symmetry

MDPI

Article

Complex Fuzzy Geometric Aggregation Operators

Lvqing Bi [1], Songsong Dai [2,*] and Bo Hu [2,3]

[1] School of Electronics and Communication Engineering, Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China; bilvqing108@163.com

[2] School of Information Science and Engineering, Xiamen University, Xiamen 361005, China; hb.gznu@gmail.com

[3] School of Mechanical and Electrical Engineering, Guizhou Normal University, Guiyang 550025, China

* Correspondence: ssdai@stu.xmu.edu.cn; Tel.: +86-592-258-0135

Received: 15 June 2018; Accepted: 28 June 2018; Published: 2 July 2018

check for updates

Abstract: A complex fuzzy set is an extension of the traditional fuzzy set, where traditional [0,1]-valued membership grade is extended to the complex unit disk. The aggregation operator plays an important role in many fields, and this paper presents several complex fuzzy geometric aggregation operators. We show that these operators possess the properties of rotational invariance and reflectional invariance. These operators are also closed on the upper-right quadrant of the complex unit disk. Based on the relationship between Pythagorean membership grades and complex numbers, these operators can be applied to the Pythagorean fuzzy environment.

Keywords: complex fuzzy sets; aggregation operator; complex fuzzy geometric operators; rotational invariance; reflectional invariance

1. Introduction

Ramot et al. [1] introduced the innovative concept of complex fuzzy sets (CFSs), which is an extension of the traditional fuzzy sets [2] where traditional unit interval [0,1]-valued membership degrees are extended to the complex unit disk. CFSs are completely distinct from the fuzzy complex numbers discussed by Buckley [3–5]. The complex-valued membership grade has an amplitude term with the addition of a phase term. The phase term of complex-valued membership grade is the key feature which essentially distinguishes complex fuzzy sets from other extensions of fuzzy sets. Ramot et al. [1,6] then introduced several operators of CFSs and a novel framework for complex fuzzy reasoning. Hu et al. [7] introduced the orthogonality relation for CFSs. Bi et al. [8] proposed the parallelity of CFSs and the parallelity-preserving operators. Zhang et al. [9] proposed the δ-equalities for CFSs. Alkouri and Salleh [10] and Hu et al. [11] defined several distances between CFSs. Tamir et al. [12] proposed a new interpretation of complex membership degree. They [13] then proposed complex fuzzy propositional and first-order logics. Dick [14] proposed the concept of rotational invariance for complex fuzzy operators. Recently, several scholars have developed extensions of CFSs. Greenfield et.al [15,16] introduced interval-valued complex fuzzy sets. Alkouri and Saleh [17] proposed complex intuitionistic fuzzy sets. Ali and Smarandache [18] introduced complex neutrosophic sets. Recently, CFSs and their extensions have been successfully applied in many fields, such as time series prediction [19–22], decision making [23], signal processing [1,7,9], and image restoration [24].

Yager and Abbsocv [25] discussed the relationship between CFSs and Pythagorean fuzzy sets (PFSs), which was developed by Yager [26,27] as an extension of Atanssov's intuitionistic fuzzy sets [28]. They showed that Pythagorean fuzzy membership grades can be viewed as complex numbers on the upper-right quadrant of the complex unit disk, named $\Pi - i$ numbers.

Dick, Yager, and Yazdanbahksh [29] then discussed several lattice-theoretic properties of PFSs and CFSs. Quantum information processing also allows for meaningful aggregation using complex numbers. Since qubits can be represented by unit vectors in the two-dimensional complex Hilbert space, geometric information or vector aggregation are used for meaningful clustering [30,31].

The information aggregation operator plays an important role in many fields of decision making. In the past several decades, many aggregation techniques for decision making have been developed. The ordered weighted averaging (OWA) operator introduced by Yager [32] is one of the well-known aggregation operators. Many different aggregation techniques have been applied in many different fuzzy environments, such as intuitionistic [33–35], Pythagorean [36–38], neutrosophic [39–41], interval-valued intuitionistic [42–45], and hesitant fuzzy environments [46–48].

As mentioned in [19], CFSs are suitable to represent information with uncertainty and periodicity, and thus this information aggregation procedure needs to simultaneously process the uncertainty and periodicity in the data. However, comparatively few aggregation techniques have been made in the complex fuzzy environment. Ramot et al. [6] defined the complex fuzzy aggregation operations as vectors aggregation. In particular, the complex weights are used in their definition. Ma et al. [19] developed a product-sum aggregation operator for multiple periodic factor prediction problems. They proved the continuity of this operator. However, they did not focus on techniques for complex fuzzy information aggregation in these two articles.

In this paper, we study aggregation operators in the complex fuzzy environment. Dick's [14] concept of rotational invariance is an intuitive and desirable feature for complex fuzzy operators. This feature is examined for complex fuzzy aggregation operators. This paper proposes a novel feature for complex fuzzy aggregation operators called *reflectional invariance*. Moreover, we study the aggregation operators of complex numbers in the upper-right quadrant of the complex unit disk.

The main contributions of the study include: (1) A concept of reflectional invariance for complex fuzzy operators. (2) Several complex fuzzy weighted geometric operators; we also show that these operators can also be used in a Pythagorean fuzzy environment. (3) A target location method which involves the complex fuzzy aggregation operators.

This paper is organized as follows. In Section 2, we review some basic and fundamental concepts of CFSs, rotational invariance, reflectional invariance, and Ramot et al.'s [6] complex fuzzy aggregation operators. In Section 3 we study the complex fuzzy weighted geometric (CFWG) operator on CFSs and its properties. In Section 4, we develop the complex fuzzy ordered weighted geometric (CFOWG) operator based on the traditional partial ordering by the modulus of a complex number, and study its properties. In Section 5, we study these operators in the domain of $\Pi - i$ numbers which belong to the upper-right quadrant of the complex unit disk. In Section 6, we present an application example in a target location. Conclusions are made in Section 7.

2. Preliminaries

In this section, we present some basic material, including the concepts of CFSs [1], rotational invariance [14], reflectional invariance, and complex fuzzy aggregation operators [1].

2.1. Complex Fuzzy Sets

Ramot et al. [1] defined the concept of CFSs as follows.

Definition 1 ([1]). *Let X be a universe, D be the set of complex numbers whose modulus is less than or equal to 1, i.e.,*

$$D = \{a \in \mathcal{C} \,|\, |a| \leq 1\},$$

a complex fuzzy set A defined on X is a mapping: $X \to D$, which can be denoted as below:

$$A = \{ < x, t_A(x) \cdot e^{j\nu_A(x)} > |x \in X\}, \tag{1}$$

where $j = \sqrt{-1}$, *the amplitude term* $t_A(x)$ *and the phase term* $\nu_A(x)$ *are both real-valued, and* $t_A(x) \in [0,1]$.

For convenience, we only consider the complex numbers on D, called complex fuzzy values (CFVs). Let $a = t_a \cdot e^{j\nu_a}$ be a CFV, then the amplitude of a is denoted by t_a and the phase of a is denoted by ν_a. They are both real-valued, $t_a \in [0,1]$. The modulus of a is t_a, denoted by $|a|$.

Let $a = t_a \cdot e^{j\nu_a}$ and $b = t_b \cdot e^{j\nu_b}$ be two CFVs, then we have the following two commonly used binary operations.

(i) Algebraic product:

$$a \cdot b = t_a \cdot t_b \cdot e^{j(\nu_a+\nu_b)}. \tag{2}$$

(ii) Average:

$$\frac{a+b}{2} = \frac{t_a \cos\nu_a + t_b \cos\nu_b}{2} + j\frac{t_a \sin\nu_a + t_b \sin\nu_b}{2}. \tag{3}$$

The partial ordering of CFVs is the traditional partial ordering by the modulus of a complex number, that is, $a \le b$ if and only if $|a| \le |b|$, equivalently, $t_a \le t_b$.

2.2. Rotational Invariance and Reflectional Invariance

Let $a = t_a \cdot e^{j\nu_a}$ be a CFV, then we have the following two commonly used unary operations:

(i) the rotation of a by θ radians, denoted $Rot_\theta(a)$, is defined as

$$Rot_\theta(a) = t_a \cdot e^{j(\nu_a+\theta)}; \tag{4}$$

(ii) the reflection of a, denoted $Ref(a)$, is defined as

$$Ref(a) = t_a \cdot e^{j-\nu_a}. \tag{5}$$

Then, based on the rotation operation, Dick [14] introduced the concept of rotational invariance for complex fuzzy operators.

Definition 2 ([14])**.** *A function* $f : D^2 \to D$ *is rotationally invariant if and only if*

$$f\big(Rot_\theta(a), Rot_\theta(b)\big) = Rot_\theta\big(f(a,b)\big), \tag{6}$$

for any θ.

We extend the above concept to multivariate operators.

Definition 3. *Let* $f : D^n \to D$ *be an n-order function.* f *is rotationally invariant if and only if*

$$f\big(Rot_\theta(a_1), Rot_\theta(a_2), ..., Rot_\theta(a_n)\big) = Rot_\theta\big(f(a_1,a_2,...,a_n)\big), \tag{7}$$

for any θ.

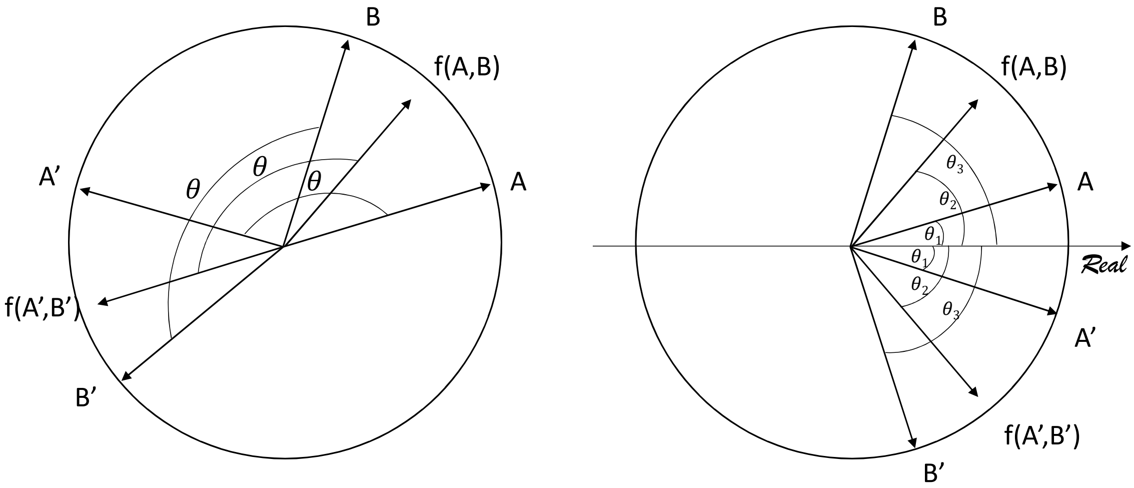
In particular, since the periodicity of complex-valued membership grade, that is, $Rot_{2\pi}(x) = x$ for any $x \in D$, we have $f\big(Rot_{2\pi}(a_1), Rot_{2\pi}(a_2), ..., Rot_{2\pi}(a_n)\big) = f(a_1,a_2,...,a_n) = Rot_{2\pi}\big(f(a_1,a_2,...,a_n)\big)$. This is a special case of rotational invariance.

Similar to the above definition, we introduce the concept of reflectional invariance for complex fuzzy operators based on the reflection operation.

Definition 4. *Let* $f : D^n \to D$ *be an n-order function.* f *is reflectionally invariant if and only if*

$$f\big(Ref(a_1), Ref(a_2), ..., Ref(a_n)\big) = Ref\big(f(a_1,a_2,...,a_n)\big). \tag{8}$$

Rotational invariance and reflectional invariance are intuitive properties for complex fuzzy operators. It makes a great deal of sense that a operator is invariant under a rotation or a reflection. If we rotate two vectors by a common value, rotational invariance states that an aggregation of those vectors will be rotated by the same value, as shown in Figure 1a. If we reflect two vectors, reflectional invariance states that an aggregation of those vectors will be reflected as well, as shown in Figure 1b.



(**a**) (**b**)

Figure 1. (**a**) Rotational invariance and (**b**) reflectional invariance.

Reflectional invariance and rotational invariance are two properties which are only concerned with the phase of CFVs.

These two properties of the algebraic product and average operators were examined, and the results are given as follows.

Theorem 1 ([14])**.** *The algebraic product is not rotationally invariant.*

Theorem 2. *The algebraic product is reflectionally invariant.*

Proof. For any $a, b \in D$, we have

$$Ref(a) \cdot Ref(b) = t_a \cdot e^{j-\nu_a} \cdot t_b \cdot e^{j-\nu_b} = t_a \cdot t_b \cdot e^{j(-\nu_a-\nu_b)},$$

$$Ref(a \cdot b) = Ref\left(t_a \cdot t_b \cdot e^{j(\nu_a+\nu_b)}\right) = t_a \cdot t_b \cdot e^{j(-\nu_a-\nu_b)},$$

then $Ref(a) \cdot Ref(b) = Ref(a \cdot b)$. □

Theorem 3. *The average operator is reflectionally invariant and rotationally invariant.*

Proof. (i) Let $a = r_a + j\omega_a, b = r_b + j\omega_b \in D$. We have

$$\frac{Ref(a) + Ref(b)}{2} = \frac{r_a + r_b}{2} + j\frac{-\omega_a - \omega_b}{2} = \frac{r_a + r_b}{2} - j\frac{\omega_a + \omega_b}{2} = Ref(\frac{a+b}{2}).$$

Then, the average operator is reflectionally invariant.

(ii) For any $a, b \in D$, we have

$$\frac{a \cdot e^{j\theta} + b \cdot e^{j\theta}}{2} = \frac{(a+b) \cdot e^{j\theta}}{2} = \frac{(a+b)}{2} \cdot e^{j\theta}.$$

Then, the average operator is rotationally invariant. □

2.3. Complex Fuzzy Aggregation Operators

Ramot et al. [6] defined the aggregation operation on CFSs as vector aggregation:

Definition 5 ([6]). *Let $A_1, A_2, ..., A_n$ be CFSs defined on X. Vector aggregation on $A_1, A_2, ..., A_n$ is defined by a function v.*

$$v: \quad D^n \to D. \tag{9}$$

The function v produces an aggregate CFS A by operating on the membership grades of $A_1, A_2, ..., A_n$ for each $x \in X$. For all $x \in X$, v is given by

$$\begin{aligned} \mu_A(x) &= v\Big(t_{A_1} \cdot e^{j\nu_{A_1}}, t_{A_2} \cdot e^{j\nu_{A_2}}, \cdots, t_{A_n} \cdot e^{j\nu_{A_n}}, \Big) \\ &= \sum_{i=1}^{n} \Big(w_i \cdot t_{A_i} \cdot e^{j\nu_{A_i}}\Big), \end{aligned} \tag{10}$$

where $w_i \in D$ for all i, and $\sum_{i=1}^{n} |w_i| = 1$.

The complex weights are used in Ramot et al.'s definition for the purpose of maintaining a definition that is as general as possible. In this paper, we only discuss the complex fuzzy aggregation operations with real-valued weights.

We notice that the above definition of Ramot et al's [6] aggregation operator is a complex fuzzy weighted arithmetic (CFWA) operator. For convenience, let $a_1, a_2, ..., a_n$ be CFVs. The CFWA operator is given as

$$CFWA(a_1, a_2, ..., a_n) = \sum_{i=1}^{n} (w_i \cdot a_i), \tag{11}$$

where $w_i \in [0,1]$ for all i, and $\sum_{i=1}^{n} w_i = 1$.

When $a_1, a_2, ..., a_n \in [0,1]$, the CFWA operator can reduce to a traditional fuzzy weighted arithmetic operator.

When $w_i = 1/n$ for all i, then the CFWA operator is the arithmetic average of $(a_1, a_2, ..., a_n)$, denoted by the complex fuzzy arithmetic average (CFAA) operator. That is,

$$CFAA(a_1, a_2, ..., a_n) = \frac{1}{n} \cdot \sum_{i=1}^{n} a_i. \tag{12}$$

When $a_1, a_2, ..., a_n \in [0,1]$ and $w_i = 1/n$ for all i, the CFAA operator is the arithmetic mean of numbers on [0,1].

As a special case of the CFWA operator, note that the average operator on CFVs is reflectionally invariant and rotationally invariant (see Theorem 3). We show that the CFWA operator also possesses these two properties.

Theorem 4. *The CFWA operator is reflectionally invariant and rotationally invariant.*

Proof. (i) Let $a_1 = r_{a_1} + j\omega_{a_1}, a_2 = r_{a_2} + j\omega_{a_2}, \cdots, a_n = r_{a_n} + j\omega_{a_n}$ be CFVs. We have

$$\begin{aligned} Ref\big(CFWA(a_1, a_2, ..., a_n)\big) &= Ref\Big(\sum_{i=1}^{n}\big(w_i \cdot r_{a_i}\big) + j\sum_{i=1}^{n}\big(w_i \cdot \omega_{a_i}\big)\Big) \\ &= \sum_{i=1}^{n}\big(w_i \cdot r_{a_i}\big) - j\sum_{i=1}^{n}\big(w_i \cdot \omega_{a_i}\big) \\ &= \sum_{i=1}^{n}\big(w_i \cdot (r_{a_i} - \omega_{a_i})\big) \\ &= \sum_{i=1}^{n}\big(w_i \cdot Ref(a_i)\big). \end{aligned}$$

Then, the CFWA operator is reflectionally invariant.

(ii) For any CFVs $a_1, a_2, ..., a_n$, we have

$$\begin{aligned} CFWA(a_1 \cdot e^{j\theta}, a_2 \cdot e^{j\theta}, ..., a_n \cdot e^{j\theta}) &= w_1 \cdot a_1 \cdot e^{j\theta} + w_2 \cdot a_2 \cdot e^{j\theta} + ... + w_n \cdot a_n \cdot e^{j\theta} \\ &= (w_1 \cdot a_1 + w_2 \cdot a_2 + ... + w_n \cdot a_n) \cdot e^{j\theta} \\ &= CFWA(a_1, a_2, ..., a_n) \cdot e^{j\theta}. \end{aligned}$$

Then, the CFWA operator is rotationally invariant. □

3. Complex Fuzzy Weighted Geometric Operators

In this section, we introduce the weighted geometric aggregation operators in a complex fuzzy environment and discuss their fundamental characteristics.

Definition 6. *Let $a_1, a_2, ..., a_n$ be CFVs, a complex fuzzy weighted geometric (CFWG) operator is defined as:*

$$CFWG(a_1, a_2, ..., a_n) = \prod_{i=1}^{n} a_i^{w_i}, \tag{13}$$

where $w_i \in [0, 1]$ for all i, and $\sum_{i=1}^{n} w_i = 1$.

Denoting $CFWG(a_1, a_2, ..., a_n) = t \cdot e^{j\nu}$, we have a weighted geometric aggregation (WGA) operator on [0,1], that is, $t = \prod_{i=1}^{n} t_{a_i}^{w_i}$ and a weighted arithmetic aggregation (WAA) operator on $\mathbb{R}$, that is, $\nu = \sum_{i=1}^{n} w_i \cdot \nu_{a_i}$.

When $a_1, a_2, ..., a_n \in [0, 1]$, the CFWG operator can reduce to a traditional fuzzy weighted geometric operator.

When $w_i = 1/n$ for all i, then $t = \sqrt[n]{t_{a_1} \cdot t_{a_2} \cdots t_{a_n}}$ is the geometric mean of real numbers on unit interval [0,1], $\nu = \frac{1}{n} \cdot \sum_{i=1}^{n} \nu_{a_i}$ is the arithmetic mean of real numbers on $\mathbb{R}$.

When $a_1, a_2, ..., a_n \in [0, 1]$ and $w_i = 1/n$ for all i, the CFWG operator is the geometric mean of real numbers on unit interval [0,1].

Theorem 5. *Let $a_1, a_2, ..., a_n$ be CFVs, then the aggregated value $CFWG(a_1, a_2, ..., a_n)$ is also a complex fuzzy value.*

Proof. Since $|CFWG(a_1, a_2, ..., a_n)| = \prod_{i=1}^{n} t_{a_i}^{w_i}$, which is a weighted arithmetic aggregation operator on unit interval [0,1], we have $|CFWG(a_1, a_2, ..., a_n)| \leq 1$. □

Similar to Theorem 4, the CFWA operator is reflectionally invariant and rotationally invariant. We show that the CFWG operator also possesses these two properties.

Theorem 6. *The CFWG operator is reflectionally invariant and rotationally invariant.*

Proof. (i) For any CFVs $a_1, a_2, ..., a_n$, we have

$$\begin{aligned} Ref\big(CFWG(a_1, a_2, ..., a_n)\big) &= Ref\Big(\prod_{i=1}^{n} t_{a_i}^{w_i} \cdot e^{j\sum_{i=1}^{n} w_i \cdot \nu_{a_i}}\Big) \\ &= \prod_{i=1}^{n} t_{a_i}^{w_i} \cdot e^{j-\sum_{i=1}^{n} w_i \cdot \nu_{a_i}} \\ &= \prod_{i=1}^{n} t_{a_i}^{w_i} \cdot e^{j\sum_{i=1}^{n} w_i \cdot (-\nu_{a_i})} \\ &= \prod_{i=1}^{n} Ref(a_i)^{w_i} \\ &= CFWG\big(Ref(a_1), Ref(a_2), ..., Ref(a_n)\big); \end{aligned}$$

(ii) and

$$\begin{aligned} CFWG(a_1 \cdot e^{j\theta}, a_2 \cdot e^{j\theta}, ..., a_n \cdot e^{j\theta}) &= a_1^{w_1} \cdot e^{jw_1\theta} \cdot a_2^{w_1} \cdot e^{jw_2\theta} \cdot ... \cdot a_n^{w_n} \cdot e^{jw_n\theta} \\ &= \Big(\prod_{i=1}^{n} a_i^{w_i}\Big) \cdot e^{j(w_1\cdot\theta + w_2\cdot\theta + ... + w_n\cdot\theta)} \\ &= CFWG(a_1, a_2, ..., a_n) \cdot e^{j\theta}. \end{aligned}$$

□

Idempotency, boundedness, and monotonicity are three important properties of aggregation operators. The CFWG operator satisfies the following properties.

Theorem 7. *Let $a_1, a_2, ..., a_n$, $b_1, b_2, ..., b_n$ be CFVs, CFWG weights are real values, that is, $w_i \in [0, 1]$ for all i, and $\sum_{i=1}^{n} w_i = 1$. Then, we have the following:*

(1) (Idempotency): If $a_1 = a_2 = ... = a_n$ then

$$CFWG(a_1, a_2, ..., a_n) = a_1.$$

(2) (Amplitude boundedness):

$$\big|CFWG(a_1, a_2, ..., a_n)\big| \le a,$$

where $a = \max_i |a_i|$.

(3) (Amplitude monotonicity): If $|a_i| \le |b_i|$ $i = 1, 2, ..., n$, then

$$|CFWAA(a_1, a_2, ..., a_n)| \le |CFWAA(b_1, b_2, ..., b_n)|.$$

Proof. (1) Trivial form the facts that both WAA operator on [0,1] and WGA operator on $\mathbb{R}$ satisfy the property of idempotency.

(2) Trivial form the facts that $|CFWG(a_1, a_2, ..., a_n)| = \prod_{i=1}^{n} t_{a_i}^{w_i}$ and WGA operator on $\mathbb{R}$ satisfy the property of boundedness.

(3) Trivial form the facts that $|CFWG(a_1, a_2, ..., a_n)| = \prod_{i=1}^{n} t_{a_i}^{w_i}$ and WGA operator on $\mathbb{R}$ satisfy the property of monotonicity. □

In this paper, for complex fuzzy aggregation operators, boundedness and monotonicity are restricted exclusively to the amplitude of CFVs. They are two properties which are only concerned with the amplitude of CFVs. Idempotency is a property that is concerned with both the phase and amplitude of CFVs.

It is easy to prove that the CFWA operator satisfies idempotency and amplitude boundedness, but it does not satisfy the property of amplitude monotonicity.

Example 1. *Let* $a_1 = 0.4$, $a_2 = 0.4 \cdot e^{j2\pi/3}$, $b_1 = b_2 = 0.3$ *and weights be* $w_1 = w_2 = 0.5$. *Then,*

$$\begin{aligned} CFWA(a_1, a_2) &= 0.5 \cdot 0.4 + 0.5 \cdot 0.4 \cdot e^{j2\pi/3} \\ &= 0.2 \cdot e^{j\pi/3} \end{aligned}$$

and $CFWA(b_1, b_2) = 0.3$. *Then,* $|a_1| \geq |b_1|$, $|a_2| \geq |b_2|$, *but* $|CFWA(a_1, a_2)| \leq |CFWA(b_1, b_2)|$.

4. Complex Fuzzy Ordered Weighted Geometric Operators

Based on the partial ordering of complex numbers, we propose a complex fuzzy ordered weighted geometric (CFOWG) operator as follows:

Definition 7. *Let* $a_1, a_2, ..., a_n$ *be CFVs, a CFOWG operator is defined as*

$$CFOWG(a_1, a_2, ..., a_n) = \prod_{i=1}^{n} a_{\sigma(i)}^{w_i}, \tag{14}$$

where $w_i \in [0, 1]$ *for all i, and* $\sum_{i=1}^{n} w_i = 1$, $(\sigma(1), \sigma(2), ..., \sigma(n))$ *is a permutation of* $(1, 2, ..., n)$ *such that* $|a_{\sigma(i-1)}| \geq |a_{\sigma(i)}|$ *for all i.*

Especially when $w_i = 1/n$ for all i, then the CFOWG operator is reduced to the CFWG operator. Similar to Theorem 5, we have the following.

Theorem 8. *Let* $a_1, a_2, ..., a_n$ *be CFVs, then the aggregated value* $CFOWG(a_1, a_2, ..., a_n)$ *is also a complex fuzzy value.*

Similar to Theorem 6, the CFWG operator is reflectionally invariant and rotationally invariant. The CFOWG operator also possesses these two properties.

Theorem 9. *The CFOWG operator is reflectionally invariant and rotationally invariant.*

Similar to Theorem 7, the CFOWG operator satisfies idempotency, amplitude boundedness, and amplitude monotonicity.

Theorem 10. *Let* $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ *be CFVs, CFOWAA weights are real values, that is,* $w_i \in [0, 1]$ *for all i, and* $\sum_{i=1}^{n} w_i = 1$. *Then, we have the following:*

(1) (Idempotency): If $a_1 = a_2 = ... = a_n$, *then*

$$CFOWG(a_1, a_2, ..., a_n) = a_1.$$

(2) (Boundedness):

$$\left|CFOWG(a_1, a_2, ..., a_n)\right| \leq a,$$

where $a = \max_i |a_i|$.

(3) (Monotonicity): If $|a_i| \leq |b_i|$ $i = 1, 2, ..., n$, *then*

$$|CFOWG(a_1, a_2, ..., a_n)| \leq |CFWAA(b_1, b_2, ..., b_n)|.$$

Besides the above properties, the CFOWG operator has the following.

Theorem 11. *Let* $a_1, a_2, ..., a_n$ *be CFVs, CFOWG weights are real values, that is,* $w_i \in [0, 1]$ *for all i, and* $\sum_{i=1}^{n} w_i = 1$. *Then, we have the following:*

(1) *If* $w = (1, 0, ..., 0)$, *then*

$$|CFOWG(a_1, a_2, ..., a_n)| = \max_i |a_i|;$$

(2) *If* $w = (0, 0, ..., 1)$, *then*

$$|CFOWG(a_1, a_2, ..., a_n)| = \min_i |a_i|;$$

(3) *If* $w_i = 1, w_k = 0, k \neq i$, *then*

$$|CFOWG(a_1, a_2, ..., a_n)| = |a_{\sigma(i)}|,$$

where $a_{\sigma(i)}$ *is the i-th (modulus-based) largest of* a_1,a_2, ...,a_n.

Now we give a brief summary of the properties of the CFWG and CFOWG operators with real-valued weights. The results are summarized in Table 1, in which $\surd$ and $\times$ represent that the corresponding property holds and does not hold, respectively.

Table 1. Properties of the complex fuzzy aggregation operators. $\surd$ and $\times$ represent that the corresponding property holds and does not hold, respectively.

	Idempotency	**Amplitude Boundedness**	**Amplitude Monotonicity**	**Reflectional Invariance**	**Rotational Invariance**
CFAA	$\surd$	$\surd$	$\times$	$\surd$	$\surd$
CFWA	$\surd$	$\surd$	$\times$	$\surd$	$\surd$
CFWG	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
CFOWG	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

5. Complex Fuzzy Values and Pythagorean Fuzzy Numbers

Yager and Abbasov [25] showed that Pythagorean membership grades can be expressed using complex numbers, called $\Pi - i$ numbers, which belong to the upper-right quadrant of the complex unit disk. Essentially, the CFWAA and CFOWAA operators are used to deal with special complex numbers, which belong to the complex unit disk.

In this section, we consider the CFWG and CFOWG operators in the domain of $\Pi - i$ numbers.

We first recall the concepts of Pythagorean fuzzy sets (PFSs) and $\Pi - i$ numbers.

Definition 8 ([25]). *Let X be a universe. A PFS A is defined by*

$$A = \{ < x, p_A(x), \nu_A(x), > | x \in X\}, \tag{15}$$

where $p_A(x) \in [0, 1]$ *and* $\nu_A(x) \in [0, 1]$ *respectively represent the membership grade and nonmembership grade of the element x to set A, such that*

$$0 \leq (p_A(x))^2 + (\nu_A(x))^2 \leq 1$$

for all $x \in X$. *The degree of indeterminacy of the element x to set A is* $\pi_A(x)$, *defined by*

$$\pi_A(x) = \sqrt{1 - (p_A(x))^2 - (\nu_A(x))^2}.$$

For convenience, Zhang and Xu [49] referred to $(p_A(x), \nu_A(x))$ as a Pythagorean fuzzy number (PFN) simply denoted by $a = (p_a, \nu_a)$, where $p_a \in [0, 1], \nu_a \in [0, 1]$ and $(p_a)^2 + (\nu_a)^2 \leq 1$.

Yager and Abbasov [25] discussed the relationship between Pythagorean membership grades and complex numbers. They showed that the complex numbers of the form $z = r \cdot e^{j\theta}$ with conditions

$r \in [0,1]$ and $\theta \in [0, \pi/2]$ can be interpretable as PFNs $(r\cos\theta, r\sin\theta)$. They referred to these complex numbers as "$\Pi - i$ numbers", which are complex numbers in the upper-right quadrant of the complex unit disk.

As explained in [25], we should consider which aggregation operation is closed under $\Pi - i$ numbers.

Let us consider the closeness of $\Pi - i$ numbers under the CFWG and CFOWG operations. For the CFWG operator, we have the following result.

Theorem 12. *Let $z_1, z_2, ..., z_n$ be $\Pi - i$ numbers, and the CFWG weights are real values, that is, $w_i \in [0,1]$ for all i, and $\sum_{i=1}^n w_i = 1$. Then, the aggregated value $CFWG(z_1, z_2, ..., z_n)$ is also a $\Pi - i$ number.*

Proof. Denoting $CFWG(z_1, z_2, ..., z_n) = t \cdot e^{j\nu} = \prod_{i=1}^n t_{z_i}^{w_i} \cdot e^{j\sum_{i=1}^n w_i \cdot \nu_{z_i}}$, from Theorem 2, we have $t = |CFWG(z_1, z_2, ..., z_n)| \leq 1$.

Since $\nu = \sum_{i=1}^n w_i \cdot \nu_{z_i}$ is a weighted geometric aggregation (WGA) operator of real numbers on $[0, \pi/2]$, then we have $\nu \in [0, \pi/2]$. Thus, $CFWG(z_1, z_2, ..., z_n)$ is also a $\Pi - i$ number. □

Similar to the above Theorem, we have the following.

Theorem 13. *Let $z_1, z_2, ..., z_n$ be $\Pi - i$ numbers, and the CFOWG weights are real values, that is, $w_i \in [0,1]$ for all i, and $\sum_{i=1}^n w_i = 1$. Then, the aggregated value $CFOWG(z_1, z_2, ..., z_n)$ is also a $\Pi - i$ number.*

The above theorems show us that the CFWG and the CFOWG operators are closed under $\Pi - i$ numbers. When PFNs are interpreted as $\Pi - i$ numbers, then we can aggregate these PFNs to a PFN by using the CFWG or CFOWG operator.

From the above theorems, the CFWG and the CFOWG operators are closed on the upper-right quadrant of complex unit disk.

Consider other quadrants of the complex unit disk. Let

$$D_k = \left\{ z = r \cdot e^{j\theta} | r \in [0,1], \theta \in [\frac{(k-1)\pi}{2}, \frac{k\pi}{2}] \right\}$$

for $k = 1$ to 4. D_1 is the set of $\Pi - i$ numbers.

Now, we discuss the closeness of the CFWG and the CFOWG operators on other quadrants of the complex unit disk. Plainly, we have the following.

Theorem 14. *For any $k \in \{1, 2, 3, 4\}$, if $z_1, z_2, ..., z_n \in D_k$, and the weights are real values, that is, $w_i \in [0,1]$ for all i, and $\sum_{i=1}^n w_i = 1$. Then, we have*

$$CFWG(z_1, z_2, ..., z_n) \in D_k,$$

$$CFOWG(z_1, z_2, ..., z_n) \in D_k.$$

Proof. Similar to Theorem 13. □

Theorem 15. *For any $k \in \{1, 2, 3, 4\}$, if $z_1, z_2, ..., z_n, y_1, y_2, ..., y_n \in D_k$, and the weights are real values, that is, $w_i \in [0,1]$ for all i, and $\sum_{i=1}^n w_i = 1$. Then, we have the following:*

(1) (Idempotency): If $z_1 = z_2 = ... = z_n$, then

$$CFWG(z_1, z_2, ..., z_n) = z_1,$$

$$CFOWG(z_1, z_2, ..., z_n) = z_1.$$

(2) *(Amplitude boundedness):*

$$\left|CFWG(z_1, z_2, \ldots, z_n)\right| \leq z,$$
$$\left|CFOWG(z_1, z_2, \ldots, z_n)\right| \leq z,$$

where $z = \max_i |z_i|$.

(3) *(Amplitude monotonicity): If* $|z_i| \leq |y_i|$ $i = 1, 2, \ldots, n$, *then*

$$|CFWG(a_1, a_2, \ldots, a_n)| \leq |CFWG(y_1, y_2, \ldots, y_n)|,$$

$$|CFOWG(a_1, a_2, \ldots, a_n)| \leq |CFWG(y_1, y_2, \ldots, y_n)|.$$

Proof. Similar to Theorem 7. □

6. Example Application

In this section, we consider a target location application of the complex fuzzy aggregation operator. We do not intend to show the potential advantages of using complex fuzzy aggregation methods in comparison with existing alternative aggregation approaches in this section.

Assume the observer position is fixed. Using a position sensor and an angular sensor, the observer measures the distance and angle of the fixed target. To improve the target location accuracy, the observer repeatedly measures the same target. Then, the target position is estimated according to aggregation theory.

Assume n measurements $\left((d_1, \theta_1), (d_2, \theta_2), \ldots, (d_n, \theta_n)\right)$ have been measured by the observer. The target position is estimated in the following five stages, as illustrated in Figure 2.

Step 1 Complexification of the measured results; each measurement is represented as $c_i = d_i \cdot e^{\theta_i}$.

Step 2 (Fuzzification) Normalize the amplitudes of all measurements. Let $d = \max_i d_i$, for each c_i, the normalized result is $a_i = c_i/d$, where $t_{a_i} = d_i/d$.

Step 3 (Aggregation) Produce an aggregate result. For simplicity, using the CFWG operator with weights $(1/n, 1/n, \ldots, 1/n)$. We obtain

$$a = \prod_{i=1}^{n} a_i^{1/n}, \tag{16}$$

where $t_a = \sqrt[n]{t_{a_1} \cdot t_{a_2} \cdots t_{a_n}}$ and $\nu_a = \frac{1}{n} \cdot \sum_{i=1}^{n} \theta_i$.

Step 4 (Defuzzification) Calculate $c = a \cdot d$, where $t_c = t_a \cdot d$.

Step 5 Decomplexification (or sometimes "realification") of c. We get the target position (p, ν), where $p = t_c, \nu = \nu_a$.

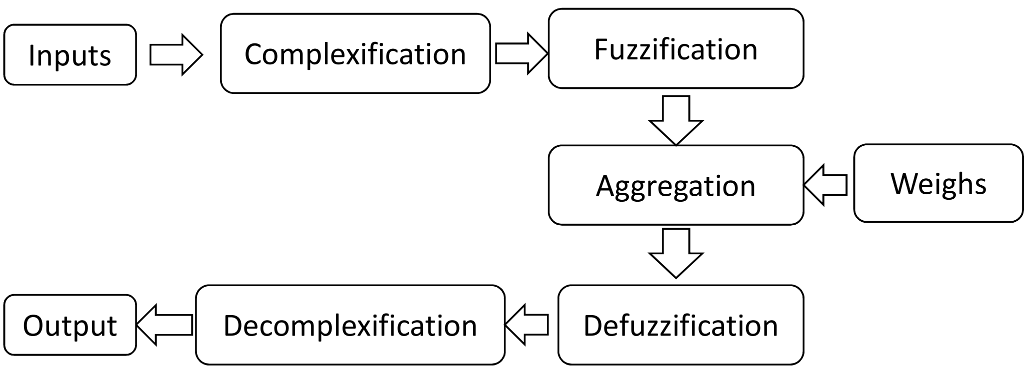


Figure 2. A simple method of target location based on complex fuzzy aggregation.

Numerical Example:Assume that the observer obtains five measurements as follows:

$$\big((865, 24^\circ), (867, 25^\circ), (871, 24^\circ), (866, 25^\circ), (869, 23^\circ)\big),$$

where (d, θ) means that the target lies on the θ degrees east of south of the observer and d metres from the observer. Then,

Step 1 Complexifications of the measured results are calculated as

$$\big((865 \cdot e^{j2\pi 336/360}), (867 \cdot e^{j2\pi 335/360}), (871 \cdot e^{j2\pi 336/360}), (866 \cdot e^{j2\pi 335/360}), (869 \cdot e^{j2\pi 337/360})\big).$$

Step 2 Normalizations of the amplitudes of all measurements are calculated as

$$\big(0.9931 \cdot e^{j2\pi 336/360}, 0.9954 \cdot e^{j2\pi 335/360}, 1 \cdot e^{j2\pi 336/360}, 0.9943 \cdot e^{j2\pi 335/360}, 0.9977 \cdot e^{j2\pi 337/360}\big).$$

Step 3 Aggregation of CFVs is calculated as

$$0.9961 \cdot e^{j2\pi 335.8/360},$$

where the weights are $(1/5, 1/5, ..., 1/5)$.

Step 4 Defuzzification of the aggregate result is calculated as $867.6 \cdot e^{j2\pi 335.8/360}$.

Step 5 Decomplexification of the above result is calculated as $(867.6, 24.2)$.

Then, the target position is estimated at $(867.6, 24.2)$. That is, it lies 24.2 degrees east of south of the observer and 867.6 m from the observer.

Note that we do not discuss how to choose complex fuzzy aggregation functions, nor their weights.

7. Conclusions

In this paper, we propose two complex fuzzy aggregation operators: the CFWG and CFOWG operators. Their main properties are summarized in Table 1. In particular, both the CFWG and the CFOWG operators are reflectionally invariant and rotationally invariant. We also showed that the CFWG and the CFOWG operators are closed under $\Pi - i$ numbers.

As we know, complex data are frequently encountered in many different applications, such as engineering, management, finance, and medicine. CFSs are suitable to represent information with uncertainty and periodicity simultaneously. Complex fuzzy information aggregation techniques may be useful in these applications.

Author Contributions: Conceptualization, S.D.; Project administration, L.B. and B.H.; Writing-original draft, L.B., S.D. and B.H.; Writing-review and editing, L.B., S.D. and B.H.

Acknowledgments: The authors would like to thank the anonymous referees and academic editor for their very valuable comments.

Funding: This research was funded by the Opening Foundation of Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing (Grant No. 2017CSOBDP0103), the Science and Technology Foundation of Guizhou Province, China (LKS [2013] 35 and LKS (2012) 34), and the Guangxi University Science and Technology Research Project (Grant No. 2013YB193).

Conflicts of Interest: The author declares no conflict of interest.

References

1. Ramot, D.; Milo, R.; Friedman, M.; Kandel, A. Complex fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 171–186. [CrossRef]
2. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
3. Buckley, J.J. Fuzzy complex numbers. *Fuzzy Sets Syst.* **1989**, *33*, 333–345. [CrossRef]
4. Buckley, J.J. Fuzzy complex analysis I: Definition. *Fuzzy Sets Syst.* **1991**, *41*, 269–284. [CrossRef]

5. Buckley, J.J. Fuzzy complex analysis II: Integration. *Fuzzy Sets Syst.* **1992**, *49*, 171–179. [CrossRef]
6. Ramot, D.; Friedman, M.; Langholz, G.; Kandel, A. Complex fuzzy logic. *IEEE Trans. Fuzzy Syst.* **2003**, *11*, 450–461. [CrossRef]
7. Hu, B.; Bi, L.; Dai, S. The Orthogonality between Complex Fuzzy Sets and Its Application to Signal Detection. *Symmetry* **2017**, *9*, 175. [CrossRef]
8. Bi, L.; Hu, B.; Li, S.; Dai, S. The Parallelity of Complex Fuzzy sets and Parallelity Preserving Operators. *J. Intell. Fuzzy Syst.* **2018**, *34*. [CrossRef]
9. Zhang, G.; Dillon, T.S.; Cai, K.-Y.; Ma, J.; Lu, J. Operation properties and delta-equalities of complex fuzzy sets. *Int. J. Approx. Reason.* **2009**, *50*, 1227–1249. [CrossRef]
10. Alkouri, A.U.M.; Salleh, A.R. Linguistic variables, hedges and several distances on complex fuzzy sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2527–2535.
11. Hu, B.; Bi, L.; Dai, S.; Li, S. Distances of Complex Fuzzy Sets and Continuity of Complex Fuzzy Operations. *J. Intell. Fuzzy Syst.* **2018**. [CrossRef]
12. Tamir, D.E.; Jin, L.; Kandel, A. A new interpretation of complex membership grade. *Int. J. Intell. Syst.* **2011**, *26*, 285–312. [CrossRef]
13. Tamir, D.E.; Kandel, A. Axiomatic theory of complex fuzzy logic and complex fuzzy classes. *Int. J. Comput. Commun. Control* **2011**, *4*, 562–576. [CrossRef]
14. Dick, S. Towards Complex Fuzzy Logic. *IEEE Trans. Fuzzy Syst.* **2005**, *13*, 405–414. [CrossRef]
15. Greenfield, S.; Chiclana, F.; Dick, S. Interval-valued complex fuzzy logic. In Proceedings of the 2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Vancouver, BC, Canada, 24–29 July 2016.
16. Greenfield, S.; Chiclana, F.; Dick, S. Join and meet operations for interval-valued complex fuzzy logic. In Proceedings of the 2016 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS), El Paso, TX, USA, 31 October–4 November 2016; pp. 1–5.
17. Alkouri, A.; Salleh, A. Complex intuitionistic fuzzy sets. In Proceedings of the AIP Conference International Conference on Fundamental and Applied Sciences, Kuala Lumpur, Malaysia, 12–14 June 2012; pp. 464–470.
18. Ali, M.; Smarandache, F. Complex neutrosophic set. *Neural Comput. Appl.* **2017**, *28*, 1–18. [CrossRef]
19. Ma, J.; Zhang, G.; Lu, J. A method for multiple periodic factor prediction problems using complex fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 32–45.
20. Chen, Z.; Aghakhani, S.; Man, J.; Dick, S. ANCFIS: A Neuro-Fuzzy Architecture Employing Complex Fuzzy Sets. *IEEE Trans. Fuzzy Syst.* **2011**, *19*, 305–322. [CrossRef]
21. Li, C. Complex Neuro-Fuzzy ARIMA Forecasting. A New Approach Using Complex Fuzzy Sets. *IEEE Trans. Fuzzy Syst.* **2013**, *21*, 567–584. [CrossRef]
22. Li, C.; Chiang, T.-W.; Yeh, L.-C. A novel self-organizing complex neuro-fuzzy approach to the problem of time series forecasting. *Neurocomputing* **2013**, *99*, 467–476. [CrossRef]
23. Al-Quran, A.; Hassan, N. The Complex Neutrosophic Soft Expert Relation and Its Multiple Attribute Decision-Making Method. *Entropy* **2018**, *20*, 101. [CrossRef]
24. Li, C.; Wu, T.; Chan, F.-T. Self-learning complex neuro-fuzzy system with complex fuzzy sets and its application to adaptive image noise canceling. *Neurocomputing* **2012**, *94*, 121–139. [CrossRef]
25. Yager, R.R.; Abbasov, A.M. Pythagorean membership grades, complex numbers and decision-making. *Int. J. Intell. Syst.* **2013**, *28*, 436–452. [CrossRef]
26. Yager, R.R. Pythagorean membership grades in multi-criteria decision making. *IEEE Trans Fuzzy Syst.* **2014**, *22*, 958–965. [CrossRef]
27. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the Joint IFSA Congress and NAFIPS Meeting, Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
28. Atanassov, K.T. Intuitionistic fuzzy sets. *IEEE Trans. Fuzzy Syst.* **1986**, *20*, 87–96. [CrossRef]
29. Dick, S.; Yager, R.R.; Yazdanbahksh, O. On Pythagorean and Complex Fuzzy Set Operations. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 1009–1021. [CrossRef]
30. Mayburov, S.N. Fuzzy Geometry of Commutative Spaces and Quantum Dynamics. *EPJ Web Conf.* **2016**, *126*, 05009. [CrossRef]
31. Bagarello, F.; Cinà, M.; Gargano, F. Projector operators in clustering. *Appl. Sci.* **2017**, *40*, 49–59. [CrossRef]
32. Yager, R.R. On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Trans. Syst. Man. Cybern.* **1988**, *18*, 183–190. [CrossRef]
33. Xu, Z.S. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.

34. Xu, Z.S.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gen. Syst.* **2006**, *35*, 417–433. [CrossRef]
35. Xu, Z.S. *Intuitionistic Fuzzy Aggregation and Clustering*; Springer: Heidelberg/Berlin, Germany, 2013.
36. Zeng, S.; Mu, Z.; Baležentis, T. A novel aggregation method for Pythagorean fuzzy multiple attribute group decision making. *Int. J. Intell. Syst.* **2018**, *33*, 573–585. [CrossRef]
37. Zhang, X.L. A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making. *Int. J. Intell. Syst.* **2016**, *31*, 593–611. [CrossRef]
38. Garg, H. A new generalized pythagorean fuzzy information aggregation using einstein operations and its application to decision making. *Int. J. Intell. Syst.* **2016**, *31*, 886–920. [CrossRef]
39. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2459–2466.
40. Tu, A.; Ye, J.; Wang, B. Multiple Attribute Decision-Making Method Using Similarity Measures of Neutrosophic Cubic Sets. *Symmetry* **2018**, *10*, 215. [CrossRef]
41. Tu, A.; Ye, J.; Wang, B. Symmetry Measures of Simplified Neutrosophic Sets for Multiple Attribute Decision-Making Problems. *Symmetry* **2018**, *10*, 144. [CrossRef]
42. Liu, P. Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 83–97. [CrossRef]
43. Liu, P. Multiple Attribute Decision-Making Methods Based on Normal Intuitionistic Fuzzy Interaction Aggregation Operators. *Symmetry* **2017**, *9*, 261. [CrossRef]
44. Yu, D.; Wu, Y.; Lu, T. Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making. *Knowl. Based Syst.* **2012**, *30*, 57–66. [CrossRef]
45. Hussain, S.A.I.; Mandal, U.K.; Mondal, S.P. Decision Maker Priority Index and Degree of Vagueness Coupled Decision Making Method: A Synergistic Approach. *Int. J. Fuzzy Syst.* **2018**, *20*, 1551–1566. [CrossRef]
46. Xia, M.; Xu, Z. Hesitant fuzzy information aggregation in decision making. *Int. J. Approx. Reason.* **2011**, *52*, 395–407. [CrossRef]
47. Zhu, B.; Xu, Z.; Xia, M. Hesitant fuzzy geometric Bonferroni means. *Inf. Sci.* **2012**, *205*, 72–85. [CrossRef]
48. Song, C.; Xu, Z.; Zhao, H. A Novel Comparison of Probabilistic Hesitant Fuzzy Elements in Multi-Criteria Decision Making. *Symmetry* **2018**, *10*, 177. [CrossRef]
49. Zhang, X.; Xu, Z. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *Int. J. Intell. Syst.* **2014**, *29*, 1061–1078. [CrossRef]