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Relation Theoretic (Θ,R) Contraction Results with Applications to Nonlinear Matrix Equations

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Abstract: Using the concept of binary relation R, we initiate a notion of Θ_R-contraction and obtain some fixed point results for such mappings in the setting of complete metric spaces. Furthermore, we establish some new results of fixed points of N-order. Consequently, we improve and generalize the corresponding known fixed point results. As an application of our main result, we provide the existence of a solution for a class of nonlinear matrix equations. A numerical example is also presented to illustrate the theoretical findings.

Keywords: complete metric space; Θ_R-contractions; fixed point; binary relation

1. Introduction and Preliminaries

The conventional Banach contraction principle (BCP), which declares that a contraction on a complete metric space has a unique fixed point and plays an intermediate role in nonlinear analysis. Because of its significance and accessibility, various authors have established numerous interesting supplements and modifications of the BCP; see References [1–28] and the references therein. Edelstein [13] obtained the following result for compact metric space.

Theorem 1 ([13]). *Let (M,d) be a compact metric space and let $F : M \to M$ be a self-mapping. Assume that*

$$d(Fu, Fv) < d(u,v) \tag{1}$$

holds for all $u, v \in M$ with $u \neq v$. Then, there exists a unique u^* in M such that $u^* = F(u^*)$.

Jleli et al. [20] initiated a new version of the contraction which is known as an Θ-contraction and proved the new results for such contractions in the setting of generalized metric spaces.

Definition 1. *Let $\Theta : \mathbb{R}^+ \to (1,\infty)$ be a mapping satisfying:*

(Θ_1) Θ is nondecreasing;
(Θ_2) for any sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1 \Longleftrightarrow \lim_{n\to\infty}(\alpha_n) = 0$;
(Θ_3) there exists $0 < h < 1$ and $l \in (0,\infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^h} = l$.

A self mapping $F : M \to M$ is an Θ-contraction if there exists a function Θ satisfying (Θ_1)–(Θ_3) and a constant $\lambda \in (0,1)$ such that

$$d(Fu, Fv) \neq 0 \Longrightarrow \Theta(d(Fu, Fv)) \leq [\Theta(d(u,v))]^{\lambda} \tag{2}$$

for all $u, v \in M$.

Theorem 2 ([20]). *Let (M,d) be a complete metric space and $F: M \to M$ be an Θ-contraction, then there exists a unique u^* in M such that $u^* = F(u^*)$.*

The authors in Reference [20] manifested that a Banach contraction is a specific case of an Θ-contraction although there are many Θ-contractions which need not be Banach contractions. We express by the Ω, the set of all functions $\Theta : (0,\infty) \to (1,\infty)$ satisfying the above conditions (Θ_1)–(Θ_3).

Recently, Sawangsup et al. [28] defined a F_R-contraction and proved some fixed point theorems including binary relations. Now we give some definitions regarding binary relation.

Definition 2 ([22]). *A binary relation on M is a nonempty subset R of $M \times M$. It is transitive if $(u,w) \in R$ for all $u,v,w \in M$ whenever $(u,v) \in R$ and $(v,w) \in R$.*

If $(u,v) \in R$, then we express it by uRv and it is said that "u is related to v". Throughout this paper, we take R as a binary relation on a nonempty subset M and (M,d) as a metric space equipped with a binary relation R.

Definition 3 ([8]). *If $F: M \to M$ is a self mapping. Then, R is said to be F-closed if for each $u,v \in M$, $(u,v) \in R$ implies $(Fu,Fv) \in R$.*

According to Reference [26], the foregoing property F-closed holds if F is nondecreasing.

Definition 4 ([21]). *For $u,v \in M$, a path of length k in R from u to v (where k is a natural number) is a finite sequence $\{w_0, w_1, w_2, \dots, w_k\} \subseteq M$ satisfying the following assertions:*

(i) $w_0 = u$ and $w_k = v$;
(ii) $(w_j, w_{j+1}) \in R$ for all $j \in \{0,1,2,3,4,\dots,k-1\}$.
We express by $\Upsilon(u,v,R)$ the family of all paths in binary relations R from u to v.

Definition 5 ([26]). *A (M,d) is said to be R-nondecreasing-regular if for any $\{u_n\} \subseteq M$,*

$$\left.\begin{array}{r}(u_n, u_{n+1}) \in R, \forall n \in \mathbb{N}, \\ u_n \to u^* \in M\end{array}\right\} \Longrightarrow (u_n, u^*) \in R, \forall n \in \mathbb{N}.$$

Definition 6 ([27]). *Let $F: M^N \to M$. An element $(u_1, u_2, \dots, u_N) \in M^N$ is a fixed point of the mapping F of N-order if*

$$\begin{cases} F(u_1, u_2, \dots, u_N) = u_1 \\ F(u_2, u_3, \dots, u_N, u_1) = u_2 \\ \quad . \\ \quad . \\ \quad . \\ F(u_N, u_1, \dots, u_{N-1}) = u_N. \end{cases}$$

Let $F: M \to M$ be a self mapping. We express by $M(F,R) = \{u \in M : (u, Fu) \in R\}$.

The purpose of this article is to introduce the idea of an Θ_R-contraction where R is a binary relation and then establish some results in this way. We also apply our main results to examine a family of nonlinear matrix equation as an application.

2. Results

We begin this section by defining an Θ_R-contraction for the class of functions Ω and obtain confident results involving a binary relation.

Definition 7. *For (M,d) and R, let*

$$\mathcal{Y} = \{(u,v) \in \mathcal{R} : d(Fu,Fv) > 0\}.$$

A self-mapping $F: M \to M$ is said to be an Θ_R-contraction if there are $\Theta \in \Omega$ and $\lambda \in (0,1)$ such that

$$\Theta(d(Fu,Fv)) \leq [\Theta(d(u,v))]^{\lambda} \tag{3}$$

for all $(u,v) \in \mathcal{Y}$.

Now, we present our main result.

Theorem 3. *Let $F : (M,d) \to (M,d)$ be a self-mapping satisfying the following properties:*

(i) $M(F,R) \neq \emptyset$;
(ii) *R is F-closed';*
(iii) *F is continuous;*
(iv) *F is a Θ_R-contraction.*

Then, there exists u^* in M such that $u^* = F(u^*)$.

Proof. Let $u_0 \in M(F,R)$ be an arbitrary point. For such u_0, we construct the sequence $\{u_n\}$ by $u_n = F^n u_0 = Fu_{n-1}$ for all $n \in \mathbb{N}_0$. If there exists $n_0 \in \mathbb{N}_0$ such that $u_{n_0} = u_{n_0+1}$, then u_{n_0} is a fixed point of F and we are done. Hence, we suppose, $u_n \neq u_{n+1}$ and so $d(Fu_{n-1}, Fu_n) > 0$ for all $n \in \mathbb{N}_0$. As $(u_0,Fu_0) \in R$ and R is F-closed, so we have $(u_n, u_{n+1}) \in R$ for all $n \in \mathbb{N}_0$. Thus, $(u_n, u_{n+1}) \in \mathcal{Y}$ for all $n \in \mathbb{N}_0$. Since F is a Θ_R-contraction, we get

$$\begin{aligned} 1 &< \Theta(d(u_n,u_{n+1})) = \Theta(d(Fu_{n-1},Fu_n)) \leq [\Theta(d(u_{n-1},u_n))]^{\lambda} \\ &= [\Theta(d(Fu_{n-2},Fu_{n-1}))]^{\lambda} \leq [\Theta(d(u_{n-2},u_{n-1}))]^{\lambda^2} \cdots \\ &\qquad\qquad \leq [\Theta(d(u_0,u_1))]^{\lambda^n} \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4), we get

$$\lim_{n\to\infty} \Theta(d(u_n,u_{n+1})) = 1 \Longleftrightarrow \lim_{n\to\infty} d(u_n,u_{n+1}) = 0. \tag{5}$$

By (Θ_3), there exist $0 < h < 1$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(Fu_n,Fu_{n+1})) - 1}{d(Fu_n,Fu_{n+1})^h} = l. \tag{6}$$

Let $l < \infty$. In this case, let $\beta = \frac{l}{2} > 0$. So there exists $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\Theta(d(Fu_n,Fu_{n+1})) - 1}{d(Fu_n,Fu_{n+1})^h} - l\right| \leq \beta$$

for all $n > n_1$. This implies that

$$\frac{\Theta(d(Fu_n,Fu_{n+1})) - 1}{d(Fu_n,Fu_{n+1})^h} \geq l - \beta = \frac{l}{2} = \beta$$

for all $n > n_1$. Then,

$$nd(Fu_n,Fu_{n+1})^h \leq \gamma n[\Theta(d(Fu_n,Fu_{n+1})) - 1] \tag{7}$$

for all $n > n_1$, where $\gamma = \frac{1}{\beta}$. Now, we suppose that $l = \infty$. Let $\beta > 0$ be an arbitrary positive number. Then, there exists $n_1 \in \mathbb{N}$ such that

$$\beta \leq \frac{\Theta(d(Fu_n, Fu_{n+1})) - 1}{d(Fu_n, Fu_{n+1})^h}$$

for all $n > n_1$. This implies that

$$nd(Fu_n, Fu_{n+1})^h \leq \gamma n[\Theta(d(Fu_n, Fu_{n+1})) - 1]$$

for all $n > n_1$, where $\gamma = \frac{1}{\beta}$. Hence, in all ways, there exist $\gamma > 0$ and $n_1 \in \mathbb{N}$ such that

$$nd(Fu_n, Fu_{n+1})^h \leq \gamma n[\Theta(d(Fu_n, Fu_{n+1})) - 1] \tag{8}$$

for all $n > n_1$. Thus, by Equations (4) and (8), we get

$$nd(Fu_n, Fu_{n+1})^h \leq \gamma n([(\Theta d(u_0, u_1))]^{\lambda^n} - 1). \tag{9}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} nd(Fu_n, Fu_{n+1})^h = 0.$$

Thus, there exists $n_2 \in \mathbb{N}$ such that

$$d(Fu_n, Fu_{n+1}) \leq \frac{1}{n^{1/h}} \tag{10}$$

for all $n > n_2$. For $m > n > n_2$ we obtain

$$d(u_n, u_m) \leq \sum_{i=n}^{m-1} d(u_i, u_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/h}}. \tag{11}$$

Since $0 < \lambda < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$ converges. Therefore, $d(u_n, u_m) \to 0$ as $m, n \to \infty$. Thus, we proved that $\{u_n\}$ is a Cauchy sequence in M. The completeness of M assures that there exists $u^* \in M$ such that, $\lim_{n\to\infty} u_n = u^*$. Now, by the continuity of F, we get $Fu^* = u^*$ and so u^* is a fixed point of F. □

Remark 1. *From the proof of Theorem 3, we observe that for each $u_0 \in M(F, R)$, the Picard sequence $\{F^n u_0\}$ converges to the fixed point of F.*

By avoiding the continuity of F, we have the following result.

Theorem 4. *Theorem 3 also holds if we replace hypotheses (iii) with following one:*

(iii)$'$ (M, d) is R-nondecreasing-regular.

Proof. By Theorem 3, we have proved that there exists $u^* \in M$ such that, $\lim_{n\to\infty} u_n = u^*$. As $(u_n, u_{n+1}) \in R$ for all $n \in \mathbb{N}$, then $(u_n, u^*) \in R$ for all $n \in \mathbb{N}$. We review the following two cases counting on set $M = \{n \in \mathbb{N} : Fu_n = Fu^*\}$. □

- If M=finite, then there exists $n_0 \in \mathbb{N}$ such that $Fu_n \neq Fu^*$ for all $n \geq n_0$. Specifically, $u_n \neq u^*$, $d(u_n, u^*) > 0$ and $d(Fu_n, Fu^*) > 0$ for all $n \geq n_0$, so

$$1 < \Theta(d(Fu_n, Fu^*)) \leq [\Theta(d(u_n, u^*))]^{\lambda}$$

for each $n \geq n_0$. As $d(u_n, u^*) \searrow 0^+$, axiom (Θ_2) implies that $\Theta(d(Fu_n, Fu^*)) \to 1$. Hence, $\Theta(d(Fu_n, Fu^*)) \to 1$, so $d(Fu_n, Fu^*) \to 0$. Thus, $Fu^* = u^*$.

- If the set M is not finite, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $u_{n(k)+1} = Fu_{n(k)} = Fu^*$ for all $k \in \mathbb{N}$. As $u_n \to u^*$, then $Fu^* = u^*$. In both cases, u^* is a fixed point of F.

Now, we prove that the obtained fixed point in Theorems 3 and 4 is unique.

Theorem 5. *Suppose that the binary relation R is transitive on M and $\Upsilon(u, v, R)$ is nonempty, for all $u, v \in Fix(F) := \{w \in M : w$ is a fixed point of $F\}$ is as an addition to the hypotheses of Theorem 3 (respectively, Theorem 4). Then, u^* is unique.*

Proof. Let u and v be such that

$$F(u) = u, F(v) = v \text{ and } u \neq v. \tag{12}$$

Then, $d(Fu, Fv) > 0$. Since $\Upsilon(u, v, R) \neq \emptyset$. So there exists a $\{w_0, w_1, w_2, \ldots, w_k\}$ from u to v in R , so that

$$w_0 = u, \quad w_k = v, \quad (w_i, w_{i+1}) \in R \quad \text{for each } i = 1, 2, \ldots, k-1.$$

As R is transitive, so we have

$$(u, w_1) \in R, (w_1, w_2) \in R, \ldots, (w_{k-1}, v) \in R \quad \Longrightarrow (u, v) \in R.$$

Thus from Equation (12), we have

$$\Theta(d(u, v)) = \Theta(d(Fu, Fv)) \leq [\Theta(d(u, v))]^\lambda$$

a contradiction because $\lambda < 1$. Thus, $u = v$. □

3. Multidimensional Results

Now we establish some multidimensional theorems from the above-mentioned results by identifying some very easy tools. We express by R^N the binary relation on M^N defined by

$$((u_1, u_2, \ldots, u_N), (v_1, v_2, \ldots, v_N)) \in R^N$$

$$\Longleftrightarrow$$

$$(u_1, v_1) \in R, (u_2, v_2) \in R, \ldots, (u_N, v_N) \in R.$$

If $F : M^N \to M$, let us express by $M^N(F, R^N)$ the class of all points $(u_1, u_2, \ldots, u_N) \in M^N$ such that

$$((u_1, u_2, \ldots, u_N), (F(u_1, u_2, \ldots, u_N), F(u_2, u_3, \ldots, u_N, u_1), \ldots, F(u_N, u_1, \ldots, u_{N-1}))) \in R^N$$

that is,

$$(u_j, F(u_j, u_{j+1}, \ldots, u_N, u_1, u_2, \ldots, u_{j-1})) \in R \text{ for } \quad j \in \{1, 2, \ldots, N\}.$$

Definition 8 ([28]). *If $N \geq 2$ and $F : M^N \to M$. A binary relation R on M is said to be F_N-closed if for any $(u_1, u_2, \ldots, u_N), (v_1, v_2, \ldots, v_N) \in M^N$,*

$$\left\{\begin{array}{c}(u_1,v_1)\in R,\\(u_2,v_2)\in R,\\ \cdot\\ \cdot\\ \cdot\\(u_N,v_N)\in R\end{array}\right\}\Longrightarrow\left\{\begin{array}{c}(F(u_1,u_2,\ldots,u_N),F(v_1,v_2,\ldots,v_N))\in R,\\(F(u_2,u_3,\ldots,u_N,u_1),F(v_2,v_3,\ldots,v_N,v_1))\in R,\\ \cdot\\ \cdot\\ \cdot\\(F(u_N,u_1,\ldots,u_{N-1}),F(v_N,v_1,\ldots,v_{N-1}))\in R\end{array}\right\}.$$

Let us express by $G_F^N : M^N \to M^N$ the mapping

$$G_F^N(u_1,u_2,\ldots,u_N)=(F(u_1,u_2,\ldots,u_N),F(u_2,u_3,\ldots,u_N,u_1),\ldots,F(u_N,u_1,\ldots,u_{N-1})).$$

Lemma 1 ([28]). *Given $N \geq 2$ and $F : M^N \to M$, a point $(u_1,u_2,\ldots,u_N)\in M^N$ is a fixed point of N-order of F if it is a fixed point of G_F^N.*

Lemma 2 ([28]). *Given $N \geq 2$ and $F : M^N \to M$, then R is F_N-closed if it is G_F^N-closed defined on M^N.*

Lemma 3 ([28]). *Given $N \geq 2$ and $F : M^N \to M$, a point $(u_1,u_2,\ldots,u_N)\in M^N(F,R^N)$ if and only if $(u_1,u_2,\ldots,u_N)\in M^N(G_F^N,R^N)$.*

Lemma 4 ([28]). *Let $D^N : M^N \times M^N \to \mathbb{R}$ given by*

$$D^N(V,W)=\sum_{j=1}^{N} d(v_j,w_j)$$

for all $V=(v_1,v_2,\ldots,v_N), W=(w_1,w_2,\ldots,w_N)\in M^N$. Then, the following assertions hold.

1. (M^N,D^N) is also a metric space.
2. Let $\{V_n=(v_n^1,v_n^2,\ldots,v_n^N)\}$ be a sequence in M^N and let $V=(v_1,v_2,\ldots,v_N)\in M^N$. Then, $\{V_n\}\overset{D^N}{\to}V \Leftrightarrow \{v_n^j\}\overset{d}{\to}v_j$ for all $j\in\{1,2,3,\ldots,N\}$.
3. If $\{V_n=(v_n^1,v_n^2,\ldots,v_n^N)\}$ is a sequence in M^N, then $\{V_n\}$ is D^N-Cauchy $\Leftrightarrow \{v_n^j\}$ is Cauchy for all $j\in\{1,2,3,\ldots,N\}$.
4. (M,d) is complete $\Leftrightarrow (M^N,D^N)$ is complete.

Definition 9 ([28]). *For $(u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N)\in M^N$, a path of length k in R^N from $(u_1,u_2,\ldots,u_N)$ to $(v_1,v_2,\ldots,v_N)$ is a finite sequence $\{(w_0^1,w_0^2,\ldots,w_0^N),(w_1^1,w_1^2,\ldots,w_1^N),\ldots,(w_k^1,w_k^2,\ldots,w_k^N)\}\subset M^N$ satisfying the following conditions:*

(i) $(w_0^1,w_0^2,\ldots,w_0^N)=(u_1,u_2,\ldots,u_N)$ *and* $(w_k^1,w_k^2,\ldots,w_k^N)=(v_1,v_2,\ldots,v_N)$;
(ii) $((w_i^1,w_i^2,\ldots,w_i^N),(w_{i+1}^1,w_{i+1}^2,\ldots,w_{i+1}^N))\in R^N$ *for all* $i=0,1,2,\ldots,k-1$.

Consistent with Reference [28], we denote by $\Upsilon((u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N),R^N)$ the class of all paths in R^N from $(u_1,u_2,\ldots,u_N)$ to $(v_1,v_2,\ldots,v_N)$.

Definition 10. *Let $F : M^N \to M$ be a given mapping and let us denote*

$$\mathcal{Y}^N=\Big\{((u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N))\in R^N : d(F(u_1,u_2,\ldots,u_N),F(v_1,v_2,\ldots,v_N))>0\Big\}$$

We say that F is an Θ_{R^N}-contraction if there are some $\Theta \in \Omega$ and $\lambda \in (0,1)$ such that

$$\Theta\begin{pmatrix} d(F(u_1,u_2,\ldots,u_N),F(v_1,v_2,\ldots,v_N))+ \\ d(F(u_2,u_3,\ldots,u_N,u_1),F(v_2,v_3,\ldots,v_N,v_1))+ \\ \cdot \\ \cdot \\ \cdot \\ +d(F(u_N,u_1,\ldots,u_{N-1}),F(v_N,v_1,\ldots,v_{N-1})) \end{pmatrix} \leq \left[\Theta\left(\sum_{i=1}^{N} d(u_i,v_i)\right)\right]^{\lambda} \tag{13}$$

for each $((u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N)) \in \mathcal{Y}^N$.

Theorem 6. *Let $F: M^N \to M$ be a mapping. Suppose that the following assertions hold:*

(i) $M^N(F,R^N) = \emptyset$;
(ii) *R is F_N-closed';*
(iii) *F is continuous;*
(iv) *F is a Θ_{R^N}-contraction.*

Then, F has a fixed point of N-order.

Proof. (M^N, D^N) is a complete metric space by 1 and 4 of Lemma 4. By Lemma 2, the binary relation R^N defined on M^N is G_F^N-closed. Suppose that $(u_0^1, u_0^2, \ldots, u_0^N) \in M^N(F,R^N)$. By Lemma 3, we obtain that $(u_0^1, u_0^2, \ldots, u_0^N) \in M^N(G_F^N, R^N)$. Since F is continuous, we conclude that G_F^N is also continuous. From the Θ_{R^N} -contractive condition of F, we conclude that G_F^N is also Θ_{R^N} -contraction. By Theorem 3, there exists $\hat{M} = (\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_N}) \in M^N$ such that $G_F^N(\hat{M}) = \hat{M}$, that is $(\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_N})$ is a fixed point of G_F^N . Using Lemma 2, $(\widehat{u_1}, \widehat{u_2}, \ldots, \widehat{u_N})$ is a fixed point of F of N-order. □

Theorem 7. *Let $F: M^N \to M$ be a mapping. Assume that the following assertions hold:*

(i) $M^N(F,R^N) = \emptyset$;
(ii) *R is F_N-closed';*
(iii) *M^N is N-nondecreasing-regular;*
(iv) *F is a Θ_{R^N}-contraction.*

Then, F has a fixed point of N-order.

Theorem 8. *In addition to the hypotheses of Theorem 6 (respectively, Theorem 7), assume that R is a transitive relation on M and $\Upsilon((u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N),R^N)$ is nonempty for each*

$$\begin{aligned}(u_1,u_2,\ldots,u_N),(v_1,v_2,\ldots,v_N) \quad &\in \quad Fix(F) \\ &= \quad \left\{w \in M^N : w \text{ is a fixed point of } F \text{of } N\text{-order}\right\}.\end{aligned}$$

Then, F has a unique fixed point of N-order.

4. Applications in Relation to Nonlinear Matrix Equations

Fixed point theorems for various functions in ordered metric spaces have been broadly explored and many applications in different branches of the sciences and mathematics have been found especially relating to differential, integral, and matrix equations (see References [6,14,25] and references therein).

Let us denote $\mathcal{M}(n)$ = set of all $n \times n$ complex matrices, $\mathcal{H}(n)$ = set of all Hermitian matrices in $\mathcal{M}(n)$, $\mathcal{P}(n)$ = the family of all positive definite matrices in $\mathcal{M}(n)$, and $\mathcal{H}^+(n)$ = the class of all positive semidefinite matrices in $\mathcal{M}(n)$. For $E \in \mathcal{P}(n)(E \in \mathcal{H}^+(n))$, we write $E \succ 0(E \succeq 0)$. Furthermore, $E_1 \succ E_2(E_1 \succeq E_2)$ means $E_1 - E_2 \succ 0(E_1 - E_2 \succeq 0)$. The symbol $\|\cdot\|$ is used for the spectral norm of A defined by $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A, where A^* is the conjugate transpose of A. In addition, $\|A\|_{tr} = \sum_{k=1}^{n} s_k(A)$, where $s_k(A)$ $(1 \leq k \leq n)$ are the singular values of $A \in \mathcal{M}(n)$. Here, $(\mathcal{H}(n), \|\cdot\|_{tr})$ is complete metric space (for more details see References [11,12,25]). Moreover, the binary relation $\preceq$ on $\mathcal{H}(n)$ defined by: $E_1 \preceq E_2 \Leftrightarrow E_2 \succeq E$ for all $E_1, E_2 \in \mathcal{H}(n)$.

In this section, we apply our results to establish a solution of the nonlinear matrix equation.

$$X = Q + \sum_{k=1}^{n} A_k^* \mathcal{G}(E) A_k \tag{14}$$

where $\mathcal{G}$ is a continuous order preserving mapping with $\mathcal{G}(0) = 0$, Q is a Hermitian positive definite matrix, and A_k are any $n \times n$ matrices and A_k^* their conjugates.

Now we state the the following lemmas which are helpful in the next results.

Lemma 5 ([25]). *Let $C, E \in \mathcal{H}(n)$ such that $C \succeq 0$ and $E \succeq 0$. Then,*

$$0 \leq tr(CE) \leq \|C\|\, tr(E).$$

Lemma 6 ([23]). *If $C \in \mathcal{H}(n)$ such that $C \prec I_n$, then $\|C\| < 1$.*

Theorem 9. *Consider the matrix Equation (14). Assume that there are positive real numbers L and $\lambda \in (0,1)$ such that:*

(i) For $E_1, E_2 \in \mathcal{H}(n)$ with $E_1 \preceq E_2$ and $\sum_{i=1}^{n} A_i^* \mathcal{G}(E_1) A_i \neq \sum_{i=1}^{n} A_i^* \mathcal{G}(E_2) A_i$, we have

$$|tr(\mathcal{G}(E_2) - \mathcal{G}(E_1))| \leq \frac{\lambda^2\, |tr(E_2 - E_1)|}{L};$$

(ii) $\sum_{i=1}^{m} A_i A_i^* \prec L I_n$ and $\sum_{i=1}^{m} A_i^* \mathcal{G}(E_1) A_i \succ 0$.

Then, Equation (12) has a solution. Moreover, the iteration

$$E_n = Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(E_{n-1}) A_i$$

where $E_0 \in \mathcal{H}(n)$ satisfies $E_0 \preceq Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(E_0) A_i$ converges to the solution of Equation (12).

Proof. Define $\mathcal{J} : \mathcal{H}(n) \to \mathcal{H}(n)$ by

$$\mathcal{J}(E) = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(E) A_i$$

for all $E \in \mathcal{H}(n)$. Then, $\mathcal{J}$ is well defined, the order $\preceq$ on $\mathcal{H}(n)$ is $\mathcal{J}$-closed. Here, the solution of Equation (14) is actually a fixed point of $\mathcal{J}$ and we have to show that $\mathcal{J}$ is an $\Theta_{\preceq}$-contraction mapping due to some $\lambda \in (0,1)$ and Θ defined by

$$\Theta(t) = e^{\sqrt[\lambda]{t}}$$

for all $t \in (0, \infty)$. Let $E_1, E_2 \in \mathcal{H}(n)$ be such that $E_1 \preceq E_2$ and $\mathcal{G}(E_1) \neq \mathcal{G}(E_2)$ which further implies that $E_1 \prec E_2$. Since $\mathcal{G}$ is an order preserving , we have $\mathcal{G}(E_1) \prec \mathcal{G}(E_2)$. Thus,

$$\begin{aligned}
\|\mathcal{J}(E_2)-\mathcal{J}(E_1)\|_{tr} &= tr(\mathcal{J}(E_2)-\mathcal{J}(E_1)) \\
&= tr\left(\sum_{i=1}^{m} A_i^*(\mathcal{G}(E_2)-\mathcal{G}(E_1))A_i\right) \\
&= \sum_{i=1}^{m} tr\left(A_i^*(\mathcal{G}(E_2)-\mathcal{G}(E_1))A_i\right) \\
&= \sum_{i=1}^{m} tr\left(A_iA_i^*(\mathcal{G}(E_2)-\mathcal{G}(E_1))\right) \\
&= tr\left(\left(\sum_{i=1}^{m} A_iA_i^*\right)(\mathcal{G}(E_2)-\mathcal{G}(E_1))\right) \\
&\leq \left\|\sum_{i=1}^{m} A_iA_i^*\right\| \|\mathcal{G}(E_2)-\mathcal{G}(E_1)\|_{tr} \\
&\leq \frac{\|\sum_{i=1}^{m} A_iA_i^*\|}{L}\left(\lambda^2\|E_2-E_1\|_{tr}\right) \\
&< \lambda^2\|E_2-E_1\|_{tr}.
\end{aligned}$$

which further implies that

$$e^{\sqrt{\|\mathcal{J}(E_2)-\mathcal{J}(E_1)\|_{tr}}} \leq e^{\lambda\sqrt{\|E_2-E_1\|_{tr}}}.$$

We have

$$\Theta(\|\mathcal{J}(E_2)-\mathcal{J}(E_1)\|_{tr}) \leq [\Theta(\|E_2-E_1\|_{tr})]^{\lambda}$$

which proves that $\mathcal{J}$ is an $\Theta_{\preceq}$-contraction. By $\sum_{i=1}^{m} A_i^*\mathcal{G}(Q)A_i \succ 0$, we get $Q \preceq \mathcal{J}(Q)$. Therefore, that $Q \in \mathcal{H}(n)(\mathcal{J};\preceq)$. Thus, by Theorem 3, $\exists\, \hat{E} \in \mathcal{H}(n)$ such that $\mathcal{J}(\hat{E}) = \hat{E}$, that is, Equation (14) has a solution. $\square$

Example 1. *Consider the matrix equation*

$$E = Q + A_1^*EA_1 + A_2^*EA_2 \tag{15}$$

where Q, A_1 and A_2 are given by

$$Q = \begin{pmatrix} 7 & 5 & 3 & 1 \\ 5 & 7 & 5 & 3 \\ 3 & 5 & 7 & 5 \\ 1 & 3 & 5 & 7 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.0241 & 0.0124 & 0.0124 & 0.0241 \\ 0.0124 & 0.0241 & 0.0241 & 0.0124 \\ 0.0124 & 0.0241 & 0.0241 & 0.0124 \\ 0.0241 & 0.0124 & 0.0124 & 0.0241 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.0521 & 0.0329 & 0.0329 & 0.0521 \\ 0.0871 & 0.68 & 0.0871 & 0.68 \\ 0.0521 & 0.0329 & 0.0329 & 0.0521 \\ 0.0871 & 0.68 & 0.0871 & 0.68 \end{pmatrix}.$$

Define $\Theta : (0,\infty) \to (1,\infty)$ by

$$\Theta(t) = e^{\sqrt[\lambda]{t}}$$

for all $t \in (0,\infty)$ and $\lambda = \frac{1}{2}$ and $\mathcal{G} : \mathcal{H}(n) \to \mathcal{P}(n)$ by $\mathcal{G}(E) = \frac{E}{2}$. Then, conditions (i) and (ii) of Theorem 9 are satisfied for $L = 2$ by using the iterative sequence

$$E_{n+1} = Q + \sum_{i=1}^{2} A_i^* E_n A_i$$

with

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

After 19 *iterations, we get the unique solution*

$$\widehat{E} = \begin{pmatrix} 7.02314 & 5.01514 & 3.00561 & 1.00515 \\ 5.01514 & 13.0343 & 5.01729 & 5.58171 \\ 3.00561 & 5.01729 & 7.01166 & 5.02357 \\ 1.00515 & 5.58171 & 5.02357 & 13.0343 \end{pmatrix}$$

of the matrix Equation (15). The residual error is $R_{19} = ||\widehat{E} - \sum_{i=1}^{2} A_i^* \widehat{E} A_i|| = 6.19191 \times 10^{-6}$.

Theorem 10. *With the assumptions of Theorem 9, Equation (15) has a unique solution* $\hat{E} \in \mathcal{H}(n)$.

Proof. Since for $E_1, E_2 \in \mathcal{H}(n)$ $\exists$ a greatest lower bound and a least upper bound. So we have $\Upsilon(x, y, R) \neq \emptyset$, for each $x, y \in E$. Thus, we conclude by Theorem 5 that $\mathcal{J}$ has a unique fixed point in $\mathcal{H}(n)$ which implies that Equation (15) has a unique solution in $\mathcal{H}(n)$. □

5. Conclusions

In this paper, we introduced the concept of Θ_R-contraction and obtained some results for such contractions in the context of complete metric spaces. Additionally, we established the theorems which guarantee the existence and the uniqueness of a fixed point. As an application, we applied our principal theorem to review a class of nonlinear matrix equations. We also presented a numerical example to illustrate the theoretical findings.

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