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# On Connection between Second-Degree Exterior and Symmetric Derivations of Kähler Modules

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**Abstract:** Mathematical physics looks for ways to apply mathematical ideas to problems in physics. In differential forms, the tensor form is first defined, and the definitions of exterior and symmetric differential forms are made accordingly. For instance, M is an R-module, $M \otimes_R M$ the tensor product of M with itself and H a submodule of $M \otimes_R M$ generated by $x \otimes y - y \otimes x$, where $x, y$ in M. Then, $\vee^2(M) = M \otimes_R M/H$ is called the second symmetric power of M. A role of the exterior differential forms in field theory is related to the conservation laws for physical fields, etc. In this study, I present a new approach to emphasize the properties of second exterior and symmetric derivations on Kahler modules, and I find a connection between them. I constitute exact sequences of $\vee^2(\Omega_1(S))$ and $\Lambda^2(\Omega_1(S))$, and I describe and prove a new isomorphism in the following: Let S be an affine algebra presented by $R/I$, where $R = k[x_1, ...x_s]$ is a polynomial algebra and $I = (f_1, ...f_m)$ an ideal of $R$. Then, I have $J_1\Omega_1(S) \simeq \Omega_1(S) \oplus \vee^2(\Omega_1(S)) \oplus \Lambda^2(\Omega_1(S))$.

**Keywords:** universal module; differential operators; Kähler module; symmetric derivation; exterior derivation

## 1. Introduction

Mathematical physics seeks any approaches to carry out mathematical ideas in problems in physics. These two disciplines are very close, especially with differential forms. Indeed, in the past, it was difficult to classify Newton and Gauss as purely physicists or mathematicians. Mathematical physics has been quite closely related to ideas in calculus, especially those of differential equations. An easy, acute definition of differential forms, due to H. Flanders [1], is that differential forms are those found under the integral signs, which give the current state of the differential forms. We draw up tensor field, related to components a basis formed of tensor products of basis tangent vectors $e_i$ and one-forms $w_i$, as displayed: $T = T^{ik...}_{...mn...} e_i \otimes e_k \otimes ... w^m \otimes w^n \otimes ...$ The components could be functions of the position. One may work in the natural bases for these spaces written, for coordinates $x^i$, $e_i = \frac{\partial}{\partial x^i}$, $w^i = dx^i$. Differential forms are presented as antisymmetric covariant tensor fields, namely fields that only the $w^i$ perform and the components are the antisymmetric basis written as $w^i \wedge w^j \wedge w^k = \sum_\pi (-1)^\pi \pi[w^i \otimes w^j \otimes w^k]$ composed of the antisymmetric tensor product of $w^i$. $\pi$ represents a permutation of the $w^i$. The symbol $\wedge$ is the called wedge product. The exterior derivative $d$ is a map from $p$-forms to $(p+1)$-forms. If $\gamma = f dx^i \wedge dx^j \wedge ...$, then $d$ is defined by $d\gamma = (\frac{\partial f}{\partial x^k}) dx^k \wedge dx^i \wedge dx^j \wedge ...$, and the exterior derivative satisfies the Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$. $(p = rank\alpha)$ in [2]. I now work with the set of differential forms on a commutative algebra. I give lemmas about second order exterior derivations and give definitions and lemmas about symmetric derivations of Kähler differentials, and I improve new approaches regarding second order exterior and symmetric derivatives of Kähler differentials.

## 2. Mathematical Background

One of the methods used to prove algebraic sets and their coordinate results is to study the universal modules of differential operators. Thus, problems related to algebraic sets are transferred to the module theory. For example, if p is a point in the algebraic set V, then under suitable conditions, it can be shown that p is a simple point of V if and only if $\Omega_1(R)$ is a free module including a local ring $R$ corresponding to the point p of V, and $\Omega_1(R)$ is the universal module of derivations of $R$. Universal modules of differential operators propose an alternative solution to the available criteria that helps to understand whether a coordinate ring corresponding to an irreducible finitely-generated algebra is regular or not. Universal modules were first defined by Y. Nakai in 1960 [3]. He gave the necessary and sufficient condition for an affine domain to be regular and gave an alternative idea of the Jacobian criterion for regularity. Universal modules of high-order differential operators were first described by H. Osborn in 1967 [4]. Y. Nakai [5] extended the fundamental theorems for higher derivations and some properties of the module of high order differentials. R.G. Heynemann and M.E. Sweedler also studied the same notations in [6]. They mentioned differential operators on a commutative algebra, which extends the notion of derivations. J. Johnson presented K*ä*hler differentials and differential algebra in [7]. Then, Erdogan studied universal modules of higher differential operators in 1993 [8]. The exterior and symmetric derivations of universal modules were also studied by Osborn [4], Hart [9,10], Erdogan [8], Olgun [11], Merkepçi and Olgun [12], Merkepçi and et al. [13], M.E. Sweedler [14] and Karakuş and et al [15]. Erdogan mentioned the second order exterior derivations of universal modules. Olgun gave the definition of generalized symmetric derivations on K*ä*hler modules and gave some homological properties. Merkepçi and Olgun defined some split exact sequences and isomorphisms about second order symmetric and exterior derivations on K*ä*hler modules, and they calculated the projective dimension of second order symmetric derivations on Kähler modules.

## 3. Preliminaries

Let k be a field with characteristic zero and $R$ be a commutative algebra over k. $J_1(R)$ is the universal module of first order differentials of $R$ over k. $\Omega_1(R)$ is the module of first order K*ä*hler differentials of $R$ over k. $d_1$ is the first order k-derivation $R \longrightarrow \Omega_1(R)$ of $R$.

$\Lambda^2(\Omega_1(R))$ is the second order exterior derivation of Kähler modules on $\Omega_1(R)$, and $\vee^2(\Omega_1(R))$ is the second order symmetric derivation of Kähler modules on $\Omega_1(R)$.

**Definition 1.** *Let R be any k-algebra (commutative with unit), $R \to \Omega_n(R)$ be the n-th order Kähler derivation of R and $\vee(\Omega_n(R))$ be the symmetric algebra $\bigoplus_{p\geq 0} \vee^p(\Omega_n(R))$ generated over R by $\Omega_n(R)$ [2].*

*A generalized symmetric derivation is any k-linear map D of $\vee(\Omega_n(R))$ into itself such that:*

- *(i)* $D(\vee^p(\Omega_n(R))) \subset \vee^{p+1}(\Omega_n(R))$
- *(ii)* *D is an n-th order derivation over k and*
- *(iii)* *the restriction of D to R ($R \simeq \vee^0(\Omega_n(R))$) is the Kähler derivation* $d_n : R \to \Omega_n(R)$.

**Lemma 1.** *Let R be a commutative k-algebra. Suppose that $\Omega_1(R)$ is the universal module of derivations of R with universal derivation $d{:}R \to \Omega_1(R)$ [6]. Then, the map:*

$$D{:}\Omega_1(R) \to \Lambda^2(\Omega_1(R))$$

$$D(\Sigma_i a_i d(b_i)) = \Sigma_i d(a_i) \wedge d(b_i)$$

*is a differential operator of order 1 on $\Omega_1(R)$ where $a_i, b_i$ in R.*

**Proof of Lemma 1.** (Osborn, H. Lemma (9.2). p. 155) □

**Proposition 1.** *Let S be an affine algebra presented by $R/I$. Then, the map $g : \Lambda^2(F/N) \longrightarrow \Lambda^2 F/L_N$ defined by $g(\overline{d_1(x_i)} \wedge \overline{d_1(x_j)}) = \overline{d_1(x_i) \wedge d_1(x_j)}$ is an isomorphism of S-modules, where $\Lambda^2 F$ is a free S-module* *with basis* $\{d_1(x_i) \wedge d_1(x_j) : 1 \leq i < j \leq s\}$. $L_N$ *is a submodule of* $\Lambda^2 F$ *generated by the set* $\{d(f_k) \wedge d(x_j) : k = 1, ..., m; j = 1, ..., s\}$.

**Proof of Proposition 1.** This follows by Proposition 4.1.5 in [6]. □

## 4. Main Results

**Lemma 2.** *Let S be an affine algebra presented by* $R/I$. *Then,*

$$\Omega_1(S) \xrightarrow{\hat{\Delta}_1} J_1(\Omega_1(S)) \xrightarrow{\hat{\beta}_1} \Lambda^2(\Omega_1(S)) \longrightarrow 0$$

*is an exact sequence of S-modules.*

**Proof of Lemma 2.** The following map:

$$\hat{D}_1 : \Omega_1(S) \to \Lambda^2(\Omega_1(S))$$

$$\hat{D}_1(\sum_{i,j} x_i d_1(x_j)) = \sum_{i,j} d_1(x_i) \wedge d_1(x_j)$$

is a differential operator of order one on $\Omega_1(S)$ where $x_i, x_j \in R$ and $1 \leq i, j \leq s$.

By the universal property of $J_1(\Omega_1(S))$, there is a unique R-module homomorphism:

$$\hat{\beta}_1 : J_1(\Omega_1(S)) \to \Lambda^2(\Omega_1(S))$$

such that $\hat{\beta}_1 \hat{\Delta}_1 = \hat{D}_1$ and the following diagram commutes.

$$\begin{array}{ccc} \Omega_1(S) & \xrightarrow{\hat{D}_1} & \Lambda^2((\Omega_1(S))) \\ \hat{\Delta}_1 \searrow & & \nearrow \exists! \hat{\beta}_1 \\ & J_1(\Omega_1(S)) & \end{array}$$

Since,

$$\begin{aligned} \hat{\beta}_1 \hat{\Delta}_1(x_i d_1(x_j)) &= \hat{D}_1(x_i d_1(x_j)) \\ &= d_1(x_i) \wedge d_1(x_j) \end{aligned}$$

$\hat{\beta}_1$ is surjective. Therefore, we have:

$$\Omega_1(S) \xrightarrow{\hat{\Delta}_1} J_1(\Omega_1(S)) \xrightarrow{\hat{\beta}_1} \Lambda^2(\Omega_1(S)) \longrightarrow 0$$

an exact sequence of S-modules. It is sufficient to show that the sequence is exact at $J_1(\Omega_1(S))$.

Note that $Im\hat{\Delta}_1$ is generated by $\hat{\Delta}_1(d_1(x_i))$ for $i$ = 1,..., s.

Therefore, we have:

$$\hat{\beta}_1 \hat{\Delta}_1(d_1(x_i)) = \hat{D}_1(d_1(x_i)) = d_1(1) \wedge d_1(x_i) = 0$$

Hence, $Im\hat{\Delta}_1$ is in $ker\hat{\beta}_1$. Therefore, we get $\hat{p} : J_1(\Omega_1(S))/Im\hat{\Delta}_1 \to \Lambda^2(\Omega_1(S))$ defined by:

$$\hat{p}(\overline{\hat{\Delta}_1(x_i d_1(x_j))} = d_1(x_i) \wedge d_1(x_j)$$

Now, assume that $\lambda^2 F$ and $L_N$ are as in Proposition 1. $\{\hat{\Delta}_1(x_i d_1(x_j)) : 1 \leq i < j \leq s\}$ generates $J_1(\Omega_1(S))/Im\hat{\Delta}_1$. Since $\Lambda^2 F$ is a free S-module with basis $d_1(x_i) \wedge d_1(x_j)$, we can write a map $\hat{q} : \Lambda^2 F \longrightarrow J_1(\Omega_1(S))/Im\hat{\Delta}_1$ by:

$$\hat{q}(d_1(x_i) \wedge d_1(x_j)) = \overline{\hat{\Delta}_1(x_i d_1(x_j))}$$

Thus, if $\{f_k\}$ is a generating set for I, we have:

$$\hat{q}(d_1(f_k) \wedge d_1(x_i)) = \hat{q}(\sum_i \frac{\partial f_k}{\partial x_i} d_1(x_i) \wedge d_1(x_i)) = \sum_i \frac{\partial f_k}{\partial x_i} \hat{\Delta}_1(x_i d_1(x_i) = \hat{\Delta}_1(f_k d_1(x_i)) = o$$

where $\frac{\partial}{\partial x_i} : R \longrightarrow R, \partial_i(x_j) = \delta_i, j$ for $i$, $j$ = 1,..., s and $\delta_i, j$ is the Kronecker delta. Hence, $\hat{q}(L_N) = 0$.

Therefore, $\hat{q}$ induces an S-module homomorphism: $\overline{\hat{q}} : \Lambda^2 F / L_N \longrightarrow J_1(\Omega_1(S)) / Im\hat{\Delta_1}$

$$\overline{\hat{q}}(\overline{d_1(x_i) \wedge d_1(x_j)}) = \overline{\hat{\Delta_1}(x_i d_1(x_j))}$$

It is clear that $\overline{\hat{q}}\hat{p}$ and $\hat{p}\overline{\hat{q}}$ are identities, and so, $ker\hat{p} = ker\hat{\beta}_1 / Im\hat{\Delta}_1 = 0$ and then $ker\hat{\beta}_1 = Im\hat{\Delta}_1$. Therefore, the sequence is exact. Similarly, the following lemma is given. □

**Lemma 3.** *Let S be an affine algebra presented by R/I. Then:*

$$\Omega_1(S) \xrightarrow{\check{\Delta}_1} J_1(\Omega_1(S)) \xrightarrow{\check{\beta}_1} \vee^2(\Omega_1(S)) \longrightarrow 0$$

*is an exact sequence of S-modules.*

**Proof of Lemma 3.** The following map:

$$\check{D}_1 : \Omega_1(S) \to \vee^2(\Omega_1(S))$$
$$\check{D}_1(\sum_{i,j} x_i d_1(x_j)) = \sum_{i,j} d_1(x_i) \vee d_1(x_j)$$

is a differential operator of order one on $\Omega_1(S)$ where $x_i, x_j$ in $R$ and $1 \leq i, j \leq s$. By the universal mapping property of $J_1(\Omega_1(S))$, there is a unique S-module homomorphism $\check{\beta}_1 : J_1(\Omega_1(S)) \longrightarrow \vee^2(\Omega_1(S))$ such that $\check{\beta}_1\check{\Delta}_1 = \check{D}_1$ and the following diagram commutes.

$$\begin{tikzcd} \Omega_1(S) \arrow[rr, "\check{D}_1"] \arrow[dr, "\check{\Delta}_1"'] & & \vee^2(\Omega_1(S)) \\ & J_1(\Omega_1(S)) \arrow[ur, "\exists! \check{\beta}_1"'] & \end{tikzcd}$$

Since,

$$\check{\beta}_1(\check{\Delta}_1(x_i d_1(x_j))) = \check{D}_1(x_i d_1(x_j)) = d_1(x_i) \vee d_1(x_j)$$

$\check{\beta}_1$ is surjective. Therefore, we have:

$$\Omega_1(S) \xrightarrow{\check{\Delta}_1} J_1(\Omega_1(S)) \xrightarrow{\check{\beta}_1} \vee^2(\Omega_1(S)) \longrightarrow 0$$

an exact sequence of S-modules. It is sufficient to prove that the sequence is exact at $J_1(\Omega_1(S))$.

Note that, $Im\check{\Delta}_1$ is generated by $\check{\Delta}_1(d_1(x_i))$, for $i$ = 1,..., s. Therefore, we have:

$$\check{\beta}_1\check{\Delta}_1(d_1(x_i)) = \check{D}_1(d_1(x_i)) = d_1(1) \vee d_1(x_i) = 0$$

It shows that $Im\check{\Delta}_1$ is in $ker\check{\beta}_1$. Therefore, we get $\check{p} : J_1(\Omega_1(S)) / Im\check{\Delta}_1 \to \vee^2(\Omega_1(S))$ defined by:

$$\check{p}(\overline{\check{\Delta}_1(x_i d_1(x_j))}) = d_1(x_i) \vee d_1(x_j)$$

Now, assume that $\vee^2 F$ and $L_N$ are as in Proposition 1. $\{\check{\Delta}_1(x_i d_1(x_j)) \mid 1 \leq i < j \leq s\}$ generates $J_1(\Omega_1(S))/Im\check{\Delta}_1$. Since $\vee^2 F$ is a free S-module with basis $d_1(x_i) \vee d_1(x_j)$, we can write a map $\check{q}$ : $\vee^2 F \longrightarrow J_1(\Omega_1(S))/Im\check{\Delta}_1$ by:

$$\check{q}(d_1(x_i) \vee d_1(x_j)) = \overline{\check{\Delta}_1(x_i d_1(x_j))}$$

Thus, if $\{f_k\}$ is a generating set for I, we have:

$$\check{q}(d_1(f_k) \vee d_1(x_i)) = \check{q}(\sum_i \frac{\partial f_k}{\partial x_i} d_1(x_i) \vee d_1(x_i)) = \sum_i \frac{\partial f_k}{\partial x_i} \check{\Delta}_1(x_i d_1(x_i) = \check{\Delta}_1(f_k d_1(x_i)) = o$$

where $\frac{\partial}{\partial x_i} : R \longrightarrow R, \partial_i(x_j) = \delta_i, j$ for $i, j$ = 1,..., s and $\delta_i, j$ is the Kronecker delta. Hence, $\check{q}(L_N) = 0$.

Therefore, $\check{q}$ induces an S-module homomorphism $\overline{\check{q}} : \vee^2 F/L_N \longrightarrow J_1(\Omega_1(S))/Im\check{\Delta}_1$:

$$\overline{\check{q}}(\overline{d_1(x_i) \vee d_1(x_j)}) = \overline{\check{\Delta}_1(x_i d_1(x_j))}$$

It is clear that $\overline{\check{q}}\check{p}$ and $\check{p}\overline{\check{q}}$ are identities, and so, $ker\check{p} = ker\check{\beta}_1/Im\check{\Delta}_1 = 0$ and then $ker\check{\beta}_1 = Im\check{\Delta}_1$. Therefore, the sequence is exact. □

**Theorem 1.** *S is an affine algebra presented by R/I. Then, there exists a split short exact sequence of S-modules:*

$$0 \longrightarrow ker\beta \longrightarrow J_1(\Omega_1(S)) \xrightarrow{\beta} \Omega_1(S) \longrightarrow 0$$

*and*

$$ker\beta \simeq \Lambda^2(\Omega_1(S)) \oplus \vee^2(\Omega_1(S))$$

*so, we have:*

$$J_1(\Omega_1(S)) \simeq \Omega_1(S) \oplus \Lambda^2(\Omega_1(S)) \oplus \vee^2(\Omega_1(S))$$

**Proof of Theorem 1.** The following diagram commutes such that $\beta\Delta_1 = 1_S$.

$$\begin{array}{ccc} \Omega_1(S) & \xrightarrow{1_S} & \Omega_1(S) \\ \Delta_1 \searrow & & \nearrow \exists! \beta \\ & J_1\Omega_1(S) & \end{array}$$

Since,

$$\beta(\Delta_1(\sum_{i,j} x_i d_1(x_j))) = \sum_{i,j} (x_i) d_1(x_j)$$

in $\Omega_1(S)$ for $i, j$ = 1,..., s.

$\beta$ is surjective. Therefore, we have:

$$0 \longrightarrow ker\beta \longrightarrow J_1(\Omega_1(S)) \xrightarrow{\beta} \Omega_1(S) \to 0$$

a short exact sequence of S-modules.

Now, the map:

$$\hat{\imath} : \Lambda^2(\Omega_1(S)) \longrightarrow J_1(\Omega_1(S))$$

defined by:

$$\hat{\imath}(d_1(x_i) \wedge d_1(x_j)) = 1/2[\Delta_1(x_i d_1(x_j)) + \Delta_1(x_j d_1(x_i)$$
$$+ x_i \Delta_1(d_1(x_j)) - x_j \Delta_1 d_1(x_i)]$$

Similarly, the map:

$$\check{i} : \vee^2(\Omega_1(S)) \longrightarrow J_1(\Omega_1(S))$$

defined by:

$$\check{i}(d_1(x_i) \vee d_1(x_j)) = 1/2[\Delta_1(x_i d_1(x_j)) - \Delta_1(x_j d_1(x_i))$$
$$-x_i\Delta_1(d_1(x_j)) + x_j\Delta_1 d_1(x_i)]$$

Then, we define the following map:

$$g : \Lambda^2(\Omega_1(S)) \oplus \vee^2(\Omega_1(S)) \longrightarrow J_1(\Omega_1(S))$$

$$g(x,y) = \beta\hat{i}(x) + \beta\check{i}(y)$$

$x \in \Lambda^2(\Omega_1(S))$ and $y \in \vee^2(\Omega_1(S))$.

Therefore,

$$ker\beta \cong \Lambda^2(\Omega_1(S)) \oplus \vee^2(\Omega_1(S))$$

Shortly, I prove the splitting. Let:

$$\hat{D}_1 : \Omega_1(S) \longrightarrow \Lambda^2(\Omega_1(S))$$

by:

$$\hat{D}_1(\sum_{i,j} x_i d_1(x_j)) = d_1(x_i) \wedge d_1(x_j)$$

and:

$$\check{D}_1 : \Omega_1(S) \longrightarrow \vee^2(\Omega_1(S))$$

defined by:

$$\check{D}_1(\sum_{i,j} x_i d_1(x_j)) = d_1(x_i) \vee d_1(x_j)$$

$\hat{D}_1$ and $\check{D}_1$ are first order derivations.

Therefore, by the universal property of $J_1(\Omega_1(S))$, there exists S-module homomorphisms:

$$\hat{\beta}_1 : J_1(\Omega_1(S)) \longrightarrow \Lambda^2(\Omega_1(S))$$
$$\check{\beta}_1 : J_1(\Omega_1(S)) \longrightarrow \vee^2(\Omega_1(S))$$

such that diagrams:

$$\begin{array}{ccc} \Omega_1(S) & \xrightarrow{\hat{D}_1} & \Lambda^2(\Omega_1(S)) \\ \downarrow \Delta_1 & & \downarrow 1_{\Omega_1(S)} \\ J_1(\Omega_1(S)) & \xrightarrow{\hat{\beta}_1} & \Lambda^2(\Omega_1(S)) \end{array}$$

and:

$$\begin{array}{ccc} \Omega_1(S) & \xrightarrow{\check{D}_1} & \vee^2(\Omega_1(S)) \\ \downarrow \Delta_1 & & \downarrow 1_{\Omega_1(S)} \\ J_1(\Omega_1(S)) & \xrightarrow{\check{\beta}_1} & \vee^2(\Omega_1(S)) \end{array}$$

commutes.

Therefore, I can write:

$$\hat{\beta}_1\Delta_1(\sum_{i,j} x_i d_1(x_j)) = \hat{D}_1(\sum_{i,j} x_i d_1(x_j)) = \sum_{i,j} d_1(x_i) \wedge d_1(x_j)$$

and:

$$\check{\beta}_1\Delta_1(\sum_{i,j} x_i d_1(x_j)) = \check{D}_1(\sum_{i,j} x_i d_1(x_j)) = \sum_{i,j} d_1(x_i) \vee d_1(x_j)$$

From here, I obtain:

$$\hat{\beta}_1\hat{i} = 1_{\Lambda^2(\Omega_1(S))}$$
$$\check{\beta}_1\check{i} = 1_{\vee^2(\Omega_1(S))}$$
$$\hat{\beta}_1\check{i} = 0$$
$$\check{\beta}_1\hat{i} = 0$$

Hence, the sequence splits. □

**Example 1.** *Let $R = k[x_1, x_2, x_3]$ be a polynomial algebra of dimension three. Then, $\Omega_1(R)$ is a free R-module of rank $\binom{4}{1} - 1 = 3$ with basis $\{d_1(x_1), d_1(x_2), d_1(x_3)\}$, and $\vee^2(\Omega_1(R))$ is a free R-module of rank $\binom{4}{2} = 6$ with basis $\{d_1(x_1) \vee d_1(x_1), d_1(x_1) \vee d_1(x_2), d_1(x_1) \vee d_1(x_3), d_1(x_2) \vee d_1(x_2), d_1(x_2) \vee d_1(x_3), d_1(x_3) \vee d_1(x_3)\}$. $\Lambda^2(\Omega_1(R))$ is a free R-module of rank $\binom{3}{2} = 3$ with basis $\{d_1(x_1) \wedge d_1(x_2), d_1(x_1) \wedge d_1(x_3), d_1(x_2) \wedge d_1(x_3)\}$, and $J_1(\Omega_1(R))$ is generated by:*

$$\{\Delta_1(d_1(x_1)), \Delta_1(d_1(x_2)), \Delta_1(d_1(x_3)), \Delta_1(x_1d_1(x_1)), \Delta_1(x_1d_1(x_2)), \Delta_1(x_1d_1(x_3)), \Delta_1(x_2d_1(x_1)), \Delta_1(x_2d_1(x_2)),$$
$$\Delta_1(x_2d_1(x_3)), \Delta_1(x_3d_1(x_1)), \Delta_1(x_3d_1(x_2)), \Delta_1(x_3d_1(x_3))\}$$

*so, I identify isomorphism from ranks of $\Omega_1(R), \vee^2\Omega_1(R)), \Lambda^2\Omega_1(R)$. Therefore,*

$$J_1(\Omega_1(R)) \cong \Omega_1(R) \oplus \vee^2(\Omega_1(R)) \oplus \Lambda^2(\Omega_1(R))$$

**Example 2.** *Let $S$ be the coordinate ring of the cups $x_2x_3 = x_1^3$. Then, $S = k[x_1, x_2, x_3]/(f)$ where $f = x_2x_3 - x_1^3$. $\Omega_1(S) \cong F/N$ whereF is a free S-module on $\{d_1(x_1), d_1(x_2), d_1(x_3)\}$ and $N$ is a submodule of $F$ generated by $d_1(f) = x_2d_1(x_3) + x_3d_1(x_2) - 3x_1^2d_1(x_1)$. Since rank $(\Omega_1(S)) = \binom{1+2}{2} - 1 = 2$,*

*I have rank $N$ = rank$F$ − rank$\Omega_1(S)$ = $3 - 2 = 1$. Therefore, $N$ is free S-module. Similarly,$\vee^2(\Omega_1(S)) \cong \vee^2F/L_N$ where $\vee^2F$ is a free module with basis $\{d_1(x_1) \vee d_1(x_1), d_1(x_1) \vee d_1(x_2), d_1(x_1) \vee d_1(x_3), d_1(x_2) \vee d_1(x_2), d_1(x_2) \vee d_1(x_3), d_1(x_3) \vee d_1(x_3)\}$, and $L_N$ is a submodule of $\vee^2F$ generated by:*

$$d_1(f) \vee d_1(x_1) = x_2d_1(x_3) \vee d_1(x_1) + x_3d_1(x_2) \vee d_1(x_1) - 3x^2d_1(x_1) \vee d_1(x_1)$$

$$d_1(f) \vee d_1(x_2) = x_2d_1(x_3) \vee d_1(x_2) + x_3d_1(x_2) \vee d_1(x_2) - 3x^2d_1(x_1) \vee d_1(x_2)$$

$$d_1(f) \vee d_1(x_3) = x_2d_1(x_3) \vee d_1(x_3) + x_3d_1(x_2) \vee d_1(x_3) - 3x^2d_1(x_1) \vee d_1(x_3)$$

*rank $\vee^2(\Omega_1(S)) = \binom{2+1}{2-1} = \binom{3}{1} = 3$.*

*By the same argument, $\Lambda^2(\Omega_1(S)) \cong \Lambda^2F/l_N$ where $\Lambda^2F$ is a free module with basis $\{d_1(x_1) \wedge d_1(x_2), d_1(x_1) \wedge d_1(x_3), d_1(x_2) \wedge d_1(x_3)\}$, and $l_N$ is a submodule of $\Lambda^2F$ generated by:*

$$d_1(f) \wedge d_1(x_1) = x_2d_1(x_3) \wedge d_1(x_1) + x_3d_1(x_2) \wedge d_1(x_1)$$

$$d_1(f) \wedge d_1(x_2) = x_2d_1(x_3) \wedge d_1(x_2) - 3x_1^2d_1(x_1) \wedge d_1(x_2)$$

$$d_1(f) \wedge d_1(x_3) = x_3d_1(x_2) \wedge d_1(x_3) - 3x_1^2d_1(x_1) \wedge d_1(x_3)$$

*rank* $\Lambda^2(\Omega_1(S)) = \binom{2}{2} = 1$.

*We derive the rank of* $J_1(\Omega_1(S))$ *from the ranks of* $\Omega_1(S), \vee^2(\Omega_1(S)), \Lambda^2(\Omega_1(S))$, *so* $J_1(\Omega_1(S)) \cong \Omega_1(S) \oplus \Lambda^2(\Omega_1(S)) \oplus \vee^2(\Omega_1(S))$.

## 5. Discussion

There are many studies about symmetric and exterior derivations of Kähler modules. Moreover, exterior derivations have been studied more extensively. Especially, there are articles about first order symmetric and exterior derivations of Kähler modules. Then, the question comes to mind: Is there a connection between these two subjects that are working in the same field? In this study, I examined the relationship between the symmetric and exterior derivations of Kähler modules. I am sure that the results that I have found are useful, especially in the next studies. Furthermore, I think the following questions are necessary for the future research:

(1) Can I define a new approach related to high order symmetric and exterior derivations of Kähler modules?

(2) Under which conditions can I write a connection about high order symmetric and exterior derivations on Kähler modules.

## 6. Conclusions

I know that symmetric and exterior derivations on Kähler modules play an important role both in mathematical physics and in commutative algebra. Studies on the structure of the symmetric and exterior derivations of Kähler modules present interesting aspects for not only mathematical physics, but also commutative algebra. Therefore, in this paper, I searched for new approaches for the connection of first order symmetric and exterior derivations. Finally, I defined some isomorphisms, proved them and gave special examples.

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