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On Some Statistical Approximation by (p,q)-Bleimann, Butzer and Hahn Operators

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Abstract: In this article, we propose a different generalization of (p,q)-BBH operators and carry statistical approximation properties of the introduced operators towards a function which has to be approximated where (p,q)-integers contains symmetric property. We establish a Korovkin approximation theorem in the statistical sense and obtain the statistical rates of convergence. Furthermore, we also introduce a bivariate extension of proposed operators and carry many statistical approximation results. The extra parameter p plays an important role to symmetrize the q-BBH operators.

Keywords: q–Bleimann–Butzer–Hahn operators; (p,q)-integers; (p,q)-Bernstein operators; (p,q)-Bleimann–Butzer–Hahn operators; modulus of continuity; rate of approximation; K-functional

MSC: 41A10; 41A25; 41A36

1. Introduction

The q-analog of Bleiman, Butzer and Hahn operators (BBH) [1] is defined by:

$$L_n^q(f;x) = \frac{1}{\ell_n^q(x)} \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right) q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k, \tag{1}$$

where $\ell_n^q(x) = \prod_{k=0}^{n-1}(1+q^s x)$.

For $q=1$, the sequence of q-BBH operators (1) reduces to the classical BBH-operators [2] in which authors investigated pointwise convergence properties of the BBH-operators in a compact sub-interval of $\mathbb{R}_+$.

Let H_ω denote the space of all real-valued functions f defined on the semi-axis $\mathbb{R}_+$ [3], where ω is the usual modulus of continuity satisfying

$$|f(x)-f(y)| \leq \omega\left(\left|\frac{x}{1+x} - \frac{y}{1+y}\right|\right)$$

for any $x,y \geq 0$.

Gadjiev and Çakar [3] established the Korovkin type theorem which gives the convergence for the sequence of linear positive operators (LPO) to the functions in H_ω.

Now, we recollect the following theorem:

Theorem 1 ([3]). *Let $\{A_n\}$ be the sequence of LPOs from H_ω into $C_B(\mathbb{R}_+)$ such that*

$$\lim_{n\to\infty}\left\|A_n\left(\left(\frac{t}{1+t}\right)^v;x\right)-\left(\frac{x}{1+x}\right)^v\right\|_{C_B}=0,\quad v=0,1,2,$$

then, for any function $f\in H_\omega$

$$\lim_{n\to\infty}\|A_n(f)-f\|_{C_B}=0.$$

(p,q)-calculus, also called post-quantum calculus, is a generalization of q-calculus which has lots of applications in quantum physics. In approximation theory, the very first (p,q)-type generalization of Bernstein polynomials was introduced by Mursaleen et al. [4] using (p,q)-calculus and improved the said operators (see Erratum [4]). The theory of semigroups of the linear operators is used in order to prove the existence and uniqueness of a weak solutions of boundary value problems in thermoelasticity of dipolar bodies (see [5,6]).

Recently, a very nice application and usage of extra parameter p has been discussed in [7] in the computer-aided geometric design. In that paper, authors applied these (p,q)-Bernstein bases to construct (p,q)-Bézier curves which are further generalizations of q-Bézier curves [8]. For more results on LPOs and its (p,q)-analogues, one can refer to [9–15].

Now, we provide some notations on (p,q)-calculus.

$[n]_{p,q}$ stands for (p,q)-integers defined as

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2\cdots+q^{n-1}=\begin{cases}\dfrac{p^n-q^n}{p-q} & (p\neq q\neq 1),\\[2ex] \dfrac{1-q^n}{1-q} & (p=1),\\[2ex] n & (p=q=1),\end{cases}\tag{2}$$

$$(ax+by)^n_{p,q}:=\sum_{j=0}^{n}p^{\frac{(n-j)(n-j-1)}{2}}q^{\frac{j(j-1)}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_{p,q}a^{n-j}b^jx^{n-j}y^j,$$

$$(x+y)^n_{p,q}=(x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1-x)^n_{p,q}=(1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x),$$

and the binomial coefficients in (p,q)-calculus are given by

$$\begin{bmatrix}n\\ j\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[j]_{p,q}![n-j]_{p,q}!}.$$

By easy computation, we have the relation given below:

$$q^j[n-j+1]_{p,q}=[n+1]_{p,q}-p^{n-j+1}[j]_{p,q}.$$

Authors suggest the readers [16–19].

The (p,q)-analogue of BBH operators was introduced by Mursaleen et al. in [20] as follows:

$$L_n^{p,q}(f;x) = \frac{1}{\ell_n^{p,q}(x)} \sum_{j=0}^{n} f\left(\frac{p^{n-j+1}[j]_{p,q}}{[n-j+1]_{p,q} q^j}\right) p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}} \left[\begin{array}{c} n \\ j \end{array}\right]_{p,q} x^j, \tag{3}$$

where $x \geq 0$, $0 < q < p \leq 1$, $\ell_n^{p,q}(x) = \prod_{s=0}^{n-1}(p^s + q^s x)$ and function f is defined on the semi axis $\mathbb{R}_+$. If we put $p = 1$, we get the q-BBH operators (1). In that paper, authors established different approximation properties of the sequence of operators (3).

Theorem 2 ([20])**.** *Let $p = (p_n)$, $q = (q_n)$ satisfying $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$ for $0 < q_n < p_n \leq 1$ and if $L_n^{p_n,q_n}(f;x)$ is defined by Label (3). Then, for any function $f \in H_\omega$,*

$$\lim_{n\to\infty} \| L_n^{p_n,q_n} f - f \|_{C_B} = 0.$$

Mursaleen and Nasiruzzaman constructed bivariate (p,q)-BBH operators [21] and studied many nice properties based on that sequence of operators and also given some generalization of that sequence of bivariate operators introducing one more parameter γ in the operators.

The statistical convergence is another notion of convergence, which was introduced by Fast [22] nearly fifty years ago and now this is a very active area of research. The statistical limit of a sequence is an extension of the idea of limit of sequence in an ordinary sense. The natural density of $K \subset \mathbb{N}$ is defined as:

$$\delta(K) = \lim_n \frac{1}{n}\{k \leq n : k \in K\}$$

whenever the limit exists (see [23,24]). The sequence $x = (x_k)$ is said to be statistically convergent to a number L means if, for every $\epsilon > 0$,

$$\delta\{k : \ |x_k - L| \geq \epsilon\} = 0,$$

and it is denoted by $st - \lim_k x_k = L$. It can be easily seen that every convergent sequence is statistically convergent but not conversely.

Now, we will state some preliminary results on positive linear operators:

Proposition 1 ([25])**.** *If L is an operator, linear and positive, then, for every $x \in X$, we have*

$$|Lf| \leq L(|f|). \tag{4}$$

Proposition 2 ([25])**.** *(Hölder's inequality for LPOs). Let $L : X \to Y$ be an operator, linear and positive, and let $1/p + 1/q = 1$, where $p, q > 1$ are real numbers. Then, for every $f, g \in X$*

$$L(|f \cdot g|) \leq (L(|f|^p))^{\frac{1}{p}} \cdot (L(|g|^q))^{\frac{1}{q}}. \tag{5}$$

Remark 1 ([25])**.** *A particular case of Proposition 2 is the Cauchy–Schwarz's inequality for LPOs, which is obtained from Hölder's inequality for $p = q = 2$ as:*

$$|L(f \cdot g; x)| \leq \sqrt{L(f^2; x)} \cdot \sqrt{L(g^2; x)}. \tag{6}$$

We have organized the rest of the paper as follows. In Section 2, we have constructed (p,q)-BBH operators and calculated some auxiliary results. In Sections 3 and 4, Korovkin type results and rate of convergence are established in statistical sense, respectively. Section 5 is devoted to the construction of the bivariate (p,q)-BBH operators. In Section 6, we have computed rate of statistical convergence for the bivariate (p,q)-BBH operators.

2. Construction of Operators and Moment Estimation

Ersan and Doğru [26] introduced a generalization of (1) and studied different statistical approximation properties of the operators towards a function f which has to be approximated. Inspired with the work of Ersan and Doğru [26], we construct a (p,q)-analogue generalization of the sequence of operators defined in [26] or, on the other hand, we generalize the operators introduced in [20] as follows:

$\forall x \geq 0,\ 0<q<p\leq 1$, let us define a sequence of (p,q)-BBH operators as follows:

$$\mathcal{B}_n^{p,q}(f;x) = \frac{q/p}{\zeta_n^{p,q}(x)} \sum_{j=0}^{n} f\left(\frac{p^{n-j+1}[j]_{p,q}}{[n-j+1]_{p,q}q^j}\right) p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}} \left[\begin{array}{c} n \\ j \end{array}\right]_{p,q} x^j, \tag{7}$$

where

$$\zeta_n^{p,q}(x) = \prod_{s=0}^{n-1}(p^s+q^s x) = \sum_{j=0}^{n} p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}} \left[\begin{array}{c} n \\ j \end{array}\right]_{p,q} x^j. \tag{8}$$

It is easy to verify that, if $p=q=1$, the operators turn into the classical BBH operators. The sequence of operators (7) is of course more generalized than (1), and it is more flexible than (1).

We need the following lemma to our main result:

Lemma 1. *Let the sequence of operators be given by* (7)*. Then,*

$$\mathcal{B}_n^{p,q}(1;x) = \frac{q}{p} \tag{9}$$

for any $x \geq 0$, $0<q<p\leq 1$.

Proof. The proof is obvious with the help of the relation (8), so we skip the proof. □

Lemma 2. *Let the sequence of operators be given by* (7)*. Then,*

$$\mathcal{B}_n^{p,q}\left(\frac{t}{1+t};x\right) = \frac{q[n]_{p,q}}{[n+1]_{p,q}} \frac{x}{1+x} \tag{10}$$

for any $x \geq 0$, $0<q<p\leq 1$.

Proof. Let $t=\frac{p^{n-j+1}[j]_{p,q}}{[n-j+1]_{p,q}q^j}$, then $\frac{t}{1+t}=\frac{p^{n-j+1}[j]_{p,q}}{[n+1]_{p,q}}$, so

$$\begin{aligned}
\mathcal{B}_n^{p,q}&\left(\frac{t}{1+t};x\right) \\
&= \frac{q/p}{\zeta_n^{p,q}(x)} \frac{[n]_{p,q}}{[n+1]_{p,q}} \sum_{j=1}^{n} p^{n-j+1} p^{\frac{(n-j)(n-j-1)}{2}} q^{\frac{j(j-1)}{2}} \left[\begin{array}{c} n-1 \\ j-1 \end{array}\right]_{p,q} x^j \\
&= \frac{q/p}{\zeta_n^{p,q}(x)} \frac{[n]_{p,q}x}{[n+1]_{p,q}} \sum_{j=0}^{n-1} p^{n-j} p^{\frac{(n-j-2)(n-j-1)}{2}} q^{\frac{j(j+1)}{2}} \left[\begin{array}{c} n-1 \\ j \end{array}\right]_{p,q} x^j \\
&= \frac{q/p}{\zeta_n^{p,q}(x)} \frac{p^n[n]_{p,q}x}{[n+1]_{p,q}} \sum_{j=0}^{n-1} p^{\frac{(n-j-1)(n-j-2)}{2}} q^{\frac{j(j-1)}{2}} \left[\begin{array}{c} n-1 \\ j \end{array}\right]_{p,q} \left(\frac{qx}{p}\right)^j.
\end{aligned}$$

By using (8), the result can be easily obtained. □

Lemma 3. *Let the sequence of operators be given by* (7)*. Then,*

$$\mathcal{B}_n^{p,q}\left(\left(\frac{t}{1+t}\right)^2;x\right)=\frac{pq^3[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\frac{x^2}{(1+x)(p+qx)}+\frac{p^nq[n]_{p,q}}{[n+1]_{p,q}^2}\frac{x}{1+x} \tag{11}$$

for any $x\geq 0$, $0<q<p\leq 1$.

Proof. It is easy to verify that

$$[j]_{p,q}=p^{j-1}+q[j-1]_{p,q};\qquad [j]_{p,q}^2=q[j]_{p,q}[j-1]_{p,q}+p^{j-1}[j]_{p,q}. \tag{12}$$

With the help of (12), we can have

$$\begin{aligned}
&\mathcal{B}_n^{p,q}\left(\left(\frac{t}{1+t}\right)^2;x\right)\\
&=\frac{q/p}{\zeta_n^{p,q}(x)}\left\{\frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\sum_{j=2}^{n}p^{2n-2j+2}p^{\frac{(n-j)(n-j-1)}{2}}q^{\frac{j(j-1)}{2}}\left[\begin{array}{c}n-2\\j-2\end{array}\right]_{p,q}x^j\right.\\
&\qquad\left.+\frac{[n]_{p,q}}{[n+1]_{p,q}^2}\sum_{j=1}^{n}p^{2n-2j+2}p^{j-1}p^{\frac{(n-j)(n-j-1)}{2}}q^{\frac{j(j-1)}{2}}\left[\begin{array}{c}n-1\\j-1\end{array}\right]_{p,q}x^j\right\}\\
&=\frac{q/p}{\zeta_n^{p,q}(x)}\left\{\frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}\sum_{j=0}^{n-2}p^{2n-2j-2}p^{\frac{(n-j-2)(n-j-3)}{2}}q^{\frac{(j+2)(j+1)}{2}}\left[\begin{array}{c}n-2\\j\end{array}\right]_{p,q}x^{j+2}\right.\\
&\qquad\left.+\frac{[n]_{p,q}}{[n+1]_{p,q}^2}\sum_{j=0}^{n-1}p^{2n-2j}p^jp^{\frac{(n-j-1)(n-j-2)}{2}}q^{\frac{(j+1)j}{2}}\left[\begin{array}{c}n-1\\j\end{array}\right]_{p,q}x^{j+1}\right\}\\
&=\frac{q/p}{\zeta_n^{p,q}(x)}\left\{\frac{p^{2n-2}q^2[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2}x^2\sum_{j=0}^{n-2}p^{\frac{(n-j-2)(n-j-3)}{2}}q^{\frac{j(j-1)}{2}}\left[\begin{array}{c}n-2\\j\end{array}\right]_{p,q}\left(\frac{q^2x}{p^2}\right)^j\right.\\
&\qquad\left.+\frac{p^{2n}[n]_{p,q}}{[n+1]_{p,q}^2}x\sum_{j=0}^{n-1}p^{\frac{(n-j-1)(n-j-2)}{2}}q^{\frac{j(j-1)}{2}}\left[\begin{array}{c}n-1\\j\end{array}\right]_{p,q}\left(\frac{qx}{p}\right)^j\right\}.
\end{aligned}$$

Now, using (8), we can get the desired result. □

3. Korovkin Type Statistical Approximation Properties

In this section, we obtain the Korovkin type statistical approximation theorem for our sequence of operators (7). Let us give the following theorem:

Theorem 3. *[3] Let* $\{A_n\}$ *be the sequence of LPOs from* H_ω *into* $C_B(\mathbb{R}_+)$ *such that*

$$st-\lim_n\left\|A_n\left(\left(\frac{t}{1+t}\right)^\nu;x\right)-\left(\frac{x}{1+x}\right)^\nu\right\|_{C_B}=0,\ \ \nu=0,1,2.$$

Then, for any function $f\in H_\omega$,

$$st-\lim_n\|A_n(f)-f\|_{C_B}=0.$$

Let us take $p = (p_n)$ and $q = (q_n)$ such that

$$st - \lim_n p_n = 1, \; st - \lim_n q_n = 1. \tag{13}$$

Theorem 4. *Let $\mathcal{B}_n^{p,q}(f;x)$ be the sequence of operators (7) and the sequences $p = (p_n)$ and $q = (q_n)$ satisfy the assumption (13) for $0 < q_n < p_n \leq 1$. Then, for any function $f \in H_\omega$,*

$$st - \lim_n \|\mathcal{B}_n^{p,q}(f;.) - f\| = 0.$$

Proof. For $\nu = 0$ and using (9), we can have

$$st - \lim_n \|\mathcal{B}_n^{p_n,q_n}(1;x) - 1\| = st - \lim_n \left| \frac{q_n}{p_n} - 1 \right|.$$

By (13), the following can be easily verified, which is

$$st - \lim_n \|\mathcal{B}_n^{p_n,q_n}(1;x) - 1\| = 0.$$

For $\nu = 1$ and using (10), we get

$$\left\| \mathcal{B}_n^{p_n,q_n}\left(\frac{t}{1+t};x\right) - \frac{x}{1+x} \right\| \leq \left| q_n \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} - 1 \right| = 1 - q_n \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}.$$

For a given $\epsilon > 0$, let us define the following sets:

$$U = \left\{ n : \left\| \mathcal{B}_n^{p_n,q_n}\left(\frac{t}{1+t};x\right) - \frac{x}{1+x} \right\| \geq \epsilon \right\},$$

and

$$U' = \left\{ n : 1 - q_n \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} \geq \epsilon \right\}.$$

It is easily perceived that $U \subset U'$, so we can write

$$\delta \left\{ k \leq n : \left\| \mathcal{B}_n^{p_n,q_n}\left(\frac{t}{1+t};x\right) - \frac{x}{1+x} \right\| \geq \epsilon \right\}$$
$$\leq \delta \left\{ k \leq n : 1 - q_n \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} \geq \epsilon \right\}.$$

On using (13), it is clear that

$$st - \lim_n \left(1 - q_n \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} \right) = 0.$$

Thus,

$$\delta \left\{ k \leq n : 1 - q_n \frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} \geq \epsilon \right\} = 0;$$

then,

$$st - \lim_n \left\| \mathcal{B}_n^{p_n,q_n}\left(\frac{t}{1+t};x\right) - \frac{x}{1+x} \right\| = 0.$$

Lastly, for $\nu = 2$ and using (11), we obtain

$$\begin{aligned}
&\left\| \mathcal{B}_n^{p_n,q_n}\left(\frac{t}{1+t};x\right)^2 - \left(\frac{x}{1+x}\right)^2 \right\| \\
&= \left\| \frac{p_n q_n^3 [n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{x^2}{(1+x)(p_n+q_nx)} + \frac{p_n^n q_n [n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{x}{1+x} - 1 \right\| \\
&\leq \left| \frac{p_n q_n^2 [n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} - 1 \right| + \left| \frac{p_n^n q_n [n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \right|. \qquad (14)
\end{aligned}$$

Using $[n+1]_{p_n,q_n} = p_n[n]_{p_n,q_n} + q_n^n$, the following can be easily justified that

$$\frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} = \frac{1}{p_n^3}\left(1 - \frac{2q_n^n + p_n q_n^{n-1}}{[n+1]_{p_n,q_n}} + \frac{q_n^{2n} + p_n q_n^{2n-1}}{[n+1]^2_{p_n,q_n}}\right).$$

Substituting it in (14), we can have

$$\begin{aligned}
&\left\| \mathcal{B}_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right) - \left(\frac{x}{1+x}\right)^2 \right\| \\
&\leq \left| \frac{q_n^2}{p_n^2} - 1 \right| + \left| \frac{q_n^2}{p_n^2}\left(\frac{2q_n^n + p_n q_n^{n-1}}{[n+1]_{p_n,q_n}} - \frac{q_n^{2n} + p_n q_n^{2n-1}}{[n+1]^2_{p_n,q_n}}\right) \right| + \left| \frac{p_n^n q_n [n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \right| \\
&\leq \frac{q_n^2}{p_n^2} - 1 + \frac{q_n^2}{p_n^2}\left(\frac{2q_n^n + p_n q_n^{n-1}}{[n+1]_{p_n,q_n}} - \frac{q_n^{2n} + p_n q_n^{2n-1}}{[n+1]^2_{p_n,q_n}}\right) + \frac{p_n^n}{[n+1]_{p_n,q_n}} - \frac{p_n^{2n}}{[n+1]^2_{p_n,q_n}}.
\end{aligned}$$

If we choose $\alpha_n = \frac{q_n^2}{p_n^2} - 1$, $\beta_n = \frac{q_n^2}{p_n^2}\left(\frac{2q_n^n + p_n q_n^{n-1}}{[n+1]_{p_n,q_n}} - \frac{q_n^{2n} + p_n q_n^{2n-1}}{[n+1]^2_{p_n,q_n}}\right)$, and $\gamma_n = \frac{p_n^n}{[n+1]_{p_n,q_n}} - \frac{p_n^{2n}}{[n+1]^2_{p_n,q_n}}$, then by (13), we have

$$st - \lim_n \alpha_n = st - \lim_n \beta_n = st - \lim_n \gamma_n = 0. \qquad (15)$$

For any given $\epsilon > 0$, now we define four sets as follows:

$$U = \left\{ n : \left\| \mathcal{B}_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right) - \left(\frac{x}{1+x}\right)^2 \right\| \geq \epsilon \right\},$$

$$U_1 = \left\{ n : \alpha_n \geq \frac{\epsilon}{3} \right\},\ U_2 = \left\{ n : \beta_n \geq \frac{\epsilon}{3} \right\},\ U_3 = \left\{ n : \gamma_n \geq \frac{\epsilon}{3} \right\}.$$

It is obvious that $U \subset U_1 \cup U_2 \cup U_3$. Then, we obtain

$$\begin{aligned}
&\delta\left\{ k \leq n : \left\| \mathcal{B}_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right) - \left(\frac{x}{1+x}\right)^2 \right\| \geq \epsilon \right\} \\
&\leq \delta\left\{ k \leq n : \alpha_n \geq \frac{\epsilon}{3} \right\} + \delta\left\{ k \leq n : \beta_n \geq \frac{\epsilon}{3} \right\} + \delta\left\{ k \leq n : \gamma_n \geq \frac{\epsilon}{3} \right\}.
\end{aligned}$$

It is clear that the right-hand side of the above inequality is zero by (15); then,

$$st - \lim_n \left\| \mathcal{B}_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right) - \left(\frac{x}{1+x}\right)^2 \right\| = 0.$$

Hence, the proof is completed. □

4. Rates of Statistical Convergence

This section is devoted to find rates of statistical convergence of operators (7).

The modulus of continuity for the space of functions $f \in H_\omega$ [1] is defined by

$$\tilde{\omega}(f;\delta) = \sup_{x,t\geq 0,\ \left|\frac{t}{1+t}-\frac{x}{1+x}\right|\leq\delta} |f(t)-f(x)|,$$

where $\tilde{\omega}(f;\delta)$ satisfies the following conditions: $\forall f \in H_\omega(R_+)$

1. $\lim\limits_{\delta\to 0}\tilde{\omega}(f;\delta)=0$,
2. $|f(t)-f(x)| \leq \tilde{\omega}(f;\delta)\left(\frac{\frac{t}{1+t}-\frac{x}{1+x}}{\delta}+1\right)$.

Theorem 5. *Let $p=(p_n)$ and $q=(q_n)$ be the sequences satisfying (13) and $0<q_n<p_n\leq 1$, we have*

$$|\mathcal{B}_n^{p_n,q_n}(f;x)-f(x)| \leq \tilde{\omega}(f;\sqrt{\delta_n(x)})\left(\frac{q_n}{p_n}+1\right),$$

where

$$\begin{aligned}
&\delta_n(x)\\
&=\left(\frac{x}{1+x}\right)^2\left(\frac{q_n^4(1+x)}{p_n+q_nx}\frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}-\frac{2q_n^2[n]_{p_n,q_n}}{p_n[n+1]^2_{p_n,q_n}}+\frac{q_n^2}{p_n^2}\right)\\
&\quad+\frac{p_n^{n-1}q_n^2[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\frac{x}{1+x}.
\end{aligned} \tag{16}$$

Proof.

$$\begin{aligned}
&|\mathcal{B}_n^{p_n,q_n}(f;x)-f(x)|\\
&\quad\leq \mathcal{B}_n^{p_n,q_n}(|f(t)-f(x)|;x)\\
&\quad\leq \tilde{\omega}(f;\delta)\left\{\mathcal{B}_n^{p_n,q_n}(1;x)+\frac{1}{\delta}\mathcal{B}_n^{p_n,q_n}\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|;x\right)\right\}.
\end{aligned}$$

By using the Cauchy–Schwarz inequality (see (6)) and using (9)–(11), we have

$$\begin{aligned}
&|\mathcal{B}_n^{p_n,q_n}(f;x)-f(x)|\\
&\leq \tilde{\omega}(f;\delta_n)\left(\frac{q_n}{p_n}+\frac{1}{\delta_n}\left\{\left(\mathcal{B}_n^{p_n,q_n}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2;x\right)\right\}^{\frac{1}{2}}\left(\mathcal{B}_n^{p_n,q_n}(1^2;x)\right)^{\frac{1}{2}}\right)\\
&\leq \tilde{\omega}(f;\delta_n)\left(\frac{q_n}{p_n}+\frac{1}{\delta_n}\left\{\left(\frac{x}{1+x}\right)^2\left(\frac{q_n^4(1+x)}{p_n+q_nx}\frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\right.\right.\right.\\
&\qquad\left.\left.\left.-\frac{2q_n^2[n]_{p_n,q_n}}{p_n[n+1]^2_{p_n,q_n}}+\frac{q_n^2}{p_n^2}\right)+\frac{p_n^{n-1}q_n^2[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\frac{x}{1+x}\right\}^{\frac{1}{2}}\right).
\end{aligned}$$

Thus, it is obvious that, by choosing δ_n as in (16), the theorem is proved. □

Notice that, by conditions in (13), $st-\lim\limits_{n}=0$. Then, we have

$$st-\lim_n \tilde{\omega}(f;\delta_n)=0.$$

This provides us the pointwise rate of statistical convergence of the sequence of operators $B_n^{p_n,q_n}(f;x)$ to $f(x)$.

Now, we will contribute an estimate related to the rate of approximation by means of Lipschitz type maximal functions.

Lenze [27] introduced a Lipschitz type maximal function as

$$\tilde{f}_\alpha(x) = \sup_{t>0,\, t\neq x} \frac{f(t)-f(x)}{|x-t|^\alpha}.$$

The Lipschitz type maximal function space on $E \subset \mathbb{R}_+$ is defined in [1] as follows:

$$\widetilde{W}_{\alpha,E} = \left\{ f : \sup(1+x)^\alpha \tilde{f}_\alpha(x) \leq M \frac{1}{(1+y)^\alpha}; x \geq 0 \text{ and } y \in E \right\},$$

where function f is bounded and continuous on $\mathbb{R}_+$, $0 < \alpha \leq 1$ and M is a positive constant.

Theorem 6. *If $B_n^{p_n,q_n}(f;x)$ is defined by (7), then $\forall f \in \widetilde{W}_{\alpha,E}$, we have*

$$|\mathcal{B}_n^{p_n,q_n}(f;x) - f(x)| \leq M \left(\rho_n(x)^{\frac{\alpha}{2}} \left(\frac{q_n}{p_n}\right)^{\frac{2-\alpha}{2}} + \frac{2q_n}{p_n} d(x,E) \right),$$

where

$$\rho_n(x) = \left(\frac{x}{1+x}\right)^2 \left(\frac{p_n q_n^3 (1+x)}{p_n + q_n x} \frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} - \frac{2q_n [n]_{p_n,q_n}}{p_n [n+1]^2_{p_n,q_n}} + \frac{q_n}{p_n} \right) + \frac{p_n^n q_n [n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{x}{1+x}.$$

Proof. A similar technique used in Theorem 7 in [26] will be taken to provide the proof. Letting $x \geq 0$, $(x, x_0) \in \mathbb{R}_+ \times E$, it is understood that

$$|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|.$$

Since $\mathcal{B}_n^{p_n,q_n}(f;x)$ is a linear and positive operator, $f \in \widetilde{W}_{\alpha,E}$, using the previous inequality, we have

$$\begin{aligned} &|\mathcal{B}_n^{p_n,q_n}(f;x) - f(x)| \\ &\quad \leq \mathcal{B}_n^{p_n,q_n}(|f - f(x_0)|;x) + |f(x_0) - f(x)| \mathcal{B}_n^{p_n,q_n}(1;x) \\ &\quad \leq M \left(\mathcal{B}_n^{p_n,q_n}\left(\left| \frac{t}{1+t} - \frac{x_0}{1+x_0} \right|^\alpha ; x \right) + \frac{|x-x_0|^\alpha}{(1+x)^\alpha (1+x_0)^\alpha} \mathcal{B}_n^{p_n,q_n}(1;x) \right). \end{aligned} \tag{17}$$

Consequently, we obtain

$$\begin{aligned} &\mathcal{B}_n^{p_n,q_n}\left(\left| \frac{t}{1+t} - \frac{x_0}{1+x_0} \right|^\alpha ; x \right) \\ &\quad \leq \mathcal{B}_n^{p_n,q_n}\left(\left| \frac{t}{1+t} - \frac{x}{1+x} \right|^\alpha ; x \right) + \frac{|x-x_0|^\alpha}{(1+x)^\alpha (1+x_0)^\alpha} \mathcal{B}_n^{p_n,q_n}(1;x). \end{aligned}$$

Using the Hölder's inequality (see (5)) with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ and using (9)–(11), we have

$$\begin{aligned}
&\mathcal{B}_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha};x\right) \\
&\quad \leq \mathcal{B}_n^{p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right)^{\frac{\alpha}{2}} (\mathcal{B}_n^{p_n,q_n}(1^2;x))^{\frac{2-\alpha}{2}} \\
&\quad + \frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}} \mathcal{B}_n^{p_n,q_n}(1;x) \\
&\quad = \rho_n(x)^{\frac{\alpha}{2}} \left(\frac{q_n}{p_n}\right)^{\frac{2-\alpha}{2}} + \frac{q_n}{p_n}\frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}.
\end{aligned}$$

If the above result is substituted in (17), we will get our desired result. Hence, the theorem is proved. □

Corollary 1. *If $B_n^{p_n,q_n}(f;x)$ is defined by (7) and take $E = \mathbb{R}_+$ implies $d(x,E) = 0$, then a special case of Theorem 6 can be obtained as the following result: $\forall f \in \widetilde{W}_{\alpha,R_+}$*

$$|\mathcal{B}_n^{p_n,q_n}(f;x) - f(x)| \leq M\rho_n(x)^{\frac{\alpha}{2}} \left(\frac{q_n}{p_n}\right)^{\frac{2-\alpha}{2}},$$

where $\rho_n(x)$ is the same as in Theorem 6.

5. Construction of the Bivariate Operators

In this section, we define a bivariate version of operators (7) and study their approximation properties.

For $\mathbb{R}_+^2 = [0,\infty) \times [0,\infty)$, $f : \mathbb{R}_+^2 \to \mathbb{R}$ and $0 < q_{n_1}, q_{n_2} < p_{n_1}, p_{n_2} \leq 1$, let us define the bivariate case of the operators (7) as follows:

$$\begin{aligned}
&\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x) \\
&= \frac{q_{n_1}/p_{n_1}}{\zeta_{n_1}^{p_{n_1},q_{n_1}}(x)} \frac{q_{n_2}/p_{n_2}}{\zeta_{n_2}^{p_{n_2},q_{n_2}}(y)} \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} f\left(\frac{p_{n_1}^{n_1-j_1+1}[j_1]_{p_{n_1},q_{n_1}}}{[n_1-j_1+1]_{p_{n_1},q_{n_1}} q_{n_1}^{j_1}}, \frac{p_{n_2}^{n_2-j_2+1}[j_2]_{p_{n_2},q_{n_2}}}{[n_2-j_2+1]_{p_{n_2},q_{n_2}} q_{n_2}^{j_2}}\right) \\
&\times p_{n_1}^{\frac{(n_1-j_1)(n_1-j_1-1)}{2}} q_{n_1}^{\frac{j_1(j_1-1)}{2}} p_{n_2}^{\frac{(n_2-j_2)(n_2-j_2-1)}{2}} q_{n_2}^{\frac{j_2(j_2-1)}{2}} \begin{bmatrix} n_1 \\ j_1 \end{bmatrix}_{p_{n_1},q_{n_1}} \begin{bmatrix} n_2 \\ j_2 \end{bmatrix}_{p_{n_2},q_{n_2}} x^{j_1}y^{j_2}, \qquad (18)
\end{aligned}$$

where $\zeta_{n_1}^{p_{n_1},q_{n_1}}(x) = \prod_{s=0}^{n_1-1}(p_{n_1}^s + q_{n_1}^s x)$ and $\zeta_{n_2}^{p_{n_2},q_{n_2}}(y) = \prod_{s=0}^{n_2-1}(p_{n_2}^s + q_{n_2}^s y)$.

For $K = l^2 = [0,\infty) \times [0,\infty)$, the modulus of continuity for bivariate case is defined by

$$|f(s,t) - f(x,y)| \leq \omega_2\left(f : \left|\frac{s}{1+s} - \frac{x}{1+x}\right|, \left|\frac{t}{1+t} - \frac{y}{1+y}\right|\right)$$

for each $f \in H_{\omega_2}$. Details of the modulus of continuity for the bivariate case can be found in [28].

Now, we will investigate Korovkin type approximation properties by using the following test functions:

$$e_0(u,v) = 1,\ e_1(u,v) = \frac{u}{1+u},\ e_2(u,v) = \frac{v}{1+v},\ e_3(u,v) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2.$$

Lemma 4.

1. $\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0;x,y) = \frac{q_{n_1}q_{n_2}}{p_{n_1}p_{n_2}}$,

2. $\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_1;x,y) = \frac{q_{n_1}q_{n_2}[n_1]_{p_{n_1},q_{n_1}}}{p_{n_2}[n_1+1]_{p_{n_1},q_{n_1}}}\frac{x}{1+x}$,
3. $\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_2;x,y) = \frac{q_{n_1}q_{n_2}[n_2]_{p_{n_2},q_{n_2}}}{p_{n_1}[n_2+1]_{p_{n_2},q_{n_2}}}\frac{y}{1+y}$,
4. $\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_3;x,y)$

$$= \frac{q_{n_1}^4 q_{n_2}}{p_{n_2}}\frac{[n_1]_{p_{n_1},q_{n_1}}[n_1-1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}}\frac{x^2}{(1+x)(p_{n_1}+q_{n_1}x)} + \frac{p_{n_1}^{n_1-1}q_{n_1}^2 q_{n_2}}{p_{n_2}}\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}}\frac{x}{1+x}$$
$$+ \frac{q_{n_1}q_{n_2}^4}{p_{n_1}}\frac{[n_2]_{p_{n_2},q_{n_2}}[n_2-1]_{p_{n_2},q_{n_2}}}{[n_2+1]^2_{p_{n_2},q_{n_2}}}\frac{y^2}{(1+y)(p_{n_2}+q_{n_2}y)} + \frac{p_{n_2}^{n_2-1}q_{n_1}q_{n_2}^2}{p_{n_1}}\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]^2_{p_{n_2},q_{n_2}}}\frac{y}{1+y}.$$

Let $(p_{n_1}), (p_{n_2}), (q_{n_1})$ and (q_{n_2}) be the sequences that converge statistically to 1 but not convergent in ordinary sense, so it can be written as for $0 < q_{n_1}, q_{n_2} < p_{n_1}, p_{n_2} \leq 1$,

$$st - \lim_{n_1} p_{n_1} = st - \lim_{n_2} p_{n_2} = st - \lim_{n_1} q_{n_1} = st - \lim_{n_2} q_{n_2} = 1. \tag{19}$$

Now, with condition (19), let us show the statistical convergence of the sequence of bivariate operators (18).

Theorem 7. *Let (p_{n_1}), (p_{n_2}), (q_{n_1}) and (q_{n_2}) be the sequences satisfying the condition (19) and let $\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x)$ be the sequence of bivariate positive linear operators acting from $H_{\omega_2}(\mathbb{R}^2_+)$ into $C_B(\mathbb{R}_+)$. Then, for any $f \in H_{\omega_2}$,*

$$st - \lim_{n_1,n_2} \|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f) - f\| = 0.$$

Proof. Using Lemma 4, the proof can be achieved similarly the proof of Theorem 4. □

6. Rates of Convergence of the Bivariate Operators

For $f \in H_{\omega_2}(\mathbb{R}^2_+)$, modulus of continuity for bivariate case is defined as follows [28]:

$$\tilde{\omega}(f;\delta_1,\delta_2) = \sup_{x,s\geq 0}\left\{|f(s,t)-f(x,y)|; \left|\frac{s}{1+s}-\frac{x}{1+x}\right| \leq \delta_1,\right.$$
$$\left.\left|\frac{t}{1+t}-\frac{y}{1+y}\right| \leq \delta_2, (s,t)\in R^2_+, (x,y)\in R^2_+\right\}.$$

Here, $\tilde{\omega}(f;\delta_1,\delta_2)$ satisfies the conditions:

$$\tilde{\omega}(f;\delta_1,\delta_2) \to 0 \text{ if } \delta_1 \ \ \delta_2 \text{ tend to } 0, \text{ and}$$

$$|f(s,t)-f(x,y)| \leq \tilde{\omega}(f;\delta_1,\delta_2)\left(1+\frac{\left|\frac{s}{1+s}-\frac{x}{1+x}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{t}{1+t}-\frac{y}{1+y}\right|}{\delta_2}\right). \tag{20}$$

Now, we give the rate of the statistical convergence of the bivariate operators (18) by means of modulus of continuity in H_{ω_2}:

Theorem 8. *Let (p_{n_1}), (p_{n_2}), (q_{n_1}) and (q_{n_2}) be the sequences satisfying the condition (19). Then, we have*

$$|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x,y) - f(x,y)| \leq 4\omega\left(f;\sqrt{\delta_{n_1}(x)}\sqrt{\delta_{n_2}(y)}\right), \tag{21}$$

where

$$\delta_{n_1}(x) = \mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left(\frac{s}{1+s}-\frac{x}{1+x}\right)^2;x,y\right), \tag{22}$$

$$\delta_{n_2}(y) = \mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left(\frac{t}{1+t} - \frac{y}{1+y}\right)^2; x, y\right). \tag{23}$$

Proof. By using (20), we have

$$\begin{aligned}
&|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f; x, y) - f(x, y)| \le \tilde{\omega}(f; \delta_1, \delta_2) \\
&\times \left\{\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0; x, y) + \frac{1}{\delta_{n_1}}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s} - \frac{x}{1+x}\right|; x, y\right)\right\} \\
&\times \left\{\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0; x, y) + \frac{1}{\delta_{n_2}}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{t}{1+t} - \frac{y}{1+y}\right|; x, y\right)\right\}.
\end{aligned} \tag{24}$$

Using Cauchy–Schwarz inequality (see (6)), we have

$$\begin{aligned}
&\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s} - \frac{x}{1+x}\right|; x, y\right) \\
&\le \left\{\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left(\frac{s}{1+s} - \frac{x}{1+x}\right)^2; x, y\right)\right\}^{\frac{1}{2}}\{\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0^2; x, y)\}^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{t}{1+t} - \frac{y}{1+y}\right|; x, y\right) \\
&\le \left\{\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left(\frac{t}{1+t} - \frac{y}{1+y}\right)^2; x, y\right)\right\}^{\frac{1}{2}}\{\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0^2; x, y)\}^{\frac{1}{2}}.
\end{aligned}$$

Putting above inequalities in (24) and choosing $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ as in (22) and (23), respectively, we get our desired result (21). The theorem is completed. □

In the end, we will present the rates of statistical convergence of the bivariate operators (18) by means of Lipschitz type maximal functions.

Let us give the Lipschitz type maximal function space for the bivariate case on $E \times E \subset \mathbb{R}_+ \times \mathbb{R}_+$ as

$$\begin{aligned}
\widetilde{W}_{\alpha_1,\alpha_2,E^2} = \Bigg\{ & f : \sup(1+s)^{\alpha_1}(1+t)^{\alpha_2}\tilde{f}_{\alpha_1,\alpha_2}(x, y) \le M\frac{1}{(1+x)^{\alpha_1}}\frac{1}{(1+y)^{\alpha_2}}; \\
& x, y \ge 0, (s, t) \in E^2\Bigg\}.
\end{aligned} \tag{25}$$

Here, f is a bounded and continuous function in $\mathbb{R}_+$, M is a positive constant and $0 \le \alpha_1, \alpha_2 \le 1$, and then let us define $\tilde{f}_{\alpha_1,\alpha_2}$ as follows:

$$\tilde{f}_{\alpha_1,\alpha_2}(x, y) = \sup_{s,t>0}\frac{|f(s, t) - f(x, y)|}{|s - x|^{\alpha_1}|t - y|^{\alpha_2}}.$$

Theorem 9. *Let (p_{n_1}), (p_{n_2}), (q_{n_1}) and (q_{n_2}) be the sequences satisfying the condition (19). Then, we have*

$$\begin{aligned} &|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x,y) - f(x,y)| \le M\left(\frac{q_{n_1}q_{n_2}}{p_{n_1}p_{n_2}}\right) \\ &\times \left\{ \delta_{n_1}(x)^{\frac{\alpha_1}{2}}\delta_{n_2}(y)^{\frac{\alpha_2}{2}}\left(\frac{q_{n_1}q_{n_2}}{p_{n_1}p_{n_2}}\right)^{1-\frac{\alpha_1+\alpha_2}{2}} + \delta_{n_1}(x)^{\frac{\alpha_1}{2}}d(y,E)^{\alpha_2}\left(\frac{q_{n_1}q_{n_2}}{p_{n_1}p_{n_2}}\right)^{-\frac{\alpha_1}{2}} \right. \\ &\left. + \delta_{n_2}(y)^{\frac{\alpha_2}{2}}d(x,E)^{\alpha_1}\left(\frac{q_{n_1}q_{n_2}}{p_{n_1}p_{n_2}}\right)^{-\frac{\alpha_2}{2}} + 2d(x,E)^{\alpha_1}d(y,E)^{\alpha_2} \right\}, \end{aligned}$$

where $0 < \alpha_1, \alpha_2 \le 1$, $d(x,E) = \inf\{|x-y| : y \in E\}$, $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are defined as in (22) and (23), respectively.

Proof. Let $x, y \ge 0$ and $(x_0, y_0) \in E^2$. Then, we can write

$$|f(s,t) - f(x,y)| \le |f(s,t) - f(x_0,y_0)| + |f(x_0,y_0) - f(x,y)|.$$

Applying the positive linear operators $\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x)$ on both the sides of the above inequality and using (25), we obtain

$$\begin{aligned} &|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x,y) - f(x,y)| \\ &\le \mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(|f(s,t) - f(x_0,y_0)|;x,y) \\ &\quad + |f(x_0,y_0) - f(x,y)|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0;x,y) \\ &\le M\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{t}{1+t} - \frac{y_0}{1+y_0}\right|^{\alpha_2};x,y\right) \\ &\quad + M\left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{y}{1+y} - \frac{y_0}{1+y_0}\right|^{\alpha_2}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0;x,y). \end{aligned} \tag{26}$$

It is known that $(u+v)^\alpha \le u^\alpha + v^\alpha$ and $0 \le \alpha \le 1$, so it can be written as

$$\begin{aligned} \left|\frac{s}{1+s} - \frac{x_0}{1+x_0}\right|^{\alpha_1} &\le \left|\frac{s}{1+s} - \frac{x}{1+x}\right|^{\alpha_1} + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha_1} \\ \left|\frac{t}{1+t} - \frac{y_0}{1+y_0}\right|^{\alpha_2} &\le \left|\frac{t}{1+t} - \frac{y}{1+y}\right|^{\alpha_2} + \left|\frac{y}{1+y} - \frac{y_0}{1+y_0}\right|^{\alpha_2}. \end{aligned}$$

By using the above inequalities in (26), we have

$$\begin{aligned} &|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x,y) - f(x,y)| \\ &\le \mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s} - \frac{x}{1+x}\right|^{\alpha_1}\left|\frac{t}{1+t} - \frac{y}{1+y}\right|^{\alpha_2};x,y\right) \\ &\quad + \left|\frac{y}{1+y} - \frac{y_0}{1+y_0}\right|^{\alpha_2}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s} - \frac{x}{1+x}\right|^{\alpha_1};x,y\right) \\ &\quad + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{t}{1+t} - \frac{y}{1+y}\right|^{\alpha_2};x,y\right) \\ &\quad + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha_1}\left|\frac{y}{1+y} - \frac{y_0}{1+y_0}\right|^{\alpha_2}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0;x,y). \end{aligned}$$

Using Hölder's inequality with $p_1 = \frac{2}{\alpha_1}$, $p_2 = \frac{2}{\alpha_2}$, $q_1 = \frac{2}{2-\alpha_1}$, $q_2 = \frac{2}{2-\alpha_2}$ (see (5)), we obtain

$$\begin{aligned}
&\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s}-\frac{x}{1+x}\right|^{\alpha_1}\left|\frac{t}{1+t}-\frac{y}{1+y}\right|^{\alpha_2};x,y\right)\\
&=\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{s}{1+s}-\frac{x}{1+x}\right|^{\alpha_1};x,y\right)\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\left|\frac{t}{1+t}-\frac{y}{1+y}\right|^{\alpha_2};x,y\right)\\
&\leq\left(\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\frac{s}{1+s}-\frac{x}{1+x}\right)^2;x,y\right)^{\frac{\alpha_1}{2}}\left(\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0^2;x,y)\right)^{\frac{2-\alpha_1}{2}}\\
&\quad\times\left(\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2;x,y\right)^{\frac{\alpha_2}{2}}\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(e_0^2;x,y)^{\frac{2-\alpha}{2}}.
\end{aligned}$$

If we use the above inequality in (26), we get our desired result. Thus, the proof is completed. □

Corollary 2. *If we take $E=[0,\infty)$, then because of $d(x,E)=0$ and $d(y,E)=0$, we have*

$$|\mathcal{B}_{n_1,n_2}^{p_{n_1}p_{n_2},q_{n_1}q_{n_2}}(f;x,y)-f(x,y)|\leq M\left(\frac{q_{n_1}q_{n_2}}{p_{n_1}p_{n_2}}\right)^{2-\frac{\alpha_1+\alpha_2}{2}}\delta_{n_1}(x)^{\frac{\alpha_1}{2}}\delta_{n_2}(y)^{\frac{\alpha_2}{2}},$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are same as defined as in (22) and (23), respectively.

7. Conclusions

In this paper, we have constructed (p,q)-BBH operators and calculated some auxiliary results for these newly defined operators. We also established Korovkin type results and rate of convergence in a statistical sense. Furthermore, we constructed the bivariate (p,q)-BBH operators and computed rate of statistical convergence for the bivariate (p,q)-BBH operators. Our results are more general than the results for BBH and q-BBH operators.

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References

1. Aral, A.; Doğru, O. Bleimann Butzer and Hahn operators based on q-integers. *J. Inequal. Appl.* **2007**, *2007*, 79410. [CrossRef]
2. Bleimann, G.; Butzer, P.L.; Hahn, L. A Bernstein-type operator approximating continuous functions on the semi-axis. *Indag. Math.* **1980**, *42*, 255–262. [CrossRef]
3. Gadjiev, A.D.; Çakar, Ö. On uniform approximation by Bleimann, Butzer and Hahn operators on all positive semi-axis. *Trans. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci.* **1999**, *19*, 21–26.
4. Mursaleen, M.; Ansari, K.J.; Khan, A. On (p,q)-analogue of Bernstein operators. *Appl. Math. Comput.* **2015**, *266*, 874–882; Erratum in **2015**, *266*, 874–882. [CrossRef]
5. Marin, M. An evolutionary equation in thermoelasticity of dipolar bodies. *J. Math. Phys.* **1999**, *40*, 1391–1399. [CrossRef]
6. Marin, M.; Stan, G. Weak solutions in Elasticity of dipolar bodies with stretch. *Carpath. J. Math.* **2013**, *29*, 33–40.
7. Khalid Khan, D.K. Lobiyal, Bézier curves based on Lupaş (p,q)-analogue of Bernstein functions in CAGD. *J. Comput. Appl. Math.* **2017**, *317*, 458–477. [CrossRef]

8. Srivastava, H.M.; Gupta, V. Rate of convergence for the Bézier variant of the Bleimann–Butzer–Hahn operators. *Appl. Math. Lett.* **2005**, *18*, 849–857. [CrossRef]
9. Ansari, K.J.; Karaisa, A. On the approximation by Chlodowsky type generalization of (p,q)-Bernstein operators. *Int. J. Nonlinear Anal. Appl.* **2017**, 8, 181–200.
10. Dattoli, G.; Lorenzutta, S.; Cesarano, C. Bernstein polynomials and operational methods. *J. Comp. Anal. Appl.* **2016**, *8*, 369–377.
11. Kadak, U.; Mishra, V.N.; Pandey, S. Chlodowsky type generalization of (p,q)-Szász operators involving Brenke type polynomials. *Rev. Real Acad. Cienc. Exact. Fís. Nat. Ser. A Mat.* **2018**, *112*, 1443–1462. [CrossRef]
12. Khan, A.; Sharma, V. Statistical approximation by (p,q)-analogue of Bernstein-Stancu operators. *Ajerbaijan J. Math.* **2018**, *8*, 100–121.
13. Mursaleen, M.; Ansari, K.J.; Khan, A. Some approximation results by (p,q)-analogue of Bernstein-Stancu operators. *Appl. Math. Comput.* **2015**, 264, 392-402; Corrigendum in **2015**, *269*, 744–746.
14. Mursaleen, M.; Alotaibi, A.; Ansari, K.J. On a Kantorovich variant of (p,q)-Szász-Mirakjan operators. *J. Funct. Spaces* **2016**, *2016*, 1035253. [CrossRef]
15. Acar, T. (p,q)-Generalization of Szász-Mirakyan operators. *Math. Meth. Appl. Sci.* **2016**, *39*, 2685–2695. [CrossRef]
16. Hounkonnou, M.N.; Désiré, J.; Kyemba, B. $\mathcal{R}(p,q)$-calculus: Dierentiation and integration. *SUT J. Math.* **2013**, *49*, 145–167.
17. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *Result. Math.* **2018**, *73*, 39. [CrossRef]
18. Sahai, V.; Yadav, S. Representations of two parameter quantum algebras and p,q-special functions. *J. Math. Anal. Appl.* **2007**, *335*, 268–279. [CrossRef]
19. Victor, K.; Pokman, C. *Quantum Calculus*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 2002.
20. Mursaleen, M.; Nasiruzzaman, M.; Khan, A.; Ansari, K.J. Some approximation results on Bleimann–Butzer–Hahn operators defined by (p,q)-integers. *Filomat* **2016**, *30*, 639–648. [CrossRef]
21. Mursaleen, M. Nasiruzzaman, Some approximation properties of bivariate Bleimann–Butzer–Hahn operators based on (p,q)-integers. *Boll. Unione Mat. Ital.* **2017**, *10*, 271–289. [CrossRef]
22. Fast, H. Sur la convergence statistique. *Colloq. Math.* **1951**, *2*, 241–244. [CrossRef]
23. Niven, I.; Zuckerman, H.S.; Montgomery, H. *An Introduction to the Theory of Numbers*, 5th ed.; Wiley: New York, NY, USA, 1991.
24. Mursaleen, M.; Nasiruzzaman, M.; Ansari, K.J.; Alotaibi, A. Generalized (p,q)-Bleimann–Butzer–Hahn operators and some approximation results. *J. Ineq. Appl.* **2017**, *2017*, 310. [CrossRef] [PubMed]
25. Ansari, K.J. Approximation Properties of Some Operators and Thier q-Analogues. Ph.D. Thesis, Aligarh Muslim University, Aligarh, India, 2015.
26. Ersan, S. Approximation properties of bivariate generalization of Bleimann, Butzer and Hahn operators based on the q-integers. In Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, 29–31 December 2007; pp. 122–127.
27. Lenze, B. Bernstein-Baskakov-Kantorovich operators and Lipschitz-type maximal functions. *Approx. Theory* **1990**, *58*, 469–496.
28. Anastassiou, G.A.; Gal, S.G. *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*; Birkhauser: Boston, MA, USA, 2000.