symmetry

MDPI

Article

Selecting the Optimal Mine Ventilation System via a Decision Making Framework under Hesitant Linguistic Environment

Weizhang Liang [1], Guoyan Zhao [1,*] and Suizhi Luo [2]

[1] School of Resources and Safety Engineering, Central South University, Changsha 410083, China; wzlian@csu.edu.cn
[2] School of Systems Engineering, National University of Defense Technology, Changsha 410073, China; szlluo@csu.edu.cn
* Correspondence: gyzhao@csu.edu.cn

Received: 15 June 2018; Accepted: 10 July 2018; Published: 13 July 2018

Abstract: Ventilation systems are amongst the most essential components of a mine. As the indicators of ventilation systems are in general of ambiguity or uncertainty, the selection of ventilation systems can therefore be regarded as a complex fuzzy decision making problem. In order to solve such problems, a decision making framework based on a new concept, the hesitant linguistic preference relation (HLPR), is constructed. The basic elements in the HLPR are hesitant fuzzy linguistic numbers (HFLNs). At first, new operational laws and aggregation operators of HFLNs are defined to overcome the limitations in existing literature. Subsequently, a novel comparison method based on likelihood is proposed to obtain the order relationship of two HFLNs. Then, a likelihood-based consistency index is introduced to represent the difference between two hesitant linguistic preference relations (HLPRs). It is a new way to express the consistency degree for the reason that the traditional consistency indices are almost exclusively based on distance measures. Meanwhile, a consistency-improving model is suggested to attain acceptable consistent HLPRs. In addition, a method to receive reasonable ranking results from HLPRs with acceptable consistency is presented. At last, this method is used to pick out the best mine ventilation system under uncertain linguistic decision conditions. A comparison and a discussion are conducted to demonstrate the validity of the presented approach. The results show that the proposed method is effective for selecting the optimal mine ventilation system, and provides references for the construction and management of mines.

Keywords: mine ventilation systems; hesitant linguistic environment; likelihood; preference relations

1. Introduction

The ventilation system is one of the most important technologies to ensure the safety of mines [1]. In the process of mining, it is necessary to provide enough fresh air and exclude harmful gases, heat and dust [2]. Then, a good working environment can be created to guarantee the health and safety of underground workers. Therefore, choosing an applicable mine ventilation system is essential and important for mines. Since there is much ambiguity and uncertainty in the evaluation process, the selection of mine ventilation systems can be deemed as a fuzzy decision making problem.

In the process of decision making, experts or decision makers (DMs) may prefer to do comparisons among each pair of systems or construct a preference matrix when expressing their opinions [3,4]. On the other hand, because of the complexity of alternatives and the fuzziness of human cognitions, many people may tend to give preference information in the form of language phrases, such as "good", "bad" and so on [5–7]. Thereafter, the decision making problems based on linguistic preference relations (LPRs) have attracted extensive attention [8–10]. However, there is a hypothesis that the membership degree of each element in LPRs is a certain number "one". It is unable to accurately depict the professionals' supporting or hesitant degrees of linguistic assessment information.

Accordingly, Rodriguez et al. [11] put forward the concept of hesitant fuzzy linguistic term set (HFLTS) to express the experts' hesitation or inconsistency. HFLTS is an orderly limited subset of linguistic terms. Different varieties of aggregation operators [12–14], measures [15,16] and decision making approaches [17–19] based on HFLTS and its extensions were proposed. For instance, Liu et al. [20] defined the distance measures for HFLTS to deal with hesitant fuzzy linguistic multi-criteria decision making problems; Adem et al. [21] proposed an integrated model using SWOT analysis and HFLTS for evaluating occupational safety risks in the life cycle of wind turbines. Besides, Zhu and Xu [22] came up with the concept of hesitant fuzzy linguistic preference relations (HFLPRs), where the basic elements are in the form of HFLTS.

Subsequently, numerous researchers had great interest in studying preference matrices under hesitant fuzzy linguistic conditions. Zhang and Wu [23] introduced the multiplicative consistency of HFLPRs based on distance measures. Wang and Xu [24] defined the additive and weak consistency of extended HFLPRs on the basis of graph theory. Wu and Xu [25] discussed the consistency and consensus of HFLPRs in a group decision environment. Gou et al. [26] proposed the compatibility measures and weights determination approach for HFLPRs, and then applied them in selecting a desirable aspect in the medical and health system reform process. Li et al. [27] recommended an approach of obtaining the interval consistency degree of HFLPRs. Xu et al. [28] constructed a group decision support model for HFLPRs to reach consistency and consensus.

Nevertheless, HFLTS cannot reflect the membership degree of an element that belongs to a specific concept [29], such as a certain ventilation system in this paper. It is only a collection of several linguistic evaluation values, and it has strong subjectivity and fuzziness [30]. In order to overcome the inherent defects of linguistic variables and HFLTS, hesitant fuzzy linguistic sets (HFLSs) were introduced by Lin et al. [31]. They can describe hesitant degrees of DMs with some membership degrees based on a given linguistic term. Compared with uncertain linguistic variables, they have the edge on describing the fuzziness [29]. HFLSs combine linguistic term sets with hesitant fuzzy sets (HFSs), which include both the quantitative and qualitative evaluation information [32]. Each element in the HFLSs can be called a hesitant fuzzy linguistic number (HFLN). For instance, half of the specialists in Group A think that vs_1 is a good ventilation system, and 80 percent in Group B think so. In this case, it can be expressed with a HFLN $< good, \{0.5, 0.8\} >$.

The motivations of this paper are mainly two-fold. (1) The mine ventilation systems selection context requires dealing with fuzzy evaluation information and building appropriate decision making models. Hesitant fuzzy linguistic numbers (HFLNs) have advantages in describing the fuzziness and hesitancy of experts [29]. Moreover, preference relations are among the most powerful tools to select the best system. (2) Currently, researches on HFLNs are relatively insufficient compared with other fuzzy sets. Wang et al. [29] developed a decision making method based on the Hausdorff distance of HFLNs. In addition, Wang et al. [30] put forward the concept of interval-valued HFLNs to deal with complex decision making issues. Yet, there are still some limitations with existent operational laws and comparison methods of HFLNs [29,31].

Taking the aforementioned motivations into account, this paper concentrates on selecting the optimal mine ventilation system under a hesitant linguistic environment.

The novelty and contributions of this paper are listed as follows.

(1) New operational laws and aggregation operators of HFLNs are presented. These new operations can reflect the relationship of the linguistic term and its corresponding membership degrees. Furthermore, a hesitant fuzzy linguistic likelihood is presented to compare two arbitrary HFLNs. It can effectively overcome the limitations of the existing comparison method based on score function and accuracy function.
(2) The concept of HLPRs is proposed to tackle decision making issues under hesitant fuzzy linguistic circumstances. A consistency index using likelihood is defined to check the consistency degree of

HLPRs and a consistency-improving model is introduced to get acceptable consistency. Besides, a likelihood-based method is adopted to obtain the final ranking result.

(3) The proposed method is applied in the engineering field of choosing appropriate mine ventilation systems. Thereafter, an in-depth comparison analysis is conducted to demonstrate the validity and merits of the presented method.

The remainder of this paper is arranged as follows: Introductory knowledge about HFLSs and preferences relations are briefly reviewed in Section 2. Section 3 proposes new operations and comparison method of HFLNs. A consistency index is put forward for checking the consistency level of HLPRs in Section 4. A consistency-improving process may be carried out when a HLPR's consistency is unacceptable. And a likelihood-based approach is presented to get the ranking results subsequently. Section 5 illustrates an example of ventilation systems selection and makes a comparative analysis to express the effectiveness of the proposed method. Necessary discussions and brief comments are informed in Section 6.

2. General Concepts

In this section, general concepts related to linguistic variables, HFSs, HFPRs and HFLSs are recalled.

2.1. Linguistic Variables

Assume lv_i stands for a possible linguistic value in a finite and entirely ordered separate term set $LV = \{lv_i | i = -t, \ldots, -1, 0, 1, \ldots, t\}$ [33,34]. It is usually required to meet the following conditions:

(1) There is an order: $lv_i > lv_j$, when $i > j$;

(2) A negation operator exists: $ne(lv_i) = lv_{-i}$.

When the aggregated information is used in the process of decision making, it usually does not go with the values in the predefined evaluation scope. To reserve all the obtained values, Xu [33] changed the preceding term set LV into a continuous one $\overline{LV} = \{lv_i | i \in [-p, p]\}$, where $p(p > t)$ is a adequately great positive integer.

Taking two linguistic terms $lv_i, lv_j \in \overline{LV}$ into account, some operations are proposed in the following:

(1) $lv_i \oplus_{Xu} lv_j = lv_{i+j}$;

(2) $lv_i \oplus_{Xu} lv_j = lv_j \oplus_{Xu} lv_i$;

(3) $\rho lv_i = lv_{\rho i}, \rho \in [0, 1]$.

2.2. Hesitant Fuzzy Sets

Since Zadeh [35] proposed fuzzy sets, it has been widely applied in various fields [36–40] and many extensions based on fuzzy set have been developed [41,42]. HFSs, as extensions of fuzzy sets, were firstly presented by Torra [32]. They are defined in coping with several numerical values permitted to indicate an element's membership degree [43–45]. The definition of HFSs is given as follows.

Definition 1 [32]. *If X is a fixed set, then a hesitant fuzzy set (HFS) on X is in relation to the function, which can go back a set of numbers between zero and one. It is described as the mathematical sign in the following:*

$$F = \{< x, h_F(x) > | x \in X\} \tag{1}$$

where $h_F(x)$ is a subset of several values between zero and one, which represents the probable membership degrees of an element $x \in X$ to a certain set F. Xia and Xu [46] believe that it is convenient to call $h_F(x)$ a hesitant fuzzy element (HFE).

Preference relations are impactful tools in respect to modeling the decision making process. On the basis of HFSs, Zhu [47] came up with the concept of hesitant fuzzy preference relations (HFPRs), which is given as follows.

Definition 2 [47]**.** *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a reference set, then a HFPR G on X is denoted by a matrix $G = (g_{ij})_{n\times n} \subset X \times X$, where $g_{ij} = \{[q_{ij}^{\sigma(l)} | l = 1, \ldots, |l_{ij}|]\}$ is a HFE expressing whole possible preference degree(s) of the object x_i over x_j. Furthermore, g_{ij} $(i, j = 1, 2, \ldots, n;\ i < j)$ should meet the following requirements:*

$$q_{ij}^{\sigma(l)} + q_{ji}^{\sigma(l)} = 1,\ q_{ii}^{\sigma(l)} = 0.5,\ |l_{ij}| = |l_{ji}| \tag{2}$$

$$q_{ij}^{\sigma(l)} < q_{ij}^{\sigma(l+1)},\ q_{ji}^{\sigma(l+1)} < q_{ji}^{\sigma(l)} \tag{3}$$

where $q_{ij}^{\sigma(l)}$ is the l-th largest element in g_{ij}, and $|l_{ij}|$ is the number of elements in g_{ij}.

2.3. Hesitant Fuzzy Linguistic Sets

The concept, operational laws and comparison method of HFLNs are recalled in this section. Moreover, the limitations of them are discussed in the corresponding places.

Definition 3 [31]**.** *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, and $lv_{\theta(x)} \in \overline{LV}$. Then, the hesitant fuzzy linguistic set (HFLS) U in X can be described as the subsequent object:*

$$U = \left\{ < x, lv_{\theta(x)}, h_U(x) > | x \in X \right\} \tag{4}$$

where $h_U(x)$ is a set of finite numbers in [0,1] and signifies the possible degrees of membership that x belongs to $lv_{\theta(x)}$.

There are two special cases of HFLNs: (1) A hesitant fuzzy linguistic number (HFLN): There is only one element in the set $X = \{x_1, x_2, \ldots, x_n\}$, and HFLS U is reduced to $< lv_{\theta(x)}, h_U(x) >$; (2) A fuzzy linguistic number: There is only one element in $h_B(x)$, like $h_U(x) = \{u\}$, and HFLS U is reduced to $< lv_{\theta(x)}, u >$. For example, $< lv_3, 0.5 >$ shows that the membership degree of x belongs to lv_3 is 0.5.

The operational laws about HFLNs are introduced in literature [31] as follows. Based on them, many aggregation operators are also presented in this paper.

Definition 4 [31]**.** *Given two HFLNs $a =< lv_{\theta(a)}, h_a >$ and $b =< lv_{\theta(b)}, h_b >$ arbitrarily, and $\lambda \in [0, 1]$, then*

(1) $a \oplus_{Lin} b =< lv_{\theta(a)+\theta(b)}, \bigcup\limits_{r_1 \in h_a, r_2 \in h_b} \{r_1 + r_2 - r_1 \cdot r_2\} >$;

(2) $\lambda a =< lv_{\lambda\cdot\theta(a)}, \bigcup\limits_{r \in h_a} \{1 - (1 - r)^\lambda\} >$.

It is clear that the operations mentioned above are not very reasonable as the linguistic values and the membership degrees are operated separately. In fact, the membership degrees should be related to the homologous linguistic values in the operation process.

Definition 5 [29]**.** *If $a =< lv_{\theta(a)}, h_a >$ is a HFLN, then the score function $E(a)$ of a can be described as follows:*

$$E(a) = s(h_a) \times f^*(lv_{\theta(a)}) \tag{5}$$

where $s(h_a) = \frac{1}{\#h_a} \sum_{r \in h_a} r$, $s(h_a)$ is the score function of h_a, $\#h_a$ is the number of values in h_a, $f^*(lv_i) = \frac{1}{2} + \frac{i}{2t}$ is one of the three different expressions of the linguistic scale function defined by Wang et al. [29], and it can be replaced by another expressions under different semantics. For more details please refer to literature [29].

Definition 6 [29]. *Let* $a =< lv_{\theta(a)}, h_a >=< lv_{\theta(a)}, \cup_{r \in h_a}\{r\} >$ *be a HFLN, and the variance function is represented as* $V^*(h_a) = \frac{1}{\# h_a}\sum_{r \in h_a}[r - s(h_a)]^2$. *Hence, the accuracy function* $V(a)$ *of* a *can be shown as follows:*

$$V(a) = f^*(lv_{\theta(a)}) \cdot [1 - V^*(h_a)] \tag{6}$$

where $\# h_\alpha$ *is the number of the values in* h_a.

The accuracy function $V(\alpha)$ is analogous to the sample variance statistically and can display the fluctuation of assessment values of h_a. The greater the volatility is, the larger the hesitation will be. Then, the ranking order of HFLNs can be derived by using the score function and accuracy function as follows.

Definition 7 [29]. *If* $a =< lv_{\theta(a)}, h_a >$ *and* $b =< lv_{\theta(b)}, h_b >$ *are two arbitrary HFLNs,* $r_a^{\sigma(l)}$ *and* $r_b^{\sigma(l)}$ *are regarded as the lth number in* h_a *and* h_b *respectively, and all membership degrees are arranged in ascending order. Then the comparison method is*

(1) *If* $lv_{\theta(a)} \leq lv_{\theta(b)}$, $r_a^{\sigma(l)} \leq_b^{\sigma(l)}$ *and* $r_a^{\sigma(\# h_a)} \leq r_b^{\sigma(\# h_b)}$, *then* $a < b$, *where at least one of "<" exists,* $r_a^{\sigma(l)} \in h_a$, $r_b^{\sigma(l)} \in h_b$, $l = 1, 2, \ldots, \min(\# h_a, \# h_b)$, $\# h_a$ *and* $\# h_b$ *are the numbers of values in* h_a *and* h_b *respectively;*
(2) *If* $E(a) < E(b)$ *but* $a < b$, *then* $a \prec b$;
(3) *If* $E(a) = E(b)$ *and* $V(a) < V(b)$, *then* $a \prec b$;
(4) *If* $E(a) = E(b)$ *and* $V(a) = V(b)$, *then* $a = b$.

Example 1. *Suppose* $a =< lv_0, \{0.1, 0.4\} >$, $b =< lv_{-3}, \{0.1, 0.4\} >$ *and* $c =< lv_0, \{0.2, 0.3\} >$ *are three HFLNs. Let* $f^*(lv_i) = \frac{1}{2} + \frac{i}{2t}$ *and* $t = 3$, *then:*

(1) $lv_{\theta(b)} = lv_{-3} < lv_{\theta(a)} = lv_0$, $r_b^{\sigma(1)} = r_a^{\sigma(1)} = 0.1$, $r_b^{\sigma(2)} = r_a^{\sigma(2)} = 0.4$, *thus* $b < a$;
(2) $E(b) = 0$, $E(c) = 0.125$, *i.e.,* $E(b) < E(c)$, *thus* $b \prec c$;
(3) $E(a) = E(c) = 0.125$, $V(a) = 0.48875$, $V(c) = 0.49875$, *i.e.,* $V(a) < V(c)$, *thus* $a \prec c$.

There is no doubt that the amounts of calculations are increased when the score function or even the accuracy function needs to be calculated. Besides, according to this comparison method, if $E(a) = E(b)$ and $V(a) = V(b)$ are true simultaneously, a conclusion is that $a = b$. It is reasonable in most conditions. However, it is not well tenable when the linguistic scale function $f^*(lv_i) = 0$ and the possible memberships in a certain HFLN are not strictly superior to the memberships in another HFLN. For instance, assume $\alpha =< lv_{-3}, \{0.1, 0.7\} >$, $\beta =< lv_{-3}, \{0.1, 0.9\} >$ and $\eta =< lv_{-3}, \{0.5, 0.6\} >$ are three HFLNs. Let $f^*(lv_i) = \frac{1}{2} + \frac{i}{2t}$ and $t = 3$, then $E(\alpha) = E(\beta) = E(\eta) = 0$ and $V(\alpha) = V(\beta) = V(\eta) = 0$ are true, a decision is that $\alpha < \beta$ according to part (1) of the comparison method, and a decision is that $\alpha = \eta$ and $\beta = \eta$ according to part (3) of this method. It is clear that these conclusions are self-contradictory and counterintuitive.

3. New Operations and Comparison Method

As mentioned in Section 2.3, there are some weaknesses in the existent operational laws and comparison method with HFLNs. Thus, new operations and comparison methods are presented in this section.

3.1. New Operational Laws and Aggregation Operators

To overcome the limitations of operations proposed in Section 2.3, some new operational laws on the HFLNs are raised in this section. Afterwards, the hesitant fuzzy linguistic weighted average (HFLWA) operator and hesitant fuzzy linguistic average (HFLA) operator based on them are presented.

Definition 8. *If* $a =< lv_{\theta(a)}, h_a >$ *and* $b =< lv_{\theta(b)}, h_b >$ *are HFLNs, and* $\lambda \in [0,1]$*, then*

(1) $a \oplus b =< lv_{\theta(a)+\theta(b)}, \bigcup_{r_1 \in h_a, r_2 \in h_b} \left\{ \frac{(\theta(a)+t)\cdot r_1 + (\theta(b)+t)\cdot r_2}{(\theta(a)+t)+(\theta(b)+t)} \right\} >$;

(2) $\lambda a =< lv_{\lambda \cdot \theta(a)}, h_a >$.

It is easily verified that all operational results mentioned above are still HFLNs. Although there are no practical meanings with the operational results, the basic operations are necessary to be defined in practice. When these operations are used together, the actual significance can be reflected in reality.

In view of Definition 8, the equivalent relations can be further acquired as follows.

(1) Commutativity: $a \oplus b = b \oplus a$;
(2) Associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(3) Distributivity: $\lambda(a \oplus b) = \lambda a \oplus \lambda b$, $\lambda \in [0,1]$;
(4) Distributivity: $\lambda_1 a \oplus \lambda_2 a = (\lambda_1 + \lambda_2)a$, $\lambda_1, \lambda_2 \in [0,1]$.

Definition 9. *Let* $a_i =< lv_{\theta(a_i)}, h_{a_i} >$ *be a group of HFLNs with* $i = 1,2,\ldots,n$*. The HFLWA operator can be denoted as follows:*

$$HFLWA(a_1, a_2, \ldots, a_n) = \omega_1 a_1 \oplus \omega_2 a_2 \oplus \cdots \oplus \omega_n a_n \tag{7}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weight vector of* a_i $(i = 1,2,\ldots,n)$, $\omega_i \in [0,1]$ *and* $\sum_{i=1}^{n} \omega_i = 1$.

Particularly, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the HFLWA operator is degenerated to the HFLA operator as follows:

$$HFLA(a_1, a_2, \ldots, a_n) = \frac{1}{n}(a_1 \oplus a_2 \oplus \cdots \oplus a_n) \tag{8}$$

Theorem 1. *Assume* $a_i =< lv_{\theta(a_i)}, h_{a_i} >$ *are a set of HFLNs, and* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weight vector of* a_i $(i = 1,2,\ldots,n)$, $\omega_i \in [0,1]$ *and* $\sum_{i=1}^{n} \omega_i = 1$*, then the aggregated result through applying the HFLWA operator is still a HFLN, and*

$$HFLWA(a_1, a_2, \ldots, a_n) =< lv_{\sum\limits_{i=1}^{n} C_i}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}, \ldots, r_n \in h_{an}} \left\{ \frac{\sum\limits_{i=1}^{n} D_i \cdot r_i}{\sum\limits_{i=1}^{n} D_i} \right\} > \tag{9}$$

where $C_i = \omega_i \theta(a_i)$ *and* $D_i = \omega_i(\theta(a_i) + t)$ *for all* $i = 1,2,\ldots,n$.

Proof. Clearly, by Definition 8, the aggregated data by exploiting the HFLWA operator remains a HFLN. Next, Equation (9) is proved through utilizing mathematical induction on n.

(1) When $n = 2$: we have $\omega_1 a_1 =< lv_{C_1}, h_{a_1} >$ and $\omega_2 a_2 =< lv_{C_2}, h_{a_2} >$, then $HFLWA(a_1, a_2) = \omega_1 a_1 \oplus \omega_2 a_2 = < lv_{C_1 + C_2}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}} \left\{ \frac{D_1 \cdot r_1 + D_2 \cdot r_2}{D_1 + D_2} \right\} > = < lv_{\sum\limits_{i=1}^{2} C_i}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}} \left\{ \frac{\sum\limits_{i=1}^{2} D_i \cdot r_i}{\sum\limits_{i=1}^{2} D_i} \right\} >$.

(2) For $n = k$: If Equation (9) holds, then $HFLA(a_1, a_2, \ldots, a_k) =< lv_{\sum\limits_{i=1}^{k} C_i}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}, \ldots, r_k \in h_{ak}} \left\{ \frac{\sum\limits_{i=1}^{k} D_i \cdot r_i}{\sum\limits_{i=1}^{k} D_i} \right\} >$. Hence, for $n = k + 1$, from Definition 8, that is $HFLA(a_1, a_2, \ldots, a_{k+1}) = < lv_{\sum\limits_{i=1}^{k} C_i}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}, \ldots, r_k \in h_{ak}} \left\{ \frac{\sum\limits_{i=1}^{k} D_i \cdot r_i}{\sum\limits_{i=1}^{k} D_i} \right\} >$

$$\oplus(\omega_{k+1} \cdot a_{k+1}), = < lv_{\sum_{i=1}^{k} C_i + C_{k+1}}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}, \ldots, r_{k+1} \in h_{a_{k+1}}} \left\{ \frac{\sum_{i=1}^{k} D_i \cdot \frac{\sum_{i=1}^{k} D_i \cdot r_i}{\sum_{i=1}^{k} D_i} + D_{k+1} \cdot r_{k+1}}{\sum_{i=1}^{k} D_i + D_{k+1}} \right\} > =$$

$$< lv_{\sum_{i=1}^{k+1} C_i}, \bigcup_{r_1 \in h_{a_1}, r_2 \in h_{a_2}, \ldots, r_{k+1} \in h_{a_{k+1}}} \left\{ \frac{\sum_{i=1}^{k+1} D_i \cdot r_i}{\sum_{i=1}^{k+1} D_i} \right\} >.$$

i.e., for $n = k + 1$, Equation (9) follows.

Therefore, combined (1) with (2), Equation (9) follows for all $n \in N$, then the proof of Theorem 1 is completed. □

3.2. Likelihood of Hesitant Fuzzy Linguistic Numbers

The likelihood-based comparison method is an effective way to compare fuzzy numbers. Inspired by literature [48,49], a new method based on likelihood to compare HFLNs is proposed. From an example, it can be seen that the limitations of the comparison method mentioned in Section 2.3 have been overcome when the proposed likelihood-based comparison method is adopted.

The likelihood between two HFLNs is described in the following:

Definition 10. *If $a =< lv_{\theta(a)}, h_a >$ and $b =< lv_{\theta(b)}, h_b >$ are two optional HFLNs, then the likelihood between a and b can be demonstrated as follows:*

$$L(a \geq b) = \begin{cases} 1, & 0v_{\theta(a)} > l_{v_{\theta(b)}}, h_a^+ > h_b^- \\ \frac{1}{\#h_a \#h_b} \sum_{i=1}^{\#h_a} \sum_{j=1}^{\#h_b} \frac{r_a^{\sigma(i)}}{r_a^{\sigma(i)} + r_b^{\sigma(j)}}, & lv_{\theta(\alpha)} = lv_{\theta(\beta)} \\ \frac{1}{\#h_a \#h_b} \sum_{i=1}^{\#h_a} \sum_{j=1}^{\#h_b} \frac{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)}}{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)} + f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}}, & lv_{\theta(a)} \neq lv_{\theta(b)} \\ 1, & lv_{\theta(a)} < lv_{\theta(a)}, h_a^- < h_b^+ \end{cases} \tag{10}$$

where $\gamma_a^{\sigma(i)}$ and $\gamma_b^{\sigma(j)}$ are the i-th and j-th largest value, $\#h_a$ and $\#h_b$ are the numbers of element in h_a and h_b respectively.

Property 1. *Suppose Ω is a set with all HFLNs, $\forall_{a,b,c} \in \Omega$, the likelihood satisfies the following properties:*

(1) $0 \leq L(a \geq b) \leq 1$;
(2) *If* $lv_{\theta(a)} \leq lv_{\theta(b)}$, $h_a^+ < h_b^-$, *then* $L(a \geq b) = 0$;
(3) *If* $lv_{\theta(a)} \geq lv_{\theta(b)}$, $h_a^- > h_b^+$, *then* $L(a \geq b) = 1$;
(4) $L(a \geq b) + L(b \geq a) = 1$;
(5) *If* $L(a \geq b) = L(b \geq a)$, *then* $L(a \geq b) = L(b \geq a) = 0.5$;
(6) *If* $L(a \geq c) \geq 0.5$, *and* $L(c \geq b) \geq 0.5$, *then* $L(a \geq b) \geq 0.5$.

Proof. We only prove (4) of Property 1 in the paper, as the other properties can be easily proven.

(1) If $lv_{\theta(a)} < lv_{\theta(b)}$, $h_a^+ < h_b^-$ or $lv_{\theta(a)} > lv_{\theta(b)}$, $h_a^- < h_b^+$, according to Definition 10, it is true that $L(a \geq b) + L(b \geq a) = 1$.

(2) If $lv_{\theta(a)} = lv_{\theta(b)}$, the following deduction can be derived: $L(a \geq b) = \frac{1}{\#h_a \#h_b} \sum_{i=1}^{\#h_a} \sum_{j=1}^{\#h_b} \frac{r_a^{\sigma(i)}}{r_a^{\sigma(i)} + r_b^{\sigma(j)}}$ and $L(a \leq b) = L(b \geq a) = \frac{1}{\#h_a \#h_b} \sum_{i=1}^{\#h_a} \sum_{j=1}^{\#h_b} \frac{r_a^{\sigma(i)}}{r_a^{\sigma(i)} + r_b^{\sigma(j)}}$, then $L(a \geq b) + L(a \leq b) = \frac{1}{\#h_a \#h_b} \sum_{i=1}^{\#h_a} \sum_{j=1}^{\#h_b} \frac{r_a^{\sigma(i)}}{r_a^{\sigma(i)} + r_b^{\sigma(j)}} + \frac{1}{\#h_a \#h_b} \sum_{j=1}^{\#h_b} \sum_{i=1}^{\#h_a} \frac{r_b^{\sigma(j)}}{r_a^{\sigma(i)} + r_b^{\sigma(j)}} = \frac{1}{\#h_a \#h_b} \sum_{j=1}^{\#h_b} \sum_{i=1}^{\#h_a} \frac{r_a^{\sigma(i)} + r_b^{\sigma(j)}}{r_a^{\sigma(i)} + r_b^{\sigma(j)}} = 1$.

(3) If $lv_{\theta(a)} \neq lv_{\theta(b)}$, similar to proof (2), we can obtain the following: $L(a \leq b) = L(b \geq a) = \frac{1}{\#h_a \#h_\beta} \sum_{j=1}^{\#h_b} \sum_{i=1}^{\#h_a} \frac{f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}}{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)} + f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}}$, $L(a \geq b) + L(a \leq b) = \frac{1}{\#h_a \#h_\beta} \sum_{i=1}^{\#h_a} \sum_{j=1}^{\#h_b} \frac{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(j)}}{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)} + f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}} + \frac{1}{\#h_a \#h_\beta} \sum_{j=1}^{\#h_b} \sum_{i=1}^{\#h_a} \frac{f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}}{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)} + f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}} = \frac{1}{\#h_a \#h_\beta} \sum_{j=1}^{\#h_b} \sum_{i=1}^{\#h_a} \frac{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)} + f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}}{f^*(lv_{\theta(a)}) \cdot r_a^{\sigma(i)} + f^*(lv_{\theta(b)}) \cdot r_b^{\sigma(j)}} = 1$.

Therefore, $L(a \geq b) + L(a \leq b) = 1$.
Now, the proof is completed. □

Definition 11. *If $a = < lv_{\theta(a)}, h_a >$ and $b = < lv_{\theta(b)}, h_b >$ are two HFLNs. The new comparison method for HFLNs can be defined as follows:*

(1) If $L(a \geq b) > 0.5$, then a is superior to b, expressed by $a > b$;
(2) If $L(a \geq b) < 0.5$, then a is inferior to b, expressed by $a < b$;
(3) If $L(a \geq b) = 0.5$, then a is indifferent to b, expressed by $a = b$.

Example 2. *Suppose that three HFLNs are the same as Example 1, the comparison results with new proposed comparison method are given as follows.*

(1) $L(a \geq b) = 1$, $L(b \geq a) = 0$, then $b < a$.
(2) $L(b \geq c) = 0$, $L(c \geq b) = 1$, then $b < c$.
(3) $L(a \geq c) = 0.455$, $L(c \geq a) = 0.5446$, then $a < c$.

It is true that the results in Examples 1 and 2 are the same, which verifies the validity of the presented comparison method. Moreover, assume $\alpha = < lv_{-3}, \{0.1, 0.7\} >$, $\beta = < lv_{-3}, \{0.1, 0.9\} >$ and $\eta = < lv_{-3}, \{0.5, 0.6\} >$ are three HFLNs, $f^*(lv_i) = \frac{1}{2} + \frac{i}{2t}$ and $t = 3$, then $L(\alpha \geq \beta) = 0.4781$, $L(\alpha \geq \eta) = 0.3578$ and $L(\beta \geq \eta) = 0.3881$. So we get a conclusion that $\alpha < \beta < \eta$, which is more reasonable than the results obtained by using the previous comparison method.

4. Decision Making Framework

In this section, a decision making framework is proposed to handle decision making problems under a hesitant linguistic environment. Original preference information is expressed by HLPRs and the consistency level is checked and improved. Then, a likelihood-based model is suggested to derive a ranking from HLPRs with acceptable consistency.

4.1. Original Preference Information

When making evaluations for some alternatives under a hesitant linguistic environment, DMs can provide original preference information with HLPRs. To facilitate the following discussions, the concepts of HLPRs and consistent HLPRs are defined as follows.

Definition 12. *If $X = \{x_1, x_2, \ldots, x_n\}$ is a set of alternatives, then the HLPR K on X can be described as a matrix $K = (k_{ij})_{n \times n} \subset X \times X$. Each element $k_{ij} = < lv_{ij}, r_{ij} >$ is a HFLN, where lv_{ij} and r_{ij} demonstrate* *respectively, the degree of x_i preferred to x_j and the possible membership degrees that x belongs to lv_{ij}. Then, for k_{ij} $(i,j = 1,2,\ldots,n,\ i<j)$, the following requirements should be met:*

$$lv_{ij} \oplus lv_{ji} = lv_0, lv_{ii} = lv_0, r_{ij}^{\sigma(l)} = r_{ji}^{\sigma(l)},\ r_{ii}^{\sigma(l)} = 1, |k_{ij}| = |k_{ji}| \tag{11}$$

where $r_{ij}^{\sigma(l)}$ is the l-th element in r_{ij}, and $|k_{ij}|$ is the number of values in k_{ij}.

Definition 13. *Let $K = (k_{ij})_{n\times n}$ be a HLPR, if*

$$r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj} = r_{ij}^{\sigma(l)} \cdot lv_{ij}\ (i,j,k = 1,2,\ldots n) \tag{12}$$

then K is a consistent HLPR.

Example 3. *Given a HLPR* $K_1 = \begin{bmatrix} < lv_0, \{1\} > & < lv_1, \{0.3, 0.9\} > & < lv_{-2}, \{0.1, 0.6\} > \\ < lv_{-1}, \{0.3, 0.9\} > & < lv_0, \{1\} > & < lv_2, \{0.4, 0.9\} > \\ < lv_2, \{0.1, 0.6\} > & < lv_{-2}, \{0.4, 0.9\} > & < lv_0, \{1\} > \end{bmatrix}$.
Since $r_{13}^{\sigma(1)} \cdot lv_{13} = lv_{-0.2}$, $r_{12}^{\sigma(1)} \cdot lv_{12} \oplus r_{23}^{\sigma(1)} \cdot lv_{23} = lv_{1.1}$, $r_{13}^{\sigma(1)} \cdot lv_{13} \neq r_{12}^{\sigma(1)} \cdot lv_{12} \oplus r_{23}^{\sigma(1)} \cdot lv_{23}$, then K_1 is not a consistent HLPR.

Theorem 2. *Assume a HLPR $K = (k_{ij})_{n\times n}$, if*

$$\max\left\{\oplus_{k=1}^{n}(r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj})\right\} < lv_0 \text{ or } \min\left\{\oplus_{k=1}^{n}(r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj})\right\} > lv_0 \quad (i,j,k = 1,2,\ldots n) \tag{13}$$

then $K = (k_{ij})_{n\times n}$ has a corresponding consistent HLPR.

Proof. The proof is straightforward. According to Equation (11), if $\min\left\{\oplus_{k=1}^{n}(r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj})\right\}$ $< lv_0$ and $\max\left\{\oplus_{k=1}^{n}(r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj})\right\} > lv_0$, some calculated membership degrees will be less than zero. Clearly, it is unreasonable. Therefore, when $\max\left\{\oplus_{k=1}^{n}(r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj})\right\} < lv_0$ or, $\min\left\{\oplus_{k=1}^{n}(r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus r_{kj}^{\sigma(l)} \cdot lv_{kj})\right\} > lv_0$, the corresponding consistent HLPR of $K = (k_{ij})_{n\times n}$ exists. □

Example 4. *Given a HLPR* $K_2 = \begin{bmatrix} < lv_0, \{1\} > & < lv_1, \{0.3, 0.5\} > & < lv_{-1}, \{0.1, 0.7\} > \\ < lv_{-1}, \{0.3, 0.5\} > & < lv_0, \{1\} > & < lv_1, \{0.4, 0.8\} > \\ < lv_1, \{0.1, 0.7\} > & < lv_{-1}, \{0.4, 0.8\} > & < lv_0, \{1\} > \end{bmatrix}$.
Since $\oplus_{k=1}^{3}(r_{1k}^{\sigma(1)} \cdot lv_{1k} \oplus r_{k3}^{\sigma(1)} \cdot lv_{k3}) = lv_{0.5} > lv_0$ and $\oplus_{k=1}^{n}(r_{1k}^{\sigma(2)} \cdot lv_{1k} \oplus r_{k3}^{\sigma(2)} \cdot lv_{k3}) = lv_{-0.1} < lv_0$, then K_2 does not have a consistent HLPR.

Note that: when a HLPR $K = (k_{ij})_{n\times n}$ does not have the corresponding consistent HLPR, it should be adjusted based on Equation (14) until a consistent HLPR exists.

Theorem 3. *Assume a HLPR $K = (k_{ij})_{n\times n}$ has the consistent HLPR, for all $i,j,k = 1,2,\ldots n$, if*

$$r_{ij}^{*\sigma(l)} \cdot lv_{ij}^{*} = \frac{1}{n} \oplus_{k=1}^{n} (r_{ik}^{\sigma(l)} \cdot lv_{ik} \oplus_{Xu} r_{kj}^{\sigma(l)} \cdot lv_{kj}), \tag{14}$$

$$lv_{ij}^{*} = \max\left\{r_{1k}^{\sigma(1)} \cdot lv_{1k} \oplus r_{k3}^{\sigma(1)} \cdot lv_{k3}\right\}\ (if\ \oplus_{k=1}^{n} (r_{1k}^{\sigma(1)} \cdot lv_{1k} \oplus r_{k3}^{\sigma(1)} \cdot lv_{k3}) > lv_0) \tag{15}$$

$$lv_{ij}^{*} = \min\left\{r_{1k}^{\sigma(2)} \cdot lv_{1k} \oplus r_{k3}^{\sigma(2)} \cdot lv_{k3}\right\}\ (if\ \oplus_{k=1}^{n}(r_{1k}^{\sigma(2)} \cdot lv_{1k} \oplus r_{k3}^{\sigma(2)} \cdot lv_{k3}) < lv_0) \tag{16}$$

then $K^* = (k^*_{ij})_{n\times n} = (lv^*_{ij}, r^*_{ij})_{n\times n}$ *is a consistent HLPR.*

Proof. Since $r^{*\sigma(l)}_{ik} \cdot lv^*_{ik} \oplus r^{*\sigma(l)} \cdot lv^*_{kj} = \frac{1}{n}(\oplus^{n}_{e=1}(r^{\sigma(l)}_{ie} \cdot lv_{ie} \oplus r^{\sigma(l)}_{ek} \cdot lv_{ek})) \oplus \frac{1}{n}(\oplus^{n}_{e=1}(r^{\sigma(l)}_{ke} \cdot lv_{ke} \oplus r^{\sigma(l)}_{ej} \cdot lv_{ej})) = \frac{1}{n}(\oplus^{n}_{e=1}(r^{\sigma(l)}_{ie} \cdot lv_{ie} \oplus r^{\sigma(l)}_{ek} \cdot lv_{ek} \oplus r^{\sigma(l)}_{ke} \cdot lv_{ke} \oplus r^{\sigma(l)}_{ej} \cdot lv_{ej})) = \frac{1}{n}(\oplus^{n}_{e=1}(r^{\sigma(l)}_{ie} \cdot lv_{ie} \oplus r^{\sigma(l)}_{ej} \cdot lv_{ej} \oplus r^{\sigma(l)}_{ek} \cdot lv_0)) = \frac{1}{n}(\oplus^{n}_{e=1}(r^{\sigma(l)}_{ie} \cdot lv_{ie} \oplus r^{\sigma(l)}_{ej} \cdot lv_{ej})) = r^{\sigma(l)}_{ij} \cdot lv^*_{ij}$ based on Definition 13, $K^* = (k^*_{ij})_{n\times n} = (lv^*_{ij}, r^*_{ij})_{n\times n}$ is a consistent HLPR. □

Example 5. *Assume a HLPR is the same in Example 3. Based on Equation (14), the consistent HLPR K^*_1 is obtained as follows:* $K^*_1 = \begin{bmatrix} <lv_0,\{1\}> & <lv_{-1},\{2/15,7/15\}> & <lv_{1.1},\{7/33,1/11\}> \\ <lv_1,\{2/15,7/15\}> & <lv_0,\{1\}> & <lv_{0.8},\{11/24,5/8\}> \\ <lv_{-1.1},\{7/33,1/11\}> & <lv_{-0.8},\{11/24,5/8\}> & <lv_0,\{1\}> \end{bmatrix}$.

4.2. Consistency Checking and Improving Models

When an initial preference matrix is constructed, checking and improving its consistency is necessary and vital [50–52]. The consistency of preference relations reflects the rationality of DMs' judgments, and inconsistent preference matrices may generate undesirable or improper conclusions. In this section, a likelihood-based consistency index is defined to test the consistency degree and a consistency-improving process is presented to modify the consistency level.

Definition 14. *Given two arbitrary HLPRs* $A = (a_{ij})_{n\times n}$ *and* $B = (b_{ij})_{n\times n}$, *then*

$$L(A \geq B) = \frac{2}{n(n-1)} \sum^{n}_{i<j} L(a_{ij} \geq b_{ij}) \tag{17}$$

is called the likelihood between two HLPRs.

The likelihood $L(A \geq B)$ satisfies Theorem 4 as follows.

Theorem 4. *Assume A and B are two HLPRs, the likelihood between them can be represented as* $L(A \geq B)$, *then*

(1) $0 \leq L(A \geq B) \leq 1$;
(2) $L(A \geq B) + L(B \geq A) = 1$;
(3) *If* $L(A \geq B) = L(B \geq A)$, *then* $L(A \geq B) = L(B \geq A) = 0.5$.

Definition 15. *Suppose a HLPR K and its corresponding consistent HLPR* K^*; *a consistency index is used to calculate the deviation between K and* K^*, *which is defined as*

$$CI(K) = \frac{1}{n(n-1)} \sum^{n}_{i\neq j} |L(k_{ij} \geq k^*_{ij}) - \frac{1}{2}| \tag{18}$$

It is true that $0 \leq CI(K) \leq \frac{1}{2}$. Based on Definition 15, a smaller value of $CI(K)$ means a more consistent HLPR K. As the DMs would be often influenced by many uncertainties when they make decisions, HLPRs provided by the DMs are not always perfectly consistent.

Definition 16. *Given a HLPR K and the corresponding threshold value* CI, *when the consistency index meets:*

$$CI(K) < CI \tag{19}$$

then K is regarded as a HLPR whose consistency is acceptable.

Note: There is an attractive subject about how to determine the value of CI. It may be confirmed in accordance with the DMs' knowledge, experience and other conditions.

In some circumstances, the HLPR K constructed by the DMs is always with unacceptable consistency due to the lack of knowledge or other reasons. Hence, a consistency-improving model is built to acquire a reasonable solution. Some critical steps in Algorithm 1 can be taken repeatedly until the predefined consistency threshold is satisfied.

The main steps of this consistency-improving process are shown as follows.

Algorithm 1. Consistency improving model of HLPRs

Input: The original HLPR $K = (k_{ij})_{n\times n}$, the threshold value $CI = CI_0$ and the maximum number of iterative times $s_{\max} \geq 1$.

Output: The adjusted HLPR K_a and its consistency index $CI(K_a)$.

Step 1: Let the iterative times $s = 0$, and the original HLPR $K = K^{(0)} = (k_{ij}^{(0)})_{n\times n}$.

Step 2: According to Equation (14), obtain the corresponding consistent HLPR $K^{*(s)} = (k_{ij}^{*(s)})_{n\times n} = (< lv_{ij}^{*(s)}, r_{ij}^{*(s)} >)_{n\times n}$ of HLPR $K^{(s)} = (k_{ij}^{(s)})_{n\times n}$.

Step 3: Based on Equation (10), calculate the likelihood $L(k_{ij}^{(s)} \geq k_{ij}^{*(s)})$ of the corresponding elements (e.g., $k_{ij}^{(s)}$ and $k_{ij}^{*(s)}$) in the HLPR $K^{(s)} = (k_{ij}^{(s)})_{n\times n}$ and its consistent HLPR $K^{*(s)} = (k_{ij}^{*(s)})_{n\times n}$. Then, construct the likelihood matrix $L^{(s)} = (l_{ij}^{(s)})_{n\times n} = (L(k_{ij}^{(s)} \geq k_{ij}^{*(s)}))_{n\times n}$ of HLPR $K^{(s)}$.

Step 4: Calculate the consistency index $CI(K^{(s)})$ of HLPR $K^{(s)}$ by Equation (18).

Step 5: If the consistency level of $K^{(s)}$ is acceptable, namely $CI(K^{(s)}) < CI_0$ or the iterative times is maximum, namely $s > s_{\max}$, then go to Step 7; or else, go to the next step.

Step 6: Find an element $l_{ij}^{(s)}$ in the likelihood matrix $L^{(s)} = (l_{ij}^{(s)})_{n\times n}$, which has the maximum deviation on the diagonal, namely $\max\left\{|l_{ij}^{(s)} - \frac{1}{2}| + |l_{ji}^{(s)} - \frac{1}{2}|\right\}$. If $l_{ij}^{(s)} + l_{ij}^{(s)} - 1 < 0$, then the DMs may increase their preference of $k_{ij}^{(s)}$; if $l_{ij}^{(s)} + l_{ij}^{(s)} - 1 > 0$, then the DMs can decrease their values of $k_{ij}^{(s)}$. And the modified HLPR is denoted as $K^{(s+1)} = (k_{ij}^{(s+1)})_{n\times n} = (< lv_{ij}^{(s+1)}, r_{ij}^{(s+1)} >)_{n\times n}$. Let $s = s + 1$, then return to Step 2.

Step 7: Let the final adjusted HLPR $K^{(s)} = K_a$, Output K_a and its consistency index $CI(K_a)$.

Theorem 5. *Given a HFPR K, which is unacceptably consistent. If $CI = CI_0$ is the consistency threshold, $\left\{K^{(s)}\right\}$ is a HFPR sequence, and $CI(K^{(s)})$ is the consistency index of $K^{(s)}$. Therefore, we can obtain that for any s: $CI(K^{(s+1)}) < CI(K^{(s)})$ and $\lim\limits_{s\to\infty} CI(K^{(s)}) = 0$.*

The proof is straightforward. There is no less than one position where $|l_{ij_1}^{(s)} - \frac{1}{2}| < |l_{ij_1}^{(s+1)} - \frac{1}{2}|$ can be obtained. It follows that $CI(K^{(s+1)}) < CI(K^{(s)})$.

Theorem 5 guarantees that any HLPR with insupportable consistency can be converted into an acceptable HLPR. The speed and times may be influenced by the values of the adjusted elements, which are recommended by the DMs or specialists according to the practical situation. How to determine the value of adjusted elements more reasonably is also a controversial issue and deserves to be further investigated.

Example 6. *Given an original HLPR* $K = \begin{bmatrix} < lv_0, \{1\} > & < lv_3, \{0.6, 0.7\} > & < lv_{-2}, \{0.8, 0.9\} > \\ < lv_{-3}, \{0.6, 0.7\} > & < lv_0, \{1\} > & < lv_1, \{0.2, 0.3\} > \\ < lv_2, \{0.8, 0.9\} > & < lv_{-1}, \{0.2, 0.3\} > & < lv_0, \{1\} > \end{bmatrix}$. *Suppose the threshold $CI_0 = 0.25$ and the maximum number of iterative times $s_{\max} = 3$, check and improve its consistency. The detailed procedures are listed as follows.*

Step 1: Let $s = 0$ and $K^{(0)} = K$.

Step 2: Based on Equation (14), obtain the consistent HLPR

$$K^{*(0)} = \begin{bmatrix} <lv_0,\{1\}> & <lv_{2.1},\{2/7,1/3\}> & <lv_{-1.8},\{2/9,2/9\}> \\ <lv_{-2.1},\{2/7,1/3\}> & <lv_0,\{1\}> & <lv_{-3.9},\{10/39,11/39\}> \\ <lv_{1.8},\{2/9,2/9\}> & <lv_{3.9},\{10/39,11/39\}> & <lv_0,\{1\}> \end{bmatrix}.$$

Step 3: Based on Equation (10), the likelihood matrix is $L^{(0)} = \begin{bmatrix} 0.5 & 1 & 0.7762 \\ 0.5249 & 0.5 & 0.9781 \\ 1 & 0.2590 & 0.5 \end{bmatrix}$.

Step 4: Based on Equation (18), calculate the consistency index $CI(K^{(0)}) \approx 0.2534$.

Step 5: Since $CI(K^{(0)}) > CI_0$*, then go to the next step.*

Step 6: Since $l_{13}^{(0)} = \max\left\{|l_{ij}^{(0)} - \frac{1}{2}| + |l_{ji}^{(0)} - \frac{1}{2}|\right\}$ *and* $l_{13}^{(0)} + l_{31}^{(0)} - 1 > 0$*, then the DMs decrease their preference. The modified HLPR is* $K^{(1)} = \begin{bmatrix} <lv_0,\{1\}> & <lv_3,\{0.6,0.7\}> & <lv_{-2},\{0.1,0.2\}> \\ <lv_{-3},\{0.6,0.7\}> & <lv_0,\{1\}> & <lv_1,\{0.2,0.3\}> \\ <lv_2,\{0.1,0.2\}> & <lv_{-1},\{0.2,0.3\}> & <lv_0,\{1\}> \end{bmatrix}$ *and* $CI(K^{(1)}) \approx 0.2230 < CI_0$.

Step 7: Let $K_a = K^{(1)}$*, Output* $K_a = \begin{bmatrix} <lv_0,\{1\}> & <lv_3,\{0.6,0.7\}> & <lv_{-2},\{0.1,0.2\}> \\ <lv_{-3},\{0.6,0.7\}> & <lv_0,\{1\}> & <lv_1,\{0.2,0.3\}> \\ <lv_2,\{0.1,0.2\}> & <lv_{-1},\{0.2,0.3\}> & <lv_0,\{1\}> \end{bmatrix}$ *and* $CI(K_a) \approx 0.2230$.

4.3. Likelihood-Based Ranking Method

As the likelihood between two HFLNs is a useful tool to make comparisons, a likelihood-based method is introduced to derive a ranking from the consistent HLPRs in this section.

Pondering over the decision making problem within the hesitant fuzzy linguistic context, assume that the DMs' plan to select the optimal alternative or get a ranking order from n objects. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of alternatives being chosen and $K = (k_{ij})_{n\times n}$ $(i, j = 1, 2, \ldots, n)$ is the preference matrix, where k_{ij} is the preference value in the form of HFLNs. The entire procedures of earning the ideal order of alternatives are shown in Algorithm 2.

Algorithm 2. Likelihood-based ranking method

Input: The initial HLPR $K = (k_{ij})_{n\times n}$.
Output: The optimal alternative x^*.
Step 1: Obtain the acceptable HLPR K_a by Algorithm 1.
Step 2: Utilize the HFLA operator based on Equation (8) to aggregate each row of the HLPR K_a, then determine the overall preference degree p_i of each alternative x_i $(i = 1, 2, \ldots, n)$.
Step 3: According to Equation (10), calculate the likelihood $l_{ij} = L(p_i \geq p_j)$ between p_i and p_j $(i = 1, 2, \ldots, n,$ $j = 1, 2, \ldots, n)$, then construct a likelihood matrix $L = (l_{ij})_{n\times n}$.
Step 4: Calculate the dominance degree $\varphi(x_i) = \frac{1}{n}\sum_{j=1}^{n} l_{ij}$ of alternative $x_i (i = 1, 2, \ldots, n)$, where $\varphi(x_i)$ represents the degree of x_i preferred to other alternatives. Obviously, the greater the value of $\varphi(x_i)$, the better the alternative x_i.
Step 5: Rank all the alternatives on the basis of the dominance degree $\varphi(x_i)$ of each alternative $x_i (i = 1, 2, \ldots, n)$. Then obtain the ranking results and the optimal alternative(s) is denoted as x^*.

5. Selection of Mine Ventilation Systems

In this section, an example of mine ventilation systems selection is afforded for voicing the application of the suggested method.

Sanshandao gold mine is the first subsea hard rock mine in China, which lies in Sanshandao Town, Laizhou City, Shandong Province, China [53]. As the mine is going into the stage of deep exploitation, the distance of ventilation becomes longer and the temperature also rises severely. Therefore, some problems are beginning to appear after using the traditional ventilation systems, for instance, the temperature is so high that laborers find it hard to work efficiently; exhaust gas emitted by diesel equipment pollutes underground air seriously; and the concentration of dust exceeds the national standard. Accordingly, a better ventilation system needs to be adopted.

After a thorough survey, four ventilation systems, i.e., $\{vs_1, vs_2, vs_3, vs_4\}$, are under consideration, and a group of professionals are invited to select the optimal ventilation system. The linguistic term set $lv = \{lv_{-4} = \textit{tremendously worse}, \ lv_{-3} = \textit{a lot worse}, lv_{-2} = \textit{worse}, lv_{-1} = \textit{a little worse}, v_0 = \textit{fair}, lv_1 = \textit{a little better}, lv_2 = \textit{better}, lv_3 = \textit{a lot better}, lv_4 = \textit{tremendously better}\}$ is used. The preference values are shown in the form of HFLNs. Suppose all DMs have a consensus on the selected linguistic term, and all teams provided their membership degrees (preference) in line with the researches of the above four systems and their preference simultaneously. Then, all of the probable membership degrees are gathered with the previous linguistic set. When a team does not give a membership degree, we consider it as 0.5. And when the same membership degrees about identical linguistic terms are given, we may regard them as different data in a HFLN.

Consequently, after a heated discussion, experts decided the threshold value $CI_0 = 0.18$ and the maximum number of iterative times $s_{\max} = 3$. Then, the preference information was given in Table 1.

Table 1. Original HLPR VS.

VS	vs_1	vs_2	vs_3	vs_4
vs_1	$< lv_0, \{1\} >$	$< lv_3, \{0.2, 0.3, 0.6\} >$	$< lv_1, \{0.4, 0.6, 0.8\} >$	$< lv_2, \{0.3, 0.4, 0.8\} >$
vs_2	$< lv_{-3}, \{0.2, 0.3, 0.6\} >$	$< lv_0, \{1\} >$	$< lv_{-2}, \{0.3, 0.4, 0.7\} >$	$< lv_3, \{0.2, 0.5, 0.6\} >$
vs_3	$< lv_{-1}, \{0.4, 0.6, 0.8\} >$	$< lv_2, \{0.3, 0.4, 0.7\} >$	$< lv_0, \{1\} >$	$< lv_{-1}, \{0.4, 0.5, 0.9\} >$
vs_4	$< lv_{-2}, \{0.3, 0.4, 0.8\} >$	$< lv_{-3}, \{0.2, 0.5, 0.6\} >$	$< lv_1, \{0.4, 0.5, 0.9\} >$	$< lv_0, \{1\} >$

5.1. Illustrative Example

Steps outlined in Section 4.3 are completed to get satisfied ventilation system(s) in this section.

Step 1: Obtain the acceptable HLPR VS_a by Algorithm 1.

Based on Equation (14), the consistent HLPR VS^* is shown in Table 2. And the likelihood matrix $L^{(0)}$ is calculated based on Equation (10), as shown in Table 3. Then, calculate the consistency index $CI(VS^{(0)}) \approx 0.1956 > 0.18$ by Equation (18). Since $l_{34}^{(0)} = \max\left\{|l_{ij}^{(0)} - \frac{1}{2}| + |l_{ji}^{(s)} - \frac{1}{2}|\right\}$ and $l_{34}^{(0)} + l_{43}^{(0)} - 1 > 0$, then the DMs decrease their preference, and the modified HLPR $VS^{(1)}$ is in Table 4. Since $CI(VS^{(1)}) \approx 0.1754 < 0.18$, let $VS_a = VS^{(1)}$.

Table 2. Consistent HLPR VS^*.

VS^*	vs_1	vs_2	vs_3	vs_4
vs_1	$< lv_0, \{1\} >$	$< lv_{2.2}, \left\{\frac{1}{4}, \frac{25}{88}, \frac{7}{11}, \frac{7}{11}\right\} >$	$< lv_{2.5}, \left\{\frac{9}{50}, \frac{13}{50}, \frac{9}{20}\right\} >$	$< lv_{3.6}, \left\{\frac{1}{6}, \frac{41}{159}, \frac{67}{159}\right\} >$
vs_2	$< lv_{-2.2}, \left\{\frac{1}{4}, \frac{25}{88}, \frac{7}{11}\right\} >$	$< lv_0, \{1\} >$	$< lv_{-1.4}, \left\{\frac{3}{14}, \frac{9}{56}, \frac{29}{56}\right\} >$	$< lv_{1.8}, \left\{\frac{1}{36}, \frac{2}{9}, \frac{11}{72}\right\} >$
vs_3	$< lv_{-2.5}, \left\{\frac{9}{50}, \frac{13}{50}, \frac{9}{20}\right\} >$	$< lv_{1.4}, \left\{\frac{3}{14}, \frac{9}{56}, \frac{29}{56}\right\} >$	$< lv_0, \{1\} >$	$< lv_{3.2}, \left\{\frac{3}{64}, \frac{15}{128}, \frac{11}{64}\right\} >$
vs_4	$< lv_{-3.6}, \left\{\frac{1}{6}, \frac{41}{159}, \frac{67}{159}\right\} >$	$< lv_{-1.8}, \left\{\frac{1}{36}, \frac{2}{9}, \frac{11}{72}\right\} >$	$< lv_{-3.2}, \left\{\frac{3}{64}, \frac{15}{128}, \frac{11}{64}\right\} >$	$< lv_0, \{1\} >$

Table 3. Likelihood matrix $L^{(0)}$.

$L^{(0)}$	vs_1	vs_2	vs_3	vs_4
vs_1	0.5000	0.5076	0.6078	0.5693
vs_2	0.3656	0.5000	0.5642	0.7844
vs_3	0.8069	0.6378	0.5000	0.6862
vs_4	0.9287	0.6195	1.0000	0.5000

Table 4. Modified HLPR $VS^{(1)}$.

$VS^{(1)}$	vs_1	vs_2	vs_3	vs_4
vs_1	$< lv_0, \{1\} >$	$< lv_3, \{0.2, 0.3, 0.6\} >$	$< lv_1, \{0.4, 0.6, 0.8\} >$	$< lv_2, \{0.3, 0.4, 0.8\} >$
vs_2	$< lv_{-3}, \{0.2, 0.3, 0.6\} >$	$< lv_0, \{1\} >$	$< lv_{-2}, \{0.3, 0.4, 0.7\} >$	$< lv_3, \{0.2, 0.5, 0.6\} >$
vs_3	$< lv_{-1}, \{0.4, 0.6, 0.8\} >$	$< lv_2, \{0.3, 0.4, 0.7\} >$	$< lv_0, \{1\} >$	$< lv_{-1}, \{0.1, 0.2, 0.3\} >$
vs_4	$< lv_{-2}, \{0.3, 0.4, 0.8\} >$	$< lv_{-3}, \{0.2, 0.5, 0.6\} >$	$< lv_1, \{0.1, 0.2, 0.3\} >$	$< lv_0, \{1\} >$

Step 2: Utilize the HFLA operator based on Equation (8) to aggregate each row of the HLPR VS_a, then the overall preference degree p_i of each alternative is acquired as follows:

$$p_1 = (lv_{1.5}, \begin{array}{l} 0.4182, 0.4455, 0.4500, 0.4636, 0.4773, 0.4909, 0.4955, 0.5091, , 0.5227, \\ 0.5364, 0.5409, 0.5455, 0.5545, 0.5682, 0.5727, 0.5864, 0.5909, 0.6000, \\ 0.6182, 0.6318, 0.6364, 0.6455, 0.6636, 0.6773, 0.6818, 0.7273, 0.7727) \end{array} ,$$

$$p_2 = (lv_{-0.5}, \begin{array}{l} 0.4429, 0.4500, 0.4571, 0.4643, 0.4714, 0.4857, 0.5000, 0.5071, 0.5286, \\ 0.5929, 0.6000, 0.6071, 0.6143, 0.6214, 0.6357, 0.6429, 0.6500, 0.6500, \\ 0.6571, 0.6571, 0.6643, 0.6714, 0.6786, 0.6857, 0.7000, 0.7071, 0.7286) \end{array} ,$$

$$p_3 = (lv_0, \begin{array}{l} 0.4563, 0.4750, 0.4938, 0.4938, 0.4938, 0.5125, 0.5125, 0.5313, 0.5313, \\ 0.5313, 0.5313, 0.5500, 0.5500, 0.5688, 0.5688, 0.5688, 0.5875, 0.6063, \\ 0.6063, 0.6250, 0.6438, 0.6438, 0.6625, 0.6813, 0.6813, 0.7000, 0.7188) \end{array} ,$$

and

$$p_4 = (lv_{-1}, \begin{array}{l} 0.4417, 0.4583, 0.4667, 0.4750, 0.4833, 0.4833, 0.4917, 0.5000, 0.5083, \\ 0.5167, 0.5250, 0.5250, 0.5250, 0.5333, 0.5417, 0.5500, 0.5500, 0.5583, \\ 0.5583, 0.5667, 0.5667, 0.5750, 0.5917, 0.6000, 0.6083, 0.6333, 0.6417) \end{array} .$$

Step 3: According to Equation (10), calculate the likelihood between p_i and p_j ($i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$), then the likelihood matrix $L = (l_{ij})_{4\times 4}$ is constructed in Table 5.

Table 5. Likelihood matrix L.

L	p_1	p_2	p_3	p_4
p_1	0.5000	0.6001	0.5756	0.6586
p_2	0.3999	0.5000	0.4745	0.5620
p_3	0.4244	0.5255	0.5000	0.5872
p_4	0.3414	0.4380	0.4128	0.5000

Step 4: Calculate the dominance degree of each alternative with $\varphi(vs_i) = \frac{1}{4}\sum_{j=1}^{4} l_{ij} (i = 1, 2, 3, 4)$ as: $\varphi(vs_1) \approx 0.5836$, $\varphi(vs_2) \approx 0.4841$, $\varphi(vs_3) \approx 0.5093$, $\varphi(vs_4) \approx 0.4231$.

Step 5: Since $\varphi(vs_1) > \varphi(vs_3) > \varphi(vs_2) > \varphi(vs_4)$, then the ranking is $vs_1 \succ vs_3 \succ vs_2 \succ vs_4$ and the optimal system is $vs^* = vs_1$.

5.2. Comparative Analysis

Since the HLPR presented in this paper is a new type of preference relation, no related researches have been conducted so far. To testify the validity and advantages of the proposed method, several methods for HFLPRs [23–28] can be made for comparisons.

Note: The definitions of HLPRs and HFLPRs are not the same. The basic elements in HLPRs are HFLNs, whereas those in HFLPRs are HFLTS. As a result, each HFLN in the HLPR should be transformed into the corresponding HFLTS by using the linguistic term multiplies the corresponding membership degrees successively. For example, $< lv_2, \{0.3, 0.5\} >$ can be converted into $\{lv_{0.6}, lv_1\}$. Then, a same illustration is applied in these methods and detailed comparisons are provided in Table 6.

Table 6. Comparisons with different methods.

Methods	Consistency Checking	Consistency Improving	Ranking Approaches	Ranking Results
Zhang and Wu [23]	Distance measure	Iterative algorithm	Score functions	$vs_1 \succ vs_3 \succ vs_4 \succ vs_2$
Wang and Xu [24]	Graph theory	Not given	Not given	Unavailable
Wu and Xu [25]	Distance measure	Feedback mechanism	Score functions	Uncertain
Gou et al. [26]	Compatibility measure	Not given	Complementary matrix	$vs_1 \succ vs_2 \succ vs_3 \succ vs_4$
Li et al. [27]	Linear programing model	Not given	Not given	Unavailable
Xu et al. [28]	Distance measure	Iterative algorithm	Score functions	$vs_1 \succ vs_3 \succ vs_2 \succ vs_4$
The proposed method	likelihood	Feedback mechanism	Likelihood matrix	$vs_1 \succ vs_3 \succ vs_2 \succ vs_4$

(1) Comparison with literature [24,25,27]

In literature [24], Wang and Xu provided a visible interpretation of additive consistency and weak consistency of extended HFLPRs based on graph theory. In literature [27], Li et al. defined an interval consistency index of HFLPRs based on the linear programming model. However, the methods of improving consistency degrees and getting ranking orders are not mentioned in literature [24,27]. Thus, the rankings are unavailable in these cases. In literature [25], Wu and Xu discussed some issues of HFLPRs on consistency and consensus, and defined a consistency index based on distance measure. Nevertheless, dissimilar ranking results may occur with different adjust preference when the feedback mechanism was adopted to improve the consistency level in this literature [25]. Note: Feedback mechanisms are presented to improve the consistency level of preference relations in both literature [25] and this paper. Different from existing feedback approaches [25], people can directly adjust their preference with our method according to the values of elements in the likelihood matrix.

(2) Comparison with literature [23,26,28]

From Table 6, it is clear that the best alternative in different methods is always vs_1, which reveals the effectiveness of the proposed method. In literature [26], Gou et al. defined the consistency index on the basis of compatibility measure and then got the ranking result based on a complementary matrix; however, the approach of improving consistency level of HFLPRs was not given. Even though the rankings obtained in literature [28] and this paper are the same, there are still some differences between these two methods. First, in literature [23,28], a consistency index based on distance measure was defined to check consistency level of HFLPRs, while a likelihood-based index is suggested in this paper. Compared with compatibility or distance measure, the largest advantage of the likelihood is that not only the deviation degree, but also that the order relationship of two elements can be directly indicated. Second, compared with automatic iterative algorithms [23,28], the feedback mechanism proposed in this paper reduces the loss of original information, and DMs can understand their current status in each round. Besides, an approach of using aggregation operators and then calculating score functions was adopted to get the ranking order in both literature [23,28]. By contrast, a likelihood matrix is constructed in this paper to avoid the second calculations and information distortions.

The advantages of the proposed approach are summarized as follows:

(1) The HFLNs can closely depict the experts' preferences as the membership degrees of a certain linguistic value are given. And they can reserve the completeness of initial information in some extents, which is the guarantee for obtaining ideal results.
(2) Only one element which greatly affects the consistency needs to be adjusted by professionals. The revised alternatives may be diverse according to the reality. Specialists make a decision in the light of a recommended direction as they are acquainted with their current positions.
(3) The experts may change the linguistic scale function under different semantics on the basis of their preferences and reality. Then different ranking results may be achieved if another linguistic scale function is applied. The flexibility and practicability of the method can be reflected.

Overall, the proposed method brings up a new and useful way to resolve complex fuzzy decision making issues under a hesitant linguistic environment, especially when experts or decision makers (DMs) readily make comparisons among each pair of alternatives but hardly provide direct evaluation information.

6. Conclusions

Ventilation systems selection is an essential decision for a mining project. However, the influence characteristics of a ventilation system are complex and of strong fuzziness or uncertainty. Since preference relations play a significant role in the decision making process, HLPRs were proposed to deal with mine ventilation systems selection problems. HLPRs can be regarded as extensions of LPRs. They provide not only the priority intensity of alternatives, but also the possible membership degrees of this priority intensity. The likelihood-based index was defined to test the consistency of experts and the improving model was constructed to modify consistency level of HLPRs. Preference information in HFPRs is in the form of HFLNs. For the accuracy of HFLNs' computation, new operational laws and the comparison method were presented after reviewing the relevant literature. Furthermore, the decision making framework based on HLPRs was built to select a proper ventilation system for mines. Finally, an illustration and some comparisons with other methods were drawn to highlight the applicability and advantages of the developed approach. In the future, more engineering applications with this proposed method could be researched or more decision making methods can be developed to address complex decision making problems in mines.

Author Contributions: W.L., G.Z. and S.L. conceived and worked together to achieve this work, W.L. wrote the paper, and G.Z. made contribution to the case study.

Funding: This work was supported by National Natural Science Foundation of China (51774321).

Acknowledgments: We sincerely thank the anonymous reviewers for their helpful and constructive suggestions and the editors for their careful and patient work.

Conflicts of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. Euler, D.S. Application of ventilation management programs for improved mine safety. *Int. J. Min. Sci. Technol.* **2017**, *27*, 647–650.
2. Geng, F.; Luo, G.; Zhou, F.B.; Zhao, P.T.; Ma, L.; Chai, H.L.; Zhang, T.T. Numerical investigation of dust dispersion in a coal roadway with hybrid ventilation system. *Powder Technol.* **2017**, *313*, 260–271. [CrossRef]
3. Xia, M.M.; Xu, Z.S.; Liao, H.C. Preference relations based on intuitionistic multiplicative information. *IEEE Trans. Fuzzy Syst.* **2013**, *21*, 113–133.
4. Luo, S.Z.; Cheng, P.F.; Wang, J.Q.; Huang, Y.J. Selecting project delivery systems based on simplified neutrosophic linguistic preference relations. *Symmetry* **2017**, *9*, 151. [CrossRef]
5. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning. *Inf. Sci.* **1974**, *8*, 199–249. [CrossRef]
6. Liang, W.Z.; Zhao, G.Y.; Hong, C.S. Selecting the optimal mining method with extended multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA) approach. *Neural Comput. Appl.* **2018**, 1–16. [CrossRef]
7. Cabrerizo, F.J.; Morente-Molinera, J.A.; Pedrycz, W.; Taghavi, A.; Herrera-Viedma, E. Granulating linguistic information in decision making under consensus and consistency. *Expert Syst. Appl.* **2018**, *99*, 83–92. [CrossRef]
8. Wang, H.; Xu, Z.S. Interactive algorithms for improving incomplete linguistic preference relations based on consistency measures. *Appl. Soft Comput.* **2016**, *42*, 66–79. [CrossRef]
9. Massanet, S.; Riera, J.V.; Torrens, J.; Herrera-Viedma, E. A model based on subjective linguistic preference relations for group decision making problems. *Inf. Sci.* **2016**, *355*, 249–264. [CrossRef]
10. Xu, Y.J.; Wang, H. A group consensus decision support model for hesitant 2-tuple fuzzy linguistic preference relations with additive consistency. *J. Intell. Fuzzy Syst.* **2017**, *33*, 41–54. [CrossRef]

11. Rodriguez, R.M.; Martinez, L.; Herrera, F. Hesitant fuzzy linguistic term sets for decision making. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 109–119. [CrossRef]
12. Zhang, Z.M.; Wu, C. Hesitant fuzzy linguistic aggregation operators and their applications to multiple attribute group decision making. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2185–2202.
13. Lee, L.W.; Chen, S.M. Fuzzy decision making based on likelihood-based comparison relations of hesitant fuzzy linguistic term sets and hesitant fuzzy linguistic operators. *Inf. Sci.* **2015**, *294*, 513–529. [CrossRef]
14. Liu, P.D.; Teng, F. Some Interval neutrosophic hesitant uncertain linguistic Bonferroni aggregation operators for multiple attribute decision-making. *Int. J. Uncertain. Quantif.* **2017**, *7*, 525–572. [CrossRef]
15. Liao, H.C.; Xu, Z.S.; Zeng, X.J. Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making. *Inf. Sci.* **2014**, *271*, 125–142. [CrossRef]
16. Lee, L.W.; Chen, S.M. Fuzzy decision making and fuzzy group decision making based on likelihood-based comparison relations of hesitant fuzzy linguistic term sets 1. *J. Intell. Fuzzy Syst.* **2015**, *29*, 1119–1137. [CrossRef]
17. Chen, S.M.; Hong, J.A. Multicriteria linguistic decision making based on hesitant fuzzy linguistic term sets and the aggregation of fuzzy sets. *Inf. Sci.* **2014**, *286*, 63–74. [CrossRef]
18. Adar, T.; Delice, E.K. Evaluating mental work load using multi-criteria hesitant fuzzy linguistic term set (HFLTS). *Int. J. Fuzzy Syst.* **2017**, *8*, 90–101.
19. Joshi, D.K.; Kumar, S. Trapezium cloud TOPSIS method with interval-valued intuitionistic hesitant fuzzy linguistic information. *Granul. Comput.* **2018**, *3*, 139–152. [CrossRef]
20. Liu, D.H.; Chen, X.H.; Peng, D. Distance measures for hesitant fuzzy linguistic sets and their applications in multiple criteria decision making. *Int. J. Fuzzy Syst.* **2018**. [CrossRef]
21. Adem, A.; Çolak, A.; Dağdeviren, M. An integrated model using SWOT analysis and hesitant fuzzy linguistic term set for evaluation occupational safety risks in life cycle of wind turbine. *Saf. Sci.* **2018**, *106*, 184–190. [CrossRef]
22. Zhu, B.; Xu, Z.S. Consistency measures for hesitant fuzzy linguistic preference relations. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 35–45. [CrossRef]
23. Zhang, Z.M.; Wu, C. On the use of multiplicative consistency in hesitant fuzzy linguistic preference relations. *Knowl.-Based Syst.* **2014**, *72*, 13–27. [CrossRef]
24. Wang, H.; Xu, Z.S. Some consistency measures of extended hesitant fuzzy linguistic preference relations. *Inf. Sci.* **2015**, *297*, 316–331. [CrossRef]
25. Wu, Z.B.; Xu, J.P. Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. *Omega* **2016**, *65*, 28–40. [CrossRef]
26. Gou, X.J.; Xu, Z.S.; Liao, H.C. Group decision making with compatibility measures of hesitant fuzzy linguistic preference relations. *Soft Comput.* **2017**, 1–17. [CrossRef]
27. Li, C.C.; Rodríguez, R.M.; Martínez, L.; Dong, Y.C.; Herrera, F. Consistency of hesitant fuzzy linguistic preference relations: An interval consistency index. *Inf. Sci.* **2018**, *432*, 347–361. [CrossRef]
28. Xu, Y.J.; Wen, X.; Sun, H.; Wang, H.M. Consistency and consensus models with local adjustment strategy for hesitant fuzzy linguistic preference relations. *Int. J. Fuzzy Syst.* **2018**, 1–18. [CrossRef]
29. Wang, J.Q.; Wu, J.T.; Wang, J.; Zhang, H.Y.; Chen, X.H. Multi-criteria decision-making methods based on the Hausdorff distance of hesitant fuzzy linguistic numbers. *Soft Comput.* **2016**, *20*, 1621–1633. [CrossRef]
30. Wang, J.Q.; Wu, J.T.; Wang, J.; Zhang, H.Y.; Chen, X.H. Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. *Inf. Sci.* **2014**, *288*, 55–72. [CrossRef]
31. Lin, R.; Zhao, X.F.; Wei, G.W. Models for selecting an ERP system with hesitant fuzzy linguistic information. *J. Intell. Fuzzy Syst.* **2014**, *26*, 2155–2165.
32. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [CrossRef]
33. Xu, Z.S. Deviation measures of linguistic preference relations in group decision making. *Omega* **2005**, *33*, 249–254. [CrossRef]
34. Liang, W.Z.; Zhao, G.Y.; Wu, H. Evaluating investment risks of metallic mines using an extended TOPSIS method with linguistic neutrosophic numbers. *Symmetry* **2017**, *9*, 149. [CrossRef]
35. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
36. Wang, H.Y.; Chen, S.M. Evaluating students' answerscripts using fuzzy numbers associated with degrees of confidence. *IEEE Trans. Fuzzy Syst.* **2008**, *16*, 403–415. [CrossRef]

37. Chen, S.M.; Wang, N.Y.; Pan, J.S. Forecasting enrollments using automatic clustering techniques and fuzzy logical relationships. *Expert Syst. Appl.* **2009**, *36*, 11070–11076. [CrossRef]
38. Chen, S.M.; Chen, C.D. Handling forecasting problems based on high-order fuzzy logical relationships. *Expert Syst. Appl.* **2011**, *38*, 3857–3864. [CrossRef]
39. Chen, S.M.; Tanuwijaya, K. Fuzzy forecasting based on high-order fuzzy logical relationships and automatic clustering techniques. *Expert Syst. Appl.* **2011**, *38*, 15425–15437. [CrossRef]
40. Chen, S.M.; Munif, A.; Chen, G.S.; Liu, H.C.; Kuo, B.C. Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights. *Expert Syst. Appl.* **2012**, *39*, 6320–6334. [CrossRef]
41. Liang, W.Z.; Zhao, G.Y.; Hong, C.S. Performance assessment of circular economy for phosphorus chemical firms based on VIKOR-QUALIFLEX method. *J. Clean. Prod.* **2018**, *196*, 1365–1378. [CrossRef]
42. Liang, W.Z.; Zhao, G.Y.; Wu, H.; Chen, Y. Assessing the risk degree of goafs by employing hybrid TODIM method under uncertainty. *Bull. Eng. Geol. Environ.* **2018**. [CrossRef]
43. Rodríguez, R.M.; Bedregal, B.; Bustince, H.; Dong, Y.C.; Farhadinia, B.; Kahraman, C.; Martinez, L.; Torra, V.; Xu, Y.J.; Xu, Z.S.; et al. A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. *Inf. Fusion* **2016**, *29*, 89–97. [CrossRef]
44. Liu, N.; Meng, S.S. Approaches to the selection of cold chain logistics enterprises under hesitant fuzzy environment based on decision distance measures. *Granul. Comput.* **2018**, *3*, 27–38. [CrossRef]
45. Alcantud, J.C.R.; Torra, V. Decomposition theorems and extension principles for hesitant fuzzy sets. *Inf. Fusion* **2018**, *41*, 48–56. [CrossRef]
46. Xia, M.M.; Xu, Z.S. Hesitant fuzzy information aggregation in decision making. *Int. J. Approx. Reason.* **2011**, *52*, 395–407. [CrossRef]
47. Zhu, B. Studies on consistency measure of hesitant fuzzy preference relations. *Proc. Comput. Sci.* **2013**, *17*, 457–464. [CrossRef]
48. Tian, Z.P.; Wang, J.; Wang, J.Q.; Zhang, H.Y. A likelihood-based qualitative flexible approach with hesitant fuzzy linguistic information. *Cogn. Comput.* **2016**, *8*, 670–683. [CrossRef]
49. Zhang, Z.M. Multi-criteria decision-making using interval-valued hesitant fuzzy QUALIFLEX methods based on a likelihood-based comparison approach. *Neural Comput. Appl.* **2017**, *28*, 1835–1854. [CrossRef]
50. Yang, Y.; Hu, J.H.; An, Q.X.; Chen, X.H. Group decision making with multiplicative triangular hesitant fuzzy preference relations and cooperative games method. *Int. J. Uncertain. Quantif.* **2017**, *7*, 271–284. [CrossRef]
51. Dong, Y.C.; Li, C.C.; Chiclana, F.; Herrera-Viedma, E. Average-case consistency measurement and analysis of interval-valued reciprocal preference relations. *Knowl.-Based Syst.* **2016**, *114*, 108–117. [CrossRef]
52. Liu, F.; Pedrycz, W.; Wang, Z.X.; Zhang, W.G. An axiomatic approach to approximation-consistency of triangular fuzzy reciprocal preference relations. *Fuzzy Set. Syst.* **2017**, *322*, 1–18. [CrossRef]
53. Li, X.B.; Li, D.Y.; Liu, Z.X.; Zhao, G.Y.; Wang, W.H. Determination of the minimum thickness of crown pillar for safe exploitation of a subsea gold mine based on numerical modelling. *Int. J. Rock Mech. Min.* **2013**, *57*, 42–56. [CrossRef]