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Determining An Unknown Boundary Condition by An Iteration Method

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Abstract: This paper investigates the boundary value in the heat conduction problem by a variational iteration method. Applying the iteration method, a sequence of convergent functions is constructed, the limit approximates the exact solution of the heat conduction equation in a few iterations using only the initial condition. This method does not require discretization of the variables. Numerical results show that this method is quite simple and straightforward for models that are currently under research.

Keywords: variational iteration method; heat conduction equation; initial and boundary condition

1. Introduction

The problem of heat conduction with unknown boundary condition is a kind of inverse problem. The inverse problem of the heat conduction equation (IPHCE) arises in some important fields of engineering and science. For instance, because of the space program, starting about 1956, some researchers in References [1–3] researched the applications of IPHCE which were related to rocket nozzles, to nose cones of missiles and probes, and other devices; in References [4,5], the researchers researched the solutions of IPHCE for testing of nuclear reactor components; Howse in Reference [6] researched the solidification of glass by the IPHCE; the problem of periodic heating in combustion chambers of internal combustion engines was solved by the IPHCE in Reference [7]. For other applications of the IPHCE the reader is referred to References [8,9].

In order to briefly introduce the IPHCE, consider the following one-dimensional heat conduction equation:

$$v_t = a_1 v_{xx} + a_2 v_x + a_3 v + F(x,t), \;\; 0 < x < 1, \;\; t > 0, \tag{1}$$

with the initial condition:

$$v(x,0) = f(x), \;\; 0 < x < 1, \tag{2}$$

and the Dirichlet or Neumann boundary conditions:

$$v(0,t) = g_1(t), v(1,t) = g_2(t), \;\; t > 0, \tag{3}$$

or

$$v_x(0,t) = h_1(t), v_x(1,t) = h_1(t), \;\; t > 0. \tag{4}$$

Here, t denotes time variable, x denotes space variable, $v = v(x,t)$ represents the temperature distribution function, $v_t = \dfrac{\partial v(x,t)}{\partial t}$, $v_x = \dfrac{\partial v(x,t)}{\partial x}$, $v_{xx} = \dfrac{\partial^2 v(x,t)}{\partial x^2}$, $F(x,t)$ represents heat source (sink), $a_1 = a_1(x,t)$ represents thermal conductivity, $a_2 = a_2(x,t)$ represents fluid velocity, and $a_3 = a_3(x,t)$ represents absorbtion (perfusion) coefficient.

There are four main types of IPHCEs, as follows:

- The reversed-time problem (RTP): $v(x,0)$ is unknown;
- the inverse boundary problem (IBP): the Dirichlet or Neumann condition is unknown;
- the inverse coefficient problem (ICP): a_1, a_2, or a_3 is unknown; and
- the inverse heat source (sink) problem (IHSP): $F(x,t)$ is unknown.

Many authors have studied the IPHCE, and some types of IPHCE have been solved by numerical methods. For instance, the authors in Reference [10] applied a numerical method to solve an IBP; in Reference [11], the boundary integral method was applied to solve an IBP; and in Reference [12,13], the authors applied the method of finite volume or boundary element to an IBP. For other methods, the reader is referred to References [14,15]. For this article, we use the variational iteration method (VIM) to investigate an IBP.

The structure of this article is as follows. We describe this inverse problem mathematically in Section 2. In Section 3, we briefly introduce the VIM and apply the method to IBP. Illustrative examples are presented in Section 4, and the conclusions are given in Section 5.

2. Problem Statement

We consider the following two IBPs in this Section.

Firstly, we consider the Equations (1)–(3) with $a_1 = 1$, $a_2 = a_3 = F(x,t) = 0$ as follows:

$$v_t = v_{xx}, \;\; 0 < x < 1, \;\; t > 0, \tag{5}$$

$$v(x,0) = f(x), \;\; 0 < x < 1, \tag{6}$$

$$v(0,t) = g_1(t), \;\; t > 0, \tag{7}$$

$$v(1,t) = g_2(t), \;\; t > 0, \tag{8}$$

with the overdetermined condition:

$$v(x^*,t) = H(t), \;\; t > 0. \tag{9}$$

Here, $x^* \in (0,1)$ is an observation point; $f(x)$ and $H(t)$ are given, continuous functions; and $g_2(t)$ is a known infinitely differentiable function. The unknown functions $v(x,t)$ and $g_1(t)$ will be determined. When $v(x,0), v(0,t)$, and $v(1,t)$ are all given functions which satisfy certain conditions, then the problem in Equations (5)–(8) is a forward problem. According to the existence and uniqueness theorem (see Reference [16]), only one bounded solution of the problem in Equations (5)–(8) exists.

Secondly, we take the Equation (1) with $a_1 = 1$, $a_2 = a_3 = 0$, Equations (2) and (4) as follows:

$$v_t = v_{xx} + F(x,t), \;\; 0 < x < 1, \;\; t > 0, \tag{10}$$

$$v(x,0) = f(x), \;\; 0 < x < 1, \tag{11}$$

$$v_x(0,t) = h_1(t), \ \ t > 0, \tag{12}$$

$$v_x(1,t) = h_2(t), \ \ t > 0, \tag{13}$$

with the overdetermined condition:

$$v(x^*,t) = H(t), \ \ t > 0. \tag{14}$$

Here, $x^* \in (0,1)$ is a fixed observation point; $F(x,t), v(x,0), v_x(1,t)$, and $H(t)$ are all given functions; and we will determine the unknown functions $v(x,t)$ and $h_1(t)$. When $F(x,t), v(x,0), v_x(0,t)$ and $v_x(1,t)$ are all given functions which satisfy certain conditions, then the problem in Equations (10)–(13) is a straightforward problem. According to the existence and uniqueness theorem (See Reference [17]), under those conditions only one bounded solution of the problem in Equations (10)–(13) exists.

3. Analysis of VIM

In [18], VIM was proposed. Using the iteration method, a sequence of convergent functions is constructed, the limit approximates the exact solution of the heat conduction equation in a few iterations by using only the initial condition. This method has been successfully applied to integro-differential equations, Bratu-like equation, and in other fields [19–22]. In Reference [23], researchers studied the convergence of the VIM via Banach's fixed point theorem.

For a brief introduction to the basic theory and the main steps of VIM, consider the following differential equation:

$$Lv + Nv = f(x). \tag{15}$$

Here, L and N respectively represent linear and nonlinear operators, and $f(x)$ represents an inhomogeneous term.

If $v_{n-1}(n \geq 1)$ is an approximate solution of Equation (15), it follows that:

$$Lv_{n-1} + Nv_{n-1} - f(x) \neq 0.$$

In order to improve its accuracy, a correction term can be added, which is written as:

$$v_n(x) = v_{n-1}(x) + \int_0^x \lambda(\xi)[Lv_{n-1}(\xi) + N\widetilde{v}_{n-1}(\xi) - f(\xi)]d\xi. \tag{16}$$

Here, the subscript n represents the n-th approximation of v, and $\lambda(\xi)$ represents a generalized Lagrange multiplier [18], which can be optimally identified in as simple form as possible using variational theory, the non-linear part must be considered as a restricted variation term $\widetilde{v}_{n-1}$ [18], then the variation of the nonlinear part is zero (i.e., $\delta\widetilde{v}_{n-1} = 0$). The stationary condition of Equation (16) requires that:

$$\delta v_n = \delta v_{n-1} + \delta \int_0^x \lambda(\xi)[Lv_{n-1}(\xi) + N\widetilde{v}_{n-1}(\xi) - f(\xi)]d\xi = 0, \tag{17}$$

for arbitrary δv_{n-1}.

Considering VIM, we can construct the correct functional for the solution in the $-t$ direction for for Equations (5) and (10) as follows:

$$v_n(x,t) = v_{n-1}(x,t) + \int_0^t \lambda(t,\tau)\left[\frac{\partial v_{n-1}(x,\tau)}{\partial \tau} - \frac{\partial^2 \widetilde{v}_{n-1}(x,\tau)}{\partial x^2}\right]d\tau, \tag{18}$$

and:

$$v_n(x,t) = v_{n-1}(x,t) + \int_0^t \lambda(t,\tau)\left[\frac{\partial v_{n-1}(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{v}_{n-1}(x,\tau)}{\partial x^2} - F(x,\tau)\right] d\tau, \tag{19}$$

respectively. Determining λ via variational theory and taking the stationarity condition for Equation (19), we have:

$$\delta v_n(x,t) = \delta v_{n-1}(x,t) + \delta \int_0^t \lambda(t,\tau)\left[\frac{\partial v_{n-1}(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{v}_{n-1}(x,\tau)}{\partial x^2} - F(x,\tau)\right] d\tau = 0. \tag{20}$$

With integration by parts, we have:

$$\delta v_n(x,t) = \delta v_{n-1}(x,t)(1+\lambda(t,t)) - \int_0^t \lambda_\tau(t,\tau)\delta v_{n-1}(x,\tau) d\tau = 0. \tag{21}$$

For arbitrary δv_{n-1}, this yields the following stationary conditions:

$$1 + \lambda(t,\tau)|_{\tau=t} = 0,$$

$$\lambda_\tau(t,\tau) = 0.$$

Hence, we find:

$$\lambda(t,\tau) = -1. \tag{22}$$

Then the iterative formulas for Equations (5) and (10) reduce to:

$$v_n(x,t) = v_{n-1}(x,t) - \int_0^t \left[\frac{\partial v_{n-1}(x,\tau)}{\partial \tau} - \frac{\partial^2 v_{n-1}(x,\tau)}{\partial x^2}\right] d\tau, \tag{23}$$

and:

$$v_n(x,t) = v_{n-1}(x,t) - \int_0^t \left[\frac{\partial v_{n-1}(x,\tau)}{\partial \tau} - \frac{\partial^2 v_{n-1}(x,\tau)}{\partial x^2} - F(x,\tau)\right] d\tau, \tag{24}$$

respectively.

By using Equation (23), a sequence of convergent of solutions v_n for Equation (5) is found by selecting the appropriate $v_0(x,t)$, which limit the exact solution v. Then, we obtain the boundary condition in Equation (7). Using a similar procedure, a sequence of convergent solutions v_n for Equation (10) are found from Equation (24) and then, we obtain the boundary condition in Equation (12).

4. Illustrative Examples

In this section, we give two examples of IBP with known exact solution to illustrate the efficiency and validity of VIM.

4.1. Example 1

Consider the IBP in Equations (5)–(9) with the following known functions:

$$f(x) = \sin(x),$$

$$g_2(t) = \exp(-t)\sin(1),$$

$$H(t) = \exp(-t)\sin(\frac{1}{2}),$$

with $x^* = \frac{1}{2}$. This problem has an exact solution as follows:

$$v = \exp(-t)\sin(x), \ \ g_1(t) = 0.$$

We choose $v_0 = f(x) = \sin(x)$. Using Equation (23), we have:

$$v_1 = \sin(x) - t\sin(x) = \sin(x)(1-t),$$

$$v_2 = \sin(x) - t\sin(x) + \frac{t^2}{2}\sin(x) = \sin(x)(1 - t + \frac{t^2}{2!}),$$

$$v_3 = \sin(x) - t\sin(x) + \frac{t^2}{2}\sin(x) - \frac{t^3}{6}\sin(x) = \sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}),$$

$$\cdots \ \cdots \ \cdots \ \cdots$$

$$v_n = \sum_{k=0}^{n} \frac{(-t)^k}{k!}\sin(x), \ n = 0, 1, \ldots.$$

Thus, the problem in Equations (5)–(9) has the solution $v = \exp(-t)\sin(x)$ when $n \to +\infty$, which satisfies the given differential equation. Then, the boundary condition $g_1(t) = 0$ is obtained. Hence, our iterative approach recovers the exact solution.

4.2. Example 2

Consider the IBP in Equations (10)–(14) with the following known functions:

$$F(x,t) = -\pi^2 x \exp(-\pi^2 t),$$

$$f(x) = x + \sin(\pi x),$$

$$h_2(t) = (1-\pi)\exp(-\pi^2 t),$$

and:

$$H(t) = (x^* + \sin(\pi x^*))\exp(-\pi^2 t),$$

with $x^* = \frac{1}{2}$. This problem has an exact solution as follows:

$$v = (x + \sin(\pi x))\exp(-\pi^2 t), \ \ h_1(t) = (1+\pi)\exp(-\pi^2 t).$$

In the iterative method, we choose $v_0 = f(x) = x + \sin(\pi x)$. Using Equation (24), we obtain:

$$v_1 = x\exp(-\pi^2 t) + \sin(\pi x) - \pi^2 t\sin(\pi x) = x\exp(-\pi^2 t) + \sin(\pi x)(1 - \pi^2 t),$$

$$v_2 = x\exp(-\pi^2 t) + \sin(\pi x) - \pi^2 t\sin(\pi x) + \frac{\pi^4 t^2}{2}\sin(\pi x) = x\exp(-\pi^2 t) + \sin(\pi x)(1 - \pi^2 t + \frac{\pi^4 t^2}{2!}),$$

$$v_3 = x\exp(-\pi^2 t) + \sin(\pi x)(1 - \pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!}),$$

$$\cdots \ \cdots \ \cdots \ \cdots$$

$$v_n = x\exp(-\pi^2 t) + \sum_{k=0}^{n} \frac{(-\pi^2 t)^k}{k!}\sin(\pi x), \ n = 1, 2, \ldots.$$

Thus, the problem in Equations (10)–(14) has the solution $v = (x + \sin(\pi x))\exp(-\pi^2 t)$ when $n \to +\infty$. Then the boundary condition $h_1(t) = (1+\pi)\exp(-\pi^2 t)$ is obtained. Again, our iterative approach recovers the exact solution.

In References [10–13], similar IBPs have been solved by other numerical methods. Here, the advantage of the VIM is that it does not need to discretize the variables, so it is not subject to computational error and has no need to face computing time and large computer memory. Additionally, the VIM solves this problem using only the initial condition, and therefore is very practical and effective for the currently researched models.

5. Conclusions

In this work, we successfully utilized VIM in a heat conduction problem with unknown boundary conditions. All the calculations could be made with simple manipulations. The method provides the solution with high precision and high efficiency in the form of a rapidly converging series by using only the initial condition. The solutions obtained through VIM recover the exact solutions as the number of iterations becomes infinite. VIM solves the IBP easily, while it does not need to discretize the variables. For this reason, it is not subject to computational error and requires neither substantial computing time nor large amounts of computer memory. The results of the preceding examples illustrate that this method is easy to operate and that it is a practical, reliable, and effective iterative technique for the IBP. Thus, we can say that the VIM is very practical and effective for models currently under study.

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Abbreviations

The following abbreviations are used in this manuscript:

IPHCE	Inverse Problem of Heat Conduction Equation
IBP	inverse boundary problem
ICP	inverse coefficient problem
IHSP	inverse heat source (sink) problem
RTP	reversed-time problem
VIM	variational iteration method

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