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# Logics for Finite UL and IUL-Algebras Are Substructural Fuzzy Logics

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**Abstract:** Semilinear substructural logics $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are logics for finite **UL** and **IUL**-algebras, respectively. In this paper, the standard completeness of $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ is proven by the method developed by Jenei, Montagna, Esteva, Gispert, Godo, and Wang. This shows that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are substructural fuzzy logics.

**Keywords:** substructural fuzzy logics; residuated lattices; semilinear substructural logics; standard completeness; fuzzy logic

**MSC:** 03B52; 06F99; 03B50; 03B47

## 1. Introduction

In [1], we constructed three semilinear substructural logics $\mathbf{UL}_\omega$, $\mathbf{IUL}_\omega$, and $\mathbf{HpsUL}^*_\omega$ by adding one simple axiom:

$$\text{(FIN)}\ \ (A \to e) \leftrightarrow (A \odot A \to e)$$

to Metcalfe and Montagna's uninorm logic **UL**, involutive uninorm logic **IUL** [2], and a suitable extension $\mathbf{HpsUL}^*$ [3] of Metcalfe, Olivetti, and Gabbay's pseudo-uninorm logic **HpsUL** [4], respectively. Especially, we show that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are complete with respect to finite **UL** and **IUL**-algebras, respectively. That is, they are logics for finite **UL** and **IUL**-algebras, respectively.

In this paper, we prove that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are standard complete by Wang's constructions in [5] and [6], which are some generalizations of the Jenei and Montagna-style approach for proving standard completeness for monoidal $t$-norm-based logic **MTL** [7] and the proof of the standard completeness for **IMTL** given by Esteva, Gispert, Godo, and Montagna in [8]. These constructions have been extended by Yang in [9–12].

Substructural logics are logics that lack some of the three basic structural rules of contraction, weakening, and exchange. For a survey, see [13]. Substructural fuzzy logics are substructural logics that are complete with respect to algebras whose lattice reduct is the real unit interval $[0,1]$, i.e., logics that are standard complete [2]. Our result in this paper thus shows that $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ are substructural fuzzy logics.

As pointed out in [6], our construction in Lemma 5 also presents a method to construct uninorms and involutive uninorms. Then, the standard completeness for $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$ gives a characterization of uninorms and involutive uninorms and their residua constructed by finite **UL** and **IUL**-algebras from our constructions. That is, the identity (Fin) holds in all these standard **UL** and **IUL**-algebras; see Definition 5. These new classes of uninorms and involutive uninorms may be used in the theory of evaluation as the aggregation operators or combining functions [14,15].

We have proven that $\mathbf{HpsUL}^*$ is standard complete in [16]. However, we are unable to prove whether $\mathbf{HpsUL}^*_\omega$ is standard complete or complete with respect to finite $\mathbf{HpsUL}^*$-algebras and left them as open problems. In addition, we have proven that **IUL** is standard complete in [17], which is a longstanding open problem in the circle of fuzzy logic. Unfortunately, such a great work has not been accepted by our community since 2015, although one referee thought that the central ideas in our proof are reasonable and could find no significant flaws in the reasoning. The referee also said that he would also not be confident that the proof is correct if the proof were his own and he had spent many months laboring over it.

## 2. $\mathbf{HpsUL}^*_\omega$, $\mathbf{UL}_\omega$, $\mathbf{IUL}_\omega$ and Algebras Involved

The Hilbert system **HpsUL** is the logic of bounded representable residuated lattices, which is based on a countable propositional language with formulas built inductively as usual from a set of propositional variables, binary connectives $\odot, \to, \rightsquigarrow, \wedge, \vee$, and constants $e, f, \bot, \top$, with definable connectives:

$$\neg\varphi := \varphi \to f,$$

$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi),$$

$$\lambda_\chi(\varphi) := (\chi \to \varphi \odot \chi) \wedge e,$$

$$\rho_\chi(\varphi) := (\chi \rightsquigarrow \chi \odot \varphi) \wedge e.$$

**Definition 1.** **HpsUL** *consists of the following axioms and rules [4]:*

($A_1$) $\vdash \varphi \to \varphi$
($A_2$) $\vdash (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$
($A_3$) $\vdash \varphi \to ((\varphi \rightsquigarrow \psi) \to \psi)$
($A_4$) $\vdash (\varphi \rightsquigarrow (\psi \to \chi)) \to (\psi \to (\varphi \rightsquigarrow \chi))$
($A_5$) $\vdash \psi \to (\varphi \to \varphi \odot \psi)$
($A_6$) $\vdash (\psi \to (\varphi \to \chi)) \to (\varphi \odot \psi \to \chi)$
($A_7$) $\vdash (\psi \rightsquigarrow \psi \odot (\psi \to \varphi)) \to (\psi \rightsquigarrow \varphi)$
($A_8$) $\vdash (\varphi \wedge t) \odot (\psi \wedge t) \to \varphi \wedge \psi$
($A_9$) $\vdash \varphi \wedge \psi \to \psi$
($A_{10}$) $\vdash \varphi \wedge \psi \to \varphi$
($A_{11}$) $\vdash (\chi \to \varphi) \wedge (\chi \to \psi) \to (\chi \to \varphi \wedge \psi)$
($A_{12}$) $\vdash \varphi \to \varphi \vee \psi$
($A_{13}$) $\vdash \psi \to \varphi \vee \psi$
($A_{14}$) $\vdash (\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$
($A_{15}$) $\vdash e$
($A_{16}$) $\vdash \varphi \to (e \to \varphi)$
($A_{17}$) $\vdash \varphi \to \top$
($A_{18}$) $\vdash \bot \to \varphi$
(PRL) $\vdash (\lambda_\chi(\varphi \vee \psi \to \varphi)) \vee (\rho_\chi(\varphi \vee \psi \to \psi))$
(MP) $\varphi, \varphi \to \psi \vdash \psi$
($ADJ_U$) $\varphi \vdash \varphi \wedge e$
($PN_\to$) $\varphi \vdash \psi \to \varphi \odot \psi$
($PN_\rightsquigarrow$) $\varphi \vdash \psi \rightsquigarrow \psi \odot \varphi$

**Definition 2** ([2,3])**.** *A logic is a schematic extension (extension for short) of* **HpsUL** *if it results from* **HpsUL** *by adding axioms in the same language. In particular,*

- $\mathbf{HpsUL}^*$ *is* **HpsUL** *plus (WCM)* $\vdash (\varphi \rightsquigarrow e) \to (\varphi \to e)$;
- **UL** *is* **HpsUL** *plus* $\vdash \varphi \odot \psi \to \psi \odot \varphi$;
- **IUL** *is* **UL** *plus* $\vdash \neg\neg\varphi \to \varphi$.

**Definition 3.** *New extensions of* **HpsUL** *are defined as follows.*

- $\mathbf{HpsUL}^*_\omega$ *is* $\mathbf{HpsUL}^*$ *plus* (FIN) $\vdash (\varphi \to e) \leftrightarrow (\varphi \odot \varphi \to e)$;
- $\mathbf{UL}_\omega$ *and* $\mathbf{IUL}_\omega$ *are* **UL** *and* **IUL** *plus* (FIN), *respectively.*

Let $\mathbf{L} \in \{\mathbf{HpsUL}^*, \mathbf{UL}, \mathbf{IUL}, \mathbf{HpsUL}^*_\omega, \mathbf{UL}_\omega, \mathbf{IUL}_\omega\}$ in the remainder of this section. A proof in $\mathbf{L}$ of a formula $\varphi$ from a set $\Gamma$ of formulas is defined as usual. We write $\Gamma \vdash_{\mathbf{L}} \varphi$ if such a proof exists.

**Definition 4** ([4])**.** *An* **HpsUL***-algebra is a bounded residuated lattice* $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top \rangle$ *with universe* $A$*, binary operations* $\wedge, \vee, \cdot, \to, \rightsquigarrow$*, and constants* $e, f, \bot, \top$ *such that:*
*(i)* $\langle A, \wedge, \vee, \bot, \top \rangle$ *is a bounded lattice with top element* $\top$ *and bottom element* $\bot$*;*
*(ii)* $\langle A, \cdot, e \rangle$ *is a monoid;*
*(iii)* $\forall x, y, z \in A, x \cdot y \leqslant z$ *iff* $x \leqslant y \rightsquigarrow z$ *iff* $y \leqslant x \to z$*;*
*(iv)* $\forall x, y, u, v \in A, (\lambda_u(x \vee y \to x)) \vee (\rho_v(x \vee y \to y)) = e$*, where, for any* $a, b \in A$*,* $\lambda_a(b) := (a \to b \cdot a) \wedge e, \rho_a(b) := (a \rightsquigarrow a \cdot b) \wedge e$*.*

We use the convention that $\cdot$ binds stronger than other binary operations, and we shall often omit $\cdot$; we will thus write $xy$ instead of $x \cdot y$, for example. Suitable classes of algebras of extensions of **HpsUL** are defined as follows.

**Definition 5** ([1,3,4])**.** *Let* $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top \rangle$ *be an* **HpsUL***-algebra. For* **L***, an extension of* **HpsUL***,* $\mathcal{A}$ *is an* **L***-algebra if all axioms of* **L** *are valid in* $\mathcal{A}$*. An* **L***-chain is an* **L***-algebra that is linearly ordered. In particular:*
- $\mathcal{A}$ *is an* $\mathbf{HpsUL}^*$*-algebra if the weak commutativity (Wcm) holds:* $xy \leq e$ *iff* $yx \leq e$ *for all* $x, y \in A$*;*
- $\mathcal{A}$ *is a* **UL***-algebra if* $xy = yx$ *for all* $x, y \in A$*;*
- $\mathcal{A}$ *is an* **IUL***-algebra if it is a* **UL***-algebra such that* $\neg\neg x = x$ *for all* $x \in A$*;*
- $\mathcal{A}$ *is an* $\mathbf{HpsUL}^*_\omega$*-algebra (*$\mathbf{UL}_\omega$ *or* $\mathbf{IUL}_\omega$*-algebra) if it is an* $\mathbf{HpsUL}^*$*-algebra (***UL** *or* **IUL***-algebra) such that the following identity* (Fin) *holds:* $x \to e = x^2 \to e$ for all $x \in A$*.*

**Definition 6** ([4])**.** *Let* $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top \rangle$ *be an* **L***-algebra. (i) An* $\mathcal{A}$*-valuation* $v$ *is a homomorphism from the term algebra determined by formulas in* **L** *to* $\mathcal{A}$*; (ii) A formula* $\varphi$ *is valid in* $\mathcal{A}$ *if* $v(\varphi) \geqslant e$ *holds for any* $\mathcal{A}$*-valuation* $v$*; (iii) The relation of semantic consequence* $\Gamma \vDash_{\mathcal{A}} \varphi$ *holds if each* $\mathcal{A}$*-evaluation that validates all formulae in a theory* $\Gamma$ *validates* $\varphi$ *as well.*

**Theorem 1** ([4,18])**.** $\Gamma \vdash_{\mathbf{L}} \varphi$ *iff* $\Gamma \vDash_{\mathcal{A}} \varphi$ *for every* **L***-chain* $\mathcal{A}$*, i.e.,* **L** *is a semilinear substructural logic.*

**Proposition 1.** *Let* $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top \rangle$ *be an* $\mathbf{HpsUL}^*$*-algebra. Then,* $\mathcal{A}$ *is an* $\mathbf{HpsUL}^*_\omega$*-algebra if and only if* $xy \leqslant e$ *iff* $xy^2 \leqslant e$ *for all* $x, y \in A$*.*

**Proof.** For the proof of the necessity part, see Lemma 2.4(i) of [1]. For the sufficiency part, assume that $xy \leqslant e$ iff $xy^2 \leqslant e$ for all $x, y \in A$. Suppose that $x \to e > x^2 \to e$. Then, $x^2(x \to e) > e$ by Definition 4 (iii), and hence, $(x \to e)x^2 > e$ by (Wcm). Therefore, $(x \to e)x > e$ by the assumption. Then, $x \to e > x \to e$, a contradiction, and thus, $x \to e \leq x^2 \to e$. $x^2 \to e \leq x \to e$ is proven by a similar way. Hence, $x \to e = x^2 \to e$ for all $x \in A$, i.e., $\mathcal{A}$ is an $\mathbf{HpsUL}^*_\omega$-algebra. □

**Lemma 1.** *Let* $\mathcal{A}$ *be an* $\mathbf{HpsUL}^*_\omega$*-chain and* $s, t, u \in A$*. Then:*

*(1)* $stu = s$ *implies* $st = s$ *and* $su = s$*;*
*(2)* $stu = u$ *implies* $su = u$ *and* $tu = u$*;*
*(3)* $st = e$ *implies* $s = t = e$*.*

**Proof.** Only (1) is proven as follows; for the others, see [1]. If $tu \leqslant e$, then $tut \leqslant e$ and $utu \leqslant e$ by Proposition 1 and (Wcm). Thus, $stut \leqslant s$ and $stutu \leqslant st$. Hence, $st \leqslant s$ and $s \leqslant st$. Therefore, $st = s$. The case of $tu > e$ is proven in the same way. □

Clearly, Lemma 1 holds for all $\mathbf{UL}_\omega$ and $\mathbf{IUL}_\omega$-chains.

## 3. Wang's Construction and Standard Completeness

In this section, let $\mathbf{L}_\omega \in \{\mathbf{UL}_\omega, \mathbf{IUL}_\omega\}$, $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \to, \rightsquigarrow, e, f, \bot, \top\rangle$ be a finite or countable $\mathbf{L}_\omega$-chain and $s, t, u$ be arbitrary elements of $A$.

**Definition 7** ([3,5]). *Let $\mathcal{A}$ be an $\mathbf{UL}_\omega$-chain. For each $s \in A$, $t$ is the immediate predecessor of $s$ in $A$ if: (i) $t \in A$, $t < s$; (ii) $\forall u \in A, u < s$ implies $u \leqslant t$. For each $s \in A$, let $s^-$ denote the immediate predecessor of $s$ in $A$ if it exists, otherwise take $s^- = s$.*

*Let $X = \{(s,1) : s \in A\} \cup \{(s,q) : s \in A, s > s^-, q \in Q \cap (0,1)\}$; we define:*
*$(s,q) \leqslant (t,r)$ iff either $s <_S t$, or $s = t$ and $q \leqslant r$ and,*

$$\begin{aligned}
I_1 &:= \{(s,t) : s,t \in A, st = s \neq t, s > s^- t\}\\
I_2 &:= \{(s,t) : s,t \in A, st = t \neq s, t > st^-\}\\
I_3 &:= \{(s,t) : s,t \in A, st = t = s, s > st^-\}\\
I_4 &:= \{(s,t) : s,t \in A, (st \neq t \text{ and } st \neq s) \text{ or}\\
&\qquad (st = s^- t = s) \text{ or } (st = st^- = t)\}.
\end{aligned}$$

*Now define, for $(s,q),(t,r) \in X$:*

$$(s,q) \circ (t,r) = \begin{cases} (s,q) & (s,t) \in I_1,\\ (t,r) & (s,t) \in I_2,\\ (s,q) \wedge_X (t,r) & (s,t) \in I_3,\\ (st,1) & (s,t) \in I_4, \end{cases}$$

*where by $\wedge_X$ and $\vee_X$ are meant $\min_X$ and $\max_X$ with respect to $\leqslant_X$, respectively. We will omit the index if it does not cause confusion.*

**Lemma 2.** *Let $\mathcal{A}$ be an $\mathbf{UL}_\omega$-chain. Then, $(s,q) \circ (t,r) \leqslant (e,1)$ iff $(s,q) \circ (t,r) \circ (t,r) \leqslant (e,1)$ for all $(s,q),(t,r)$ in $X$.*

**Proof.** Let $(s,q) \circ (t,r) \leqslant (e,1)$. Since $(s,q) \circ (t,r) = (st, \diamond)$ for some $\diamond \in \{q,r,1\}$ by Definition 7, then $st \leqslant e$. Thus, $stt \leqslant e$ by (Fin). Hence, $(s,q) \circ (t,r) \circ (t,r) \leqslant (e,1)$. The sufficiency part of the lemma is proven in the same way. □

**Definition 8** ([6,8]). *Let $\mathcal{A}$ be an $\mathbf{IUL}_\omega$-algebra. Let:*

$$I^* := \{(s,t) : s,t \in A, s^- < s, t^- < t, t = \neg s^-\},$$

$$I^{**} := \{(s,t) : s,t \in A, ss = s^- s = s = t\}.$$

*$\forall (s,q),(t,r) \in X$, define:*

$$(s,q) \vartriangle (t,r) = \begin{cases} (s,q) \circ (t^-,1) \vee (s^-,1) \circ (t,r) & \text{if } (s,t) \in I^*, q + r \leqslant 1,\\ (s, q \vee r) \circ (s^-,1) & \text{if } (s,t) \in I^{**},\\ (s,q) \circ (t,r) & \text{otherwise.} \end{cases}$$

**Lemma 3.** *Let $\mathcal{A}$ be an $\mathbf{IUL}_\omega$-chain and $s,t \in A$. (i) If $st^- \neq s$, $st^- \leqslant e$, $s^- t \leqslant e$, then $st^- t \leqslant e$; (ii) if $st^- = s^- t^-$ and $s^- t \leqslant e$, then $st^- t \leqslant e$; (iii) $(s,q) \vartriangle (t,r) \leqslant (s,q) \circ (t,r)$.*

**Proof.** (i) If $st \leqslant e$, then $stt \leqslant e$ by Proposition 1, and thus, $st^- t \leqslant stt \leqslant e$. If $t \leqslant e$, then $st^- t \leqslant t \leqslant e$ by $st^- \leqslant e$. Thus, let $st > e$ and $t > e$ in the following.

$t^- \geq e$ by $t > e$. $t^- \neq e$ by $st^- \neq s$. Then, $t^- > e$. Thus, $st^- \geq s$. Hence, $st^- > s$ by $st^- \neq s$. $st^- \neq e$ by Lemma 1(3) and $t^- > e$. Therefore, $st^- < e$ by $st^- \leqslant e$. Then, $st^- < e < t^-$. Thus, $st^- < t^-$. Hence, $s < e$.

Suppose that $st \leqslant t^-$. Then, $sst \leqslant st^- \leqslant e$. Thus, $st \leqslant e$ by Proposition 1, a contradiction, and, hence $st > t^-$. Therefore, $st \geqslant t$. $st \leqslant t$ by $s < e$. Then, $st = t$.

Suppose that $s^-t \geqslant s$, then $s^-tt \geqslant st > e$. Thus, $s^-t > e$ by Proposition 1, a contradiction, and hence, $s^-t < s$.

Therefore, $s^-t \leqslant s^-$. $s^-t \geqslant s^-$ by $t > e$. Then, $s^-t = s^-$. Then, $s^-st = s^-$ by $st = t$. Thus, $s^-s = s^-$ by Lemma 1(1).

Suppose that $ss = s$, then $st^- = sst^- \leqslant s$, a contradiction with $st^- > s$, and hence, $ss < s$ by $ss \leqslant s$. Then, $ss \leqslant s^-$.

Thus, $s^- = s^-s \leqslant ss \leqslant s^-$. Hence, $ss = s^-$. Then, $(ss)t = s^-t = s^-$ and $s(st) = st = t$. Thus, $s^- = t$ by $(ss)t = s(st)$, a contradiction with $s^- < e < t$. Thus, the case of $st > e$ and $t > e$ does not exist. This completes the proof of (i).

(ii) It follows from $s^-t \leqslant e$ that $s^-tt \leqslant e$ by Proposition 1. Then, $st^-t = s^-t^-t \leqslant s^-tt \leqslant e$ by $st^- = s^-t^-$, and thus, $st^-t \leqslant e$.

(iii) See Proposition 3.7 (2) of [6]. □

**Lemma 4.** *Let $\mathcal{A}$ be a finite $\mathbf{IUL}_\omega$-chain. Then, $(s,q) \vartriangle (t,r) \leqslant (e,1)$ if and only if $(s,q) \vartriangle (t,r) \vartriangle (t,r) \leqslant (e,1)$ for all $(s,q),(t,r)$ in $X$.*

**Proof.** Let $(s,q) \vartriangle (t,r) \leqslant (e,1)$. There are three cases to be considered.

Case 1. $(s,t) \in I^*$ and $q + r \leqslant 1$. Then, $(s,q) \vartriangle (t,r) = (s,q) \circ (t^-,1) \vee (s^-,1) \circ (t,r) \leqslant (e,1)$. Thus, $st^- \leqslant e$, $s^-t \leqslant e$. Then, $s^-tt \leqslant e$ by Proposition 1. If $(s,q) \vartriangle (t,r) = (s^-,1) \circ (t,r)$, then $(s,q) \vartriangle (t,r) \vartriangle (t,r) = ((s^-,1) \circ (t,r)) \vartriangle (t,r) \leqslant ((s^-,1) \circ (t,r)) \circ (t,r) \leqslant (s^-tt,1) \leqslant (e,1)$ by Lemma 3(iii). Let $(s,q) \vartriangle (t,r) = (s,q) \circ (t^-,1)$ in the following. If $(s,q) \circ (t^-,1) = (s,q)$, then $(s,q) \vartriangle (t,r) \vartriangle (t,r) = (s,q) \vartriangle (t,r) \leqslant (e,1)$. Otherwise, $st^- \neq s$ or $st^- = s^-t^-$. Then, $st^-t \leqslant e$ by Lemmas 3(i) and 3(ii). Thus, $(s,q) \vartriangle (t,r) \vartriangle (t,r) = ((s,q) \circ (t^-,1)) \vartriangle (t,r) \leqslant ((s,q) \circ (t^-,1)) \circ (t,r) \leqslant (st^-t,1) \leqslant (e,1)$.

Case 2. $(s,t) \in I^{**}$, then $ss = s^-s = s = t$ and $(s,q) \vartriangle (t,r) = (s, q \vee r) \circ (s^-,1) \leqslant (e,1)$. Thus, $ss^- \leqslant e$. Hence, $ss^-s \leqslant e$ by Proposition 1 and (Wcm). Therefore, $(s,q) \vartriangle (t,r) \vartriangle (t,r) = ((s, q \vee r) \circ (s^-,1)) \vartriangle (s,r) \leqslant ((s, q \vee r) \circ (s^-,1)) \circ (s,r) \leqslant (ss^-s,1) \leqslant (e,1)$.

Case 3. $(s,q) \vartriangle (t,r) = (s,q) \circ (t,r) \leqslant (e,1)$, then $st \leqslant e$. Thus, $stt \leqslant e$ by Proposition 1. Hence, by Lemma 3(iii), $(s,q) \vartriangle (t,r) \vartriangle (t,r) \leqslant (s,q) \circ (t,r) \circ (t,r) \leqslant (stt,1) \leqslant (e,1)$.

By a similar procedure, we prove that $(s,q) \vartriangle (t,r) \leqslant (e,1)$ if $(s,q) \vartriangle (t,r) \vartriangle (t,r) \leqslant (e,1)$. □

**Lemma 5.** *Let $\mathcal{A}$ be an $\mathbf{HpsUL}^*_\omega$-chain, $X$, and the binary operation $\circ$ on $X$ be as in Definition 7. The following conditions hold:*

*(a) $X$ is densely ordered and has a maximum $\top_X = (\top,1)$ and a minimum $\bot_X = (\bot,1)$.*

*(b) $\langle X, \circ, \leqslant_X, e_X \rangle$ is a linearly-ordered monoid, where $e_X = (e,1)$.*

*(c) $\circ$ is left-continuous with respect to the order topology on $\langle X, \leqslant_X \rangle$.*

*(d) There is a map $\Phi$ from $A$ into $X$ such that $\Phi$ is an embedding of the structure $\langle A, \wedge, \vee, \cdot, e, \bot, \top \rangle$ into $\langle X, \wedge_X, \vee_X, \circ, e_X, \bot_X, \top_X \rangle$, and for all $s,t \in A, \Phi(s \to t)$ is the residuum of $\Phi(s)$ and $\Phi(t)$ in $\langle X, \wedge_X, \vee_X, \circ, e_X, \bot_X, \top_X \rangle$, respectively.*

*(e) $\forall (s,q),(t,r) \in X, (s,q) \circ (t,r) \leq (e,1)$ iff $(s,q) \circ (t,r) \circ (t,r) \leq (e,1)$.*

**Proof.** Claim (e) has been proven by Lemma 2. As pointed out in [3], the associativity of $\circ$ is mainly dependent on Lemma 1(1)~(2). Other claims are proven in the same way as that of Theorem 4.5 in [3]. □

**Lemma 6.** *Every countable $\mathbf{UL}^*_\omega$-chain can be embedded into a standard $\mathbf{UL}^*_\omega$-algebra.*

**Proof.** Let $X, \mathcal{A}$, etc., be as in Definition 7. We can assume, without loss of generality, that $X = \mathbb{Q} \cap [0,1]$. Now, define for $\alpha, \beta \in [0,1]$, $\alpha * \beta = \sup\{x \circ y : x, y \in X, x \leq \alpha, y \leq \beta\}$. The proof of the weak commutativity, the monotonicity, associativity, left-continuity, etc., of $*$ is the same as that of Theorem 4.6 in [3]. The neutral element of $*$ is $e_X$ in $\mathbb{Q} \cap [0,1]$. By the left-continuity of $*$, the following property holds.

(P) $\alpha, \beta, \gamma \in [0,1]$, $\alpha * \beta * \gamma = \sup\{x \circ y \circ z : x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \gamma\}$.

We prove that $\alpha * \beta \leq e_X$ iff $\alpha * \beta * \beta \leq e_X$ for any $\alpha, \beta$ in $[0,1]$. Given $\alpha * \beta \leq e_X$, then $x \circ y \leq e_X$ for all $x, y \in X, x \leq \alpha, y \leq \beta$. Let $x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \beta$. Then, $x \circ y \leq e_X, x \circ z \leq e_X$. Thus, $x \circ y \circ y \leq e_X, x \circ z \circ z \leq e_X$ by Lemma 5(e). Hence, $x \circ y \circ z \leq \max\{x \circ y \circ y, x \circ z \circ z\} \leq e_X$. Therefore, $\alpha * \beta * \beta \leq e_X$ by (P). The sufficient part of the claim is proven in a similar way. □

By Lemma 1, Definition 8, Lemma 4, we can prove the claims similar to Lemma 5 and 6 for $\mathbf{IUL}_\omega$-algebras. As a consequence of these lemmas, and extending Theorem 3.3 of [7] in the obvious way, we obtain the following standard completeness.

**Theorem 2.** $\mathbf{UL}_\omega$ *and* $\mathbf{IUL}_\omega$ *are complete with respect to the class of standard algebras involved.*

## 4. Concluding Remarks

Roughly speaking, the methodological significance of Jenei and Montagna's proof is that it does not require a complete understanding of the structure of the **MTL**-algebras by embedding a countable **MTL**-algebra into a dense one. It is indeed different from the proof of the **BL**'s standard completeness given by Hajek, Cignoli, Esteva, Godo, Torrens et al. in [19,20]. The validation of the structure $X$ in Definitions 7, 8 and Lemmas 5, 6 is dependent on Lemma 1(1), which claims that $stu = s$ implies $st = s$. However, we are unable to prove the condition that $stu = t$ implies $st = t$ in $\mathbf{HpsUL}^*_\omega$. It seems that we need to introduce some stronger axioms into $\mathbf{HpsUL}^*_\omega$ to guarantee its completeness with respect to finite (or standard) $\mathbf{HpsUL}^*$-algebras.

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