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Limits of *it*-Soft Sets and Their Applications for Rough Sets

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Abstract: Soft set theory is a mathematical tool for handling uncertainty. This paper investigates the limits of the interval type of soft sets (*it*-soft sets). The notion of *it*-soft sets is first introduced. Then, the limits of *it*-soft sets are proposed and their properties obtained. Next, point-wise continuity of *it*-soft sets and continuous *it*-soft sets is discussed. Finally, an application for rough sets is given.

Keywords: soft set; *it*-soft set; limit; continuity; rough set

1. Introduction

To manage complicated problems in engineering, economics and social science, classical mathematical tools are not always successful as a result of various types of uncertainties existing in these problems. Probability theory, fuzzy set theory [1], interval mathematics and rough set theory [2] are mathematical tools for handling uncertainty. However, there are some difficulties in these theories. For instance, probability theory may only handle stochastic phenomena. To overcome these difficulties, Molodtsov [3] presented soft set theory for managing uncertainty.

Nowadays, works on soft set theory are progressing rapidly. Maji et al. [4,5] used this theory to deal with decision making questions. Aktas et al. [6] proposed soft groups. Jiang et al. [7] depicted a soft set by means of description logics. Feng et al. [8] studied relationships among fuzzy sets, rough sets and soft sets. Ge et al. [9] investigated relationships between topological spaces and soft sets. Li et al. [10] discussed relationships among topologies, soft sets and soft rough sets. Li et al. [11] researched the roughness of fuzzy soft sets. Li et al. [12] considered parameter reduction in soft coverings.

Rough set theory as an important tool for dealing with the fuzziness and uncertainty of knowledge was proposed by Pawlak [2]. After thirty years of development, rough set theory has been applied to knowledge discovery, intelligent systems, machine learning, pattern recognition, decision analysis, inductive reasoning, image processing, meteorology, signal analysis and expert systems [2,13–15]. An approximation space is its base. Based on an approximation space, lower approximation and upper approximation may be produced. By using these approximations, knowledge concealed in an information system can be expressed in the form of decision rules [13–15]. The rough set model is based on the completeness of available information and ignores the incompleteness of available information and the possible existence of statistical information. This model for extracting rules in uncoordinated decision information systems often seems incapable. These have motivated many researchers to investigate probabilistic generalization of rough set theory and provide new rough set models for the study of uncertain information systems.

The probabilistic rough set model is the probabilistic generalization of rough set theory. In this model, probabilistic rough approximations are dependent on parameters. Researching the infinite change trend or the limit state of these approximations in accordance with parameters is helpful for the study of probabilistic rough sets.

It is well known that calculus theory is the foundation of modern science. The limits of functions are its basic concepts, which play a significant role in the process of development [16]. Since probabilistic rough approximations and level sets of a fuzzy set are both *it*-soft sets (i.e., interval type of soft sets), we may attempt to study the infinite change trend or the limit state of *it*-soft sets. It is worth mentioning that there is no systematic research and summary for the limits of *it*-soft sets, although the limit of *it*-soft sets has been formed in [17,18].

In general, most of the uncertain mathematical theories can only deal with uncertainty problems of discreteness. If the limit theory of *it*-soft sets is established, then these theories may be used to solve uncertainty problems of continuity. The aim of this paper is to establish the preliminarily limit theory of the interval type soft set so that some uncertain mathematical theories such as rough set theory may be used to solve uncertainty problems of continuity.

The rest of this paper is arranged as follows. In Section 2, we review some notions about the limits of set sequences and rough sets. In Section 3, we introduce *it*-soft sets and related notions. In Section 4, we propose the concept of the limits of *it*-soft sets and obtain their properties. In Section 5, we discuss the continuity of *it*-soft sets including the point-wise continuity of *it*-soft sets and continuous *it*-soft sets. In Section 6, we give an application for rough sets. Section 7 summarizes this paper.

2. Preliminaries

In this section, we review some notions about the limits of a set sequence, rough sets and *it*-soft sets.

Throughout this paper, U denotes the universe, which can be a finite set or an infinite set, 2^U means the collection of all subsets of U, E expresses the set of all possible parameters, R indicates the set of all real numbers, N shows the set of all natural numbers and I means an interval in R.

2.1. Limits of Set Sequences

Definition 1. *Given that U is the universe, if for each $n \in N,\ E_n \in 2^U$, then $\{E_n\}$ is said to be a set sequence in U. Denote [19]:*

$$\overline{\lim_{n\to\infty}} E_n = \{x \in U : \{n \in N : x \in E_n\} \ is\ infinite\},$$

$$\underline{\lim_{n\to\infty}} E_n = \{x \in U : \{n \in N : x \notin E_n\} \ is\ finite\}.$$

If $\underline{\lim_{n\to\infty}} E_n = \overline{\lim_{n\to\infty}} E_n = E$, then $\{E_n : n \in N\}$ is said to have the limit E, which is denoted by $\lim_{n\to\infty} E_n$, i.e., $\lim_{n\to\infty} E_n = E$; If $\underline{\lim_{n\to\infty}} E_n \neq \overline{\lim_{n\to\infty}} E_n$, then $\{E_n : n \in N\}$ is said to have no limit.

Obviously, $\underline{\lim_{n\to\infty}} E_n \subseteq \overline{\lim_{n\to\infty}} E_n$.

Proposition 1. *Let $\{E_n : n \in N\}$ be a set sequence in U [19].*

(1) $\overline{\lim_{n\to\infty}} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

(2) $\underline{\lim_{n\to\infty}} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$.

Proposition 2. *Suppose that $\{E_n : n \in N\}$ is a set sequence in U [19].*

(1) *If $\{E_n\} \uparrow$, then $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$.*

(2) *If* $\{E_n\}\downarrow$, *then* $\lim\limits_{n\to\infty} E_n = \bigcap\limits_{n=1}^{\infty} E_n$.

2.2. Rough Sets

Suppose that R is an equivalence relation on the universe U. Then, the pair (U, R) is said to be a Pawlak approximation space. Based on (U, R), two rough approximations are defined as:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\},\ \overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Then, $\underline{R}(X)$ and $\overline{R}(X)$ are called Pawlak lower and upper approximations of X, respectively. X is called rough if $\underline{R}(X) \neq \overline{R}(X)$; X is called crisp if $\underline{R}(X) = \overline{R}(X)$.

Definition 2. *Suppose that U is a finite universe. Then, a function $P : 2^U \to [0,1]$ is called a probability measure over U, if $P(U) = 1$ and $P(A \cup B) = P(A) + P(B)$ whenever $A \cap B = \emptyset$ [17,18].*

If P is a probability measure over U, $A, B \in 2^U$ and $P(B) > 0$, then $P(A|B) = \frac{P(A\cap B)}{P(B)}$ is said to be the conditional probability of the event A when the event B occurs.

Definition 3. *Let U be a finite universe, R an equivalence relation over U and P a probability measure over U. Then, the pair (U, R, P) is called a probabilistic approximate space. Based on (U, R, P), lower approximation and upper approximations of X are defined, respectively, as [17,18]:*

$$\underline{PI}_\alpha(X) = \{x \in U : P(X|[x]) \geq \alpha\},\ \overline{PI}_\beta(X) = \{x \in U : P(X|[x]) > \beta),$$

where $0 \leq \beta < \alpha \leq 1$.

Theorem 1. *Let (U, R, P) be a probabilistic approximate space. Then, the following properties hold [17,18].*

(1) $\underline{PI}_\alpha(\emptyset) = \overline{PI}_\alpha(\emptyset) = \emptyset$, $\underline{PI}_\alpha(U) = \overline{PI}_\alpha(U) = U$.

(2) $\underline{PI}_\alpha(X) \subseteq \overline{PI}_\alpha(X)$.

(3) $\underline{PI}_\alpha(U - X) = U - \overline{PI}_{1-\alpha}(X)$, $\overline{PI}_\alpha(U - X) = U - \underline{PI}_{1-\alpha}(X)$.

(4) *If* $X \subseteq Y$, *then* $\underline{PI}_\alpha(X) \subseteq \underline{PI}_\alpha(Y)$, $\overline{PI}_\alpha(X) \subseteq \overline{PI}_\alpha(Y)$.

(5) *If* $0 < \alpha_1 \leq \alpha_2 \leq 1$, $0 \leq \beta_1 \leq \beta_2 < 1$ *then*
$\underline{PI}_{\alpha_2}(X) \subseteq \underline{PI}_{\alpha_1}(X)$, $\overline{PI}_{\beta_2}(X) \subseteq \overline{PI}_{\beta_1}(X)$.

Theorem 2. *Suppose that (U, R, P) is a probabilistic approximate space. Then, for $0 < \gamma < 1$, $X \in 2^U$ [17,18],*

(1) $\lim\limits_{\alpha\uparrow\gamma} \underline{PI}_\alpha(X) = \bigcap\limits_{\alpha\in(0,\gamma)} \underline{PI}_\alpha(X) = \underline{PI}_\gamma(X)$,
$\lim\limits_{\alpha\downarrow\gamma} \underline{PI}_\alpha(X) = \bigcup\limits_{\alpha\in(\gamma,1]} \underline{PI}_\alpha(X) = \overline{PI}_\gamma(X)$;

(2) $\lim\limits_{\alpha\uparrow\gamma} \overline{PI}_\alpha(X) = \bigcap\limits_{\alpha\in[0,\gamma)} \overline{PI}_\alpha(X) = \underline{PI}_\gamma(X)$,
$\lim\limits_{\alpha\downarrow\gamma} \overline{PI}_\alpha(X) = \bigcup\limits_{\alpha\in(\gamma,1)} \overline{PI}_\alpha(X) = \overline{PI}_\gamma(X)$.

Although the limit of *it*-soft sets has been formed in Theorem 2, there is no systematic research and summary for the limits of *it*-soft sets. Thus, the limit theory of the interval type soft set deserves deep study so that rough set theory can be used to deal with uncertainty questions of continuity.

3. Soft Sets

Definition 4. *Given $A \subseteq E$, a pair (f, A) is said to be a soft set over U, if f is a mapping given by $f : A \to 2^U$. We also denote (f, A) by f_A [3].*

That is to say, a soft set f_A over U is a parametrized collection of subsets of U. For $e \in A$, $f(e)$ may be seen as the set of e-approximate elements of f_A. Clearly, every soft set is not a set.

Definition 5. *Let f_A and g_B be two soft sets over U [4].*

(1) f_A is called a soft subset of g_B, if $A \subseteq B$, and for each $e \in A$, $f(e) = g(e)$. We denote it by $f_A \,\widetilde{\subset}\, g_B$.

(2) f_A is said to be a soft super set of g_B, if $g_B \,\widetilde{\subset}\, f_A$. We denote it by $f_A \,\widetilde{\supset}\, g_B$.

Definition 6. *Let f_A and g_B be two soft sets over U [4].*

f_A and g_B are called soft equal, if $A \subseteq B$ and for each $e \in A$, $f(e) = g(e)$. We denote it by $f_A = g_B$.

Obviously, $f_A = g_B$ if and only if $f_A \,\widetilde{\subset}\, g_B$ and $f_A \,\widetilde{\supset}\, g_B$.

Definition 7. *Let f_A be a soft set over U [4].*

(1) f_A is called null, if for each $e \in A$, $f(e) = \emptyset$. We denote it by $\widetilde{\emptyset}$.

(2) f_A is said to be absolute, if for each $e \in A$, $f(e) = U$. We denote it by $\widetilde{U}$.

(3) f_A is referred to as constant, if there exists $X \in 2^U$ such that $f(e) = X$ for each $e \in A$. We denote it by $\widetilde{X}$ or X_A.

Definition 8. *Let f_A and g_B be two soft sets over U [4].*

(1) h_C is called the intersection of f_A and g_B, if $C = A \cap B$ and for each $e \in C$, $h(e) = f(e) \cap g(e)$. We denote it by $f_A \,\widetilde{\cap}\, g_B = h_C$.

(2) h_C is said to be the union of f_A and g_B, if $C = A \cup B$ and:

$$h(e) = \begin{cases} f(e), & \text{if } e \in A - B, \\ g(e), & \text{if } e \in B - A, \\ f(e) \cup g(e), & \text{if } e \in A \cap B. \end{cases}$$

We denote it by $f_A \,\widetilde{\cup}\, g_B = h_C$.

(3) h_C is referred to as the bi-intersection of f_A and g_B, if $C = A \times B$ and for any $a \in A$ and $b \in B$, $h(a,b) = f(a) \cap g(b)$. We denote it by $f_A \bigwedge g_B = h_C$.

(4) h_C is said to be the bi-union of f_A and g_B, if $C = A \times B$ and for any $a \in A$ and $b \in B$, $h(a,b) = f(a) \cup g(b)$. We denote it by $f_A \bigvee g_B = h_C$.

Definition 9. *The relative complement of a soft set f_A is defined as $f^c : A \to 2^U$ where $f^c(e) = U - f(e)$ for each $e \in A$ [20].*

Definition 10. *Suppose that f_A is a soft set over U [8].*

(1) f_A is called full, if $\bigcup\limits_{e \in A} f(e) = U$.

(2) f_A is said to be a partition, if $\{f(e) : e \in A\}$ is a partition of U.

Definition 11. *Given that f_A is a soft set over U [10],*

(1) f_A is called topological, if $\{f(e) : e \in A\}$ is a topology on U.

(2) f_A is said to be keeping intersection, if for any a, $b \in A$, there exists $c \in A$ such that $f(a) \cap f(b) = f(c)$.

(3) f_A is referred to as keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cup f(b) = f(c)$.

(4) f_A is said to be perfect, if $f : A \to 2^U$.

(5) f_A is called having no kernel, if $\cap\{f(e) : e \in A\} = \emptyset$.

Definition 12. *Let f_A be a soft set over U.*

(1) f_A *is called strong keeping intersection, if for each* $B \subseteq A$*, there exists* $b \in A$ *such that* $\bigcap_{a \in A} f(a) = f(b)$.

(2) f_A *is said to be strong keeping union, if for each* $B \subseteq A$*, there exists* $b \in A$ *such that* $\bigcup_{a \in A} f(a) = f(b)$.

Obviously, f_A is strong keeping intersection $\Rightarrow f_A$ is keeping intersection, and f_A is strong keeping union $\Rightarrow f_A$ is keep union.

Proposition 3. *Suppose that* f_A *is a soft set over* U*. Then, the following properties hold [10].*

(1) *If* f_A *is topological, then* f_A *is full, keeping intersection and strong keeping union.*
(2) f_A *is perfect if and only if* $\{f(e) : e \in A\}$ *is a discrete topology over* U.
(3) *If* f_A *is perfect, then* f_A *is topological.*
(4) f_A *has no kernel if and only if* (f^c, A) *is full.*

Example 1. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$*. Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2, x_5\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \emptyset, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_1, x_2\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ U, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then, f_A *is topological. However,* f_A *is neither perfect nor a partition.*

Example 2. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$*. Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2, x_5\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \{x_1, x_2\}, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_3\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ \{x_3, x_4\}, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

It should be noted that $\{x_1, x_2, x_5\} \cap \{x_3\} = \emptyset \neq f(\alpha)$ $(\forall\, \alpha \in I)$*. Then,* f_A *is not keeping intersection.*

Example 3. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$*. Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \{x_1, x_4\}, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_1, x_3, x_4\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ U, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then, f_A *is full, keeping intersection and strong keeping union. However,* f_A *is not topological.*

Example 4. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$*. Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \{x_5\}, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_3\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ \{x_4\}, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then, f_A *is partition. However,* f_A *is neither topological nor perfect.*

Example 5. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2, x_5\}, & \textit{if } \alpha \in [0, \frac{1}{4}), \\ \varnothing, & \textit{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_3\}, & \textit{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ \{x_3, x_4\}, & \textit{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then, f_A *is full and strong keeping intersection. However,*

$$\{x_1, x_2, x_5\} \cup \{x_3\} = \{x_1, x_2, x_3, x_5\} \neq f(\alpha) \ (\forall\, \alpha \in I).$$

Thus, f_A *is not keeping union.*

Example 6. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

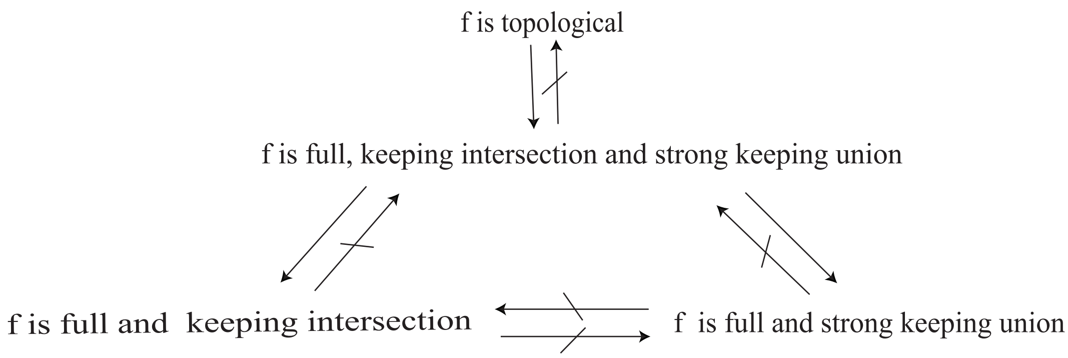
$$f(e) = \begin{cases} \{x_1\}, & \textit{if } \alpha \in [0, \frac{1}{4}), \\ \{x_2\}, & \textit{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_1, x_2\}, & \textit{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ U, & \textit{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

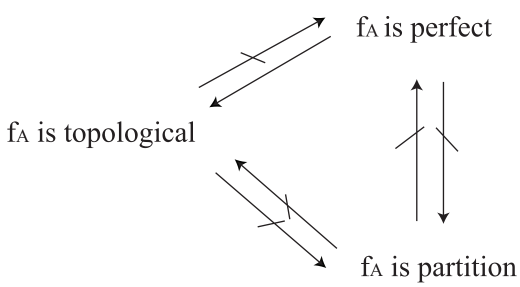
Then, f_A *is full and strong keeping union. However,*

$$\{x_1\} \cap \{x_2\} = \varnothing \neq f(\alpha) \ (\forall\, \alpha \in I).$$

Thus, f_A *is not keeping intersection.*

From Examples 1, 2, 3, 4, 5 and 6, we have thefollowing relationships:





4. Limit Theory of *it*-Soft Sets

4.1. The Concept of it-Soft Sets

Suppose that I is an interval in R. Let f_I be a soft set over U. Then, f_I is said to be an interval type of soft set (*it*-soft set) over U.

It is worth mentioning that the *it*-soft sets are different from interval soft sets in [21].

Definition 13. *Let f_I be an it-soft set over U.*

(1) *If for any $e_1, e_2 \in I, e_1 < e_2$ implies $f(e_1) \subset f(e_2)$(resp., $f(e_1) \supset f(e_2)$), then f_I is called strictly increasing (resp., strictly decreasing) on I.*

(2) *If for any $e_1, e_2 \in I, e_1 < e_2$ implies $f(e_1) \subseteq f(e_2)$(resp., $f(e_1) \supseteq f(e_2)$), then f_I is said to be increasing (resp., decreasing) on I.*

Definition 14. *Suppose that f_I is an it-soft set over U.*

(1) *If for any $e \in I$, $f(e) \subseteq f(e_0)$ ($e_0 \in I$), then $f(e_0)$ is called the maximum value of f_I.*

(2) *If for any $e \in I$, $f(e) \supseteq f(e_0)$ ($e_0 \in I$), then $f(e_0)$ is said to be the minimum value of f_I.*

4.2. Limits of it-Soft Sets

Let $e_0 \in R$, $\delta > 0$. Denote:

$$U(e_0, \delta) = \{e : |e - e_0| < \delta\},\ U^0(e_0, \delta) = \{e : 0 < |e - e_0| < \delta\}.$$

Then, $U(e_0, \delta)$ is called the δ neighborhood of e_0, $U^0(e_0, \delta)$ is said to be the δ neighborhood of e_0 having no heart, e_0 is the center of the neighborhood and δ is the radius of the neighborhood.

$U^+(e_0, \delta) = [e_0, e_0 + \delta)$ is referred to as the δ right neighborhood of e_0,

$U^-(e_0, \delta) = (e_0 - \delta, e_0]$ is said to be the δ left neighborhood of e_0.

Obviously, $U(e_0, \delta) = (e_0 - \delta, e_0 + \delta) = U^+(e_0, \delta) \cup U^-(e_0, \delta)$.

Given that f_I is an *it*-soft set over U, for $e_0 \in I$, $x \in U$, denote:

$$[x]_{f_I} = \{e \in I - \{e_0\} : x \in f(e)\},$$

$$(x)_{f_I} = \{e \in I - \{e_0\} : x \notin f(e)\}.$$

Remark 1. (1) $[x]_{f_I} \cup (x)_{f_I} = I - \{e_0\}$, $[x]_{f_I} \widetilde{\cap} (x)_{f_I} = \emptyset$.

(2) $[x]_{f_I} \cap [x]_{g_I} = [x]_{f_I \widetilde{\cap} g_I}$, $[x]_{f_I} \cup [x]_{g_I} = [x]_{f_I \widetilde{\cup} g_I}$.

(3) $(x)_{f_I} \cap (x)_{g_I} = (x)_{f_I \widetilde{\cup} g_I}$, $(x)_{f_I} \cup (x)_{g_I} = (x)_{f_I \widetilde{\cap} g_I}$.

(4) $[x]_{f_I^c} = (x)_{f_I}$, $(x)_{f_I^c} = [x]_{f_I}$.

Definition 15. *Let f_I be an it-soft set over U. For $e_0 \in I$, define:*

(1) $\overline{\lim\limits_{e \to e_0^+}}\, f(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta)\ is\ infinite\}$, *which is called the over-right limit of f_I as $e \to e_0$ (or the over limit of f_I as $e \to e_0^+$);*

(2) $\underline{\lim\limits_{e \to e_0^+}}\, f(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta)\ is\ finite\}$, *which is said to be the under-right limit of f_I as $e \to e_0$(or the under limit of f_I as $e \to e_0^+$).*

(3) $\overline{\lim\limits_{e \to e_0^-}}\, f(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^-(e_0, \delta)\ is\ infinite\}$, *which is referred to as the over-left limit of f_I as $e \to e_0$(or the over limit of f_I as $e \to e_0^-$).*

(4) $\underline{\lim\limits_{e \to e_0^-}}\, f(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^-(e_0, \delta)\ is\ finite\}$, *which is said to be the under-left limit of f_I as $e \to e_0$(or the under limit of f_I as $e \to e_0^-$).*

The following theorem shows that the limits can be characterized by δ and $\frac{1}{n}$.

Theorem 3. *Suppose that f_I is an it-soft set over U. Then, for $e_0 \in I$,*

(1) $\overline{\lim\limits_{e \to e_0^+}} f(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\}$
$= \{x \in U : \forall\, n \in N,\ [x]_{f_I} \cap U^+(e_0, \frac{1}{n}) \neq \emptyset\}$.

(2) $\underline{\lim\limits_{e \to e_0^+}} f(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\}$
$= \{x \in U : \exists\, n \in N,\ (x)_{f_I} \cap U^+(e_0, \frac{1}{n}) = \emptyset\}$.

(3) $\overline{\lim\limits_{e \to e_0^-}} f(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^-(e_0, \delta) \neq \emptyset\}$
$= \{x \in U : \forall\, n \in N,\ [x]_{f_I} \cap U^-(e_0, \frac{1}{n}) \neq \emptyset\}$.

(4) $\underline{\lim\limits_{e \to e_0^-}} f(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\}$
$= \{x \in U : \exists\, n \in N,\ (x)_{f_I} \cap U^-(e_0, \frac{1}{n}) = \emptyset\}$.

Proof. (1) Put:

$$S = \overline{\lim\limits_{e \to e_0^+}} f(e),\ T = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\},$$
$$L = \{x \in U : \forall\, n \in N,\ [x]_{f_I} \cap U^+(e_0, \tfrac{1}{n}) \neq \emptyset\}.$$

Obviously, $S \subseteq T \subseteq L$. We only need to prove $L \subseteq S$. Suppose $L \nsubseteq S$. Then, $L - S \neq \emptyset$. Pick $x \in L - S$. We have $x \notin S$. Therefore, $\exists\, \delta_0 > 0$, $[x]_{f_I} \cap U^+(e_0, \delta_0)$ is finite. Denote:

$$[x]_{f_I} \cap U^+(e_0, \delta_0) = \{e_1, e_2, \ldots, e_n\}.$$

Put $e^* = \min\{e_1, e_2, \ldots, e_n\}$, $0 < \frac{1}{n_0} < e^* - e_0$. Then:

$$0 < \frac{1}{n_0} < \delta_0,\ [x]_{f_I} \cap U^+(e_0, \frac{1}{n_0}) = \emptyset.$$

Therefore, $x \notin L$. However, $x \in L$. This is a contradiction. Thus, $L \subseteq S$.

(2) Put:

$$P = \underline{\lim\limits_{e \to e_0^+}} f(e),\ Q = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\},$$
$$K = \{x \in U : \exists\, n \in N,\ (x)_{f_I} \cap U^+(e_0, \tfrac{1}{n}) = \emptyset\}.$$

Obviously, $K \subseteq Q \subseteq P$. We only need to prove $P \subseteq K$. Suppose $P \nsubseteq K$. Then, $P - K \neq \emptyset$. Pick $x \in P - K$. Then, $x \notin K$.

Claim $\forall\, \delta$, $(x)_{f_I} \cap U^+(e_0, \delta)$ is infinite.

In fact, suppose that $\exists\, \delta > 0$, $(x)_{f_I} \cap U^+(e_0, \delta)$ is finite. Put:

$$(x)_{f_I} \cap U^+(e_0, \delta) = \{e_1, e_2, \ldots, e_n\},\ \ e^* = \min\{e_1, e_2, \ldots, e_n\},\ 0 < \frac{1}{n_0} < e^* - e_0.$$

Then, $0 < \frac{1}{n_0} < \delta$, $(x)_{f_I} \cap U^+(e_0, \frac{1}{n_0}) = \emptyset$. Therefore, $x \in K$, but $x \notin K$. This is a contradiction.

Since $\forall\, \delta > 0$, $(x)_{f_I} \cap U^+(e_0, \delta)$ is infinite, we have $x \notin P$. However, $x \in P$. This is a contradiction. Thus, $P \subseteq K$.

(3) The proof is similar to (1).

(4) The proof is similar to (2). $\square$

Example 7. *Consider Example 2, and pick $e_0 = \frac{1}{4}$. We have:*

$$[x_1]_f = [x_2]_f = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}),\ [x_3]_f = [\frac{1}{2}, 1),\ [x_4]_f = [\frac{3}{4}, 1),\ [x_5]_f = [0, \frac{1}{4}).$$

$$(x_1)_f = (x_2)_f = [\frac{1}{2}, 1),\ (x_3)_f = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}),\ (x_4)_f = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{3}{4}),\ (x_5)_f = (\frac{1}{4}, 1).$$

By Theorem 3

$$\overline{\lim_{e \to e_0^+}} f(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\} = \{x_1, x_2\};$$

$$\underline{\lim_{e \to e_0^+}} f(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\} = \{x_1, x_2\};$$

$$\overline{\lim_{e \to e_0^-}} f(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^-(e_0, \delta) \neq \emptyset\} = \{x_1, x_2, x_5\};$$

$$\underline{\lim_{e \to e_0^-}} f(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^-(e_0, \delta) = \emptyset\} = \{x_1, x_2, x_5\}.$$

Lemma 1. *Given that f_I is an it-soft set over U, then, for $e_0 \in I$,*

(1) $\overline{\lim_{e \to e_0^+}} f(e) = \bigcap_{n=1}^{\infty} \bigcap_{e \in (e_0, e_0 + \frac{1}{n}) \cap I} \bigcup_{\beta \in (e_0, e]} f(\beta).$

(2) $\underline{\lim_{e \to e_0^+}} f(e) = \bigcup_{n=1}^{\infty} \bigcup_{e \in (e_0, e_0 + \frac{1}{n}) \cap I} \bigcap_{\beta \in (e_0, e]} f(\beta).$

(3) $\overline{\lim_{e \to e_0^-}} f(e) = \bigcap_{n=1}^{\infty} \bigcap_{e \in (e_0 - \frac{1}{n}, e_0) \cap I} \bigcup_{\beta \in [e, e_0)} f(\beta).$

(4) $\underline{\lim_{e \to e_0^-}} f(e) = \bigcup_{n=1}^{\infty} \bigcup_{e \in (e_0 - \frac{1}{n}, e_0) \cap I} \bigcap_{\beta \in [e, e_0)} f(\beta).$

Proof. (1) Denote:

$$S = \overline{\lim_{e \to e_0^+}} f(e),\ \ T = \bigcap_{n=1}^{\infty} \bigcap_{e \in (e_0, e_0 + \frac{1}{n}) \cap I} \bigcup_{\beta \in (e_0, e]} f(\beta).$$

To prove $S = T$, it suffices to show that:

$$x \in S \ \Leftrightarrow\ \forall\, n \in N,\ \forall\, e \in (e_0, e_0 + \frac{1}{n}) \cap I,\ \exists\, \beta \in (e_0, e],\ x \in f(\beta).$$

" $\Rightarrow$ ". Let $x \in S$, $\forall\, n \in N$, $\forall\, e \in (e_0, e_0 + \frac{1}{n}) \cap I$. Put $\delta = e - e_0$. Then, $0 < \delta < \frac{1}{n}$.

Since $x \in S$, by Theorem 3(1), we have $[x]_{f_I} \cap U^+(e_0,\ \delta) \neq \emptyset$, pick $\beta \in\ [x]_{f_I} \cap U^+(e_0, \delta)$. Then, $\beta \in\ [x]_{f_I},\ \ \beta \in U^+(e_0, \delta)$.

This implies $x \in f(\beta),\ e_0 < \beta < e_0 + \delta = e$. Thus, $\beta \in (e_0, e]$.

" $\Leftarrow$ ". $\forall\, n \in N$, pick $e \in (e_0, e_0 + \frac{1}{n}) \cap I$.

By the condition, $\exists\, \beta \in (e_0, e]$, $x \in f(\beta)$. Then, $\beta \in U^+(e_0, \frac{1}{n}), \beta \in\ [x]_{f_I}$. Thus, $\forall\, n \in N$, $[x]_{f_I} \cap U^+(e_0, \frac{1}{n}) \neq \emptyset$.

By Theorem 3(1), $x \in S$.

(2) By (1) and Theorem 3(2),

$$\begin{aligned}
& x \notin \underline{\lim_{e \to e_0^+}} f(e) \\
& \Longleftrightarrow \forall\, n \in N,\ (x)_{f_I} \cap U^+(e_0, \tfrac{1}{n}) \neq \emptyset \\
& \Longleftrightarrow \forall\, n \in N, \{e \in I - e_0 : x \in U - f(e)\} \cap U^+(e_0, \tfrac{1}{n}) \neq \emptyset
\end{aligned}$$

$$\iff x \in \bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} (U - f(\beta))$$
$$\iff x \in U - \bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$$
$$\iff x \notin \bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta).$$

Hence, $\underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$.

(3) The proof is similar to (1).

(4) The proof is similar to (2). □

Lemma 2. *Let f_I be an it-soft set over U. Then, for $e_0 \in I$,*

(1) $\bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$.

(2) $\bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$.

(3) $\bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0-\frac{1}{n},e_0)\cap I} \bigcup_{\beta\in[e,e_0)} f(\beta) = \bigcap_{e\in(e_0-1,e_0)\cap I} \bigcup_{\beta\in[e,e_0)} f(\beta)$.

(4) $\bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0-\frac{1}{n},e_0)\cap I} \bigcap_{\beta\in[e,e_0)} f(\beta) = \bigcup_{e\in(e_0-1,e_0)\cap I} \bigcap_{\beta\in[e,e_0)} f(\beta)$.

Proof. (1) Put $E_n = \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$. Then, $\{E_n\} \uparrow$. Therefore, $\bigcap_{n=1}^{\infty} E_n = E_1$. Thus,

$\bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$.

(2) Put $F_n = \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$. Then, $\{F_n\} \downarrow$. Therefore, $\bigcup_{n=1}^{\infty} F_n = F_1$.

Thus,

$\bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$.

(3) It is similar to the proof of (1).

(4) It is similar to the proof of (2). □

Theorem 4. *Suppose that f_I is an it-soft set over U. Then, for $e_0 \in I$,*

(1) $\overline{\lim}_{e\to e_0^+} f(e) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$*; if f_I is increasing, then:*

$$\overline{\lim}_{e\to e_0^+} f(e) = \bigcap_{e\in(e_0,e_0+1)\cap I} f(e).$$

(2) $\underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$*; if f_I is decreasing, then:*

$$\underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{e\in(e_0,e_0+1)\cap I} f(e).$$

(3) $\overline{\lim}_{e\to e_0^-} f(e) = \bigcap_{e\in(e_0-1,e_0)\cap I} \bigcup_{\beta\in[e,e_0)} f(\beta)$*; if f_I is decreasing, then:*

$$\overline{\lim}_{e\to e_0^-} f(e) = \bigcap_{e\in(e_0-1,e_0)\cap I} f(e).$$

(4) $\underline{\lim}_{e \to e_0^-} f(e) = \bigcup_{e \in (e_0 - 1, e_0) \cap I} \bigcap_{\beta \in [e, e_0)} f(\beta)$; *if* f_I *is increasing, then:*

$$\underline{\lim}_{e \to e_0^-} f(e) = \bigcup_{e \in (e_0 - 1, e_0) \cap I} f(e).$$

Proof. This holds by Lemmas 1 and 2. □

Definition 16. *Given that* f_I *is an it-soft set over* U*, then, for* $e_0 \in I$*,*

(1) *If* $\underline{\lim}_{e \to e_0^+} f(e) = \overline{\lim}_{e \to e_0^+} f(e) = S$*, then* f_I *is said to have the limit* S *as* $e \to e_0^+$ *(or has the right-limit* S *as* $e \to e_0$*), which is denoted by* $\lim_{e \to e_0^+} f(e)$*, i.e.,* $\lim_{e \to e_0^+} f(e) = S$*;*

if $\underline{\lim}_{e \to e_0^+} f(e) \neq \overline{\lim}_{e \to e_0^+} f(e)$*, then* f_I *is said to have no limit as* $e \to e_0^+$ *(or has no right-limit as* $e \to e_0$*).*

(2) *If* $\underline{\lim}_{e \to e_0^-} f(e) = \overline{\lim}_{e \to e_0^-} f(e) = S$*, then* f_I *is said to have the limit* S *as* $e \to e_0^-$ *(or has the left-limit* S *as* $e \to e_0$*), which is denoted by* $\lim_{e \to e_0^-} f(e)$*, i.e.,* $\lim_{e \to e_0^-} f(e) = S$*;*

if $\underline{\lim}_{e \to e_0^+} f(e) \neq \overline{\lim}_{e \to e_0^+} f(e)$*, then* f_I *is said to have no limit as* $e \to e_0^+$ *(or has no left-limit as* $e \to e_0$*).*

(3) *If* $\lim_{e \to e_0^-} f(e) = \lim_{e \to e_0^+} f(e) = S$*, then* f_I *is said to have the limit* S *as* $e \to e_0$*, which is denoted by* $\lim_{e \to e_0} f(e)$*, i.e.,* $\lim_{e \to e_0} f(e) = S$*;*

if $\lim_{e \to e_0^-} f(e) \neq \lim_{e \to e_0^+} f(e))$*, then* f_I *is said to have no limit as* $e \to e_0$*.*

Definition 17. *Let* f_I *be an it-soft set over* U*. Then, for* $e_0 \in I$*,*

(1) *If* $\overline{\lim}_{e \to e_0^-} f(e) = \overline{\lim}_{e \to e_0^+} f(e) = S$*, then* f_I *is said to have the over-limit* S *as* $e \to e_0$*, which is denoted by* $\overline{\lim}_{e \to e_0} f(e)$*, i.e.,* $\overline{\lim}_{e \to e_0} f(e) = S$*;*

if $\overline{\lim}_{e \to e_0^-} f(e) \neq \overline{\lim}_{e \to e_0^+} f(e)$*, then* f_I *is said to have no over-limit as* $e \to e_0^+$*.*

(2) *If* $\underline{\lim}_{e \to e_0^-} f(e) = \underline{\lim}_{e \to e_0^+} f(e) = S$*, then* f_I *is said to have the under-limit* S *as* $e \to e_0$*, which is denoted by* $\underline{\lim}_{e \to e_0} f(e)$*, i.e.,* $\underline{\lim}_{e \to e_0} f(e) = S$*;*

if $\underline{\lim}_{e \to e_0^-} f(e) \neq \underline{\lim}_{e \to e_0^+} f(e)$*, then* f_I *is said to have no under-limit as* $e \to e_0$*.*

(3) *If* $\underline{\lim}_{e \to e_0} f(e) = \overline{\lim}_{e \to e_0} f(e) = S$*, then* f_I *is said to have the limit as* $e \to e_0$*, which is denoted by* $\lim_{e \to e_0} f(e)$*, i.e.,* $\lim_{e \to e_0} f(e) = S$*;*

if $\underline{\lim}_{e \to e_0} f(e) \neq \overline{\lim}_{e \to e_0} f(e)$*, then* f_I *is said to have no limit as* $e \to e_0$*.*

Remark 2. *The limit in Definition 16(3) and the limit in Definition 17(3) are consistent.*

Example 8. *Let* X_I *be a constant it-soft set over* U *where* $X \in 2^U$*. Then, for* $e_0 \in I$*,* $\lim_{e \to e_0} X(e) = X$*.*

Obviously, $[x]_{X_I} = \begin{cases} I - \{e_0\}, & x \in X \\ \varnothing, & x \notin X \end{cases}$, $(x)_{X_I} = \begin{cases} I - \{e_0\}, & x \notin X \\ \varnothing, & x \in X \end{cases}$.

By Theorem 3,

$$\overline{\lim}_{e \to e_0^+} X(e) = \{x \in U : \forall\, \delta > 0,\ [x]_{\widetilde{A}} \cap U^+(e_0, \delta) \neq \varnothing\},$$

$$\underline{\lim}_{e \to e_0^+} X(e) = \{x \in U : \exists\, \delta > 0,\ (x)_{\widetilde{A}} \cap U^+(e_0, \delta) = \emptyset\}.$$

Then, $\overline{\lim}_{e \to e_0^+} X(e) = X$, $\underline{\lim}_{e \to e_0^+} X(e) = X$.

Similarly, $\overline{\lim}_{e \to e_0^-} X(e) = X$, $\underline{\lim}_{e \to e_0^-} X(e) = X$.

Thus, $\lim_{e \to e_0} X(e) = X$.

Other types of limits of *it*-soft sets are proposed by the following definition, and these limits can be discussed in a similar way.

Definition 18. *Let $(f, (-\infty, +\infty))$ be an it-soft set over U. Define:*

$$(1)\quad \overline{\lim}_{e \to +\infty} f(e) = \overline{\lim}_{e \to 0^+} f(\frac{1}{e}), \quad \overline{\lim}_{e \to -\infty} f(e) = \overline{\lim}_{e \to 0^-} f(\frac{1}{e}),$$

$$\overline{\lim}_{e \to \infty} f(e) = \overline{\lim}_{e \to 0} f(\frac{1}{e}).$$

$$(2)\quad \underline{\lim}_{e \to +\infty} f(e) = \underline{\lim}_{e \to 0^+} f(\frac{1}{e}), \quad \underline{\lim}_{e \to -\infty} f(e) = \underline{\lim}_{e \to 0^-} f(\frac{1}{e}),$$

$$\underline{\lim}_{e \to \infty} f(e) = \underline{\lim}_{e \to 0} f(\frac{1}{e}).$$

$$(3)\quad \lim_{e \to +\infty} f(e) = \lim_{e \to 0^+} f(\frac{1}{e}), \quad \lim_{e \to -\infty} f(e) = \lim_{e \to 0^-} f(\frac{1}{e}),$$

$$\lim_{e \to \infty} f(e) = \lim_{e \to 0} f(\frac{1}{e}).$$

4.3. Properties of Limits of it-Soft Sets

Proposition 4. *For the over-right limit, the following properties hold:*

(1) *If $f(e) \subseteq g(e) (\forall\, e \in (e_0, e_0 + \delta_0))$, then $\overline{\lim}_{e \to e_0^+} f(e) \subseteq \overline{\lim}_{e \to e_0^+} g(e)$.*

(2) $\overline{\lim}_{e \to e_0^+} (f(e) \cup g(e)) = \overline{\lim}_{e \to e_0^+} f(e) \cup \overline{\lim}_{e \to e_0^+} g(e)$.

(3) $\overline{\lim}_{e \to e_0^+} (U - f(e)) = U - \underline{\lim}_{e \to e_0^+} f(e)$.

(4) *If $\overline{\lim}_{e \to e_0^+} f(e) = \triangle \subset B$, then $\exists\, \delta > 0, \forall\, e \in (e_0, e_0 + \delta)$, $f(e) \subset B$.*

(5) 1) $\overline{\lim}_{e \to e_0^+} (f(e) \times g(e)) \subseteq \overline{\lim}_{e \to e_0^+} f(e) \times \overline{\lim}_{e \to e_0^+} g(e)$;

2) $\overline{\lim}_{e \to e_0^+} f(e) \times \overline{\lim}_{e \to e_0^+} g(e) = \bigcap_{e \in (e_0, e_0+1) \cap I} \ \bigcup_{\beta, \gamma \in (e_0, e]} (f(\beta) \times g(\gamma))$.

Proof. (1) Denote:

$$[x]_{f_I} = \{e \in I - \{e_0\} : x \in f(e)\},\ [x]_{g_I} = \{e \in I - \{e_0\} : x \in g(e)\}.$$

$\forall\, x \in \overline{\lim}_{e \to e_0^+} f(e)$, by Theorem 3(1), $\forall\, \delta > 0$, $[x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$. Pick $e_\delta \in [x]_{f_I} \cap U^+(e_0, \delta)$. Then, $x \in f(e_\delta)$, $e_\delta \in U^+(e_0, \delta)$.

1) If $\delta \leq \delta_0$, then $e_\delta \in U^+(e_0, \delta_0)$. By the condition, $f(e_\delta) \subseteq g(e_\delta)$. Then, $x \in g(e_\delta)$. This implies $e_\delta \in (x)_{f_I} \cap U^+(e_0, \delta)$. Therefore, $(X)_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$.

2) If $\delta > \delta_0$, then $U^+(e_0, \delta_0) \subseteq U^+(e_0, \delta)$. Therefore, $(x)_{f_I} \cap U^+(e_0, \delta_0) \subseteq (X)_{f_I} \cap U^+(e_0, \delta)$. Since $e_{\delta_0} \in (X)_{f_I} \cap U^+(e_0, \delta_0)$, we have $(x)_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$.

By 1) and 2), $\forall\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$. By Theorem 3(1), $x \in \overline{\lim\limits_{e \to e_0^+}}\, g(e)$.

Thus,

$$\overline{\lim_{e \to e_0^+}}\, f(e) \subseteq \overline{\lim_{e \to e_0^+}}\, g(e).$$

(2) " $\supseteq$ ". This holds by (1).

" $\subseteq$ ". Suppose $\overline{\lim\limits_{e \to e_0^+}}\,(f(e) \cup g(e)) \nsubseteq \overline{\lim\limits_{e \to e_0^+}}\, f(e) \cup \overline{\lim\limits_{e \to e_0^+}}\, g(e)$. Then:

$$\overline{\lim_{e \to e_0^+}}\,(f(e) \cup g(e)) - \overline{\lim_{e \to e_0^+}}\, f(e) \cup \overline{\lim_{e \to e_0^+}}\, g(e) \neq \emptyset.$$

Pick $x \in \overline{\lim\limits_{e \to e_0^+}}\,(f(e) \cup g(e)) - \overline{\lim\limits_{e \to e_0^+}}\, f(e) \cup \overline{\lim\limits_{e \to e_0^+}}\, g(e)$. We have:

$$x \in \overline{\lim_{e \to e_0^+}}\,(f(e) \cup g(e)),\ \ x \notin \overline{\lim_{e \to e_0^+}}\, f(e) \text{ and } x \notin \overline{\lim_{e \to e_0^+}}\, g(e).$$

By Theorem 3, $\exists\, \delta_1, \delta_2 > 0,\ [x]_f \cap U^+(e_0, \delta_1) = \emptyset,\ \ [x]_g \cap U^+(e_0, \delta_2) = \emptyset$.

Pick δ_3=min$\{\delta_1, \delta_2\}$. Then, $[x]_f \cap U^+(e_0, \delta_3) = \emptyset$ and $[x]_g \cap U^+(e_0, \delta_3) = \emptyset$. It follows that:

$$([x]_f \cup [x]_{g_I}) \cap U^+(e_0, \delta_3) = ([x]_f \cap U^+(e_0, \delta_3)) \cup ([x]_g \cap U^+(e_0, \delta_3)) = \emptyset.$$

By Remark 1, $[x]_{f \cup g} \cap U^+(e_0, \delta_3) = \emptyset$.

Thus,

$x \notin \overline{\lim\limits_{e \to e_0^+}}\,(f \cup g)(e) = \overline{\lim\limits_{e \to e_0^+}}\,(f(e) \cup g(e))$. This is a contradiction.

(3) $\forall\, x \in \overline{\lim\limits_{e \to e_0^+}}\,(U - f(e))$. Then, $x \in \overline{\lim\limits_{e \to e_0^+}}\, f^c(e)$. By Theorem 3, $\forall\, \delta > 0$, $[x]_{f^c} \cap U^+(e_0, \delta) \neq \emptyset$. By Remark 1, $(x)_f \cap U^+(e_0, \delta) \neq \emptyset$. Thus,

$$x \in U - \underline{\lim_{e \to e_0^+}}\, f(e).$$

Conversely, the proof is similar.

(4) Suppose that $\forall\, \delta > 0,\ \exists\, e \in (e_0, e_0 + \delta)$, $f(e) \nsubseteq B$ or $f(e) = B$.

1) If $f(e) \nsubseteq B$, then $f(e) - B \neq \emptyset$. Pick $x \in f(e) - B$.

We have:

$$x \in f(e), x \notin B, e \in\ [x]_{f_I}.$$

Since $e \in (e_0, e_0 + \delta)$. Then, $[x]_{f_I} \cap (e_0, e_0 + \delta) \neq \emptyset$. Therefore, $x \in \overline{\lim\limits_{e \to e_0^+}}\, f(e)$.

Thus, $x \in B$. This is a contradiction.

2) If $f(e) = B$, then $\triangle - B = \emptyset$. Therefore, $\exists\, x \in B, x \notin \triangle$.

Since $x \in f(e)$, we have $x \in\ [x]_{f_I}$, $[x]_{f_I} \cap (e_0, e_0 + \delta) \neq \emptyset$. Therefore,

$$x \in \overline{\lim_{e \to e_0^+}}\, f(e) = \triangle.$$

This is a contradiction.

(5) 1) Put:

$$H_{f \times g}(e) = \bigcup_{\beta \in (e_0, e]} (f(\beta) \times g(\beta)).$$

By Theorem 4(1),

$$\overline{\lim_{e \to e_0^+}}\,(f(e) \times g(e)) = \bigcap_{e \in (e_0, e_0+1) \cap I} H_{f \times g}(e).$$

$\forall\ (x,y) \in \overline{\lim\limits_{e\to e_0^+}}\,(f(e)\times g(e))$, we have $(x,y) \in \bigcap\limits_{e\in(e_0,e_0+1)\cap I} H_{f\times g}(e)$. Since:

$$H_{f\times g}(e) = \bigcup_{\beta\in(e_0,e]} (f(\beta)\times g(\beta)),$$

we have $\forall\ e \in (e_0,e_0+1)\cap I$, $\exists\ \beta_e \in (e_0,e]$, $(x,y)\in f(\beta_e)\times g(\beta_e)$. It follows that $x\in f(\beta_e)$, $y\in g(\beta_e)$. Then, $x\in H_f(e)$ and $y\in H_g(e)$. Therefore,

$$x \in \bigcap_{e\in(e_0,e_0+1)\cap I} H_f(e) = \overline{\lim_{e\to e_0^+}}\, f(e),\ \ y \in \bigcap_{e\in(e_0,e_0+1)\cap I} H_g(e) = \overline{\lim_{e\to e_0^+}}\, g(e).$$

Thus, $(x,y) \in \overline{\lim\limits_{e\to e_0^+}}\, f(e) \times \overline{\lim\limits_{e\to e_0^+}}\, g(e)$.

Thus,

$$\overline{\lim_{e\to e_0^+}}\,(f(e)\times g(e)) \subseteq \overline{\lim_{e\to e_0^+}}\, f(e) \times \overline{\lim_{e\to e_0^+}}\, g(e).$$

2) $\forall\ (x,y) \in \overline{\lim\limits_{e\to e_0^+}}\, f(e) \times \overline{\lim\limits_{e\to e_0^+}}\, g(e)$, we have:

$$x \in \overline{\lim_{e\to e_0^+}}\, f(e) = \bigcap_{e\in(e_0,e_0+1)\cap I}\ \bigcup_{\beta\in(e_0,e]} f(\beta),\ y \in \overline{\lim_{e\to e_0^+}}\, g(e) = \bigcap_{e\in(e_0,e_0+1)\cap I}\ \bigcup_{\beta\in(e_0,e]} g(\beta).$$

Then, $\forall\ e \in (e_0,e_0+1)\cap I$, $\exists\ \beta_e,\gamma_e \in (e_0,e]$, $x\in f(\beta_e)$, $y \in g(\gamma_e)$. Then, $(x,y)\in f(\beta_e)\times g(\gamma_e)$. Therefore,

$$(x,y) \in \bigcap_{e\in(e_0,e_0+1)\cap I}\ \bigcup_{\beta,\gamma\in(e_0,e]} (f(\beta)\times g(\gamma)).$$

Conversely, the proof is similar.

Thus,

$$\overline{\lim_{e\to e_0^+}}\, f(e) \times \overline{\lim_{e\to e_0^+}}\, g(e) = \bigcap_{e\in(e_0,e_0+1)\cap I}\ \bigcup_{\beta,\gamma\in(e_0,e]} (f(\beta)\times g(\gamma)). \quad \square$$

Proposition 5. *For the under-right limit, the following properties hold.*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\ e \in (e_0, e_0+\delta_0))$, *then* $\underline{\lim\limits_{e\to e_0^+}}\, f(e) \subseteq \underline{\lim\limits_{e\to e_0^+}}\, g(e)$.

(2) $\underline{\lim\limits_{e\to e_0^+}}\,(f(e)\cap g(e)) = \underline{\lim\limits_{e\to e_0^+}}\, f(e) \cap \underline{\lim\limits_{e\to e_0^+}}\, g(e)$.

(3) $\underline{\lim\limits_{e\to e_0^+}}\,(U - f(e)) = U - \overline{\lim\limits_{e\to e_0^+}}\, f(e)$.

(4) *If* $\underline{\lim\limits_{e\to e_0^+}}\, f(e) = \triangle \supset A$, *then* $\exists\ \delta>0$, $\forall\ e\in(e_0,e_0+\delta)$, $f(e)\supset A$.

(5) $\underline{\lim\limits_{e\to e_0^+}}\,(f(e)\times g(e)) = \underline{\lim\limits_{e\to e_0^+}}\, f(e) \times \underline{\lim\limits_{e\to e_0^+}}\, g(e)$.

Proof. (1) It is similar to the proof of Proposition 4(1).

(2) " $\subseteq$ ". This holds by (1).

" $\supseteq$ ". Suppose $\underline{\lim\limits_{e\to e_0^+}}\, f(e) \cap \underline{\lim\limits_{e\to e_0^+}}\, g(e) \not\subseteq \underline{\lim\limits_{e\to e_0^+}}\,(f(e)\cap g(e))$. Then, $\underline{\lim\limits_{e\to e_0^+}}\, f(e) \cap \underline{\lim\limits_{e\to e_0^+}}\, g(e) - \underline{\lim\limits_{e\to e_0^+}}\,(f(e)\cap g(e)) \neq \emptyset$. Pick $x \in \underline{\lim\limits_{e\to e_0^+}}\, f(e) \cap \underline{\lim\limits_{e\to e_0^+}}\, g(e) - \underline{\lim\limits_{e\to e_0^+}}\,(f(e)\cap g(e))$. We have:

$$x \in \underline{\lim_{e\to e_0^+}}\, f(e),\ x \in \underline{\lim_{e\to e_0^+}}\, g(e) \text{ and } x \notin \underline{\lim_{e\to e_0^+}}\,(f(e)\cap g(e)).$$

By Theorem 3,

$$\exists\ \delta_1,\delta_2>0,\ (x)_f \cap U^+(e_0,\delta_1) = \emptyset,\ (x)_g \cap U^+(e_0,\delta_2) = \emptyset.$$

Pick δ_3=min$\{\delta_1, \delta_2\}$. Then, $(x)_f \cap U^+(e_0, \delta_3) = \emptyset$, $(x)_g \cap U^+(e_0, \delta_3) = \emptyset$. It follows that:

$$((x)_f \cup (x)_{g_I}) \cap U^+(e_0, \delta_3) = ((x)_f \cap U^+(e_0, \delta_3)) \cup ((x)_g \cap U^+(e_0, \delta_3)) = \emptyset.$$

By Remark 1, $(x)_{f \cap g} \cap U^+(e_0, \delta_3) = \emptyset$.

Thus,

$x \in \underline{\lim}_{e \to e_0^+} (f \cap g)(e) = \underline{\lim}_{e \to e_0^+} (f(e) \cap g(e))$. This is a contradiction.

(3) $\forall\, x \in \underline{\lim}_{e \to e_0^+} (U - f(e))$. Then, $x \in \underline{\lim}_{e \to e_0^+} f^c(e)$. By Theorem 3, $\exists\, \delta > 0$, $(x)_{f^c} \cap U^+(e_0, \delta) = \emptyset$. By Remark 1, $[x]_f \cap U^+(e_0, \delta) = \emptyset$.

Thus, $x \in U - \overline{\lim}_{e \to e_0^+} f(e)$.

Conversely, the proof is similar.

(4) By Proposition 4(3),

$$\overline{\lim}_{e \to e_0^+} (U - f(e)) = U - \underline{\lim}_{e \to e_0^+} f(e).$$

Since $\underline{\lim}_{e \to e_0^+} f(e) = \triangle \supset A$, we have $\overline{\lim}_{e \to e_0^+} (U - f(e)) \subset U - A$.

By Proposition 4(4), $\exists\, \delta > 0, \forall\, e \in (e_0, e_0 + \delta)$, $U - f(e) \subset U - A$.

Thus,

$$\exists\, \delta > 0, \forall\, e \in (e_0, e_0 + \delta),\ f(e) \supset A.$$

(5) $\forall\, (x, y) \in \underline{\lim}_{e \to e_0^+} (f(e) \times g(e))$, by Theorem 4(2),

$$(x, y) \in \bigcup_{e \in (e_0, e_0+1) \cap I} \bigcap_{\beta \in (e_0, e]} (f(\beta) \times g(\beta)).$$

Then, $\exists\, e \in (e_0, e_0 + 1) \cap I, \forall\, \beta \in (e_0, e]$, $(x, y) \in f(\beta) \times g(\beta)$. It follows that $x \in f(\beta), y \in g(\beta)$. Then,

$$x \in \bigcup_{e \in (e_0, e_0+1) \cap I} \bigcap_{\beta \in (e_0, e]} f(\beta),\ \ y \in \bigcup_{e \in (e_0, e_0+1) \cap I} \bigcap_{\beta \in (e_0, e]} g(\beta).$$

By Theorem 4(2), $x \in \underline{\lim}_{e \to e_0^+} f(e), y \in \underline{\lim}_{e \to e_0^+} g(e)$. Thus, $(x, y) \in \underline{\lim}_{e \to e_0^+} f(e) \times \underline{\lim}_{e \to e_0^+} g(e)$.

$\forall\, (x, y) \in \underline{\lim}_{e \to e_0^+} f(e) \times \underline{\lim}_{e \to e_0^+} g(e)$, By Theorem 4(2),

$$x \in \underline{\lim}_{e \to e_0^+} f(e) = \bigcup_{e \in (e_0, e_0+1) \cap I} \bigcap_{\beta \in (e_0, e]} f(\beta),\ \ y \in \underline{\lim}_{e \to e_0^+} g(e) = \bigcup_{e \in (e_0, e_0+1) \cap I} \bigcap_{\beta \in (e_0, e]} g(\beta).$$

Then, $\exists\, e_1, e_2 \in (e_0, e_0 + 1) \cap I, \forall\, \beta \in (e_0, e_1], \forall\, \gamma \in (e_0, e_2], x \in f(\beta), y \in g(\gamma)$.

Put $e^* = \min\{e_1, e_2\}$. Then, $e^* \in (e_0, e_0 + 1) \cap I$, $(e_0, e^*] \subseteq (e_0, e_1] \cap (e_0, e_2]$. Then, $\forall\, \beta \in (e_0, e^*]$, $x \in f(\beta), y \in g(\beta)$. It follows that $(x, y) \in f(\beta) \times g(\beta)$. Therefore,

$$(x, y) \in \bigcup_{e \in (e_0, e_0+1) \cap I} \bigcap_{\beta, \gamma \in (e_0, e]} (f(\beta) \times g(\beta)).$$

By Theorem 4(2), $(x, y) \in \underline{\lim}_{e \to e_0^+} (f(e) \times g(e))$.

Thus,

$$\underline{\lim}_{e \to e_0^+} (f(e) \times g(e)) = \underline{\lim}_{e \to e_0^+} f(e) \times \underline{\lim}_{e \to e_0^+} g(e).$$

□

Proposition 6. *For the over-left limit, the following properties hold:*
(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in (e_0 - \delta_0, e_0))$, *then* $\overline{\lim\limits_{e \to e_0^-}} f(e) \subseteq \overline{\lim\limits_{e \to e_0^-}} g(e)$.
(2) $\overline{\lim\limits_{e \to e_0^-}} (f(e) \cup g(e)) = \overline{\lim\limits_{e \to e_0^-}} f(e) \cup \overline{\lim\limits_{e \to e_0^-}} g(e)$.
(3) $\overline{\lim\limits_{e \to e_0^-}} (U - f(e)) = U - \underline{\lim\limits_{e \to e_0^-}} f(e)$.
(4) *If* $\overline{\lim\limits_{e \to e_0^-}} f(e) = \triangle \subset B$, *then* $\exists\, \delta > 0, \forall\, e \in (e_0 - \delta, e_0)$, $f(e) \subset B$.
(5) 1) $\overline{\lim\limits_{e \to e_0^-}} (f(e) \times g(e)) \subseteq \overline{\lim\limits_{e \to e_0^-}} f(e) \times \overline{\lim\limits_{e \to e_0^-}} g(e)$.
2) $\overline{\lim\limits_{e \to e_0^-}} f(e) \times \overline{\lim\limits_{e \to e_0^-}} g(e) = \bigcap\limits_{e \in (e_0 - 1, e_0) \cap I} \bigcup\limits_{\beta, \gamma \in [e, e_0)} (f(\beta) \times g(\gamma))$.

Proof. The proof is similar to Proposition 4. □

Proposition 7. *For the under-left limit, the following properties hold:*
(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in (e_0 - \delta_0, e_0))$, *then* $\underline{\lim\limits_{e \to e_0^-}} f(e) \subseteq \underline{\lim\limits_{e \to e_0^-}} g(e)$.
(2) $\underline{\lim\limits_{e \to e_0^-}} (f(e) \cap g(e)) = \underline{\lim\limits_{e \to e_0^-}} f(e) \cap \underline{\lim\limits_{e \to e_0^-}} g(e)$.
(3) $\underline{\lim\limits_{e \to e_0^-}} (U - f(e)) = U - \overline{\lim\limits_{e \to e_0^-}} f(e)$.
(4) *If* $\underline{\lim\limits_{e \to e_0^-}} f(e) = \triangle \supset A$, *then* $\exists\, \delta > 0, \forall\, e \in (e_0 - \delta, e_0)$, $f(e) \supset A$.
(5) $\underline{\lim\limits_{e \to e_0^-}} (f(e) \times g(e)) = \underline{\lim\limits_{e \to e_0^-}} f(e) \times \underline{\lim\limits_{e \to e_0^-}} g(e)$.

Proof. The proof is similar to Proposition 5. □

Corollary 1. *Suppose that f_I is an it-soft set over U and $A \in 2^U$. For $e_0 \in I$,*
(1) *If* $f(e) \subseteq A$ *or* $f(e) \subset A$ $(\forall\, e \in (e_0, e_0 + \delta_0))$, *then:*

$$\overline{\lim_{e \to e_0^+}} f(e) \subseteq A, \quad \underline{\lim_{e \to e_0^+}} f(e) \subseteq A.$$

(2) *If* $f(e) \subseteq A$ *or* $f(e) \subset A$ $(\forall\, e \in (e_0 - \delta_0, e_0))$, *then:*

$$\overline{\lim_{e \to e_0^-}} f(e) \subseteq A, \quad \underline{\lim_{e \to e_0^-}} f(e) \subseteq A.$$

Proof. This holds by Propositions 4, 5, 6 and 7. □

Corollary 2. *Given that f_I is an it-soft set over U and $A \in 2^U$, for $e_0 \in I$,*
(1) *If* $f(e) \supseteq A$ *or* $f(e) \supset A$ $(\forall\, e \in (e_0, e_0 + \delta_0))$, *then:*

$$\overline{\lim_{e \to e_0^+}} f(e) \supseteq A, \quad \underline{\lim_{e \to e_0^+}} f(e) \supseteq A.$$

(2) *If* $f(e) \supseteq A$ *or* $f(e) \supset A$ $(\forall\, e \in (e_0 - \delta_0, e_0))$, *then:*

$$\overline{\lim_{e \to e_0^-}} f(e) \supseteq A, \quad \underline{\lim_{e \to e_0^-}} f(e) \supseteq A.$$

Proof. This follows from Propositions 4, 5, 6 and 7. □

Theorem 5. *For the over limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in U^0(e_0, \delta_0))$, *then* $\overline{\lim\limits_{e\to e_0}} f(e) \subseteq \overline{\lim\limits_{e\to e_0}} g(e)$.
(2) $\overline{\lim\limits_{e\to e_0}}(f(e) \cup g(e)) = \overline{\lim\limits_{e\to e_0}} f(e) \cup \overline{\lim\limits_{e\to e_0}} g(e)$.
(3) $\overline{\lim\limits_{e\to e_0}}(U - f(e)) = U - \underline{\lim\limits_{e\to e_0}} f(e)$.
(4) *If* $\overline{\lim\limits_{e\to e_0}} f(e) = \triangle \subset B$, *then* $\exists\, \delta > 0, \forall\, e \in U^0(e_0, \delta)$, $f(e) \subset B$.
(5) $\overline{\lim\limits_{e\to e_0}}(f(e) \times g(e)) \subseteq \overline{\lim\limits_{e\to e_0}} f(e) \times \overline{\lim\limits_{e\to e_0}} g(e)$.

Proof. This is a direct result from Propositions 4 and 6. □

Theorem 6. *For the under limit, the following properties hold:*
(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in U^0(e_0, \delta_0))$, *then* $\underline{\lim\limits_{e\to e_0}} f(e) \subseteq \underline{\lim\limits_{e\to e_0}} g(e)$.
(2) $\underline{\lim\limits_{e\to e_0}}(f(e) \cap g(e)) = \underline{\lim\limits_{e\to e_0}} f(e) \cap \underline{\lim\limits_{e\to e_0}} g(e)$.
(3) $\underline{\lim\limits_{e\to e_0}}(U - f(e)) = U - \overline{\lim\limits_{e\to e_0}} f(e)$.
(4) *If* $\underline{\lim\limits_{e\to e_0}} f(e) = \triangle \supset A$, *then* $\exists\, \delta > 0, \forall\, e \in U^0(e_0, \delta)$, $f(e) \supset A$.
(5) $\underline{\lim\limits_{e\to e_0}}(f(e) \times g(e)) = \underline{\lim\limits_{e\to e_0}} f(e) \times \underline{\lim\limits_{e\to e_0}} g(e)$.

Proof. This holds by Propositions 5 and 7. □

Lemma 3. *Let* f_I *be an it-soft set over* U. *For* $e_0 \in I$, *denote:*
$W = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$,
$S = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\}$,
$T = \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^-(e_0, \delta) \neq \emptyset\}$.
Then,
$$W = S \cup T.$$

Proof. Suppose $W \nsubseteq S \cup T$. Then, $W - S \cup T \neq \emptyset$.

Pick $x \in W - S \cup T$. Then, $x \notin S,\ x \notin T$. Therefore, $\exists\, \delta_1, \delta_2 > 0$,
$$[x]_{f_I} \cap U^+(e_0, \delta_1) = \emptyset,\ [x]_{f_I} \cap U^-(e_0, \delta_2) = \emptyset.$$

Put $\delta^* = \min\{\delta_1, \delta_2\}$. Then, $\delta^* > 0$, $[x]_{f_I} \cap U^+(e_0, \delta^*) = \emptyset$, $[x]_{f_I} \cap U^-(e_0, \delta^*) = \emptyset$. It follows that $[x]_{f_I} \cap U(e_0, \delta^*) = \emptyset$. Then, $x \notin W$. This is a contradiction.

Thus, $W \subseteq S \cup T$.

On the other hand, suppose $S \cup T \nsubseteq W$; we have $S \cup T - W \neq \emptyset$.

Pick $x \in S \cup T - W$. Then, $x \notin W$. Therefore, $\exists\, \delta^* > 0$, $[x]_{f_I} \cap U(e_0, \delta^*) = \emptyset$. This implies $[x]_{f_I} \cap U^+(e_0, \delta^*) = \emptyset$, $[x]_{f_I} \cap U^-(e_0, \delta^*) = \emptyset$. Then, $x \notin S,\ x \notin T$. Therefore, $x \notin S \cup T$. This is a contradiction.

Thus, $S \cup T \subseteq W$.

Hence, $W = S \cup T \nsubseteq W$.

□

Theorem 7. *Suppose that* f_I *is an it-soft set over* U. *Then, for* $e_0 \in I$,
(1) $\{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta)\ is\ infinite\}$
$= \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$
$= \underline{\lim\limits_{e\to e_0^+}} f(e) \cup \underline{\lim\limits_{e\to e_0^-}} f(e)$.
(2) $\{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U(e_0, \delta)\ is\ finite\}$
$= \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\}$

$= \underline{\lim}_{e \to e_0^+} f(e) \cap \underline{\lim}_{e \to e_0^-} f(e)$.

Proof. (1) Similar to the proof of Theorem 3(1), we have:

$\{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\}$
$= \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta)\ is\ infinite\}$.

By Lemma 3,

$\{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$
$= \overline{\lim}_{e \to e_0^+} f(e) \cup \overline{\lim}_{e \to e_0^-} f(e)$.

(2) Similar to the proof of Theorem 3(2), we have:

$\{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\}$
$= \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta)\ is\ finite\}$.

By Proposition 4(3), $\underline{\lim}_{e \to e_0^+} f(e) = U - \overline{\lim}_{e \to e_0^+} (U - f(e))$.

By Proposition 6(3), $\underline{\lim}_{e \to e_0^-} f(e) = U - \overline{\lim}_{e \to e_0^-} (U - f(e))$.

By (1),

$\underline{\lim}_{e \to e_0^+} f(e) \cap \underline{\lim}_{e \to e_0^-} f(e)$
$= [U - \overline{\lim}_{e \to e_0^+} (U - f(e))] \cap [U - \overline{\lim}_{e \to e_0^-} (U - f(e))]$
$= U - [\overline{\lim}_{e \to e_0^+} (U - f(e)) \cup \overline{\lim}_{e \to e_0^-} (U - f(e))]$
$= U - \{x \in U : \forall\, \delta > 0,\ (x)_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$
$= \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U(e_0, \delta) = \emptyset\}$.

□

Theorem 8. *Given that f_I is an it-soft set over U, then, for $e_0 \in I$,*

(1) $\{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta)\ is\ infinite\}$
$= \{x \in U : \forall\, \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$
$= \overline{\lim}_{e \to e_0} f(e)$.

(2) $\{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta)\ is\ finite\}$
$= \{x \in U : \exists\, \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\}$
$= \underline{\lim}_{e \to e_0} f(e)$.

Proof. This follows from Theorem 7. □

Theorem 9. *For the right limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in (e_0, e_0 + \delta_0))$, *then* $\lim_{e \to e_0^+} f(e) \subseteq \lim_{e \to e_0^+} g(e)$.

(2) *If* $\lim_{e \to e_0^+} f(e) = \triangle$, $A \subset \triangle \subset B$, *then* $\exists\, \delta > 0, \forall\, e \in (e_0, e_0 + \delta)$, $A \subset f(e) \subset B$.

(3) $\lim_{e \to e_0^+} (f(e) \times g(e)) \subseteq \lim_{e \to e_0^+} f(e) \times \lim_{e \to e_0^+} g(e)$.

Proof. This holds by Propositions 4 and 5. □

Theorem 10. *For the left limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in (e_0 - \delta_0, e_0))$, *then* $\lim_{e \to e_0^-} f(e) \subseteq \lim_{e \to e_0^-} g(e)$.

(2) *If* $\lim_{e \to e_0^-} f(e) = \triangle$, $A \subset \triangle \subset B$, *then* $\exists\, \delta > 0, \forall\, e \in (e_0 - \delta, e_0)$, $A \subset f(e) \subset B$.

(3) $\lim_{e \to e_0^-} (f(e) \times g(e)) \subseteq \lim_{e \to e_0^-} f(e) \times \lim_{e \to e_0^-} g(e)$.

Proof. This holds by Propositions 6 and 7. □

Theorem 11. *For the limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\, e \in U^0(e_0, \delta_0))$, *then* $\lim\limits_{e \to e_0} f(e) \subseteq \lim\limits_{e \to e_0} g(e)$.

(2) *If* $\lim\limits_{e \to e_0^-} f(e) = \triangle$, $A \subset \triangle \subset B$, *then* $\exists\, \delta > 0, \forall\, e \in U^0(e_0, \delta_0)$, $A \subset f(e) \subset B$.

(3) $\lim\limits_{e \to e_0} (f(e) \times g(e)) \subseteq \lim\limits_{e \to e_0} f(e) \times \lim\limits_{e \to e_0} g(e)$.

Proof. This follows from Theorems 9 and 10. □

5. Continuity of *it*-Soft Sets

5.1. Point-Wise Continuity of it-Soft Sets

Definition 19. *Suppose that* f_I *is an it-soft set over* U. *Then, for* $e_0 \in I$,

(1) f_I *is called over-right continuous at* e_0, *if* $\overline{\lim\limits_{e \to e_0^+}} f(e) = f(e_0)$.

(2) f_I *is said to be under-right continuous at* e_0, *if* $\underline{\lim\limits_{e \to e_0^+}} f(e) = f(e_0)$.

(3) f_I *is referred to as over-left continuous at* e_0, *if* $\overline{\lim\limits_{e \to e_0^-}} f(e) = f(e_0)$.

(4) f_I *is called under-left continuous at* e_0, *if* $\underline{\lim\limits_{e \to e_0^-}} f(e) = f(e_0)$.

Definition 20. *Let* f_I *be an it-soft set over* U. *Then, for* $e_0 \in I$,

(1) f_I *is called over-continuous at* e_0, *if* f_I *is both over-left and over-right continuous at* e_0.

(2) f_I *is said to be under-continuous at* e_0, *if* f_I *is both under-left and under-right continuous at* e_0.

(3) f_I *is referred to as continuous at* e_0, *if* f_I *is both over-continuous and under-continuous at* e_0.

Definition 21. *Given that* f_I *is an it-soft set over* U. *Then, for* $e_0 \in I$,

(1) f_I *is called right-continuous at* e_0, *if* f_I *is both over-right and under-right continuous at* e_0.

(2) f_I *is said to be left-continuous at* e_0, *if* f_I *is both over-left and under-left continuous at* e_0.

(3) f_I *is referred to as continuous at* e_0, *if* f_I *is both left-continuous and right-continuous at* e_0.

Remark 3. *The point-wise continuity in Definition 19(3) and the point-wise continuity in Definition 20(3) is consistent.*

Denote:

$$C^{or}(e_0) = \{f_I : f_I \text{ is over-right continuous at } e_0\},$$

$$C^{ur}(e_0) = \{f_I : f_I \text{ is under-right continuous at } e_0\},$$

$$C^{ol}(e_0) = \{f_I : f_I \text{ is over-left continuous at } e_0\},$$

$$C^{ul}(e_0) = \{f_I : f_I \text{ is under-left continuous at } e_0\};$$

$$C^{o}(e_0) = \{f_I : f_I \text{ is over-continuous at } e_0\},\ \ C^{u}(e_0) = \{f_I : f_I \text{ is under-continuous at } e_0\};$$

$$C^{l}(e_0) = \{f_I : f_I \text{ is left-continuous at } e_0\},\ \ C^{r}(e_0) = \{f_I : f_I \text{ is right-continuous at } e_0\};$$

$$C(e_0 = \{f_I : f_I \text{ is continuous at } e_0\}.$$

Proposition 8. (1) $C^o(e_0) = C^{ol}(e_0) \cap C^{or}(e_0)$.

(2) $C^u(e_0) = C^{ul}(e_0) \cap C^{ur}(e_0)$.

(3) $C^l(e_0) = C^{ol}(e_0) \cap C^{ul}(e_0)$.

(4) $C^r(e_0) = C^{or}(e_0) \cap C^{ur}(e_0)$.

(5) $C(e_0) = C^o(e_0) \cap C^u(e_0) = C^l(e_0) \cap C^r(e_0)$.

Proof. This is obvious. □

Proposition 9. *Let f_I and g_I be two it-soft sets over U. Then, for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{or}(e_0)$, then $f_I \widetilde{\cup} g_I \in C^{or}(e_0)$.*
(2) *If $f_I \in C^{or}(e_0)$, then $f_I^c \in C^{ur}(e_0)$.*

Proof. This holds by Proposition 4. □

Proposition 10. *Let f_I and g_I be two it-soft sets over U. Then, for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{ur}(e_0)$, then $f_I \widetilde{\cap} g_I \in C^{ur}(e_0)$.*
(2) *If $f_I \in C^{ur}(e_0)$, then $f_I^c \in C^{or}(e_0)$.*
(3) *If $f_I, g_I \in C^{ur}(e_0)$, then $f_I \widetilde{\times} g_I \in C^{ur}(e_0)$.*

Proof. This holds by Proposition 5. □

Proposition 11. *Let f_I and g_I be two it-soft sets over U. Then, for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{ol}(e_0)$, then $f_I \widetilde{\cup} g_I \in C^{ol}(e_0)$.*
(2) *If $f_I \in C^{ol}(e_0)$, then $f_I^c \in C^{ul}(e_0)$.*

Proof. This follows from Proposition 6. □

Proposition 12. *Let f_I and g_I be two it-soft sets over U. Then, for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{ul}(e_0)$, then $f_I \widetilde{\cap} g_I \in C^{ul}(e_0)$.*
(2) *If $f_I \in C^{ul}(e_0)$, then $f_I^c \in C^{ol}(e_0)$.*
(3) *If $f_I, g_I \in C^{ul}(e_0)$, then $f_I \widetilde{\times} g_I \in C^{ul}(e_0)$.*

Proof. This is a direct result from Proposition 7. □

Theorem 12. *Let f_I and g_I be two it-soft sets over U. Then, for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^o(e_0)$, then $f_I \widetilde{\cup} g_I \in C^o(e_0)$.*
(2) *If $f_I \in C^o(e_0)$, then $f_I^c \in C^u(e_0)$.*

Proof. This holds by Propositions 8 and 10. □

Theorem 13. *Let f_I and g_I be two it-soft sets over U. Then, for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^u(e_0)$, then $f_I \widetilde{\cap} g_I \in C^u(e_0)$.*
(2) *If $f_I \in C^u(e_0)$, then $f_I^c \in C^o(e_0)$.*
(3) *If $f_I, g_I \in C^u(e_0)$, then $f_I \widetilde{\times} g_I \in C^u(e_0)$.*

Proof. This follows from Propositions 9 and 11. □

5.2. Continuous it-Soft Sets

Definition 22. *Suppose that f_I is an it-soft set over U.*
(1) *f_I is called over-continuous, if $\forall\, e_0 \in I$, f_I is over-continuous at e_0.*
(2) *f_I is said to be under-continuous, if $\forall\, e_0 \in I$, f_I under-continuous at e_0.*
(3) *f_I is referred to as left-continuous, if $\forall\, e_0 \in I$, f_I is left-continuous at e_0.*
(4) *f_I is called right-continuous, if $\forall\, e_0 \in I$, f_I right-continuous at e_0.*
(5) *f_I is said to be continuous, if $\forall\, e_0 \in I$, f_I continuous at e_0.*

Denote:

$$C^{or}(e_0) = \{f_I : f_I \text{ is over-right continuous}\},$$

$$C^{ur}(e_0) = \{f_I : f_I \text{ is under-right continuous}\},$$

$$C^{ol}(e_0) = \{f_I : f_I \text{ is over-left continuous}\},$$

$$C^{ul}(e_0) = \{f_I : f_I \text{ is under-left continuous}\};$$

$$C^{o}(I) = \{f_I : f_I \text{ is over-continuous}\},\ C^{u}(I) = \{f_I : f_I \text{ is under-continuous}\};$$

$$C^{l}(I) = \{f_I : f_I \text{ is left-continuous}\},\ C^{r}(I) = \{f_I : f_I \text{ is right-continuous }\};$$

$$C(I) = \{f_I : f_I \text{ is continuous}\}.$$

Proposition 13. (1) $C^o(I) = C^{ol}(I) \cap C^{or}(I)$.
(2) $C^u(I) = C^{ul}(I) \cap C^{ur}(I)$.
(3) $C^l(I) = C^{ol}(I) \cap C^{ul}(I)$.
(4) $C^r(I) = C^{or}(I) \cap C^{ur}(I)$.
(5) $C(I) = C^o(I) \cap C^u(I) = C^l(I) \cap C^r(I)$.

Proof. This is obvious. □

Theorem 14. *Let f_I and g_J be two it-soft sets over U.*
(1) If $f_I \in C^o(I)$, $g_J \in C^o(J)$, then $f_I \widetilde{\cup} g_I \in C^o(I \cup J)$.
(2) If $f_I \in C^o(I)$, then $f_I^c \in C^u(I)$.

Proof. This holds by Theorem 12. □

Theorem 15. *Let f_I and g_J be two it-soft sets over U.*
(1) If $f_I \in C^u(I)$, $g_J \in C^u(J)$, then $f_I \widetilde{\cap} g_J \in C^u(I \cap J)$.
(2) If $f_I \in C^u(I)$, then $f_I^c \in C^o(I)$.

Proof. This holds by Theorem 13. □

Theorem 16. *Let $(f, [a, b])$ be an it-soft set over U.*
(1) If $(f, [a, b])$ is strong keeping union or increasing, then $(f, [a, b])$ has the maximum value.
(2) If $(f, [a, b])$ is strong keeping intersection or decreasing, then $(f, [a, b])$ has the minimum value.

Corollary 3. *If $(f, [a, b])$ is a perfect it-soft set over U, then $(f, [a, b])$ has the maximum and minimum value.*

Proof. This is obvious. □

Lemma 4. *Let $f_I \in C^o(e_0)$. If $\lim\limits_{n\to\infty} e_n = e_0$, then $\overline{\lim\limits_{n\to\infty}} f(e_n) \subseteq f(e_0)$.*

Proof. Since $\overline{\lim\limits_{n\to\infty}} f(e_n) = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} f(e_k)$,
we only need to prove that:

$$if\ \forall\ n \in N,\ \exists\ k \geq n,\ x \in f(e_k),\ \text{then } x \in f(e_0).$$

$\forall\ \delta$, $\exists\ n \in N_1$, $\frac{1}{n_1} < \delta$. It follows $U(e_0, \frac{1}{n_1}) \subset U(e_0, \delta)$.
Since $\lim\limits_{n\to\infty} e_n = e_0$, $\exists\ n \in N_2$, when $n > n_2$, we have $e_n \in U(e_0, \frac{1}{n_1})$.
Put $n_3 = n_1 + n_2$. Then, for n_3, $\exists\ k \geq n_3$, $x \in f(e_k)$. Therefore, $e_k \in\ [x]_{f_I}$.

$k \geq n_3 > n_2$ implies:

$$e_k \in U(e_0, \frac{1}{n_1}) \subset U(e_0, \delta).$$

Then, $e_k \in [x]_{f_I} \cap U(e_0, \delta)$. Therefore, $\forall\, \delta,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset$.
By Theorem 7, $x \in \overline{\lim\limits_{e \to e_0}} f(e)$.
Since $f \in C^o(e_0)$, we have $f(e_0) = \overline{\lim\limits_{e \to e_0}} f(e)$.
Hence, $x \in f(e_0)$. □

Theorem 17. *Let $(f, [a,b]) \in C([a,b])$.*
(1) *Suppose $f(a) \subset f(b)$, then $\forall\, \mu : f(a) \subseteq \mu \subseteq f(b)$, $\exists\, e_0 \in [a,b]$, $f(e_0) = \mu$. Moreover, if $f(a) \subset \mu \subset f(b)$, then $\exists\, e_0 \in (a,b)$, $f(e_0) = \mu$.*
(2) *Suppose $f(b) \subset f(a)$, then $\forall\, \mu : f(b) \subseteq \mu \subseteq f(a)$, $\exists\, e_0 \in [a,b]$, $f(e_0) = \mu$. Moreover, if $f(b) \subset \mu \subset f(a)$, then $\exists\, e_0 \in (a,b)$, $f(e_0) = \mu$.*

Proof. (1) It suffices to show that:

$$if\ f(a) \subset \mu \subset f(b),\ then\ \exists\, e_0 \in (a,b),\ f(e_0) = \mu.$$

Denote $E = \{e \in [a,b] : f(e) \supset \mu\}$. Put $e_0 = \inf E$. Then,

$$\exists\, \{e_n : n \in N\} \subseteq E - \{e_0\},\ \lim_{n \to \infty} e_n = e_0.$$

Since $\forall\, n \in N,\ f(e_n) \supset \mu$, we have $\overline{\lim\limits_{n \to \infty}} f(e_n) = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} f(e_k) \supseteq \mu$. Since $f \in C^o(e_0)$, by Lemma 4,

$$f(e_0) \supseteq \overline{\lim_{n \to \infty}} f(e_n) \supseteq \mu.$$

It should be noted that $f(a) \subset \mu$. Then, $e_0 \neq a$.
We assert $e_0 \neq b$. Suppose $e_0 = b$. Since:

$$\mu \subset f(b) = \lim_{e \to b^-} f(e) = \underline{\lim_{e \to b^-}} f(e),$$

by Proposition 7(4), then

$$\exists\, \delta,\ \forall\, e \in (b - \delta, b), f(e) \supset \mu.$$

Put $e_1 \in (b - \delta, b)$. Then, $f(e_1) \supset \mu$. We have $e_1 \in E$. This implies $e_1 \geq e_0$. However, $e_1 < b = e_0$. This is a contradiction.
Thus, $e_0 \in (a,b)$.
We claim $f(e_0) \not\supset \mu$. Suppose $f(e_0) \supset \mu$. Since $f \in C^u(e_0)$, we have:

$$\mu \subset f(e_0) = \lim_{e \to e_0} f(e) = \underline{\lim_{e \to e_0}} f(e).$$

By Theorem 6(4),

$$\exists\, \delta,\ \forall\, e \in U^0(e_0, \delta),\ f(e) \supset \mu.$$

Put $e_1 \in (e_0 - \delta, e_0)$. Then, $f(e_1) \supset \mu$. We have $e_1 \in E$. This implies $e_1 \geq e_0$. This is a contradiction.
It should be noted that $f(e_0) \supseteq \mu$. Thus, $f(e_0) = \mu$.
(2) The proof is similar to (1).
□

6. An Application for Rough Sets

Definition 23. *Let (U, R, P) be a probabilistic approximate space. For $e \in [0,1]$, $X \in 2^U$, denote:*

$$f_X(e) = \underline{PI}_e(X), \ \ g_X(e) = \overline{PI}_e(X).$$

Then, $(f_X, [0,1])$ and $(g_X, [0,1])$ are two it-soft sets over U, which are called the it-soft sets induced by the lower and upper approximations of X, respectively.

Theorem 18. *Suppose that (U, R, P) is a probabilistic approximate space. Then, for $e_0 \in (0,1)$, $X \in 2^U$,*

(1) 1) $\overline{\lim\limits_{e \to e_0^+}} f_X(e) = \bigcap\limits_{e \in (e_0,1]} \bigcup\limits_{\beta \in (e_0,e]} f_X(\beta)$;

2) $\overline{\lim\limits_{e \to e_0^-}} f_X(e) = \bigcap\limits_{e \in [0,e_0)} f_X(e) = f_X(e_0)$;

3) $\underline{\lim\limits_{e \to e_0^+}} f_X(e) = \bigcup\limits_{e \in (e_0,1]} f_X(e) = g_X(e_0)$;

4) $\underline{\lim\limits_{e \to e_0^-}} f_X(e) = \bigcup\limits_{e \in [0,e_0)} \bigcap\limits_{\beta \in [e,e_0)} f_X(\beta)$.

(2) 1) $\overline{\lim\limits_{e \to e_0^+}} g_X(e) = \bigcap\limits_{e \in (e_0,1]} \bigcup\limits_{\beta \in (e_0,e]} g_X(\beta)$;

2) $\overline{\lim\limits_{e \to e_0^-}} g_X(e) = \bigcap\limits_{e \in [0,e_0)} g_X(e) = f_X(e_0)$;

3) $\underline{\lim\limits_{e \to e_0^+}} g_X(e) = \bigcup\limits_{e \in (e_0,1]} g_X(e) = g_X(e_0)$;

4) $\underline{\lim\limits_{e \to e_0^-}} g_X(e) = \bigcup\limits_{e \in [0,e_0)} \bigcap\limits_{\beta \in [e,e_0)} g_X(\beta)$.

(3) 1) $f_{U-X}(e) = U - g_X(1-e)$,

2) $g_{U-X}(e) = U - f_X(1-e)$.

Proof. This holds by Theorems 1, 2 and 4. □

Corollary 4. *Given that (U, R, P) is a probabilistic approximate space. Then, for $X \in 2^U$,*

$$(f_X, [0,1]) \in C^{ol}((0,1)), \ (g_X, [0,1]) \in C^{ur}((0,1)).$$

Proof. This holds by Theorems 18. □

Example 9. *Let $U = \{x_i : 1 \le i \le 20\}$, $P(X) = \frac{|X|}{|U|}$ $(X \in 2^U)$, $U/R = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ where*

$X_1 = \{x_1, x_2, x_3, x_4, x_5\}$, $X_2 = \{x_6, x_7, x_8\}$, $X_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$,

$X_4 = \{x_{13}, x_{14}\}$, $X_5 = \{x_{15}, x_{16}, x_{17}, x_{18}\}$, $X_6 = \{x_{19}, x_{20}\}$.

Put:

$$X^* = \{x_6, x_7, x_8, x_{13}, x_{17}\}.$$

By Example 4.9 in [17] or Example 8.1 in [18],

$$f_{X^*}(0.5) = X_2 \cup X_4, \ \ g_{X^*}(0.5) = X_2.$$

By Theorem 2,

$$\underline{\lim\limits_{e \to 0.5^+}} f_{X^*}(e) = g_{X^*}(0.5) \neq f_{X^*}(0.5).$$

By Theorem 2,

$$\overline{\lim\limits_{e \to 0.5^-}} g_{X^*}(e) = f_{X^*}(e_0) \neq g_{X^*}(0.5).$$

Thus,

$$(f_{X^*}, [0,1]) \notin C^{ur}(0.5),\ (g_{X^*}, [0,1]) \notin C^{ol}(0.5).$$

This example illustrates that

$$(f_{X^*}, [0,1]) \notin C^{ur}((0,1)),\ (g_{X^*}, [0,1]) \notin C^{ol}((0,1)).$$

Example 10. *Let* $U = \{x_i : 1 \leq i \leq 10\}$, $P(X) = \frac{|X|}{|U|}$ $(X \in 2^U)$, $U/R = \{X_1, X_2, X_3, X_4\}$ *where*

$X_1 = \{x_1, x_3\}$, $X_2 = \{x_2, x_4, x_5, x_7\}$, $X_3 = \{x_6, x_8\}$, $X_4 = \{x_9, x_{10}\}$.

(1) *Put* $X^* = \{x_1, x_5, x_6, x_8\}$. *Then:*

$$f_{X^*}(e) = \begin{cases} X_1 \cup X_2 \cup X_3, & \text{if } e \in (0, \frac{1}{4}], \\ X_1 \cup X_3, & \text{if } e \in (\frac{1}{4}, \frac{1}{2}], \\ X_3, & \text{if } e \in (\frac{1}{2}, 1]; \end{cases}$$

$$g_{X^*}(e) = \begin{cases} X_1 \cup X_2 \cup X_3, & \text{if } e \in [0, \frac{1}{4}), \\ X_1 \cup X_3, & \text{if } e \in [\frac{1}{4}, \frac{1}{2}), \\ X_3, & \text{if } e \in [\frac{1}{2}, 1). \end{cases}$$

Therefore, $\overline{\lim\limits_{e \to 0.5^+}} f_{X^*}(e) = \bigcap\limits_{e \in (0.5,1]} \bigcup\limits_{\beta \in (0.5,e]} f_{X^*}(\beta) = X_3 \neq X_1 \cup X_3 = f_{X^*}(0.5)$,

$\underline{\lim\limits_{e \to 0.5^-}} g_X(e) = \bigcup\limits_{e \in [0,0.5)} \bigcap\limits_{\beta \in [e,0.5)} g_X(\beta) = X_1 \cup X_3 \neq X_3 = g_{X^*}(0.5)$.

Thus,

$$(f_{X^*}, [0,1]) \notin C^{or}(0.5),\ (g_{X^*}, [0,1]) \notin C^{ul}(0.5).$$

(2) *Put* $Y^* = \{x_2, x_9, x_{10}\}$. *Then:*

$$f_{Y^*}(e) = \begin{cases} X_2 \cup X_4, & \text{if } e \in (0, \frac{1}{4}], \\ X_4, & \text{if } e \in (\frac{1}{4}, 1]. \end{cases}$$

Therefore, $\underline{\lim\limits_{e \to 0.5^-}} f_{Y^*}(e) = \bigcup\limits_{e \in [0,0.5)} \bigcap\limits_{\beta \in [e,0.5)} f_{Y^*}(\beta) = X_2 \cup X_4 \neq X_4 = f_{Y^*}(0.5)$.

Thus,

$$(f_{Y^*}, [0,1]) \notin C^{ul}(0.5).$$

(3) *Put*

$$Z^* = U - Y^*.$$

By Proposition 4(3) and Theorem 2,

$$\begin{aligned} \overline{\lim\limits_{e \to 0.5^+}} g_{Z^*}(e) &= \overline{\lim\limits_{e \to 0.5^+}} (U - f_{Y^*}(1-e)) \\ &= U - \underline{\lim\limits_{e \to 0.5^+}} f_{Y^*}(1-e) \\ &= U - \underline{\lim\limits_{1-e \to 0.5^-}} f_{Y^*}(1-e). \end{aligned}$$

It should be noted that $\underline{\lim\limits_{e \to 0.5^-}} f_{Y^*}(e) \neq f_{Y^*}(0.5)$. *Then, by Theorem 2,*

$$\overline{\lim\limits_{e \to 0.5^+}} g_{Z^*}(e) \neq U - f_{Y^*}(0.5) = g_{Z^*}(0.5).$$

Thus,

$$(g_{Z^*}, [0,1]) \notin C^{or}(0.5).$$

This example illustrates that

$$(f_{X^*}, [0,1]) \notin C^{or}((0,1)),\ (g_{X^*}, [0,1]) \notin C^{ul}((0,1));$$

$$(f_{Y^*}, [0,1]) \notin C^{ul}((0,1));\ (g_{Z^*}, [0,1]) \notin C^{or}((0,1)).$$

7. Conclusions

In this paper, limits of *it*-soft sets have been proposed. Point-wise continuity of *it*-soft sets and continuous *it*-soft sets have been investigated. An application for rough sets has been given. These results will be helpful for the study of soft sets. In the future, we will further study applications of these limits in information science.

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