C} : \|y\| < \mathcal{M}\}$ and observe that the operator $\mathcal{G} : \overline{V} \to \mathcal{C}$ is continuous and completely continuous. By the definition of V, there does not exist any $y \in \partial V$ satisfying $y = \lambda\mathcal{G}(y)$ for some $\lambda \in (0,1)$. Hence, we deduce by the nonlinear alternative of Leray–Schauder type [40] that $\mathcal{G}$ has a fixed point $y \in \overline{V}$ that is indeed a solution of the problem (1). This completes the proof. □

In the next result, we prove the existence of solutions for problem (1) by applying Krasnoselskii's fixed point theorem [41].

Theorem 2. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following assumptions hold:*

(H_3) $|f(t,x) - f(t,y)| \le L\|x - y\|$, $\quad \forall t \in [0,T]$, $L > 0$, $x, y \in \mathbb{R}$;
(H_4) $|f(t,y)| \le \Phi(t)$, $\forall (t,y) \in [0,T] \times \mathbb{R}$, *and* $\Phi \in C([0,T], \mathbb{R}^+)$.

Then, problem (1) has at least one solution on $[0,T]$, provided that

$$L\left(\frac{T^\rho}{\rho|\Omega|}\left\{|B_2|\sum_{j=1}^k |\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_2|\left[\sum_{i=1}^m |\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\right]\right\}\right) < 1. \tag{28}$$

Proof. Let us fix $\bar r \ge \|\Phi\|\Lambda + \frac{|\kappa|(\rho|A_1|+T^\rho|A_2|)}{\rho|\Omega|}$, where $\|\Phi\| = \sup_{t\in J}|\Phi(t)|$ and consider $B_{\bar r} = \{y \in \mathcal{C} : \|y\| \le \bar r\}$. Let us split the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ defined by (25) on $B_{\bar r}$ as $\mathcal{G} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A}(t) = {}^\rho I^\alpha_{0^+}f(t,y(t)) + \frac{1}{\Omega}\Big\{-B_1\sum_{j=1}^k \mu_j{}^\rho I^\alpha_{0^+}f(\xi_j,y(\xi_j)) + A_1\Big[\sum_{i=1}^m \sigma_i\,{}^\rho I^{\alpha+\beta}_{0^+}f(\eta_i,y(\eta_i)) - {}^\rho I^\alpha_{0^+}f(T,y(T)) + \kappa\Big]\Big\},$$

and

$$\mathcal{B}(t)=\frac{t^\rho}{\rho\Omega}\Big\{B_2\sum_{j=1}^{k}\mu_j{}^\rho I_{0^+}^{\alpha}f(\xi_j,y(\xi_j))+A_2\Big[\sum_{i=1}^{m}\sigma_i\,{}^\rho I_{0^+}^{\alpha+\beta}f(\eta_i,y(\eta_j))-{}^\rho I_{0^+}^{\alpha}f(T,y(t))+\kappa\Big]\Big\}.$$

For $x,y\in B_{\bar r}$, we find that

$$\begin{aligned}\|\mathcal{A}x+\mathcal{B}y\| &\le \sup_{t\in J}\Bigg\{{}^\rho I_{0^+}^{\alpha}|f(t,x(t))|+\frac{1}{\Omega}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^\rho I_{0^+}^{\alpha}f(\xi_j,x(\xi_j))+A_1\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I_{0^+}^{\alpha+\beta}|f(\eta_i,x(\eta_i))|\\
&\quad+{}^\rho I_{0^+}^{\alpha}|f(T,x(T))|+|\kappa|\Big]\Big\}+\frac{t^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I_{0^+}^{\alpha}|f(\xi_j,y(\xi_j))|\\
&\quad+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I_{0^+}^{\alpha+\beta}|f(\eta_i,y(\eta_j))|+{}^\rho I_{0^+}^{\alpha}|f(T,y(T))|+|\kappa|\Big]\Bigg\}\\
&\le \|\Phi\|\Bigg\{\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\\
&\quad+|A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}+\frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\\
&\quad+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Bigg\}+\frac{|\kappa|(\rho|A_1|+T^\rho|A_2|)}{\rho|\Omega|}\\
&\le \|\Phi\|\Lambda+\frac{|\kappa|(\rho|A_1|+T^\rho|A_2|)}{\rho|\Omega|}<\bar r.\end{aligned}$$

Thus, $\mathcal{A}x+\mathcal{B}y\in B_{\bar r}$. Now, for $x,y\in B_{\bar r}$ and for each $t\in J$, we obtain

$$\begin{aligned}\|\mathcal{B}x-\mathcal{B}y\| &\le \sup_{t\in J}\Big\{\frac{t^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|{}^\rho I_{0^+}^{\alpha}|f(\xi_j,x(\xi_j))-f(\xi_j,y(\xi_j))|\\
&\quad+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I_{0^+}^{\alpha+\beta}|f(\eta_i,x(\eta_i))-f(\eta_i,y(\eta_i))|+{}^\rho I_{0^+}^{\alpha}|f(T,x(T))-f(T,y(T))|\Big]\Big\}\Big\}\\
&\le L\Big(\frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Big)\|x-y\|,\end{aligned}$$

which, together with condition (28), implies that $\mathcal{B}$ is a contraction. Continuity of f implies that the operator $\mathcal{A}$ is continuous. In addition, $\mathcal{A}$ is uniformly bounded on $B_{\bar r}$ as

$$\begin{aligned}\|\mathcal{A}y\| &\le \|\Phi\|\Big(\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}+|A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\\
&\quad+\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Big)+\frac{|A_1||\kappa|}{|\Omega|}.\end{aligned}$$

In order to show the compactness of the operator $\mathcal{A}$, let $\sup_{(t,y)\in J\times B_{\bar r}}|f(t,y)|=\bar f<\infty$. Consequently, for $t_1,t_2\in J$, $t_1<t_2$, we have

$$\begin{aligned}\|(\mathcal{A}y)(t_2)-(\mathcal{A}y)(t_1)\| &\le \Big\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\Big[\int_0^{t_1}s^{\rho-1}[(t_2^\rho-s^\rho)^{\alpha-1}-(t_1^\rho-s^\rho)^{\alpha-1}]f(s,y(s))ds\\
&\quad+\int_{t_1}^{t_2}s^{\rho-1}(t_2^\rho-s^\rho)^{\alpha-1}f(s,y(s))ds\Big]\Big\|\\
&\le \frac{\bar f}{\rho^\alpha\Gamma(\alpha+1)}\Big\{2(t_2^\rho-t_1^\rho)^\alpha+|t_2^{\rho\alpha}-t_1^{\rho\alpha}|\Big\}.\end{aligned}$$

As the right-hand side of the above inequality tends to zero independently of $y \in B_{\bar{r}}$ when $t_2 \to t_1$, therefore $\mathcal{A}$ is equicontinuous. Thus, $\mathcal{A}$ is relatively compact on $B_{\bar{r}}$. Hence, the conclusion of Arzelá-Ascoli theorem applies and that $\mathcal{A}$ is compact on $B_{\bar{r}}$. Since all the conditions of Krasnoselskii's fixed point theorem hold true, it follows by Krasnoselskii's fixed point theorem that problem (1) has at least one solution on J. $\square$

Our final result in this section is concerned with the uniqueness of solutions for problem (1) and is based on a Banach fixed point theorem.

Theorem 3. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and the condition (H_3) holds. Then, problem (1) has a unique solution on J if*

$$L\Lambda < 1, \tag{29}$$

where Λ is defined by (26).

Proof. In view of the condition (29), consider a set $B_{\tilde{r}} = \{y \in \mathcal{C} : \|y\| \le \tilde{r}\}$, where

$$\tilde{r} > \frac{f_0\Lambda + \frac{|\kappa|(\rho|A_1|+T^\rho|A_2|)}{\rho|\Omega|}}{1 - L\Lambda}, \quad \sup_{t\in[0,T]} |f(t,0)| = f_0$$

and show that $\mathcal{G}B_{\tilde{r}} \subset B_{\tilde{r}}$ ($\mathcal{G}$ is defined by (25)). For $y \in B_{\tilde{r}}$, using (H_3), we get

$$\begin{aligned}
& |\mathcal{G}(y)(t)| \\
\le\ & {}^\rho I^\alpha_{0^+}[|f(t,y(t)) - f(t,0)| + |f(t,0)|] + \frac{1}{|\Omega|}\Big\{|B_1| \sum_{j=1}^{k} |\mu_j| {}^\rho I^\alpha_{0^+}[|f(\xi_j,y(\xi_j)) - f(\xi_j,0)| + |f(\xi_j,0)|] \\
& + A_1\Big[\sum_{i=1}^{m} |\sigma_i|\, {}^\rho I^{\alpha+\beta}_{0^+}[|f(\eta_i,y(\eta_i)) - f(\eta_i,0)| + |f(\eta_i,0)|] + {}^\rho I^\alpha_{0^+}[|f(T,y(T)) - f(T,0)| + |f(T,0)|\Big] \\
& + |\kappa|\Big\} + \frac{t^\rho}{\rho|\Omega|}\Big\{|B_2| \sum_{j=1}^{k} \mu_j {}^\rho I^\alpha_{0^+}[|f(\xi_j,y(\xi_j)) - f(\xi_j,0)| + |f(\xi_j,0)|] \\
& + |A_2|\Big[\sum_{i=1}^{m} \sigma_i\, {}^\rho I^{\alpha+\beta}_{0^+}[|f(\eta_i,y(\eta_i)) - f(\eta_i,0)| + |f(\eta_i,0)|] + {}^\rho I^\alpha_{0^+}[|f(T,y(T)) - f(T,0)| + |f(T,0)|] \\
& + |\kappa|\Big]\Big\} \\
\le\ & (L\|y\| + f_0)\Bigg[\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Big\{|B_1| \sum_{j=1}^{k} |\mu_j| \frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_1|\Big[\sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \\
& + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\} + \frac{T^\rho}{\rho|\Omega|}\Big\{|B_2| \sum_{j=1}^{k} |\mu_j| \frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_2|\Big[\sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \\
& + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Bigg] + \frac{|\kappa|(\rho|A_1| + T^\rho|A_2|)}{\rho|\Omega|} \\
\le\ & (L\tilde{r} + f_0)\Lambda + \frac{|\kappa|(\rho|A_1| + T^\rho|A_2|)}{\rho|\Omega|} \le \tilde{r},
\end{aligned}$$

which, on taking the norm for $t \in J$, yields $\|\mathcal{G}(y)\| \le \tilde{r}$. This shows that $\mathcal{G}$ maps $B_{\tilde{r}}$ into itself. Now, we establish that the operator $\mathcal{G}$ is a contraction. For that, let $y, z \in \mathcal{C}$. Then, we get

$$\begin{aligned}
|\mathcal{G}(y)(t) - \mathcal{G}(z)(t)| \le\ & {}^\rho I^\alpha_{0^+}|f(t,y(t)) - f(t,z(t))| + \frac{1}{|\Omega|}\Big\{|B_1| \sum_{j=1}^{k} |\mu_j| {}^\rho I^\alpha_{0^+}|f(\xi_j,y(\xi_j)) - f(\xi_j,z(\xi_j))| \\
& + A_1\Big[\sum_{i=1}^{m} |\sigma_i|\, {}^\rho I^{\alpha+\beta}_{0^+}|f(\eta_i,y(\eta_i)) - f(\eta_i,z(\eta_i))| + {}^\rho I^\alpha_{0^+}|f(T,y(T)) - f(T,z(T))|\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|^{\rho}I_{0^+}^{\alpha}|f(\xi_j,y(\xi_j))-f(\xi_j,z(\xi_j))| \\
&+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}|f(\eta_i,y(\eta_i))-f(\eta_i,z(\eta_i))|+{}^{\rho}I_{0^+}^{\alpha}|f(T,y(T))-f(T,z(T))|\Big]\Big\} \\
\leq\ & L\Lambda\|y-z\|.
\end{aligned}
$$

Consequently, we obtain

$$\|\mathcal{G}(y)-\mathcal{G}(z)\|\leq L\Lambda\|y-z\|,$$

which implies that $\mathcal{G}$ is a contraction by the condition (29). Hence, $\mathcal{G}$ has a unique fixed point by a Banach fixed point theorem. Equivalently, we deduce that problem (1) has a unique solution on J. The proof is completed. □

Example 1. *Consider the following boundary value problem*

$$
\begin{cases}
{}^{1/3}{}_{c}D_{0^+}^{7/5}y(t)=f(t,y(t)),\ t\in[0,2], \\
y(2)=1/2\,{}^{1/3}I^{3/5}y(1/4)+2/3\,{}^{1/3}I^{3/5}y(3/4)+2/9, \\
\delta y(0)=2/5\,y(1)+4/5\,y(3/2),
\end{cases}
\tag{30}
$$

where $\rho=1/3$, $\alpha=7/5$, $\sigma_1=1/2$, $\sigma_2=2/3$, $\beta=3/5$, $\eta_1=1/4$, $\eta_2=3/4$, $\mu_1=2/5$, $\mu_2=4/5$, $\kappa=2/9$, $\xi_1=1$, $\xi_2=3/2$, $T=2$ *and* $f(t,y(t))$ *will be fixed later.*

Using the given data, we find that $|A_1|=2.947314182$, $|A_2|=1.2$, $|B_1|=0.491608875$, $|B_2|=1.181571585$, $|\Omega|=4.072393340$, and $\Lambda=27.12293267$ (A_i,B_i $(i=1,2),\Omega$ and Λ are respectively given by Equations (12), (13), (14) and (26)).

For illustrating Theorem 1, we take

$$f(t,y)=\frac{(1+t)}{60}\Big(\frac{|y|}{|y|+1}+y+\frac{1}{8}\Big). \tag{31}$$

Clearly, $f(t,x)$ is continuous and satisfies the condition (H_1) with $\phi(t)=\frac{(1+t)}{60}$, $\psi(\|y\|)=||y||+\frac{9}{8}$. By the condition (H_2), we find that $\mathcal{M}>2.390158$. Thus, all conditions of Theorem 1 are satisfied and, consequently, there exists at least one solution for problem (30) with $f(t,y(t))$ given by (31) on $[0,2]$.

In order to illustrate Theorem 2, we choose

$$f(t,y)=\frac{\tan^{-1}y+e^{-t}}{4\sqrt{81+\sin t}}. \tag{32}$$

It is easy to check that $f(t,x)$ is continuous and satisfies the conditions (H_3) and (H_4) with $L=1/36$ and $\Phi(t)=\frac{\pi+2e^{-t}}{8\sqrt{81+\sin t}}$. In addition,

$$L\Big(\frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\Big\}\Big)\approx 0.420512<1.$$

Thus, all of the conditions of Theorem 2 hold true. Thus, by the conclusion of Theorem 2, problem (30) has at least one solution on $[0,2]$.

With $L\Lambda\approx 0.753415<1$, one can note that the assumptions of Theorem 3 are also satisfied. Hence, the conclusion of Theorem 3 applies and the problem (30) with $f(t,y)$ given (32) has a unique solution on $[0,2]$.

4. Existence Results for the Problem (2)

This section is devoted to the existence of solutions for problem (2).

Definition 4. *A function $y \in C([0,T],\mathbb{R})$ possessing Liouville–Caputo-type generalized deravative of order α is said to be a solution of the boundary value problem (2) if $y(T) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I_{0+}^{\beta} y(\eta_i) + \kappa$, $\delta y(0) = \sum_{j=1}^{k} \mu_j y(\xi_j)$ and there exists function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. on $[0,T]$ and*

$$\begin{aligned} y(t) &= {}^{\rho}I_{0+}^{\alpha} v(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0+}^{\alpha} v(\xi_j) + A_1 \Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0+}^{\alpha} v(T) + \kappa \Big] \Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega} \Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0+}^{\alpha} v(\xi_j) + A_2 \Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0+}^{\alpha} v(T) + \kappa \Big] \Big\}. \end{aligned} \tag{33}$$

4.1. The Carathéodory Case

Here, we present an existence result for problem (2) when F has convex values and is of the Carathéodory type. The main tool of our study is a nonlinear alternative of Leray–Schauder type [40].

Theorem 4. *Assume that:*

- (C_1) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory, where $\mathcal{P}_{cp,c}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) :$ Y is compact and convex$\}$;*
- (C_2) *there exists a continuous nondecreasing function $\Psi : [0,\infty) \to (0,\infty)$ and a function $\Phi \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,y)\|_{\mathcal{P}} := \sup\{|x| : x \in F(t,y)\} \le \Phi(t)\Psi(\|y\|) \ \text{for each} \ (t,y) \in [0,T] \times \mathbb{R};$$

- (C_3) *there exists a constant $W > 0$ such that*

$$\frac{\|W\|}{\Psi(\|W\|)\Lambda_2 + \dfrac{|\kappa|(\rho|A_1| + T^{\rho}|A_2|)}{\rho|\Omega|}} > 1,$$

where

$$\begin{aligned} \Lambda_2 &= {}^{\rho}I_{0+}^{\alpha}\Phi(T) + \frac{1}{|\Omega|}\Big\{ |B_1| \sum_{j=1}^{k} |\mu_j| {}^{\rho}I_{0+}^{\alpha}\Phi(\xi_j) + |A_1| \Big[\sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0+}^{\alpha+\beta}\Phi(\eta_i) + {}^{\rho}I_{0+}^{\alpha}\Phi(T) \Big] \Big\} \\ &\quad + \frac{T^{\rho}}{\rho|\Omega|}\Big\{ |B_2| \sum_{j=1}^{k} |\mu_j| {}^{\rho}I_{0+}^{\alpha}\Phi(\xi_j) + |A_2| \Big[\sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0+}^{\alpha+\beta}\Phi(\eta_i) + {}^{\rho}I_{0+}^{\alpha}\Phi(T) \Big] \Big\}. \end{aligned} \tag{34}$$

Then, problem (2) has at least one solution on $[0,T]$.

Proof. In order to convert problem (2) into a fixed point problem, we introduce an operator $\mathcal{N} : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C})$ by

$$\mathcal{N}(y) = \{h \in \mathcal{C} : h(t) = \mathcal{F}(y)(t)\}, \tag{35}$$

where

$$\begin{aligned} \mathcal{F}(y)(t) = \; & {}^{\rho}I_{0+}^{\alpha} v(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0+}^{\alpha} v(\xi_j) + A_1 \Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0+}^{\alpha} v(T) + \kappa \Big] \Big\} \\ & + \tfrac{t^{\rho}}{\rho\Omega}\Big\{ B_2 \textstyle\sum_{j=1}^{k} \mu_j {}^{\rho}I_{0+}^{\alpha} v(\xi_j) + A_2 \Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0+}^{\alpha} v(T) + \kappa \Big] \Big\}, \end{aligned}$$

for $v \in S_{F,y}$. Obviously, the fixed points of the operator $\mathcal{N}$ correspond to solutions of the problem (2).

It will be shown in several steps that the operator $\mathcal{N}$ satisfies the assumptions of the Leray–Schauder nonlinear alternative [40].

Step 1. $\mathcal{N}(y)$ is convex for each $y \in \mathcal{C}$.

This step is obvious since $S_{F,y}$ is convex (F has convex values).

Step 2. $\mathcal{N}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$.

For a positive number R, let $B_R = \{y \in \mathcal{C} : \|y\| \leq R\}$ be a bounded ball in $\mathcal{C}$. Then, for each $h \in \mathcal{N}(y)$, $y \in B_R$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) = \;& {}^{\rho}I_{0^+}^{\alpha} v(t) + \frac{1}{\Omega}\Big\{ -B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} v(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v(T) + \kappa\Big]\Big\} \\ & + \tfrac{t^{\rho}}{\rho\Omega}\Big\{B_2 \textstyle\sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} v(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v(T) + \kappa\Big]\Big\}. \end{aligned}$$

Then, for $t \in [0, T]$, we have

$$\begin{aligned} |h(t)| \leq\;& {}^{\rho}I_{0^+}^{\alpha} |v(t)| + \frac{1}{|\Omega|}\Big\{|B_1| \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} |v(\xi_j)| + |A_1|\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} |v(\eta_i)| + {}^{\rho}I_{0^+}^{\alpha} |v(T)| + |\kappa|\Big]\Big\} \\ & + \frac{t^{\rho}}{\rho|\Omega|}\Big\{|B_2| \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} |v(\xi_j)| + |A_2|\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} |v(\eta_i)| + {}^{\rho}I_{0^+}^{\alpha} |v(T)| + |\kappa|\Big]\Big\} \\ \leq\;& \Psi(\|y\|)\Big({}^{\rho}I_{0^+}^{\alpha} \Phi(T) + \frac{1}{|\Omega|}\Big\{|B_1| \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} \Phi(\xi_j) + |A_1|\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} |\Phi(\eta_i)| + {}^{\rho}I_{0^+}^{\alpha} \Phi(T)\Big]\Big\} \\ & + \frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2| \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} \Phi(\xi_j) + |A_2|\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} \Phi(\eta_i) + {}^{\rho}I_{0^+}^{\alpha} \Phi(T)\Big]\Big\}\Big) \\ & + \frac{|\kappa|(\rho|A_1| + T^{\rho}|A_2|)}{\rho|\Omega|}. \end{aligned}$$

Thus,

$$\begin{aligned} \|h\| \leq\;& \Psi(R)\Big({}^{\rho}I_{0^+}^{\alpha} \Phi(T) + \frac{1}{|\Omega|}\Big\{|B_1| \sum_{j=1}^{k} |\mu_j| {}^{\rho}I_{0^+}^{\alpha} \Phi(\xi_j) + |A_1|\Big[\sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0^+}^{\alpha+\beta} \Phi(\eta_i) + {}^{\rho}I_{0^+}^{\alpha} \Phi(T)\Big]\Big\} \\ & + \frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2| \sum_{j=1}^{k} |\mu_j| {}^{\rho}I_{0^+}^{\alpha} \Phi(\xi_j)| + |A_2|\Big[\sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0^+}^{\alpha+\beta} \Phi(\eta_i) + {}^{\rho}I_{0^+}^{\alpha} \Phi(T)\Big]\Big\}\Big) \\ & + \frac{|\kappa|(\rho|A_1| + T^{\rho}|A_2|)}{\rho|\Omega|} := \tilde{\ell}. \end{aligned}$$

Step 3. $\mathcal{N}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, and let $y \in B_R$. Then,

$$\begin{aligned} & |h(t_2) - h(t_1)| \\ \leq\;& \Big|{}^{\rho}I_{0^+}^{\alpha} v(t_2) - {}^{\rho}I_{0^+}^{\alpha} v(t_1)\Big| + \frac{|t_2^{\rho} - t_1^{\rho}|}{\rho|\Omega|}\Big\{|B_2| \sum_{j=1}^{k} |\mu_j| {}^{\rho}I_{0^+}^{\alpha} |v(\xi_j| \\ & + |A_2|\Big[\sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0^+}^{\alpha+\beta} |v(\eta_i)| + {}^{\rho}I_{0^+}^{\alpha} |v(T)| + |\kappa|\Big]\Big\} \\ \leq\;& \frac{\rho^{1-\alpha}\Psi(R)}{\Gamma(\alpha)}\Big|\int_0^{t_1}\Big[\frac{s^{\rho-1}}{(t_2^{\rho} - s^{\rho})^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^{\rho} - s^{\rho})^{1-\alpha}}\Big]\Phi(s)ds + \int_{t_1}^{t_2} \frac{s^{\rho-1}}{(t_2^{\rho} - s^{\rho})^{1-\alpha}}\Phi(s)ds\Big| \\ & + \frac{|t_2^{\rho} - t_1^{\rho}|}{\rho|\Omega|}\Big\{\Psi(R)\Big(|B_2| \sum_{j=1}^{k} |\mu_j| {}^{\rho}I_{0^+}^{\alpha} \Phi(\xi_j) + |A_2|\Big[\sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0^+}^{\alpha+\beta} \Phi(\eta_i) + {}^{\rho}I_{0^+}^{\alpha} \Phi(T)\Big]\Big) + |A_2||\kappa|\Big\} \end{aligned}$$

$$\to \quad 0 \text{ as } t_2 - t_1 \to 0,$$

independently of $y \in B_R$. In view of the foregoing steps, the Arzelá–Ascoli theorem applies and that the operator $\mathcal{N} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is completely continuous.

In our next step, we show that $\mathcal{N}$ is u.s.c. We just need to establish that $\mathcal{N}$ has a closed graph as it is already shown to be completely continuous [42] (Proposition 1.2).

Step 4. $\mathcal{N}$ has a closed graph.

Let $y_n \to y_*, h_n \in \mathcal{N}(y_n)$ and $h_n \to h_*$. Then, we have to show that $h_* \in \mathcal{N}(y_*)$. Associated with $h_n \in \mathcal{N}(y_n)$, we have that $v_n \in S_{F,y_n}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_n(t) &= {}^{\rho}I^{\alpha} v_n(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v_n(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v_n(\eta_i) - {}^{\rho}I^{\alpha} v_n(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} v_n(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v_n(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v_n(T) + \kappa\Big]\Big\}. \end{aligned}$$

Thus, it is sufficient to establish that there exists $v_* \in S_{F,y_*}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= {}^{\rho}I_{0^+}^{\alpha} v_*(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v_*(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v_*(\eta_i) - {}^{\rho}I^{\alpha} v_*(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} v_*(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v_*(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v_*(T) + \kappa\Big]\Big\}. \end{aligned}$$

Next, we introduce the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to \mathcal{C}$ as

$$\begin{aligned} v \mapsto \Theta v(t) &= {}^{\rho}I_{0^+}^{\alpha} v(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v(T) + \kappa\Big]\Big\}. \end{aligned}$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$. Therefore, by a closed graph result obtained in [43], $\Theta \circ S_F$ is a closed graph operator. Moreover, we have that $h_n(t) \in \Theta(S_{F,y_n})$. As $y_n \to y_*$, we have that

$$\begin{aligned} h_*(t) &= {}^{\rho}I_{0^+}^{\alpha} v_*(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v_*(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v_*(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v_*(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v_*(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v_*(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v_*(T) + \kappa\Big]\Big\} \end{aligned}$$

for some $v_* \in S_{F,y_*}$.

Step 5. There exists an open set $\mathcal{U} \subseteq C([0,T],\mathbb{R})$ with $y \notin \lambda\mathcal{N}(y)$ for any $\lambda \in (0,1)$ and all $y \in \partial\mathcal{U}$.

Let $\lambda \in (0,1)$ and $y \in \lambda\mathcal{N}(y)$. Then, we can find $v \in L^1([0,T],\mathbb{R})$ and $v \in S_{F,y}$ such that, for $t \in [0,T]$, we have

$$\begin{aligned} y(t) &= \lambda {}^{\rho}I_{0^+}^{\alpha} v(t) + \frac{\lambda}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} v(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v(T) + \kappa\Big]\Big\} \\ &\quad + \lambda\frac{t^{\rho}}{\rho\Omega}\Big\{ B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I_{0^+}^{\alpha} v(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i\, {}^{\rho}I_{0^+}^{\alpha+\beta} v(\eta_i) - {}^{\rho}I_{0^+}^{\alpha} v(T) + \kappa\Big]\Big\}. \end{aligned}$$

As in Step 2, one can find that

$$\begin{aligned}|y(t)| &\leq {}^{\rho}I^{\alpha}_{0^+}|v(t)| + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|^{\rho}I^{\alpha}_{0^+}|v(\xi_j)| + |A_1|\Big[\sum_{i=1}^{m}\sigma_i\,{}^{\rho}I^{\alpha+\beta}_{0^+}|v(\eta_i)| + {}^{\rho}I^{\alpha}_{0^+}|v(T)| + |\kappa|\Big]\Big\}\\
&+\frac{t^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|^{\rho}I^{\alpha}_{0^+}|v(\xi_j)| + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}|v(\eta_i)| + {}^{\rho}I^{\alpha}_{0^+}|v(T)| + |\kappa|\Big]\Big\}\\
&\leq \Psi(y)\Big({}^{\rho}I^{\alpha}\Phi(T) + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|^{\rho}I^{\alpha}\Phi(\xi_j) + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}|\Phi(\eta_i)| + {}^{\rho}I^{\alpha}|\Phi(T)|\Big]\Big\}\\
&+\frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|^{\rho}I^{\alpha}\Phi(\xi_j) + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}\Phi(\eta_i) + {}^{\rho}I^{\alpha}\Phi(T)\Big]\Big\}\Big)\\
&+\frac{|\kappa|(\rho|A_1| + T^{\rho}|A_2|)}{\rho|\Omega|},\end{aligned}$$

which implies that

$$\frac{\|y\|}{\Psi(\|y\|)\Lambda_2 + \dfrac{|\kappa|(\rho|A_1| + T^{\rho}|A_2|)}{\rho|\Omega|}} \leq 1.$$

By the assumption (C_3), there exists W such that $\|y\| \neq W$. Let us set

$$U = \{y \in C(J,\mathbb{R}) : \|y\| < W\}.$$

Observe that the operator $\mathcal{N} : \overline{\mathcal{U}} \to \mathcal{P}(C(J,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of $\mathcal{U}$, there does not exist $y \in \partial U$ satisfying $y \in \lambda\mathcal{N}(y)$ for some $\lambda \in (0,1)$. In consequence, we deduce by the nonlinear alternative of Leray–Schauder type [40] that the operator $\mathcal{N}$ has a fixed point $y \in \overline{\mathcal{U}}$ that is a solution of problem (2). This completes the proof. □

4.2. The Lipschitz Case

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Define $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}\cup\{\infty\}$ as $H_d(P,Q) = \max\{\sup_{p\in P} d(p,Q), \sup_{q\in Q} d(P,q)\}$, where $d(P,q) = \inf_{p\in P} d(p;q)$ and $d(p,Q) = \inf_{q\in Q} d(p;q)$. Then, $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space [16]. (Here, $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$).

In the following result, we apply a fixed point theorem (If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$, where $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$) due to Covitz and Nadler [44].

Theorem 5. *Let the following conditions hold:*

(C_4) *$F : [0,T]\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,y) : [0,T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$, where $\mathcal{P}_{cp}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ is compact}\}$;*
(C_5) *$H_d(F(t,y),F(t,\bar{y})) \leq \theta(t)|y-\bar{y}|$ for almost all $t \in [0,T]$ and $y,\bar{y} \in \mathbb{R}$ with $\theta \in C([0,T],\mathbb{R}^+)$ and $d(0,F(t,0)) \leq \theta(t)$ for almost all $t \in [0,T]$, where*

Then, problem (2) has at least one solution on $[0,T]$ if $\|\theta\|\Lambda < 1$, i.e.,

$$\begin{aligned}K &:= \|\theta\|\Big[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\\
&+|A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\Big\}\\
&+\frac{T^{\rho}}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big]\Big\}\Big] < 1.\end{aligned} \tag{36}$$

Proof. By the assumption (C_4), it is clear that the set $S_{F,y}$ is nonempty for each $y \in \mathcal{C}$ and thus there exists a measurable selection for F (see Theorem III.6 [45]). Firstly, it will be shown that $\mathcal{N}(y) \in \mathcal{P}_{cl}(\mathcal{C})$ for each $y \in \mathcal{C}$, where the operator $\mathcal{N}$ is defined by (35). Let $\{u_n\}_{n\geq 0} \in \mathcal{F}(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{C}$. Then, $u \in \mathcal{C}$ and we can find $v_n \in S_{F,y_n}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) &= {}^{\rho}I^{\alpha}_{0^+}v_n(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha} v_n(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{\eta_0} v_n(\eta_i) - {}^{\rho}I^{\alpha}v_n(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v_n(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v_n(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v_n(T) + \kappa\Big]\Big\}. \end{aligned}$$

Since F has compact values, we pass onto a subsequence (if necessary) such that v_n converges to v in $L^1([0,T],\mathbb{R})$, which implies that $v \in S_{F,y}$ and for each $t \in [0,T]$, we have

$$\begin{aligned} u_n(t) \to u(t) &= {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v_n(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v(T) + \kappa\Big]\Big\}. \end{aligned}$$

Thus, $u \in \mathcal{N}(y)$.

Next, it will be shown that there exists $K < 1$ (defined by (36)) such that

$$H_d(\mathcal{N}(y), \mathcal{N}(\bar{y})) \leq K\|y - \bar{y}\| \ \text{ for each } \ y, \bar{y} \in \mathcal{C}.$$

Let $y, \bar{y} \in \mathcal{C}$ and $h_1 \in \mathcal{F}(y)$. Then, there exists $v_1(t) \in F(t, y(t))$ for each $t \in [0,T]$ and that

$$\begin{aligned} h_1(t) &= {}^{\rho}I^{\alpha}_{0^+}v_1(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v_1(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v_1(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v_1(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v_1(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v_1(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v_1(T) + \kappa\Big]\Big\}. \end{aligned}$$

By (C_5), $H_d(F(t,y), F(t,\bar{y})) \leq \theta(t)|y(t) - \bar{y}(t)|$ and that there exists $w \in F(t, \bar{y}(t))$ satisfying the inequality: $|v_1(t) - w| \leq \theta(t)|y(t) - \bar{y}(t)|, \ \ t \in [0,T]$.

Next, we introduce $\mathcal{S} : [0,T] \to \mathcal{P}(\mathbb{R})$ as

$$\mathcal{S}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq \theta(t)|y(t) - \bar{y}(t)|\}.$$

By Proposition III.4 [45], the multivalued operator $\mathcal{S}(t) \cap F(t, \bar{y}(t))$ is measurable. Thus, there exists a function $v_2(t)$, which is a measurable selection for $\mathcal{S}$ and that $v_2(t) \in F(t, \bar{y}(t))$. Thus, for each $t \in [0,T]$, we have $|v_1(t) - v_2(t)| \leq \theta(t)|y(t) - \bar{y}(t)|$.

Next, we define

$$\begin{aligned} h_2(t) &= {}^{\rho}I^{\alpha}_{0^+}v_2(t) + \frac{1}{\Omega}\Big\{ - B_1 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v_2(\xi_j) + A_1\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v_n(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v_2(T) + \kappa\Big]\Big\} \\ &\quad + \frac{t^{\rho}}{\rho\Omega}\Big\{B_2 \sum_{j=1}^{k} \mu_j {}^{\rho}I^{\alpha}_{0^+} v_2(\xi_j) + A_2\Big[\sum_{i=1}^{m} \sigma_i \, {}^{\rho}I^{\alpha+\beta}_{0^+} v_2(\eta_i) - {}^{\rho}I^{\alpha}_{0^+}v_2(T) + \kappa\Big]\Big\} \end{aligned}$$

for each $t \in [0, T]$. Then,

$$
\begin{aligned}
&|h_1(t) - h_2(t)| \\
&\le {}^\rho I^{\alpha}_{0^+}|v_1(t) - v_2(t)| + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|{}^\rho I^{\alpha}_{0^+}|v_1(\xi_j) - v_2(\xi_j)| + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\,{}^\rho I^{\alpha+\beta}_{0^+}|v_1(\eta_i) - v_2(\eta_i)| \\
&+{}^\rho I^{\alpha}_{0^+}|v_1(T) - v_2(T)|\Big]\Big\} + \frac{t^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}\mu_j{}^\rho I^{\alpha}_{0^+}|v_2(\xi_j) - v_1(\xi_j)| + |A_2|\Big[\sum_{i=1}^{m}\sigma_i\,{}^\rho I^{\alpha+\beta}_{0^+}|v_2(\eta_i) - v_1(\eta_i)| \\
&+{}^\rho I^{\alpha}_{0^+}|v_2(T) - v_1(T)|\Big]\Big\} \\
&\le \|\theta\|\Big[\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\} \\
&+\frac{T^\rho}{\rho|\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Big]\|y - \bar{y}\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|h_1 - h_2\| \;\le\; & \|\theta\|\Big[\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \\
&+\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\} + \frac{T^\rho}{|\rho\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \\
&+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Big]\|y - \bar{y}\|.
\end{aligned}
$$

Analogously, interchanging the roles of y and $\overline{y}$, we find that

$$
\begin{aligned}
H_d(\mathcal{N}(y), \mathcal{N}(\bar{y})) \;\le\; & \|\theta\|\Big[\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Big\{|B_1|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} + |A_1|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \\
&+\frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\} + \frac{T^\rho}{|\rho\Omega|}\Big\{|B_2|\sum_{j=1}^{k}|\mu_j|\frac{\xi_j^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \\
&+|A_2|\Big[\sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big]\Big\}\Big]\|y - \bar{y}\|.
\end{aligned}
$$

This shows that $\mathcal{N}$ is a contraction. Therefore, the operator $\mathcal{N}$ has a fixed point y by Covitz and Nadler [44], which corresponds to a solution of problem (2). This completes the proof. □

Example 2. *Consider the following inclusions problem:*

$$
\begin{cases}
{}^{1/3}{}_cD^{7/5}_{0^+}y(t) \in F(t, y(t)),\ t \in [0, 2], \\
y(2) = 1/2\,{}^{1/3}I^{3/5}y(1/4) + 2/3\,{}^{1/3}I^{3/5}y(3/4) + 2/9, \\
\delta y(0) = 2/5\,y(1) + 4/5\,y(3/2),
\end{cases}
\tag{37}
$$

where $F(t, y(t))$ will be defined later.

The values of $|A_1|$, $|A_2|$, $|B_1|$, $|B_2|$, $|\Omega|$ and Λ are the same as those in Example 1. For illustrating Theorem 4, we take

$$
F(t, y(t)) = \Big[\frac{e^{-t}}{\sqrt{4000+t}}\Big(\sin y + \frac{1}{2}\Big),\ \frac{(1+t)}{60}\Big(\tan^{-1} y + y + \frac{1}{4}\Big)\Big]. \tag{38}
$$

It is easy to check that $F(t, y(t))$ is L^1-Carathéodory. In view of (C_2), we find that $\Phi(t) = \dfrac{(1+t)}{60}, \Psi(\|y\|) = ||y|| + \dfrac{2\pi+1}{4}$ and the condition (C_3) implies that $W > 3.289470$. Thus, all hypotheses of Theorem 4 hold true and the conclusion of Theorem 4 applies to problem (37) with $F(t, y(t))$ given by (38) on $[0, 2]$.

Now, we illustrate Theorem 5 by considering

$$F(t, y(t)) = \Big[\frac{(t+1)}{4\sqrt{900+t}}\Big(\tan^{-1} y + \sin t\Big), \frac{e^{-t}\cos t}{250}\Big(\frac{|y|}{|y|+1} + \frac{1}{8}\Big)\Big]. \tag{39}$$

Clearly,

$$H_d(F(t, y), F(t, \bar{y})) \leq \frac{(t+1)}{120}\|y - \bar{y}\|.$$

Letting $\theta(t) = \dfrac{(t+1)}{120}$, we observe that $d(0, F(t, 0)) \leq \theta(t)$ for almost all $t \in [0, 2]$ and that $K \approx 0.6780733168 < 1$ (K is given by (36)). As the assumptions of Theorem 5 hold true, there exists at least one solution for problem (37) with $F(t, y(t))$ given by (39) on $[0, 2]$.

5. Conclusions

We have developed the existence theory for fractional differential equations and inclusions involving the Liouville–Caputo-type generalized derivative, supplemented with nonlocal generalized fractional integral and multipoint boundary conditions. Our results are based on the modern techniques of the functional analysis. In case of a single valued problem (1), we have obtained three results: the first two results deal with the existence of solutions while the third one is concerned with the uniqueness of solutions for the given problem. The first existence result relies on a Leray–Schauder nonlinear alternative, which allows the nonlinearity $f(t, y)$ to behave like $|f(t, y)| \leq \phi(t)\psi(\|y\|)$ (see (H_1)) and the second results, depending on Krasnoselskii's fixed point theorem, handles the nonlinearity $f(t, y)$ of the form described by the conditions (H_3) and (H_4). The third result provides a criterion ensuring a unique solution of the problem at hand by requiring the nonlinear function $f(t, y)$ to satisfy the classical Lipschitz condition and is based on a Banach fixed point theorem. The tools of the fixed point theory chosen for our case are easy to apply and extend the scope of the obtained results in the scenario of simplicity of the assumptions. Again, for the inclusion problem (2), the idea is to assume a simple set of conditions to establish the existence of solutions for problem (2) involving both convex and nonconvex valued maps. As a matter of fact, the fixed point theorems chosen to solve the multivalued problem (2) are standard and popular in view of their applicability. Concerning the choice of the method to solve a given problem, one needs to loook at the set of assumptions satisfied by the single and multivalued maps involved in the problem, which decides the selection of the tool to be employed. As an application of the present work, the generalization of the Feynman and Wiener path integrals developed by Laskin [46], in the context of fractional quantum mechanics and fractional statistical mechanics, can be enhanced further. We emphasize that we obtain new results associated with symmetric solutions of a second-order ordinary differential equation equipped with nonlocal fractional integral and multipoint boundary conditions if we take $0 < \eta_1 < \cdots < \eta_i < \cdots < \eta_m < \xi_1 < \cdots < \xi_j < \cdots < \xi_k < T/2$ and $f(t, x)$ to be symmetric on the interval $[0, T]$ for all $x \in \mathbb{R}$ when $\alpha \to 2^-$ $(\rho = 1)$.

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