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A Method to Identify Simple Graphs by Special Binary Systems

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Abstract: In this paper, we discuss some relations between the semigroup, $Bin(X)$, of all groupoids $(X, *)$ and graphs. We discuss minimum (mutual) covering sets in several groupoids and discuss distances of graphs with groupoids. Finally, we obtain some results on frame graphs with groupoids.

Keywords: binary system (groupoid); minimum (mutual) covering set; (mutual) shortest distance; (di)frame graph; d/BCK -algebra

1. Introduction

J. Neggers [1] defined the notion of a pogroupoid, obtained a functorial connection between pogroupoids and posets and associated structure mappings. J. Neggers and H. S. Kim [2] proved that a pogroupoid (X, \cdot) is modular if and only if its associated poset (X, \leq) is $(C_2 + \underline{1})$ -free. Given a pogroupoid (X, \cdot) , the relation, \triangleright_μ , is transitive for any fuzzy subset, μ , of X if and only if the associated poset, (X, \leq) , of (X, \cdot) is $(C_2 + \underline{1})$ -free [3]. In a sequence of papers, Nebeský [4–6] discussed various properties of a graph (V, E) with its associated groupoid $(V, *)$. Although it is not identical in terms of outlook, there are some similarities between Nebeský's work and results which are discussed in this paper.

As we note below, our point of view permits us to consider arbitrary (simple) graphs as particular groupoids. By using this model, it is possible to assign to groupoids of a particular (locally zero) type to certain simple graphs as well. Using the viewpoint developed, we can then assign the theoretical parameters of graphs to groupoids in a meaningful way. How this is done is the topic of what follows. We concentrate on two such parameters: "covering" and "shortest distance". It is clear that a great deal more work remains to be done and can be done in a straightforward manner.

2. Preliminaries

A non-empty set, X , with a constant 0 and a binary operation " $*$ ", satisfying the following axioms, (i) $x * x = 0$, (ii) $0 * x = 0$, and (iii) $x * y = 0$, where $y * x = 0$ implies $x = y$ for all $x, y \in X$ is said to be a d -algebra [7]. For general references on d -algebras, we refer to [8–11].

A d -algebra $(X, 0, *)$ is said to be a BCK -algebra [12] if it satisfies the following additional axioms:

- (iv) $(x * y) * (x * z) * (z * y) = 0$,
- (v) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Example 1. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following table [11]:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

Then, $(X, *, 0)$ is a d -algebra which is not a BCK-algebra.

For general references on BCK-algebras, we refer to references [12–14]. Given a non-empty set, X , we denote the collection of all groupoids $(X, *)$, where $*$: $X \times X \rightarrow X$ is a map, by $Bin(X)$. Given elements $(X, *)$ and (X, \bullet) of $Bin(X)$, define a product “ \square ” on these groupoids as follows:

$$(X, *) \square (X, \bullet) = (X, \square),$$

where

$$x \square y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 1. $(Bin(X), \square)$ [15] is a semigroup, i.e., the operation “ \square ”, as defined in general, is associative. Furthermore, the left-zero-semigroup $(X, *)$ for which $x * y = x$ for all $x, y \in X$ is the identity for this operation.

Another element with interesting properties in $(Bin(X), \square)$ is the right-zero-semigroup, i.e., the groupoid $(X, *)$ for which $x * y = y$ for all $x, y \in X$. Fayoumi [16] is not referred in main text, please check. showed that a groupoid $(X, *)$ commutes relative to the operation, \square , of $(Bin(X), \square)$ if and only if any two element subset of $(X, *)$ is a subgroupoid which is either a left-zero-semigroup or a right-zero-semigroup. Thus, $(X, *)$ is an element of the center, $ZBin(X)$, of $(Bin(X), \square)$ if and only if for any pair of elements $x, y \in X, x * y = x, y * x = y$ or $x * y = y, y * x = x$. Therefore, among these groupoids, the left-zero-semigroup on X and the right-zero-semigroup on X are extreme cases. Furthermore, it is easily seen that $(ZBin(X), \square)$ is itself a semigroup with identity—a subsemigroup of the semigroup $(Bin(X), \square)$.

3. Graphs and Binary Systems

Let $(X, *)$ be an element of $ZBin(X)$. Suppose, in addition, that we construct a graph, Γ_X , as follows: $V(\Gamma_X) = X$ and $(x, y) \in E(\Gamma_X)$ —the edge set of Γ_X provided that $x \neq y, y * x = y, x * y = x$. Thus, if $(x, y) \in E(\Gamma_X)$, then $(y, x) \in E(\Gamma_X)$ as well, and we identify $(x, y) = (y, x)$ as an undirected edge of Γ_X . Similarly, since $(X, *)$ is an element of $ZBin(X)$, if $(x, y) \notin E(\Gamma_X)$, then $x * y = y, y * x = x$ and $(x, y) = (y, x)$ determines the absence of an edge, directed or otherwise. Since $x * x = x$ in any case, we do not consider this to be of particular graphical interest. The mapping $(X, *) \leftrightarrow \Gamma_X$ accomplishes the following:

Theorem 2. If $G = (V(G), E(G))$ is a simple graph with vertex set $V(G) = X$ and edge set $E(G)$, then G determines a unique groupoid, $(X, *)$, in the center of the semigroup $(Bin(X), \square)$ by defining the binary operation “ $*$ ” as $x * y = x, y * x = y$ if $(x, y) = (y, x) \in E(G)$ and $x * y = y, y * x = x$ if $(x, y) = (y, x) \notin E(G)$.

Theorem 3. If $G = (V(G), E(G))$ is a simple graph and $(X, *)$ is defined as in Theorem 2, then $V(\Gamma_X) = X = V(G)$ and $E(\Gamma_X) = E(G)$, so that $G = \Gamma_X$.

Proof. The proofs of both theorems are straightforward. \square

Proposition 1. If $(X, *)$ is the left-zero-semigroup, then Γ_X is the complete graph on X . If $|X| = n < \infty$, then $\Gamma_X = K_n$.

Proof. From the definition, it is clear that if $\{x, y\} \subseteq X, x \neq y$, then $x * y = x, y * x = y$, so that $(x, y) = (y, x) \in E(\Gamma_X)$, and the conclusion follows. \square

Proposition 2. If $(X, *)$ is the right-zero-semigroup, then Γ_X is the null(empty) graph on X , i.e., $E(\Gamma_X) = \emptyset$, while $V(\Gamma_X) = X$.

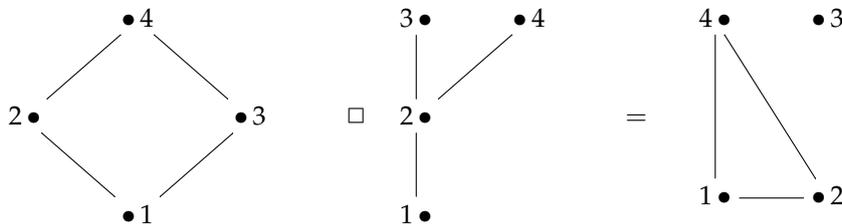
Proof. From the definition, it is clear that if $\{x, y\} \subseteq X, x \neq y$, then $x * y = y, y * x = x$, so that $(x, y) = (y, x) \notin E(\Gamma_X)$, i.e., $E(\Gamma_X) = \emptyset$ and the conclusion follows. \square

To illustrate the close relationship between simple graphs and groupoids, we note the following.

Theorem 4. If $G = (V(G), E(G))$ is a simple graph and $(X, *)$, $X = V(G)$, is the groupoid associated with G , then for the right-zero-semigroup (X, \bullet) defined on X , $(X, \square) = (X, *) \square (X, \bullet) = (X, \bullet) \square (X, *)$ defines (X, \square) as the complementary graph of G .

Proof. Suppose that $(x, y) = (y, x) \in E(G)$. Then, $x * y = x, y * x = y$. Hence, $x \square y = (x * y) \bullet (y * x) = y * x = y$ and $y \square x = (y * x) \bullet (x * y) = x * y = x$. Thus, $(x, y) = (y, x) \notin E(\Gamma_{(X, \square)})$. Similarly, if $(x, y) = (y, x) \notin E(G)$, then $(x, y) = (y, x) \in E(\Gamma_{(X, \square)})$, so that (X, \square) is the opposite groupoid of $(X, *)$, with $x \square y = y * x$, while $\Gamma_{(X, \square)}$ is the complementary graph of G , with $(x, y) = (y, x) \in E(\Gamma_{(X, \square)})$ if and only if $(x, y) = (y, x) \notin E(G)$. \square

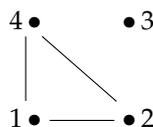
Example 2. The operation “ \square ” on $\text{Bin}(X)$ induces an operation “ \square ” on the graphs with vertex set X as well as edges illustrated in the proof of Theorem 4. Let $X = \{1, 2, 3, 4\}$ and consider two simple graphs on vertex set X . Then, we have



Based on the fact that $(\text{Bin}(X), \square)$ is a semigroup, it follows that this product is associative as well. For groupoids, we have tables and a resultant (product) table:

$*$	1	2	3	4	\bullet	1	2	3	4	\square	1	2	3	4
1	1	1	1	4	1	1	1	3	4	1	1	1	3	1
2	2	2	3	2	2	2	2	2	2	2	2	2	3	2
3	3	2	3	3	3	1	3	3	4	3	1	2	3	4
4	1	4	4	4	4	1	4	3	4	4	4	4	3	4

Hence, by reconstructing the graph, $\Gamma_{(X, \square)}$, we obtain

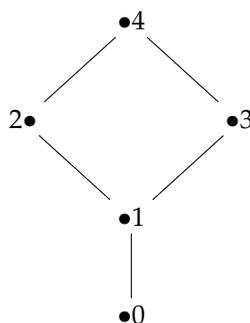


as pictured.

4. Graphs with Groupoids

If $G = (V(G), E(G))$ is a simple graph, then the *covering number*, $\gamma(G)$ is the cardinal number of a minimal covering set which is the smallest among these cardinal numbers. A covering set is a set of vertices such that any vertex not in the set is connected to an element in the set via an edge. A covering set is *minimal* if no vertex can be deleted from the set and still maintain the property of being a covering set. Thus, e.g., $\gamma(K_n) = \gamma(K_{1,n-1}) = 1$, since $\gamma(G) = 1$ for any n -graph containing a $K_{1,n-1}$, i.e., a star or a complete bipartite graph partitioned into two classes containing 1 and $n - 1$ elements respectively. The number $\gamma(G)$ is an example of a graph parameter which can be directly taken over by groupoids. Indeed, we shall consider an element, x , of a groupoid to *cover* an element, y , of a groupoid if $x * y = x$ and mutually y covers x if $y * x = y$ as well. In the setting of graphs with the constructions made above, it follows that if $G = (X, E(G))$, $X = V(G)$ is a simple graph, then if $x * y = x$, it follows that $y * x = y$ as well whenever $(x, y) = (y, x) \in E(G)$. Thus, for simple graphs, the two notions are equivalent. However, in the context of groupoids, they are not the same.

Example 3. Suppose that $(X, \leq, 0)$ is a poset with a minimal element of 0. The standard BCK-algebra $(X, *, 0)$ for this poset is defined by setting $x * y = 0$ if $x \leq y$ and $x * y = x$ otherwise. Now $x * 0 = x$ and $0 * x = 0$ for all $x \in X$, and thus, 0 mutually covers every element x of X . On the other hand, if $0 \notin \{x, y\}$ and $x \neq y$, then if $x \not\leq y$, $x * y = x$ and x covers y . If $y \not\leq x$, then $y * x = y$, and x and y mutually cover each other. Thus, we may consider the Hasse diagram of the poset in terms of this relationship, for example, the poset $(X := \{0, 1, 2, 3, 4\}, \leq)$



has a standard BCK-algebra table, as follows:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

In terms of the covering relation, we obtain

$$\mathcal{R}_C = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)\},$$

while the mutual covering relation yields

$$\mathcal{R}_{MC} = \{(0, 1), (1, 0), (0, 2), (2, 0), (0, 3), (3, 0), (0, 4), (4, 0), (2, 3), (3, 2)\}.$$

Following the analogy pathway, it is clear that in this case, if $\gamma_C(X, *, 0)$ is the cardinal of a minimum covering set (i.e., a minimal covering set of the smallest cardinal number), then $\gamma_C(X, *, 0) = 1$. If $\gamma_{MC}(X, *, 0)$ is the cardinal number of a minimum mutual covering set, then $\gamma_{MC}(X, *, 0) = 1$ as well.

Proposition 3. *If $(X, *)$ is a groupoid for which a mutual covering set exists, then*

$$\gamma_C(X, *) \leq \gamma_{MC}(X, *).$$

Proof. Suppose that $S \subseteq X$ is a mutual covering set for $(X, *)$. Then, S is also a covering set for $(X, *)$. Hence, if S is a minimum mutual covering set for $(X, *)$, then it is a covering set for $(X, *)$, and hence, it has a cardinal at least as large as $\gamma_C(X, *)$. \square

Example 4. *If $(X, *)$ is a left-zero-semigroup, then $x * y = x$ and $y * x = y$ implies that elements x and y mutually cover each other. Thus, any singleton $\{x\}$ is a minimal mutual covering set, and $\gamma_{MC}(X, *) = 1$, so that $\gamma_C(X, *) = 1$ as well.*

Proposition 4. *If $(X, *, 0)$ is a BCK-algebra, then $\gamma_{MC}(X, *, 0) = 1$.*

Proof. This follows from the fact that $0 * x = 0, x * 0 = x$ for all $x \in X$. Hence, $\{0\}$ is a mutual covering set. \square

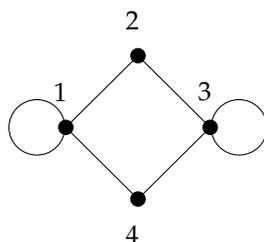
Proposition 5. *If $(X, *, 0)$ is a d -algebra, then $\gamma_C(X, *, 0) = 1$.*

Proof. This follows on from the fact that $0 * x = 0$ for all $x \in X$. \square

Example 5. *Let K be a field and let $x * y = x(x - y)$ for all $x, y \in K$. Then, $(K, *, 0)$ is a d -algebra. Since $0 * x = 0$ for all $x \in X$, $\{0\}$ is a minimal covering set, and thus, $\gamma_C(K, *, 0) = 1$. Let $x \neq 0$ in K . Then, $x * 0 = x^2 \neq x$, and thus, $\{0\}$ is not a mutual covering set. If $\{u\}$ is a covering set, then $u * x = u$ for all $x \in K$. It follows that $x = u - 1$ for all $x \in K$ —a contradiction. Hence, $\{0\}$ is the only minimal covering set of $(K, *, 0)$.*

Proposition 6. *If $G = (X, E(G)), X = V(G)$, is a simple graph and $(X, *)$ is the associated groupoid. Then, $\gamma_C(X, *) = \gamma_{MC}(X, *) = \gamma(G)$, the covering number of G .*

Example 6. *Given a simple graph, as shown below, we see that $\{1, 3\}$ covers $V(G) = \{1, 2, 3, 4\}$.*



Since 1 does not cover 3, it follows that $\gamma_C(G) = 2$. Since both 2 and 4 cover 1 and 3, $\{1, 3\}$ is a mutual covering set which is minimal, i.e., $\gamma_{MC}(G) = 2$.

Proposition 7. *If $(X, *, e)$ is a group, then $\gamma_C(X, *, e) + 1 = \gamma_{MC}(X, *, e) = |X|$, the cardinal of X .*

Proof. If $x * y = x$, then $y = e$. Hence, it follows that the smallest covering set of $(X, *, e)$ is $X \setminus \{e\}$. Now, if $x * y = x$ and $y * x = y$, then $x = y = e$. Hence $X \setminus \{e\}$ is not a mutual covering set, i.e., the smallest mutual covering set is X itself, i.e., $\gamma_{MC}(X, *, e) = \gamma_C(X, *, e) + 1 = |X|$, as claimed. \square

Proposition 8. *Let $(X, *)$ be a groupoid with $\gamma_{MC}(X, *) = 1$. The defined set $X_1 = \{u | \{u\} \text{ is a minimum mutual covering set, } \}$. Then, $(X_1, *)$ is a subsemigroup of the groupoid $(X, *)$ and a left-zero-semigroup.*

Proof. Suppose $u, v \in X_1$. Then, $u * v = u, v * u = v$, since $u \in X_1$. It follows that $(X_1, *)$ is a left-zero-semigroup, as claimed, and as such, it is a subsemigroup of $(X, *)$. \square

Corollary 1. *If $(X, *, 0)$ is a BCK-algebra, then $X_1 = \{0\}$.*

Proof. This follows immediately from Propositions 4 and 8. \square

5. Distances of Graphs with Groupoids

Given a groupoid $(X, *)$ we consider the *shortest distance*, $d(x, y)$, to be $n + 1$, if, for u_1, u_2, \dots, u_n , we have $x * u_1 = x, u_i * u_{i+1} = u_i, i = 1, 2, \dots, n - 1, u_n * y = u_n$, where $x \neq y, x, y \in X$, and there is no set with fewer elements having this property. The *mutual shortest distance*, $md(x, y)$, is $n + 1$ for elements $x \neq y, x, y \in X$, if, for u_1, u_2, \dots, u_n , we have $x * u_1 = x, u_1 * x = u_1, u_i * u_{i+1} = u_i, u_{i+1} * u_i = u_{i+1}, i = 1, 2, \dots, n - 1, u_n * y = u_n, y * u_n = y$, and there is no set with fewer elements having this property. We set $d(x, x) = md(x, x) = 0$ for any $x \in X$.

Proposition 9. *If $(X, *, 0)$ is a BCK-algebra, then $d(x, y) \leq 2$ for all $x, y \in X$.*

Proof. If $x \neq y$, then $x * 0 = x, 0 * y = 0$, and thus, $d(x, y) \leq 2$. \square

Example 7. *If $(X, \leq, 0)$ is a poset with a standard BCK-algebra $(X, *, 0)$, then $x \not\leq y$ implies $x * y = x$, and thus, $d(x, y) = 1$, since $x \neq y$. If $x \leq y, y \leq x$, then $x * y = x$ and $y * x = y$, so that $md(x, y) = 1$ in that situation as well.*

Proposition 10. *If $(X, *)$ is any groupoid for which the (mutual) shortest distance exists for all $x, y, z \in X$, then*

- (i) $d(x, y) \leq md(x, y)$,
- (ii) $d(x, z) \leq d(x, y) + d(y, z)$,
- (iii) $md(x, y) \leq md(x, y) + md(y, z)$,
- (iv) $md(x, y) = md(y, x)$.

Proof. This follows immediately from the definition. \square

Given a groupoid $(X, *)$ and $x, y \in X$, the *cycle number* $c(x, y)$ is the sum $c(x, y) = d(x, y) + d(y, x)$ if $d(x, y)$, and $d(y, x)$ exists. Since, $md(x, y) = md(y, x)$, the *mutual cycle number*, $mc(x, y)$, is simply $2md(x, y)$.

Proposition 11. *If $(X, *)$ is any groupoid with the property that $(x * y) * z = (x * z) * y$ and if $d(x, z)$ exists and is finite, then $d(x * y, z) \leq d(x, z)$.*

Proof. Suppose $d(x, z) = n + 1$, with intermediate elements u_1, u_2, \dots, u_n . Then, $(x * y) * u_1 = (x * u_1) * y = x * y, (u_1 * y) * u_2 = (u_1 * u_2) * y = u_1 * y, \dots, (u_n * y) * z = (u_n * z) * y = u_n * y$, so that $u_1 * y, \dots, u_n * y$ could serve as a set of intermediate elements for $x * y$ and z . The proposition follows. \square

Given a poset (X, \leq) , it is said to be *strongly connected* if $d(x, y) < \infty$ for all $x, y \in X$. It is said to be *strongly mutually connected* if $md(x, y) < \infty$ for all $x, y \in X$. In Proposition 10(i), $d(x, y) \leq md(x, y)$, so strongly mutually connected groupoids are also strongly connected.

Proposition 12. *If $(X, *)$ is a commutative (abelian) groupoid, i.e., $x * y = y * x$ for all $x, y \in X$, then $x * y = x$ implies $y * x = x$. Hence, if $x \neq y$, then $md(x, y) = \infty$.*

Proof. If $x \neq y$ and u_1, \dots, u_n is a set of intermediate elements for $md(x, y) = n$, then $x * u_1 = x, u_1 * x = u_1$ implies $x = u_1$, and by continuation, $x = u_1 = u_2 = \dots = u_n = y$ —a contradiction. It follows that $md(x, y) = \infty$. \square

Example 8. Let $DG = (\{x, y\}, x \rightarrow y)$ be the digraph with diagram $x \bullet \longrightarrow \bullet y$. Then, the associated

groupoid $(X, *)$ has the following Cayley table: $\begin{array}{c|cc} * & x & y \\ \hline x & x & x \\ y & x & y \end{array}$. It follows that $(X, *)$ is commutative, and $d(x, y) = 1, d(y, x) = md(x, y) = \infty$. If $DG = (\{x, y, z\}, x \bullet \longrightarrow \bullet y, y \bullet \longrightarrow \bullet z, z \bullet \longrightarrow \bullet x)$,

then $(X, *)$ has the Cayley table $\begin{array}{c|ccc} * & x & y & z \\ \hline x & x & x & z \\ y & x & y & y \\ z & z & y & z \end{array}$ so that $(X, *)$ is a commutative groupoid. We have $d(x, y) = d(y, z) = 1, d(x, z) = d(y, x) = d(z, y) = 2$, so that $(X, *)$ is strongly connected, but not strongly mutually connected.

Proposition 13. Let $(X, *)$ be a groupoid such that $x * y = x$ implies $y * x = y$ and $d(x, y)$ exists. Then, $d(x, y) = md(x, y)$.

Proof. Suppose that $d(x, y) = n$ and u_1, \dots, u_n is a set of intermediate elements. Then, $x * u_1 = x$ implies $u_1 * x = u_1, u_i * u_{i+1} = u_i$ which implies $u_{i+1} * u_i = u_{i+1}, u_n * z = u_n$ which implies $z * u_n = z$. Hence, u_1, \dots, u_n is also a set of intermediate elements for $md(x, y)$, i.e., $md(x, y) \leq d(x, y)$. From Proposition 10(i), $d(x, y) \leq md(x, y)$, and thus, $d(x, y) = md(x, y)$. \square

We consider the groupoids of Proposition 13 to be *undirected groupoids*.

Proposition 14. If $G = (X, E(G)), X = V(G)$ is an (undirected) simple graph, then the associated groupoid $(X, *)$ is an undirected groupoid.

Proof. This follows from the definition. \square

Example 9. Consider the groupoid $(X, *)$ with $X = \{a, b, c\}$ and Cayley table $\begin{array}{c|ccc} * & a & b & c \\ \hline a & a & a & b \\ b & b & b & b \\ c & b & c & c \end{array}$. Then, $a * c = c * a = b, a * b = a, b * a = b, c * b = c, b * c = b$, and $(X, *)$ is an undirected groupoid, since $a * b = a$ implies $b * a = b$. However, it does not correspond to a simple graph. If we compute $md(a, b) = 1, md(b, c) = 1, md(a, c) = 2$, we obtain an undirected graph as follows: $a \bullet \text{---} \bullet b \text{---} \bullet c$. Thus, the undirected groupoid $(X, *)$ is also strongly mutually connected.

6. Frame Graphs with Groupoids

Given a groupoid $(X, *)$, we consider the *frame* of $(X, *)$ to be the collection of all pairs $\{x, y\}$ such that $md(x, y) = md(y, x) = 1$. From the frame of $(X, *)$ we may construct a simple graph, $F(X, *)$, where $VF(X, *) = \{x | \{x, y\} \text{ is a pair in the frame for some } y \text{ in } X\}$, and $EF(X, *) = \{(x, y) = (y, x) | \{x, y\} \text{ is a pair in the frame}\}$. $F(X, *)$ is the *frame graph* of $(X, *)$, and every groupoid has a frame graph.

From the diframe of $(X, *)$, we may construct a simple digraph, $DF(X, *)$, with $VDF(X, *) = \{x | (x, y) \text{ or } (y, x) \text{ belongs to the diframe of } (X, *) \text{ for some } y \in X\}$, and $EDF(X, *) = \{(x, y) | (x, y) : \text{an ordered pair in the diframe}\}$. $DF(X, *)$ is the *diframe graph* of $(X, *)$ and every groupoid has such a diframe graph. The frame graph, $F(X, *)$, is seen to be a subdigraph of the diframe graph, $DF(X, *)$, in a natural way, i.e., the undirected edge, $\{x, y\}$, of $F(X, *)$ generates a pair of directed edges, (x, y) and (y, x) , in $DF(X, *)$.

Given a groupoid $(X, *)$ with a frame $F(X, *)$, a groupoid $(VF(X, *), \bullet)$ is generated, where $x, y \in VF(X, *)$ implies $x \bullet y = x, y \bullet x = y$ if $md(x, y) = md(y, x) = 1$ and $x \bullet y = y, y \bullet x = x$ otherwise, i.e., the groupoid $(VF(X, *), \bullet)$ is an element of $ZBin(VF(X, *))$, the frame groupoid of $(X, *)$.

Example 10. The groupoid $(X, *)$ in Example 9 has a frame $a \bullet \text{---} \bullet b \text{---} \bullet c$, $(VF(X, *), \bullet)$ has the

Cayley table $\begin{array}{c|ccc} \bullet & a & b & c \\ \hline a & a & a & c \\ b & b & b & b \\ c & a & c & c \end{array}$. If, for the groupoid $(X, *)$, we consider the diframe graph, $DF(X, *)$, then the

associated algebraic structure is the groupoid $(VDF(X, *), \bullet)$, where $x * y = x$ if $(x, y) \in EDF(X, *)$, i.e., if $d(x, y) = 1$. Since $md(x, y) = md(y, x) = 1$ implies $d(x, y) = d(y, x) = 1$ as well, it follows that in such a case that $x \rightarrow y$ and $y \rightarrow x$ represent the edge $x \bullet \text{---} \bullet y$. Hence, in this sense, the frame graph, $F(X, *)$, is a subdigraph of the diframe graph, $DF(X, *)$.

Example 11. In Example 9, the condition $a * c = c * ab$ yields $(a, c), (c, a)$ as not being elements of $EDF(X, *)$, and thus, $a * c = c, c * a = a$ in $DF(X, *)$, i.e., $(VF(X, *), \bullet)$ and $(VDF(X, *), \bullet)$ have exactly the same Cayley table.

Example 12. Suppose that $(X, *)$ is the groupoid with $X = \{a, b, c\}$, and $DF(X, *)$ is the digraph

$a \bullet \text{---} \bullet b \text{---} \bullet c$. Then, $F(X, *)$ is the frame graph, $a \bullet \text{---} \bullet b$. The table for $(X, *)$ can be

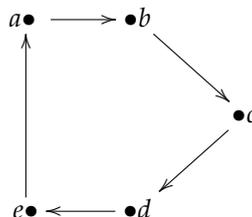
filled in as $\begin{array}{c|ccc} * & a & b & c \\ \hline a & \checkmark & a & c \\ b & b & \checkmark & b \\ c & a & b & \checkmark \end{array}$, where \checkmark can be filled with any element of X . For $(VDF(X, *), \bullet)$ the table is

$\begin{array}{c|ccc} \bullet & a & b & c \\ \hline a & a & a & c \\ b & b & b & b \\ c & a & b & c \end{array}$. Also, for $(VF(X, *), \bullet)$, the Cayley table is $\begin{array}{c|cc} \bullet & a & b \\ \hline a & a & a \\ b & b & b \end{array}$.

Example 13. Let $(X, *)$ be a groupoid with the following general table:

*	a	b	c	d	e
a	√	a	¬a	¬a	¬a
b	¬b	√	b	¬b	¬b
c	¬c	¬c	√	c	¬c
d	¬d	¬d	¬d	√	d
e	e	¬e	¬e	¬e	√

where $\neg x$ means "not x ", i.e., anything but x in X , and \checkmark can be filled with any element of X . Then, $DF(X, *)$ is the following graph:



i.e., for any $x, y \in \{a, b, c, d, e\}$, $c(x, y) = d(x, y) + d(y, x) = 5$. Also, $F(X, *)$ has $VF(X, *) = \emptyset$, since there is no pair $\{x, y\}$ with $(x, y), (y, x)$ elements of $EDF(X, *)$.

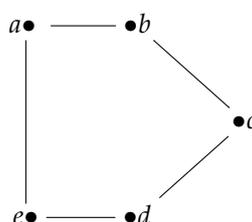
The groupoid $(VDF(X, *), \bullet)$ has the associated table:

\bullet	a	b	c	d	e
a	a	a	c	d	e
b	a	b	b	d	e
c	a	b	c	c	e
d	a	b	c	d	d
e	e	b	c	d	e

Example 14. Let $(X, *)$ be a groupoid with the following general table:

$*$	a	b	c	d	e
a	\surd	a	$\neg a$	$\neg a$	a
b	b	\surd	b	$\neg b$	$\neg b$
c	$\neg c$	c	\surd	c	$\neg c$
d	$\neg d$	$\neg d$	d	\surd	d
e	e	$\neg e$	$\neg e$	e	\surd

where $\neg x$ means “not x ”, i.e., anything but x in X , and \surd can be filled with any element of X . Then, $F(X, *) = DF(X, *)$ is the simple graph:



The corresponding groupoid $(VF(X, *), \bullet)$ (or $(VDF(X, *), \bullet)$) has the following multiplication table:

\bullet	a	b	c	d	e
a	a	a	c	d	a
b	b	b	b	d	e
c	a	c	c	c	e
d	a	b	d	d	d
e	e	b	c	e	e

The question is then how to represent the undirected cyclical nature of the groupoid $(X, *)$ in the most elegant way, such as we were able to do in Example 14.

Theorem 5. Let $f : (X, *) \rightarrow (Y, \bullet)$ be an onto homomorphism for groupoids. Define a binary operation “ \rightsquigarrow ” on X by $x \rightsquigarrow y$ with $f(x * y) = f(x)$ for any $x, y \in X$. If we define $DEF_f(X, *) := \{(x, y) \mid x \rightsquigarrow y\}$, then the graph, $DF(X, *)$, is a subdigraph of $DF_f(X, *)$.

Proof. If $f : (X, *) \rightarrow (Y, \bullet)$ is an onto homomorphism for groupoids, then $x * y = x$ implies $f(x) \bullet f(y) = f(x)$. Thus, either $f(x) = f(y)$ or $x \bullet \longrightarrow \bullet y$ maps to $f(x) \bullet \longrightarrow \bullet f(y)$ from $DF(X, *)$ to $DF(Y, \bullet)$. Furthermore, if $u \bullet \longrightarrow \bullet v$ is an arrow (directed edge) in $DF(Y, \bullet)$, then $f(x) \bullet f(y) = f(x)$ if $u = f(x), v = f(y)$. Hence, $f(x * y) = f(x)$, i.e., $x \rightsquigarrow y$. Thus, if we consider the digraph $DF_f(X, *)$ to have the vertex set $DVF_f(X, *) = X$ and edge set $DEF_f(X, *) = \{(x, y) \mid x \rightsquigarrow y\}$, then f induces a mapping $\hat{f} : DF_f(X, *) \rightarrow DF(Y, \bullet)$ by setting $\hat{f}(x \rightsquigarrow y) = f(x) \rightarrow f(y)$. The mapping \hat{f} is a surjection of graphs (or a graph epimorphism). Hence the graph $DF(X, *)$ is naturally a subdigraph of $DF_f(X, *)$. \square

Example 15. Let $f : (\mathbf{R}, +) \rightarrow (\mathbf{R}, +)$ be a mapping defined by $f(x) = 2x$ for all $x \in \mathbf{R}$. Then, $f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y)$, and f is a group(oid) epimorphism. Now, $x + y = x$ means $y = 0$, and thus, $f(x) + f(y) = f(x) + 0 = f(x)$. Thus, $x \rightsquigarrow y$ means $y = 0$ and $x \rightsquigarrow 0$ for all x , while $y \neq 0$

means $x \not\sim y$. Hence, $DEF_f(X, *) = \{(x, y) | f(x + y) = f(x)\} = \{(x, 0) | x \in \mathbf{R}\} = \{(x, y) | x \rightsquigarrow y\}$. The mapping $\hat{f} : DF_f(\mathbf{R}, +) \rightarrow DF(\mathbf{R}, +)$ given by $\hat{f}(x \rightsquigarrow 0) = f(x) \rightarrow f(0) = 2x \rightarrow 0$, as a mapping of graphs maps an edge, $x \rightarrow 0$, to an edge, $2x \rightarrow 0$, and it is an epimorphism of graphs, in fact, it is an isomorphism. Note that $2(x + y) = 2x$ implies that $x + y = x$ (i.e., $y = 0$), and $u = 2x, v = 2y, u \rightarrow v$ implies $v = 0$ and $2x \rightarrow 0$ implies $x \rightarrow 0$ as well.

7. Final Considerations

From the constructions made above, we see clearly that we may now infuse the theory of $(Bin(X), \square)$ with a multitude of graph theoretical notions. As a further illustration, we may consider the eccentricity, $e(x)$, of a vertex, x , of a groupoid $(X, *)$ as the maximum, $\max\{d(x, y) | x, y \in X\}$, and if the maximum is finite, assign that number to the vertex, x . From the eccentricity function, we may now derive the concepts of radius and diameter. Incidentally we may do the same with respect to the mutual distance to obtain the mutual eccentricity function. After having done so, types of groupoids can be discussed in terms of these parameters. We have a selection of possibilities:

- (1) Groupoids in $Bin(X)$, such that $\min e$ (i.e., radius) = $\max e$ (i.e., diameter) or equivalently e is a constant;
- (2) Groupoids in $Bin(X)$, such that $e(x) = me(x)$ for $x \in A \subseteq X$, where A is an interesting subset of X , e.g., $A = X$;
- (3) Groupoids in $Bin(X)$ with a small $\min e$ but $\max e$ being as large as possible;
- (4) Groupoids in $Bin(X)$ with $e(x) + me(x) = \text{constant } K$.

Obviously, there are many other interesting parameters of a graph's theoretical nature which may be introduced. Hence, it may be of interest in certain applications to pursue the subject further. In the interest of producing an article which is clear but manageable in size, we have decided not to go into more detail in this paper.

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