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A Novel Comparison of Probabilistic Hesitant Fuzzy Elements in Multi-Criteria Decision Making

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Abstract: The probabilistic hesitant fuzzy element is a common tool to deal with multi-criteria decision-making problems when the decision makers are irresolute in providing their evaluations. The existing methods for ranking probabilistic hesitant fuzzy elements are limited and not reasonable in practical applications. The main purpose of this paper is to find a more precise and appropriate comparison method so that we can deal with multi-criteria decision-making problems more efficiently. We first propose a chart technique to analyze the structure of a probabilistic hesitant fuzzy element. After that, we propose a novel possibility degree formula to rank probabilistic hesitant fuzzy elements. Last but not least, we provide a useful process to solve the actual multi-criteria decision-making problems, and make a real case study which demonstrates that our method is feasible and reliable.

Keywords: probabilistic hesitant fuzzy elements; multi-criteria decision making; possibility degree formula; fuzzy ranking

1. Introduction

Zadeh [1] introduced the fuzzy set in 1965. Since then, many experts have studied some other extended forms of fuzzy set; for example, intuitionistic fuzzy set (IFS) [2], probabilistic linguistic term set (PLTS) [3] and hesitant fuzzy set (HFS) [4]. Actually, the HFS, which aims at solving the difficulty in describing the hesitance in practical evaluation, has been used widely in multi-criteria decision-making (MCDM) problems [5,6]. The main reason is that it can be confronted with situations in which people are hesitant to provide their preferences in the process of decision making by permitting the experts to provide their preferences with several possible values between 0 and 1. Torra [3] introduced some basic operations of HFSs. Many experts have done large amounts of work to develop the theory of HFSs [7,8] and aided in development in uncertain decision-making problems [9,10]. Meng [7] and Li [8] proposed some correlation coefficients and a variety of distance measures for HFSs. They also investigated applications based on the correlation coefficients and distance measures.

However, there is an obvious problem: that every possible hesitant fuzzy evaluation value provided by the experts has the same weight in the current approaches. This is not appropriate. When facing group decision-making problems, the experts may not use the HFSs correctly to represent the preferences over the given objects. For example, if every value in the hesitant fuzzy elements (HFEs) (note: each HFE is the basic component of the HFS) has a probability distribution, then the HFE not only includes several possible values but also the corresponding probabilistic information. If we ignore the probabilistic information, it may result in errors. For example, if there are two experts studying the preferences for a scheme, and one assigns 0.6 and another assigns 0.7, then the preferences can be expressed as $\{0.6, 0.7\}$. However, if one assigns 0.6 and 0.7, and another expert assigns 0.7 and 0.8, how can we express their preferences using an HFE now? It is obvious that taking the preferences as $\{0.6, 0.7, 0.8\}$ is not exactly suitable, because this loses a preference value 0.7. Therefore, avoiding the information loss of HFSs in the decision-making process becomes an important problem. Recently, some research has solved this issue [11,12]. In order to improve the performances of the HFSs in group decision making, Zhu [13] brought probability to the HFS and proposed the probabilistic hesitant fuzzy set (P-HFS), which can overcome the defect of HFS to a great extent. As a result, it can remain the experts' evaluation information and describe their preferences better. In the above example, we can express their preferences by a probabilistic hesitant fuzzy element (P-HFE) as $\{0.6(0.25), 0.7(0.5), 0.8(0.25)\}$ (note: each P-HFE is the basic component of the P-HFS).

In fact, some other scholars [14] have carried on further research and conducted deeper study on the P-HFSs. Recently, Zhang et al. [15] further developed the operations and integrations of the P-HFSs. In many practical applications, the experts cannot give complete assessment information on the considered objects because they may lack related knowledge or they are not sure enough about the problem. Furthermore, when there is more than one expert, there is a special case that not all the experts can give their evaluations about every aspect, which obviously will lose partial information. Thus, Zhang et al. [15] improved the definition of the probabilistic hesitant fuzzy element (P-HFE). In this research, they made a study on the properties of the improved P-HFEs and defined their operations. Compared with other concepts which can characterize the hesitant information, we can find that the improved P-HFEs allow the decision makers (DMs) to give one or more possible HFEs with probabilistic information and can even deal with incomplete evaluated information.

In order to understand the existing main hesitant fuzzy models easily, we summarize their features, and then discuss their advantages and disadvantages. The results are listed in Table 1. There are three hesitant fuzzy models—the HFS, the original P-HFS and the improved P-HFS—which have been widely applied in different situations.

Table 1. A summary on the hesitant fuzzy models for decision making. DMs: decision makers; HFS: hesitant fuzzy set; P-HFS: probabilistic HFS.

Different Models	DMs Have more Choices	Retain most Decision-Making Information	DMs Have more Space to Hesitate
The HFS [4]	Yes	No	No
The original P-HFS [13]	Yes	Yes	No
The improved P-HFS [15]	Yes	Yes	Yes

However, the method of comparing the P-HFEs is still an important question in MCDM problems [16]. Zhang et al. [15] provided the basic way to rank P-HFEs. The main idea of Zhang et al.'s comparison method is based on the score function and the deviation degree of every P-HFE. It is obvious that their method sorts the P-HFEs with the absolute priorities, which is not very accurate and logical. In the actual assessment problems [17,18], it is not reasonable to say that one P-HFE is absolutely superior to another if they have some common or intersecting values. In a sense, this kind of sorting method for the P-HFEs is not precise.

In order to overcome the deficiencies discussed above, we shall propose a novel comparison method for P-HFEs. Inspired by the chart technique [19] to analyze the structures of P-HFEs, we come up with the new method to compare P-HFEs based on a possibility degree formula. Our new comparison method is more precise, especially when facing situations that different P-HFEs have common or intersecting values. At the same time, the proposed possibility degree formula can realize the optimal sorting under hesitant fuzzy environment and reduce the complexity of the computation effectively. After that, we put forward a more efficient model to rank the alternatives for the MCDM problems.

To do that, the reminder of this paper is organized as follows: Section 2 reviews the basic concepts of the P-HFE. In Section 3, we propose the chart technique to describe the structures of P-HFEs, based on which we put forward the novel possibility degree formula and prove some properties of the formula in detail. In Section 4, based on the proposed sorting method, we give an algorithm to deal with the MCDM problems with probabilistic hesitant fuzzy information. Section 5 presents a practical case that can illustrate the advantages of our method. Section 6 ends this paper with some conclusions.

2. Preliminaries

In this section, we first introduce some basic concepts and knowledge related to the P-HFEs.

2.1. Concept of P-HFE

Definition 1. *Let H be a probabilistic hesitant fuzzy set, which is expressed as $H = \{x, h_x(p_x) | x \in X\}$ [15], where both h_x and p_x are two sets of some values in $[0, 1]$.*

For convenience, Zhang et al. [15] first defined the probabilistic hesitant fuzzy element (P-HFE) as follows: $h(p) = \{\gamma_l(p_l) | l = 1, 2, \ldots, |h(p)|\}$, where p_l is the probability of the membership degree γ_l, satisfying $\sum_{l=1}^{|h(p)|} p_l = 1$, $\gamma_l(p_l)$ is called a term of the P-HFE and $|h(p)|$ is the number of all the different membership degrees in the element $h(p)$. For convenience, we assume that the values of γ_l are ascending ordered.

As a matter of fact, some DMs cannot afford integrated information, because they do not have enough knowledge related to the problem domain. Zhang et al. [15] generalized the original P-HFE to the weak P-HFE, changing the condition from $\sum_{l=1}^{|h(p)|} p_l = 1$ to $\sum_{l=1}^{|h(p)|} p_l \leq 1$. To eliminate the ignorance of the probability distribution in a P-HFE defined above with the condition $\sum\limits_{l=1}^{|h(p)|} p_l < 1$, the normalization method of a P-HFE is provided as follows:

Definition 2. *If a P-HFE $h(p)$ is given by the condition $\sum_{l=1}^{|h(p)|} p_l < 1$, then a new P-HFE associated with the original one is defined as $\dot{h}(p) = \left\{\gamma_l\left(\dot{p}_l\right) | l = 1, 2, \ldots, |h(p)|\right\}$, where $\dot{p}_l = p_l / \sum_{l=1}^{|h(p)|} p_l$, $l = 1, 2, \ldots, |h(p)|$, which is called the normalized P-HFE [15].*

Obviously, Definition 2 is an efficacious and precise mean to assess the probabilistic information. In order to introduce briefly and facilitate understanding, in the rest of the paper, the term P-HFE must be regarded as the normalized P-HFE. In order to understand it easily, we still denote the normalized P-HFE $\dot{h}(p)$ as $h(p)$.

Example 1. *Let H be a set composed of two P-HFEs $h_1(p)$ and $h_2(p)$, where*

$$h_1(p) = \{0.4(0.3), 0.5(0.3), 0.6(0.3)\}, h_2(p) = \{0.4(0.4), 0.5(0.6)\}$$

Based on Definition 2, we can easily obtain $\dot{h}_1(p) = \{0.4(0.33), 0.5(0.33), 0.6(0.33)\}$. Then, the term $0.4(0)$ in $h_2(p)$ is added so that we can obtain $\dot{h}_2(p) = \{0.4(0), 0.4(0.4), 0.5(0.6)\}$. Last but not least, we obtain the normalized P-HFEs as follows:

$$h_1(p) = \{0.4(0.33), 0.5(0.33), 0.6(0.33)\}, h_2(p) = \{0.4(0), 0.4(0.4), 0.5(0.6)\}$$

Inspired by the operations of HFEs, Zhang et al. [15] have defined some basic operations of the P-HFEs, which are listed as follows:

Definition 3. *Let $h(p)$, $h_1(p)$ and $h_2(p)$ be three normalized* P-HFEs *[15], $\lambda > 0$, then*

(1) $\lambda h(p) = \bigcup_{\gamma_l \in h} \left\{ \left[1 - (1 - \gamma_l)^\lambda\right](p_l) \right\}$;

(2) $h^\lambda(p) = \bigcup_{\gamma_l \in h} \left\{ \left[\gamma_l{}^\lambda\right](p_l) \right\}$;

(3) $h_1(p) \oplus h_2(p) = \bigcup_{\gamma_{1_l} \in h_1, \gamma_{2_k} \in h_2} \{[\gamma_{1_l} + \gamma_{2_k} - \gamma_{1_l}\gamma_{2_k}](p_{1_l} \cdot p_{2_k})\}$;

(4) $h_1(p) \otimes h_2(p) = \bigcup_{\gamma_{1_l} \in h_1, \gamma_{2_k} \in h_2} \{[\gamma_{1_l}\gamma_{2_k}](p_{1_l} \cdot p_{2_k})\}$.

2.2. The Ranking Method of the P-HFEs

In order to compare P-HFEs with each other, Zhang et al. [15] defined the score function and deviation degrees of a P-HFE as follows:

Definition 4. *For a P-HFE $h(p)$, its score is:*

$$s(h(p)) = \left(\sum_{l=1}^{|h_1(p)|} \gamma_l \cdot p_l\right) \Big/ \left(\sum_{l=1}^{|h_1(p)|} p_l\right)$$

Definition 5. *If the score of $h(p)$ is denoted as $\overline{\gamma}$, then the deviation degree of $h(p)$ is:*

$$d(h(p)) = \sum_{l=1}^{|h(p)|} (p_l(\gamma_l - \overline{\gamma}))^2 \Big/ \sum_{l=1}^{|h(p)|} p_l$$

Using the score and deviation degrees of a P-HFE, Zhang et al. [19] put forward a method to compare two P-HFEs $h_1(p)$ and $h_2(p)$:

(1) If $s(h_1(p)) > s(h_2(p))$, then $h_1(p) > h_2(p)$;
(2) If $s(h_1(p)) < s(h_2(p))$, then $h_1(p) < h_2(p)$;
(3) If $s(h_1(p)) = s(h_2(p))$ and $d(h_1(p)) < d(h_2(p))$, then $h_1(p) > h_2(p)$;
(4) If $s(h_1(p)) = s(h_2(p))$ and $d(h_1(p)) > d(h_2(p))$, then $h_1(p) < h_2(p)$;
(5) If $s(h_1(p)) = s(h_2(p))$ and $d(h_1(p)) = d(h_2(p))$, then we define that $h_1(p)$ is equivalent to $h_2(p)$, denoted as $h_1(p) \sim h_2(p)$.

In order to understand this easily, below we will give an illustrative example:

Example 2. *Let $h_i(p)(i = 1, 2, 3)$ be three P-HFEs, and $h_1(p) = \{0.6(0.4), 0.8(0.5)\}$, $h_2(p) = \{0.5(0.5), 0.8(0.5)\}$, $h_3(p) = \{0.6(0.5), 0.7(0.5)\}$.*

After that, we could calculate the scores and deviation degrees of the three P-HFEs by Definitions 4 and 5 as follows:

$$s(h_1(p)) = \frac{0.6 \times 0.4 + 0.8 \times 0.5}{0.4 + 0.5} = 0.711$$

$$s(h_2(p)) = \frac{0.5 \times 0.5 + 0.8 \times 0.5}{0.5 + 0.5} = 0.65$$

$$s(h_3(p)) = \frac{0.6 \times 0.5 + 0.7 \times 0.5}{0.5 + 0.5} = 0.65$$

$$d(h_1(p)) = \frac{0.4 \times (0.6 - 0.711)^2 + 0.5 \times (0.8 - 0.711)^2}{0.4 + 0.5} = 0.0099$$

$$d(h_2(p)) = \frac{0.5 \times (0.5 - 0.65)^2 + 0.5 \times (0.8 - 0.65)^2}{0.5 + 0.5} = 0.0225$$

$$d(h_3(p)) = \frac{0.5 \times (0.6 - 0.65)^2 + 0.5 \times (0.7 - 0.65)^2}{0.5 + 0.5} = 0.0025$$

From the calculation results above, we can easily obtain $h_1(p) > h_3(p) > h_2(p)$. Thus, $h_1(p)$ is absolutely superior to the other two, and $h_3(p)$ is absolutely better than $h_2(p)$. Nevertheless, we believe that it may be not so reasonable to say that $h_2(p)$ is absolutely worse under the criterion "portability" when there are hesitant fuzzy elements to be assessed. Therefore, we will come up with a new comparison method for P-HFEs rating in the next section, which can successfully overcome the disadvantage of Zhang et al.'s [15] method.

3. Possibility Degree Formula for Ranking P-HFEs

We have reviewed some basic knowledge of the P-HFEs above. As known to us all, we could compare different P-HFEs to choose the best alternative by the score function and the deviation degrees that Zhang et al. [15] defined. However, it is limited and not reasonable in practical applications. In this paper, we will come up with a novel possibility degree formula for P-HFE rating. To understand our method easily, we first review some common methods to rank fuzzy numbers. After that, we put forward our idea and propose a novel method to compare the P-HFEs.

3.1. The Different Methods for Ranking Fuzzy Numbers

Many experts have studied how to rank fuzzy numbers in recent years. For example, there are minimizing and maximizing set-based methods [20], the distance-based method [21], the area-based method [22] and so on [23,24]. These methods could not avoid the loss of information to some extent. Some scholars proposed possibility distribution-based methods [25] and tried to overcome the shortcoming of information loss. Yet those ranking methods could be much too complicated for P-HFEs' ranking because they always focus on interval comparisons, and sometimes the ranking results may be unsuitable. Thus, we will put forward a new concrete formula to compare the P-HFEs with possibility degree in the next subsection.

3.2. A Concrete Formula for Ranking P-HFEs

In order to understand the new formula better, we first give some basic definitions to develop our diagram method so as to analyze the structure of the P-HFE. Our main idea is to develop a visualization method to understand it.

Definition 6. *Let $h(p) = \left\{\gamma^k(p_k) | k = 1, 2, \ldots, |h(p)|\right\}$ be a P-HFE, $\gamma^+ = max\left(\gamma^k(p_k)\right)$ and $\gamma^- = min\left(\gamma^k(p_k)\right)$ be the largest and least values of the probabilistic hesitant fuzzy element $h(p)$, respectively. p^+ and p^- are the corresponding probability. Then we define the upper area $a(h)^+$ $a(h)^+$ and the lower area $a(h)^-$ of $h(p)$ in Figure 1.*

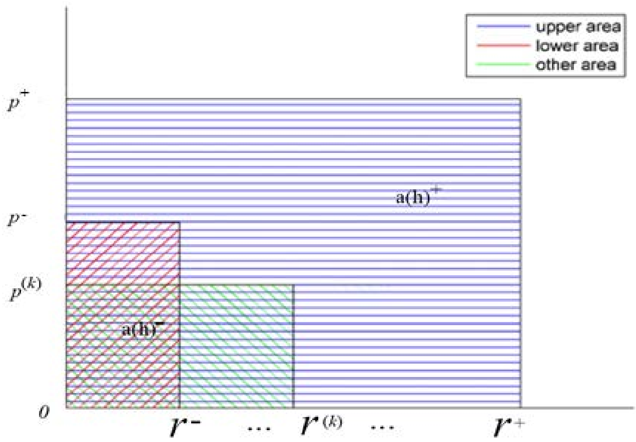

Figure 1. The lower area, upper area and other area of the probabilistic hesitant fuzzy element.

From Definition 6 and Figure 1, and motivated by [19], we put forward a concrete formula for the possibility degree when comparing the P-HFEs as follows:

Definition 7. *Let $h_1(p)$ and $h_2(p)$ be two P-HFEs. The possibility degree of $h_1(p)$ being not less than $h_2(p)$ is defined as:*

$$p(h_1(p) \geq h_2(p)) = 0.5 \cdot \left(1 + \frac{\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_1)^+ - a(h_2)^+\right)}{\left|a(h_1)^- - a(h_2)^-\right| + \left|a(h_1)^+ - a(h_2)^+\right| + a(h_1 \cap h_2)}\right) \tag{1}$$

If $\gamma^-(h_1) > \gamma^-(h_2)$, $\left(a(h_1)^- - a(h_2)^-\right) = |\gamma^-(h_1) \cdot p^-(h_1) - \gamma^-(h_2) \cdot p^-(h_2)|$; if $\gamma^-(h_1) = \gamma^-(h_2)$, then $\left(a(h_1)^- - a(h_2)^-\right) = \gamma^-(h_1) \cdot p^-(h_1) - \gamma^-(h_2) \cdot p^-(h_2)$; if $\gamma^-(h_1) < \gamma^-(h_2)$, then $\left(a(h_1)^- - a(h_2)^-\right) = -|\gamma^-(h_1) \cdot p^-(h_1) - \gamma^-(h_2) \cdot p^-(h_2)|$.

If $\gamma^+(h_1) > \gamma^+(h_2)$, $\left(a(h_1)^+ - a(h_2)^+\right) = |\gamma^+(h_1) \cdot p^+(h_1) - \gamma^+(h_2) \cdot p^+(h_2)|$; if $\gamma^+(h_1) = \gamma^+(h_2)$, then $\left(a(h_1)^+ - a(h_2)^+\right) = \gamma^+(h_1) \cdot p^+(h_1) - \gamma^+(h_2) \cdot p^+(h_2)$; if $\gamma^+(h_1) < \gamma^+(h_2)$, then $\left(a(h_1)^+ - a(h_2)^+\right) = -|\gamma^+(h_1) \cdot p^+(h_1) - \gamma^+(h_2) \cdot p^+(h_2)|$.

What is more, $a(h_1 \cap h_2)$ represents the area of the intersection between $h_1(p)$ and $h_2(p)$.

Remark 1.

(1) *With Equation (1), if two P-HFEs $h_1(p)$ and $h_2(p)$ have no common values in hesitant fuzzy sets, then $p(h_1(p) \geq h_2(p)) = 1$ or $p(h_1(p) \geq h_2(p)) = 0$; if $h_1(p) = h_2(p)$, then we get $p(h_1(p) \geq h_2(p)) = 0.5$.*

(2) *The main innovations of our new method for the P-HFEs' ranking are as follows:*

- *It is based on the structure of the P-HFEs and it considers their full information so that it can avoid the loss of information.*
- *The comparison result can show the relationship between different P-HFEs.*

Definition 8. *If $p(h_1(p) \geq h_2(p)) > p(h_2(p) \geq h_1(p))$, then $h_1(p)$ is superior to $h_2(p)$ with the degree of $p(h_1(p) \geq h_2(p))$, denoted as $h_1(p) \succ^{p(h_1(p) \geq h_2(p))} h_2(p)$; if $p(h_1(p) \geq h_2(p)) = 1$, then $h_1(p)$ is absolutely superior to $h_2(p)$; if $p(h_1(p) \geq h_2(p)) = 0.5$, then $h_1(p)$ is equal to $h_2(p)$, denoted as $h_1(p) \sim h_2(p)$.*

For example, if $h_1(p) = \{0.6(0.4), 0.8(0.6)\}$ and $h_2(p) = \{0.5(0.6), 0.7(0.4)\}$, then by Equation (1), we can obtain $p(h_1(p) \geq h_2(p)) = 0.5 \times (1 + \frac{0.06+0.2}{0.06+0.2+0.04}) = 0.933$. The comparison result implies that $h_1(p)$ is not absolutely superior to $h_2(p)$ and is consistent with our analysis in Section 2.

In the following, we discuss some properties of the possibility degree method:

Property 1 (Complementarity). *If $h_1(p)$ and $h_2(p)$ are two P-HFEs, then $p(h_1(p) \geq h_2(p)) + p(h_2(p) \geq h_1(p)) = 1$; especially, if $h_1(p) = h_2(p)$, then $p(h_1(p) \geq h_2(p)) = p(h_2(p) \geq h_1(p)) = 0.5$.*

Proof. From Equation (1), $p(h_1(p) \geq h_2(p)) + p(h_2(p) \geq h_1(p)) =$

$$= 0.5 \cdot \left(1 + \frac{\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_1)^+ - a(h_2)^+\right)}{\left|a(h_1)^- - a(h_2)^-\right| + \left|a(h_1)^+ - a(h_2)^+\right| + a(h_1 \cap h_2)}\right) + 0.5 \cdot \left(1 + \frac{\left(a(h_2)^- - a(h_1)^-\right) + \left(a(h_2)^+ - a(h_1)^+\right)}{\left|a(h_2)^- - a(h_1)^-\right| + \left|a(h_2)^+ - a(h_1)^+\right| + a(h_2 \cap h_1)}\right)$$

$$= 0.5 \cdot \left(2 + \frac{\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_1)^+ - a(h_2)^+\right) + \left(a(h_2)^- - a(h_1)^-\right) + \left(a(h_2)^+ - a(h_1)^+\right)}{\left|a(h_1)^- - a(h_2)^-\right| + \left|a(h_1)^+ - a(h_2)^+\right| + a(h_1 \cap h_2)}\right).$$

If $\gamma^-(h_1) > \gamma^-(h_2)$, $\left(a(h_1)^- - a(h_2)^-\right) = |\gamma^-(h_1)\cdot p^-(h_1) - \gamma^-(h_2)\cdot p^-(h_2)|$, $\left(a(h_2)^- - a(h_1)^-\right) = |\gamma^-(h_2)\cdot p^-(h_2) - \gamma^-(h_1)\cdot p^-(h_1)| = -|\gamma^-(h_1)\cdot p^-(h_1) - \gamma^-(h_2)\cdot p^-(h_2)|$, then $\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_2)^- - a(h_1)^-\right) = 0$;

If $\gamma^-(h_1) = \gamma^-(h_2)$, $\left(a(h_1)^- - a(h_2)^-\right) = \gamma^-(h_1)\cdot p^-(h_1) - \gamma^-(h_2)\cdot p^-(h_2)$, $\left(a(h_2)^- - a(h_1)^-\right) = \gamma^-(h_2)\cdot p^-(h_2) - \gamma^-(h_1)\cdot p^-(h_1)$, the $\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_2)^- - a(h_1)^-\right) = 0$;

If $\gamma^-(h_1) < \gamma^-(h_2)$, $\left(a(h_1)^- - a(h_2)^-\right) = -|\gamma^-(h_1)\cdot p^-(h_1) - \gamma^-(h_2)\cdot p^-(h_2)|$, $\left(a(h_2)^- - a(h_1)^-\right) = |\gamma^-(h_2)\cdot p^-(h_2) - \gamma^-(h_1)\cdot p^-(h_1)| = |\gamma^-(h_1)\cdot p^-(h_1) - \gamma^-(h_2)\cdot p^-(h_2)|$, then $\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_2)^- - a(h_1)^-\right) = 0$.

If $\gamma^+(h_1) > \gamma^+(h_2)$, $\left(a(h_1)^+ - a(h_2)^+\right) = |\gamma^+(h_1)\cdot p^+(h_1) - \gamma^+(h_2)\cdot p^+(h_2)|$, $\left(a(h_2)^+ - a(h_1)^+\right) = -|\gamma^+(h_2)\cdot p^+(h_2) - \gamma^+(h_1)\cdot p^+(h_1)| = -|\gamma^+(h_1)\cdot p^+(h_1) - \gamma^+(h_2)\cdot p^+(h_2)|$, then $\left(a(h_1)^+ - a(h_2)^+\right) + \left(a(h_2)^+ - a(h_1)^+\right) = 0$;

If $\gamma^+(h_1) = \gamma^+(h_2)$, $\left(a(h_1)^+ - a(h_2)^+\right) = \gamma^+(h_1)\cdot p^+(h_1) - \gamma^+(h_2)\cdot p^+(h_2)$, $\left(a(h_2)^+ - a(h_1)^+\right) = \gamma^+(h_2)\cdot p^+(h_2) - \gamma^+(h_1)\cdot p^+(h_1)$, then $\left(a(h_1)^+ - a(h_2)^+\right) + \left(a(h_2)^+ - a(h_1)^+\right) = 0$;

If $\gamma^+(h_1) < \gamma^+(h_2)$, $\left(a(h_1)^+ - a(h_2)^+\right) = -|\gamma^+(h_1)\cdot p^+(h_1) - \gamma^+(h_2)\cdot p^+(h_2)|$, $\left(a(h_2)^+ - a(h_1)^+\right) = |\gamma^+(h_2)\cdot p^+(h_2) - \gamma^+(h_1)\cdot p^+(h_1)| = |\gamma^+(h_1)\cdot p^+(h_1) - \gamma^+(h_2)\cdot p^+(h_2)|$, then $\left(a(h_1)^+ - a(h_2)^+\right) + \left(a(h_2)^+ - a(h_1)^+\right) = 0$.

From the analysis above, we can obtain that $p(h_1(p) \geq h_2(p)) + p(h_2(p) \geq h_1(p)) = 1$. □

Property 2 (Transitivity). *Let $h_1(p)$, $h_2(p)$ and $h_3(p)$ be three P-HFEs.*

If $p(h_1(p) \geq h_2(p)) > 0.5$ and $p(h_2(p) \geq h_3(p)) \geq 0.5$ or $p(h_1(p) \geq h_2(p)) \geq 0.5$ and $p(h_2(p) \geq h_3(p)) > 0.5$, then $p(h_1(p) \geq h_3(p)) > 0.5$.

If $p(h_1(p) \geq h_2(p)) = 0.5$ and $p(h_2(p) \geq h_3(p)) = 0.5$, then $p(h_1(p) \geq h_3(p)) = 0.5$.

Proof. If $p(h_1(p) \geq h_2(p)) > 0.5$ and $p(h_2(p) \geq h_3(p)) \geq 0.5$, or $p(h_1(p) \geq h_2(p)) \geq 0.5$ and $p(h_2(p) \geq h_3(p)) > 0.5$, then from Equation (1), we can get $\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_1)^+ - a(h_2)^+\right) > 0$ and $\left(a(h_2)^- - a(h_3)^-\right) + \left(a(h_2)^+ - a(h_3)^+\right) \geq 0$, or $\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_1)^+ - a(h_2)^+\right) \geq 0$ and $\left(a(h_2)^- - a(h_3)^-\right) + \left(a(h_2)^+ - a(h_3)^+\right) > 0$, thus, $p(h_1(p) \geq h_3(p)) = 0.5 \cdot \left(1 + \frac{\left(a(h_1)^- - a(h_3)^-\right) + \left(a(h_1)^+ - a(h_3)^+\right)}{\left|a(h_1)^- - a(h_3)^-\right| + \left|a(h_1)^+ - a(h_3)^+\right| + a(h_1 \cap h_3)}\right) = 0.5 \cdot \left(1 + \frac{\left(a(h_1)^- - a(h_2)^-\right) + \left(a(h_2)^- - a(h_3)^-\right) + \left(a(h_1)^+ - a(h_2)^+\right) + \left(a(h_2)^+ - a(h_3)^+\right)}{\left|a(h_1)^- - a(h_3)^-\right| + \left|a(h_1)^+ - a(h_3)^+\right| + a(h_1 \cap h_3)}\right) > 0.5$.

Obviously, if $p(h_1(p) \geq h_2(p)) = 0.5$, $p(h_2(p) \geq h_3(p)) = 0.5$, then we can obtain that: $h_1(p) = h_2(p)$,$h_2(p) = h_3(p)$. So $h_1(p) = h_3(p)$, $p(h_1(p) \geq h_3(p)) = 0.5$. □

Example 3. *Suppose there are three different bands of computers A, B and C, then we use P-HFEs to represent the criterion "performance":*

$$h_1(p) = \{0.5(0.3), 0.6(0.5), 0.7(0.2)\},\ h_2(p) = \{0.6(0.4), 0.7(0.6)\},\ h_3(p) = \{0.4(0.5), 0.5(0.5)\}$$

Based on Equation (1), we can obtain:

$$p(h_1(p) \geq h_2(p)) = 0.5 \times \left(1 + \frac{(-0.09) + (-0.28)}{0.09 + 0.28 + 0.08}\right) = 0.089$$

$$p(h_1(p) \geq h_3(p)) = 0.5 \times \left(1 + \frac{0.05 + 0.11}{0.05 + 0.11 + 0.1}\right) = 0.808$$

$$p(h_2(p) \geq h_3(p)) = 0.5 \times \left(1 + \frac{0.04 + 0.17}{0.04 + 0.17 + 0}\right) = 1$$

Then according to the above analysis, it is easy to obtain

$$p(h_2(p) \geq h_1(p)) = 1 - p(h_1(p) \geq h_2(p)) = 0.5 \times \left(1 + \frac{0.09 + 0.28}{0.09 + 0.28 + 0.08}\right) = 0.911$$

Thus, the ranking results can be derived:

$$h_2(p) \succ^{0.911} h_1(p) \succ^{0.808} \succ h_3(p)$$

The ranking results imply that the computer B is not absolutely superior to the computer A. In fact, the computer B is superior to A with the possibility degree of 0.911. What is more, the computer A is superior to C in some extent with the possibility degree of 0.808 under the criterion "performance".

4. The Novel Ranking Method Based on the Possibility Degree Formula for P-HFEs

At first, we describe the MCDM issues. We define that $x = \{x_1, x_2, \ldots, x_m\}$ is a finite set of alternatives, $c = \{c_1, c_2, \ldots, c_n\}$ is a set of criteria. After that, we invite an expert to express his/her preferences on the alternatives $x = \{x_1, x_2, \ldots, x_m\}$ under the criteria $c = \{c_1, c_2, \ldots, c_n\}$ by the P-HFEs:

$$h_{ij}(p) = \left\{ h_{ij}^{(k)}\left(p_{ij}^{(k)}\right) \middle| k = 1, 2, \ldots, |h(p)|, \sum_{k=1}^{|h(p)|} p_{ij}^{(k)} = 1 \right\}$$

where $h_{ij}^{(k)}$ is the kth value of $h_{ij}(p)$ and $p_{ij}^{(k)}$ is the associated probability. Then we collect all the P-HFEs and make up the probabilistic hesitant fuzzy decision matrix R:

$$R = \left[h_{ij}(p)\right]_{m \times n} = \begin{bmatrix} h_{11}(p) & h_{12}(p) & \ldots & h_{1n}(p) \\ h_{21}(p) & h_{22}(p) & \ldots & h_{2n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1}(p) & h_{m2}(p) & \ldots & h_{mn}(p) \end{bmatrix}$$

Our main purpose is to get the best alternative based on the ranking results. Thus, we use the aggregation operators for P-HFEs and the possibility degree formula. Then, we put forward a new ranking method by the following steps:

Step 1 If the weighting vector is given to us, $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then we could use the probabilistic hesitant fuzzy weighted averaging (PHFWA) operator [15] to aggregate the P-HFEs of the alternatives $x_i\{i = 1, 2, \ldots, m\}$:

$$h_i(p) = PHFWA(h_{i1}(p), h_{i2}(p), \ldots, h_{in}(p)) = w_1 h_{i1}(p) \oplus w_2 h_{i2}(p) \oplus \ldots \oplus w_n h_{in}(p) \quad (2)$$

If we do not know the weights of the criteria, then we can get the weighting vector by the weighting technique for the OWA operator [26]. After that, we can use the PHFWA operator to aggregate the P-HFEs.

Step 2 Construct a possibility degree matrix P by computing $p_{ij} = p\left(h_i(p) \geq h_j(p)\right)$ using Equation (1):

$$P = \left(p_{ij}\right)_{m \times n} = \begin{bmatrix} 0.5 & p_{12} & \ldots & p_{1m} \\ p_{21} & 0.5 & \ldots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \ldots & 0.5 \end{bmatrix}$$

On the one hand, due to the complementarity of the novel possibility degree formula for the P-HFEs, P is a fuzzy complementary judgment matrix. On the other hand, it also has acceptable consistency owing to the transitivity of the new method.

Step 3 Derive the priorities from P for its complementary judgment by employing the exact solution [27]:

$$v = (v_1, v_2, \ldots, v_m)^T = \left(\frac{1}{\sum_{i=1}^{m}(p_{i1}/p_{1i})}, \frac{1}{\sum_{i=1}^{m}(p_{i2}/p_{2i})}, \ldots, \frac{1}{\sum_{i=1}^{m}(p_{im}/p_{1m})}\right)^T \tag{3}$$

Step 4 Let $v' = \left(v_{k_1}, v_{k_2}, \ldots, v_{k_m}\right)^T$ be the descending order of v, thus we can obtain the ranking results of $h_i(p)$, $i = 1, 2, \ldots, m$:

$$h_{k_1}(p) \succ^{p(h_{k_1}(p) \geq h_{k_2}(p))} h_{k_2}(p) \succ \ldots \succ^{p(h_{k_{m-1}}(p) \geq h_{k_m}(p))} \succ h_{k_m}(p)$$

Step 5 Based on the equation above, the ranking results of the alternatives are shown as follows:

$$x_{k_1} \succ^{p(h_{k_1}(p) \geq h_{k_2}(p))} x_{k_2} \succ \ldots \succ^{p(h_{k_{m-1}}(p) \geq h_{k_m}(p))} \succ x_{k_m}$$

5. A Case Study

In this part, we conduct an actual case to prove the reasonability of the novel possibility degree formula for the P-HFEs and the procedure to solve the MCDM problems.

Example 4. *In the background of the limited medical resources and aging tendency of the population in China, we have to analyze which hospital is the best one* [19]*. In this paper, we mainly consider three criteria: the environment of health service* (c_1)*; the treatment optimization* (c_2)*; and the social resource allocation and health services* (c_3)*. The weight vector of the above three factors is* $w = (0.2, 0.1, 0.7)^T$*. We take the following four hospitals as examplse: the West China Hospital of Sichuan University* (h_1)*, the Huashan Hospital of Fudan University* (h_2)*, the Union Medical College Hospital* (h_3) *and the Chinese PLA General Hospital* (h_4)*. As the influence factors are too complicated to be described in just one number, the experts are invited to use the HFSs to express their preferences for four hospitals with respect to the three main criteria. In order to overcome the information loss problem mentioned in the introduction, the experts' preferences can be represented by P-HFEs, which can provide a better description for all their preferences and remain the original information provided by the DMs to the maximum. At last, we build the probabilistic hesitant fuzzy decision matrix* $R = \left[h_{ij}(p)\right]_{m \times n}$ *shown in Table 2.*

Table 2. The evaluations of the four hospitals with P-HFEs.

	c_1	c_2	c_3
h_1	$\{0.5(0.4), 0.7(0.6)\}$	$\{0.9(1)\}$	$\{0.3(0.2), 0.5(0.8)\}$
h_2	$\{0.8(0.3), 0.9(0.7)\}$	$\{0.5(1)\}$	$\{0.8(0.4), 0.9(0.6)\}$
h_3	$\{0.5(1)\}$	$\{0.7(0.5), 0.9(0.5)\}$	$\{0.8(0.6), 0.9(0.4)\}$
h_4	$\{0.8(0.5), 0.9(0.5)\}$	$\{0.3(0.5), 0.6(0.5)\}$	$\{0.7(1)\}$

In this paper, we will evaluate the four hospitals by the procedure mentioned in Section 4.

Step 1 As the weighting vector has already been provided to us, we can use the PHFWA operator [15] to aggregate the evaluation information of the hospitals $h_i (i = 1, 2, 3, 4)$:

$$h_1(p) = \{0.461(0.08), 0.514(0.12), 0.574(0.32), 0.616(0.48)\}$$

$$h_2(p) = \{0.781(0.12), 0.809(0.28), 0.865(0.18), 0.882(0.42)\}$$

$$h_3(p) = \{0.75(0.3), 0.776(0.3), 0.845(0.2), 0.862(0.2)\}$$

$$h_4(p) = \{0.698(0.25), 0.715(0.25), 0.738(0.25), 0.752(0.25)\}$$

Step 2 Build the possibility degree matrix by contrasting each pair of P-HFEs based on Equation (1).

$$P = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.838 & 1 \\ 1 & 0.162 & 0.5 & 0.819 \\ 1 & 0 & 0.181 & 0.5 \end{bmatrix}$$

Step 3 According to the possibility degree matrix P above, we get the rank of $h_i(p)$:

$$h_2(p) \succ^{0.838} \succ h_3(p) \succ^{0.819} h_4(p) \succ^{1} h_1(p)$$

From the result above, we can see that the hospital h_2 is the best alternative.

In order to illustrate that our method is more reasonable and precise, we make a comparison with the traditional ranking method for P-HFEs. According to Zhang et al. [15], for a P-HFE $h(p)$, its score is defined as follows:

$$s(h(p)) = \left(\sum_{l=1}^{|h_1(p)|} \gamma_l \cdot p_l \right) / \left(\sum_{l=1}^{|h_1(p)|} p_l \right)$$

Then, after aggregating the evaluation information of the hospitals $h_i (i = 1, 2, 3, 4)$, we can calculate the scores of the four hospitals. The results are shown as

$$s(h_1) = 0.461 \times 0.08 + 0.514 \times 0.12 + 0.574 \times 0.32 + 0.616 \times 0.48 = 0.578$$

$$s(h_2) = 0.781 \times 0.12 + 0.809 \times 0.28 + 0.865 \times 0.18 + 0.882 \times 0.42 = 0.846$$

$$s(h_3) = 0.75 \times 0.3 + 0.776 \times 0.3 + 0.845 \times 0.2 + 0.862 \times 0.2 = 0.799$$

$$s(h_4) = 0.698 \times 0.25 + 0.715 \times 0.25 + 0.738 \times 0.25 + 0.752 \times 0.25 = 0.726$$

According to the scores of the four hospitals, it is obvious that $s(h_2) > s(h_3) > s(h_4) > s(h_1)$. Then, we can get the ranking of $h_i(p)$: $h_2 \succ h_3 \succ h_4 \succ h_1$.

The ranking results based on the two methods are shown in Table 3. Compared with our ranking results, we can get the same optimal alternative with that of Zhang et al.'s method [15]. However, it is clear that our result contains much more probabilistic information. At the same time, our method is relatively easier. Zhang et al.'s method [15] for P-HFEs' ranking has some deficiencies when two projects have common preference. However, our method is much more reasonable because in most situations we cannot say that one project is absolutely better than another. In other words, our method for comparing P-HFE is more reliable.

Table 3. The ranking results based on the two methods.

	Ranking Order	The Optimal Alternative
The score and deviation method	$h_2 \succ h_3 \succ h_4 \succ h_1$	h_2
The novel possibility degree method	$h_2(p) \succ^{0.838} h_3(p) \succ^{0.819} h_4(p) \succ^{1} h_1(p)$	h_2

6. Conclusions

The P-HFSs have been used widespread in MCDM problems, which include not only several possible values but also the corresponding probabilistic information. It can solve the difficulty in describing the sets of the possible evaluation values with probabilistic information when people are hesitant to provide their preferences in the process of decision making. The existing comparison methods for P-HFEs based on the score function and the deviation degree of every P-HFE are limited and not so precise, because it is not reasonable to say that one P-HFE is absolutely superior to another if they have some common or intersecting values.

In order to overcome the deficiencies discussed above, inspired by the chart technique to analyze the structures of P-HFEs, we propose a novel comparison method for P-HFEs based on a possibility degree formula. Our new comparison method is more precise when facing the situations that two P-HFEs have common or intersecting values. At the same time, the proposed possibility degree formula can realize the optimal sorting under hesitant fuzzy environment and reduce the complexity of the computation effectively. Last but not least, we provide a useful and efficient process to rank alternatives and solve the actual MCDM problems. The results show that our method is not only relatively easier and more efficacious, but also is more precise and can contain much more probabilistic information.

In the future, we will focus on some more useful comparison methods for P-HFEs to improve the degree of differentiation among the P-HFEs. We will also consider how to combine the new comparison method for P-HFEs with other decision-making methods to solve MCDM problems in practical applications.

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