

Article

On Meromorphic Functions Defined by a New Operator Containing the Mittag–Leffler Function

Suhila Elhaddad  and Maslina Darus * 

Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia; suhila.e@yahoo.com

* Correspondence: maslina@ukm.edu.my

Received: 26 December 2018; Accepted: 30 January 2019; Published: 12 February 2019



Abstract: This study defines a new linear differential operator via the Hadamard product between a q -hypergeometric function and Mittag–Leffler function. The application of the linear differential operator generates a new subclass of meromorphic function. Additionally, the study explores various properties and features, such as convex properties, distortion, growth, coefficient inequality and radii of starlikeness. Finally, the work discusses closure theorems and extreme points.

Keywords: differential operator; q -hypergeometric functions; meromorphic function; Mittag–Leffler function; Hadamard product

2010 MSC: 30C45

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{j=1}^{\infty} a_j z^j, \tag{1}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

Let $\Sigma^*(\rho)$ and $\Sigma_k(\rho)$ denote the subclasses of Σ that are meromorphically starlike functions of order ρ and meromorphically convex functions of order ρ respectively. Analytically, a function f of the form (1) is in the class $\Sigma^*(\rho)$ if it satisfies

$$\operatorname{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} > \rho \quad (z \in \mathbb{U}^*),$$

and $f \in \Sigma_k(\rho)$ if satisfies

$$\operatorname{Re} \left\{ -\left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \rho \quad (z \in \mathbb{U}^*).$$

The Hadamard product for two functions $f \in \Sigma$, defined by (1) and

$$g(z) = z^{-1} + \sum_{j=1}^{\infty} b_j z^j,$$

is given by

$$f(z) * g(z) = z^{-1} + \sum_{j=1}^{\infty} a_j b_j z^j. \tag{2}$$

For the two functions $f(z)$ and $g(z)$ analytic in \mathbb{U} , we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathbb{U}$), if there exists a Schwarz function $w(z)$ in \mathbb{U} with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$), such that $f(z) = g(w(z))$, ($z \in \mathbb{U}$).

For complex parameters a_i, b_k, q ($i = 1, \dots, l, k = 1, \dots, r, b_k \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$) the basic hypergeometric function (or q -hypergeometric function) ${}_l\Psi_r(z)$ is defined by:

$${}_l\Psi_r(a_1, \dots, a_l; b_1, \dots, b_r; q, z) = \sum_{j=0}^{\infty} \frac{(a_1, q)_j \dots (a_l, q)_j}{(q, q)_j (b_1, q)_j \dots (b_r, q)_j} \times [(-1)^j q^{\binom{j}{2}}]^{1+r-l} z^j, \tag{3}$$

with $\binom{j}{2} = j(j-1)/2$, where $q \neq 0$ when $l > r + 1$ ($l, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}$), and $(a, q)_j$ is the q -analogue of the Pochhammer symbol $(a)_j$ defined by:

$$(a, q)_j = \begin{cases} (1-a)(1-aq)(1-aq^2)\dots(1-aq^{j-1}), & j = 1, 2, 3, \dots, \\ 1, & j = 0. \end{cases}$$

The hypergeometric series defined by (3) was initially introduced by Heine in 1846 and referred to as the Heines series. More details on q -theory are available in [1–3] for readers to refer to.

It is clear that

$$\lim_{q \rightarrow 1^-} [{}_l\Psi_r(q^{a_1}, \dots, q^{a_l}; q^{b_1}, \dots, q^{b_r}; q, (q-1)^{1+r-l}z)] = {}_lF_r(a_1, \dots, a_l; b_1, \dots, b_r; z),$$

where ${}_lF_r(a_1, \dots, a_l; b_1, \dots, b_r; z)$ represents the generalised hypergeometric function (as shown in [4]).

Riemann, Gauss, Euler and others have conducted extensive studies of hypergeometric functions some hundreds years ago. The focus on this area is based mostly on the structural beauty and distinctive applications that this theory has, which include dynamic systems, mathematical physics, numeric analysis and combinatorics. Based on this, hypergeometric functions are utilised in various disciplines and this includes geometric function theory. One example that can be associated with the hypergeometric functions is the well-known Dziok–Srivastava operator [5,6] defined via the Hadamard product.

Now for $z \in \mathbb{U}$, $|q| < 1$, and $l = r + 1$, the q -hypergeometric function defined in (3) takes the following form:

$${}_l\Psi_r(a_1, \dots, a_l; b_1, \dots, b_r; q, z) = \sum_{j=0}^{\infty} \frac{(a_1, q)_j \dots (a_l, q)_j}{(q, q)_j (b_1, q)_j \dots (b_r, q)_j} z^j, \tag{4}$$

which converges absolutely in the open unit disk \mathbb{U} .

In reference to the function ${}_l\Psi_r(a_1, \dots, a_l; b_1, \dots, b_r; q, z)$ for meromorphic functions $f \in \Sigma$ that consist of functions in the form of (1), (see Aldweby and Darus [7], Murugusundaramoorthy and Janani [8]), as illustrated below, have recently introduced the q -analogue of the Liu–Srivastava operator

$$\begin{aligned} {}_lY_r(a_1, \dots, a_l; b_1, \dots, b_r; q, z)f(z) &= z^{-1} {}_l\Psi_r(a_1, \dots, a_l; b_1, \dots, b_r; q, z) * f(z) \\ &= z^{-1} + \sum_{j=1}^{\infty} \frac{\prod_{i=1}^l (a_i, q)_{j+1}}{(q, q)_{j+1} \prod_{k=1}^r (b_k, q)_{j+1}} a_j z^j. \end{aligned} \tag{5}$$

For convenience, we write

$${}_lY_r(a_1, \dots, a_l; b_1, \dots, b_r; q, z)f(z) = {}_lY_r(a_i, b_k; q, z)f(z).$$

Before going further, we state the well-known Mittag–Leffler function $E_\alpha(z)$, put forward by Mittag–Leffler [9,10], as well as Wiman’s generalisation [11] $E_{\alpha,\beta}(z)$ given respectively as follows:

$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad (6)$$

and

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad (7)$$

where $\alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

There has been a growing focus on Mittag–Leffler-type functions in recent years based on the growth of possibilities for their application for probability, applied problems, statistical and distribution theory, among others. Further information about how the Mittag–Leffler functions are being utilised can be found in [12–18]. In most of our work related to Mittag–Leffler functions, we study the geometric properties, such as the convexity, close-to-convexity and starlikeness. Recent studies on the $E_{\alpha,\beta}(z)$ Mittag–Leffler function can be seen in [19]. Additionally, Ref. [20] presents findings related to partial sums for $E_{\alpha,\beta}(z)$.

The function given by (7) is not within the class Σ . Based on this, the function is then normalised as follows:

$$\begin{aligned} \Omega_{\alpha,\beta}(z) &= z^{-1} \Gamma(\beta) E_{\alpha,\beta}(z) \\ &= z^{-1} + \sum_{j=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(j+1) + \beta)} z^j. \end{aligned} \quad (8)$$

Having use of the function $\Omega_{\alpha,\beta}(z)$ given by (8), a new operator $\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda] : \Sigma \rightarrow \Sigma$ is defined, in terms of Hadamard product, as follows:

$$\begin{aligned} \mathfrak{D}_\beta^{\alpha,0}[a_l, b_r, \lambda]f(z) &= {}_l Y_r(a_i, b_k; q, z)f(z) * \Omega_{\alpha,\beta}(z), \\ \mathfrak{D}_\beta^{\alpha,1}[a_l, b_r, \lambda]f(z) &= (1 - \lambda)({}_l Y_r(a_i, b_k; q, z)f(z) * \Omega_{\alpha,\beta}(z)) + \lambda z({}_l Y_r(a_i, b_k; q, z)f(z) * \Omega_{\alpha,\beta}(z))', \\ &\vdots \\ \mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z) &= \mathfrak{D}_\beta^{\alpha,1}(\mathfrak{D}_\beta^{\alpha,m-1}[a_l, b_r, \lambda]f(z)). \end{aligned} \quad (9)$$

If $f \in \Sigma$, then from (9) we deduce that

$$\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z) = z^{-1} + \sum_{j=1}^{\infty} [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) a_j z^j, \quad (10)$$

where

$$\nabla_{(j+1,\alpha,\beta)}(a_l, b_r) = \frac{\prod_{i=1}^l (a_i, q)_{j+1}}{(q, q)_{j+1} \prod_{k=1}^r (b_k, q)_{j+1}} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(j+1) + \beta)} \right). \quad (11)$$

Remark 1. It can be seen that, when specialising the parameters $\lambda, l, r, m, \alpha, \beta, q, a_1, \dots, a_l$ and b_1, \dots, b_r , it is observed that the defined operator $\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)$ leads to various operators. Examples are presented for further illustration.

- For $\lambda = 1, l = 1, r = 0, \beta = 1, \alpha = 0, a_1 = q$ and $q \rightarrow 1$ we get the operator $I^m f(z)$ studied by El–Ashwah and Aouf [21].
- For $m = 0, \alpha = 0, \beta = 1, a_i = q^{a_i}, b_k = q^{b_k}, a_i > 0, b_k > 0, (i = 1, \dots, l; k = 1, \dots, r, l = r + 1)$ and $q \rightarrow 1$ we get the operator $\mathcal{H}_{l,r}[a_i, b_k]f(z)$ which was investigated by Liu and Srivastava [22].

- For $m = 0, l = 2, r = 1, \beta = 1, \alpha = 0, a_2 = q$ and $q \rightarrow 1$ we get the operator $\mathcal{N}[a_1, b_1]f(z)$ studied by Liu and Srivastava [23].
- For $m = 0, l = 1, r = 0, \beta = 1, \alpha = 0, a_1 = \lambda + 1$ and $q \rightarrow 1$ we get the operator $\mathcal{D}^\lambda f(z) = (1/z(1 - z)^{\lambda+1}) * f(z)$ ($\lambda > -1$) was introduced by Ganigi and Uralegaddi [24], and then it was generalised by Yang [25].

A range of meromorphic function subclasses have been explored by, for example, Challab et al. [26], Elrifai et al. [27], Lashin [28], Liu and Srivastava [22] and others. These works have inspired our introduction of the new subclass $\mathcal{T}_{\alpha,\beta}^m(a_1, b_r, \lambda; D, H, d)$ of Σ , which involves the operator $\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z)$, and is shown as follows:

Definition 1. For $-1 \leq H < D \leq 1$, the function $f \in \Sigma$ is in the class $\mathcal{T}_{\alpha,\beta}^m(a_1, b_r, \lambda; D, H, d)$ if it satisfies the inequality

$$1 - \frac{1}{d} \left\{ \frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z))'}{\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z)} + 1 \right\} < \frac{1 + Dz}{1 + Hz}, \tag{12}$$

or, equivalently, to:

$$\left| \frac{\frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z))'}{\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z)} + 1}{H \frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z))'}{\mathfrak{D}_\beta^{\alpha,m}[a_1, b_r, \lambda]f(z)} + [(D - H)d + H]} \right| < 1 \tag{13}$$

Let Σ^* denote the subclass of Σ consisting of functions of the form:

$$f(z) = z^{-1} + \sum_{j=1}^{\infty} |a_j|z^j. \tag{14}$$

Now, we define the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$ by

$$\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d) = \mathcal{T}_{\alpha,\beta}^m(a_1, b_r, \lambda; D, H, d) \cap \Sigma^*.$$

2. Main Result

This section presents work to acquire sufficient conditions in which (14) gives the function f within the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$, as well as demonstrates that this condition is required for functions which belong to this class. In addition, linear combinations, growth and distortion bounds, closure theorems and extreme points are presented for the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$.

In our first theorem, we begin with the necessary and sufficient conditions for functions f in $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$.

Theorem 1. Let the function $f(z)$ be of the form (14). Then the function $f(z) \in \mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$ if and only if

$$\sum_{j=1}^{\infty} [(j + 1)(1 - H) - |d|(D - H)] [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_1, b_r) |a_j| \leq |d|(D - H). \tag{15}$$

Proof. Suppose that the inequality (15) holds true, we obtain

$$\begin{aligned} & \left| \frac{\frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z))'}{\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)} + 1}{H\frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z))'}{\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)} + [(D - H)d + H]} \right| \\ &= \left| \frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z))' + \mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)}{Hz(\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z))' + [d(D - H) + H]\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)} \right| \\ &= \left| \frac{\sum_{j=1}^\infty (j + 1) [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) |a_j| z^{j+1}}{d(D - H) + \sum_{j=1}^\infty [H(j + 1) + d(D - H)] [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) |a_j| z^{j+1}} \right| \\ &< 1 \quad (z \in \mathbb{U}^*). \end{aligned}$$

Then, by the maximum modulus theorem, we have $f(z) \in \mathcal{T}_{\alpha,\beta}^{m,*}(a_l, b_r, \lambda; D, H, d)$.

Conversely, assume that $f(z)$ is in the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_l, b_r, \lambda; D, H, d)$ with $f(z)$ of the form (14), then we find from (13) that

$$\begin{aligned} & \left| \frac{z(\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z))' + \mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)}{Hz(\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z))' + [d(D - H) + H]\mathfrak{D}_\beta^{\alpha,m}[a_l, b_r, \lambda]f(z)} \right| \\ &= \left| \frac{\sum_{j=1}^\infty (j + 1) [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) |a_j| z^{j+1}}{d(D - H) + \sum_{j=1}^\infty [H(j + 1) + d(D - H)] [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) |a_j| z^{j+1}} \right| \tag{16} \\ &< 1, \end{aligned}$$

since the above inequality is genuine for all $z \in \mathbb{U}^*$, choose values of z on the real axis. After clearing the denominator in (16) and letting $z \rightarrow 1^-$ through real values, we get

$$\sum_{j=1}^\infty [(j + 1)(1 - H) - |d|(D - H)] [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) |a_j| \leq |d|(D - H).$$

Thus, we obtain the desired inequality (15) of Theorem 1. \square

Corollary 1. If the function f of the form (14) is in the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_l, b_r, \lambda; D, H, d)$ then

$$|a_j| \leq \frac{|d|(D - H)}{[(j + 1)(1 - H) - |d|(D - H)] [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r)} \quad (j \geq 1),$$

the result is sharp for the function

$$f(z) = z^{-1} + \frac{|d|(D - H)}{[(j + 1)(1 - H) - |d|(D - H)] [1 + (j - 1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r)} z^j \quad (j \geq 1). \tag{17}$$

Growth and distortion bounds for functions belonging to the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_l, b_r, \lambda; D, H, d)$ will be given in the following result:

Theorem 2. If a function f given by (14) is in the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$ then for $|z| = r$, we have:

$$\frac{1}{r} - \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)} r \leq |f(z)| \leq \frac{1}{r} + \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)} r,$$

and

$$\frac{1}{r^2} - \frac{|d|(D-H)}{[2(1-B) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)}.$$

Proof. By Theorem 1,

$$\begin{aligned} & [2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r) \sum_{j=1}^{\infty} |a_j| \\ & \leq \sum_{j=1}^{\infty} [(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_1, b_r) |a_j| \leq |d|(D-H), \end{aligned}$$

which yields:

$$\sum_{j=1}^{\infty} |a_j| \leq \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)}.$$

Therefore,

$$|f(z)| \leq \frac{1}{|z|} + |z| \sum_{j=1}^{\infty} |a_j| \leq \frac{1}{|z|} + \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)} |z|,$$

and

$$|f(z)| \geq \frac{1}{|z|} - |z| \sum_{j=1}^{\infty} |a_j| \geq \frac{1}{|z|} - \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)} |z|.$$

Now, by differentiating both sides of (14) with respect to z , we get:

$$|f'(z)| \leq \frac{1}{|z|^2} + \sum_{j=1}^{\infty} |a_j| \leq \frac{1}{|z|^2} + \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)},$$

and

$$|f'(z)| \geq \frac{1}{|z|^2} - \sum_{j=1}^{\infty} |a_j| \geq \frac{1}{|z|^2} - \frac{|d|(D-H)}{[2(1-H) - |d|(D-H)]\nabla_{(2,\alpha,\beta)}(a_1, b_r)}.$$

□

Next, we determine the radii of meromorphic starlikeness and convexity of order ρ for functions in the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$.

Theorem 3. Let the function f given by (14) be in the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$. Thus, we have:

(i) f is meromorphically starlike of order ρ in the disc $|z| < r_1$, that is

$$\operatorname{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} > \rho \quad (|z| < r_1, 0 \leq \rho < 1),$$

where

$$r_1 = \inf_{j \geq 1} \left\{ \frac{(1-\rho)[(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_1, b_r)}{|d|(D-H)(j+\rho)} \right\}^{\frac{1}{j+1}}. \quad (18)$$

(ii) f is meromorphically convex of order ρ in the disc $|z| < r_2$, that is

$$\operatorname{Re} \left\{ - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \rho \quad (|z| < r_2, 0 \leq \rho < 1),$$

where

$$r_2 = \inf_{j \geq 1} \left\{ \frac{(1-\rho)[(j+1)(1-H) - |d|(D-H)][1+(j-1)\lambda]^m \nabla_{(j+1, \alpha, \beta)}(a_l, b_r)}{j|d|(D-H)(j+\rho)} \right\}^{\frac{1}{j+1}}. \quad (19)$$

Proof. (i) From the definition (14), we can get:

$$\left| \frac{\frac{zf'(z)}{f(z)} + 1}{\frac{zf'(z)}{f(z)} - 1 + 2\rho} \right| \leq \frac{\sum_{j=1}^{\infty} (j+1)|a_j||z|^{j+1}}{2(1-\rho) - \sum_{j=1}^{\infty} (j-1+2\rho)|a_j||z|^{j+1}}.$$

Then, we have:

$$\left| \frac{\frac{zf'(z)}{f(z)} + 1}{\frac{zf'(z)}{f(z)} - 1 + 2\rho} \right| \leq 1 \quad (0 \leq \rho < 1),$$

if

$$\sum_{j=1}^{\infty} \left(\frac{j+\rho}{1-\rho} \right) |a_j||z|^{j+1} \leq 1. \quad (20)$$

Thus, by Theorem 1, the inequality (20) will be true if

$$\left(\frac{j+\rho}{1-\rho} \right) |z|^{j+1} \leq \frac{[(j+1)(1-H) - |d|(D-H)][1+(j-1)\lambda]^m \nabla_{(j+1, \alpha, \beta)}(a_l, b_r)}{|d|(D-H)},$$

then

$$|z| \leq \left\{ \frac{(1-\rho)[(j+1)(1-H) - |d|(D-H)][1+(j-1)\lambda]^m \nabla_{(j+1, \alpha, \beta)}(a_l, b_r)}{|d|(D-H)(j+\rho)} \right\}^{\frac{1}{j+1}}.$$

The last inequality leads us immediately to the disc $|z| < r_1$, where r_1 is given by (18).

(ii) In order to prove the second affirmation of Theorem 3, we find from (14) that:

$$\left| \frac{2 + \frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 2\rho} \right| \leq \frac{\sum_{j=1}^{\infty} j(j+1)|a_j||z|^{j+1}}{2(1-\rho) - \sum_{j=1}^{\infty} j(j-1+2\rho)|a_j||z|^{j+1}}.$$

Thus, we have the desired inequality:

$$\left| \frac{2 + \frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 2\rho} \right| \leq 1 \quad (0 \leq \rho < 1),$$

if

$$\sum_{j=1}^{\infty} j \left(\frac{j+\rho}{1-\rho} \right) |a_j||z|^{j+1} \leq 1. \quad (21)$$

Thus, by Theorem 1, the inequality (21) will be true if

$$j \left(\frac{j+\rho}{1-\rho} \right) |z|^{j+1} \leq \frac{[(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_1, b_r)}{|d|(D-H)},$$

then

$$|z| \leq \left\{ \frac{(1-\rho) [(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_1, b_r)}{j|d|(D-H)(j+\rho)} \right\}^{\frac{1}{j+1}}.$$

The last inequality readily yields the disc $|z| < r_2$, where r_2 is given by (19). \square

The closure theorems and extreme points of the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$ will now be determined.

Theorem 4. The class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$ is closed under convex linear combinations.

Proof. Assume that the functions

$$f_i(z) = z^{-1} + \sum_{j=1}^{\infty} |a_{j,i}| z^j \quad (i = 1, 2),$$

are in $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$. It suffices to show that the function h defined by

$$h(z) = (1-c)f_1(z) + cf_2(z) \quad (0 \leq c \leq 1),$$

is in the class $\mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$, since

$$h(z) = z^{-1} + \sum_{j=1}^{\infty} [(1-c)|a_{j,1}| + c|a_{j,2}|] z^j \quad (0 \leq c \leq 1).$$

In view of Theorem 1, we have:

$$\begin{aligned} & \sum_{j=1}^{\infty} [(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)} \cdot \{(1-c)|a_{j,1}| + c|a_{j,2}|\} \\ &= (1-c) \sum_{j=1}^{\infty} [(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)} |a_{j,1}| \\ & \quad + c \sum_{j=1}^{\infty} [(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)} |a_{j,2}| \\ & \leq (1-c)|d|(D-H) + c|d|(D-H) = |d|(D-H), \end{aligned}$$

which shows that $h(z) \in \mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$. \square

Theorem 5. Let $f_0(z) = \frac{1}{z}$ and

$$f_j(z) = \frac{1}{z} + \frac{|d|(D-H)}{[(j+1)(1-H) - |d|(D-H)] [1 + (j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}} z^j \quad (j \geq 1).$$

Then $f \in \mathcal{T}_{\alpha,\beta}^{m,*}(a_1, b_r, \lambda; D, H, d)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=0}^{\infty} v_j f_j(z), \quad (22)$$

where

$$v_j \geq 0 \quad \text{and} \quad \sum_{j=0}^{\infty} v_j = 1.$$

Proof. Let the function $f(z)$ be expressed in the form given by (22), then

$$f(z) = z^{-1} + \sum_{j=1}^{\infty} v_j \frac{|d|(D-H)}{[(j+1)(1-H) - |d|(D-H)] [1+(j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}} z^j$$

and for this function, we have:

$$\begin{aligned} & \sum_{j=1}^{\infty} [(j+1)(1-H) - |d|(D-H)] [1+(j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}(a_l, b_r) \\ & \quad \times v_j \frac{|d|(D-H)}{[(j+1)(1-H) - |d|(D-H)] [1+(j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}} \\ & \quad = \sum_{j=1}^{\infty} v_j |d|(D-H) = |d|(D-H)(1-v_0) \leq |d|(D-H) \end{aligned}$$

The condition (15) is satisfied. Thus, $f \in \mathcal{T}_{\alpha,\beta}^{m,*}(a_l, b_r, \lambda; D, H, d)$

Conversely, we suppose that $f \in \mathcal{T}_{\alpha,\beta}^{m,*}(a_l, b_r, \lambda; D, H, d)$, since

$$|a_j| \leq \frac{|d|(D-H)}{[(j+1)(1-H) - |d|(D-H)] [1+(j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}} \quad (j \geq 1),$$

we set

$$v_j = \frac{[(j+1)(1-H) - |d|(D-H)] [1+(j-1)\lambda]^m \nabla_{(j+1,\alpha,\beta)}}{|d|(D-H)} |a_j|, \quad (j \geq 1),$$

and

$$v_0 = 1 - \sum_{j=1}^{\infty} v_j,$$

so it follows that

$$f(z) = \sum_{j=0}^{\infty} v_j f_j(z).$$

This completes the assertion of Theorem 5. \square

3. Conclusions

Studying the theory of analytical functions has been an area of concern for many researchers. A more specific field is the study of inequalities in complex analysis. Literature review indicates lots of studies based on the classes of analytical functions. The interplay of geometry and analysis represents a very important aspect in complex function theory study. This rapid growth is directly linked to the relation that exists between analytical structure and geometric behaviour. Motivated by this approach, in the current study, we have introduced a new meromorphic function subclass which is related to both the Mittag–Leffler function and q -hypergeometric function, and we have obtained sufficient and necessary conditions in relation to this subclass. Linear combinations, distortion theory and other properties are also explored. For further research we could study the certain classes related to functions with respect to symmetric points associated with hypergeometric and Mittag–Leffler functions.

Author Contributions: Funding acquisition, M.D.; Investigation, S.E.; Methodology, S.E.; Supervision, M.D.; Writing—review & editing, M.D.

Funding: This research was funded by Universiti Kebangsaan Malaysia, grant number GUP-2017-064.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Exton, H. *q-Hypergeometric Functions and Applications*, Ellis Horwood Series: Mathematics and Its Applications; Ellis Horwood: Chichester, UK, 1983.
2. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*; Cambridge University Press: Cambridge, UK, 1990.
3. Srivastava, H.M. Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inf. Sci.* **2011**, *5*, 390–444.
4. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Ellis Horwood: Chichester, UK, 1985.
5. Dziok, J.; Srivastava, H.M. Classes of analytic functions associated with the generalized hypergeometric function. *Appl. Math. Comput.* **1999**, *103*, 1–13. [[CrossRef](#)]
6. Dziok, J.; Srivastava, H.M. Certain subclasses of analytic functions associated with the generalized hypergeometric function. *Integral Transforms Spec. Funct.* **2003**, *14*, 7–18. [[CrossRef](#)]
7. Aldweby, H.; Darus, M. Integral operator defined by q -analogue of Liu–Srivastava operator. *Stud. Univ. Babeş-Bolyai Math.* **2013**, *58*, 529–537.
8. Murugusundaramoorthy, G.; Janani, T. Meromorphic parabolic starlike functions associated with q -hypergeometric series. *ISRN Math. Anal.* **2014**, *2014*, 923607. [[CrossRef](#)]
9. Mittag–Leffler, G.M. Sur la nouvelle fonction $E_\alpha(x)$. *CR Acad. Sci. Paris* **1903**, *137*, 554–558.
10. Mittag–Leffler, G.M. Sur la représentation analytique d’une branche uniforme d’une fonction monogène. *Acta Math.* **1905**, *29*, 101–181. [[CrossRef](#)]
11. Wiman, A. Über den fundamentalsatz in der theorie der funktionen $E_\alpha(x)$. *Acta Math.* **1905**, *29*, 191–201. [[CrossRef](#)]
12. Attiya, A.A. Some applications of Mittag–Leffler function in the unit disk. *Filomat* **2016**, *30*, 2075–2081. [[CrossRef](#)]
13. Gupta, I.S.; Debnath, L. Some properties of the Mittag–Leffler functions. *Integral Transforms Spec. Funct.* **2007**, *18*, 329–336. [[CrossRef](#)]
14. Răducanu, D. Third-Order differential subordinations for analytic functions associated with generalized Mittag–Leffler functions. *Mediterr. J. Math.* **2017**, *14*, 167. [[CrossRef](#)]
15. Rehman, H.; Darus, M.; Salah, J. Coefficient properties involving the generalized k -Mittag–Leffler functions. *Transylv. J. Math. Mech.* **2017**, *9*, 155–164.
16. Salah, J.; Darus, M. A note on generalized Mittag–Leffler function and application. *Far East J. Math. Sci.* **2011**, *48*, 33–46.
17. Srivastava, H.M.; Frasin, B.A.; Pescar, V. Univalence of integral operators involving Mittag–Leffler functions. *Appl. Math. Inf. Sci.* **2017**, *11*, 635–641. [[CrossRef](#)]
18. Srivastava, H.M.; Tomovski, Ž. Fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel. *Appl. Math. Comput.* **2009**, *211*, 198–210. [[CrossRef](#)]
19. Bansal, D.; Prajapat, J.K. Certain geometric properties of the Mittag–Leffler functions. *Complex Var. Elliptic Equ.* **2016**, *61*, 338–350. [[CrossRef](#)]
20. Răducanu, D. On partial sums of normalized Mittag–Leffler functions. *An. Șt. Univ. Ovidius Constanța* **2017**, *25*, 123–133. [[CrossRef](#)]
21. El-Ashwah, R.M.; Aouf, M.K. Hadamard product of certain meromorphic starlike and convex functions. *Comput. Math. Appl.* **2009**, *57*, 1102–1106. [[CrossRef](#)]
22. Liu, J.-L.; Srivastava, H.M. Classes of meromorphically multivalent functions associated with the generalized hypergeometric function. *Math. Comput. Model.* **2004**, *39*, 21–34. [[CrossRef](#)]
23. Liu, J.-L.; Srivastava, H.M. A linear operator and associated families of meromorphically multivalent functions. *J. Math. Anal. Appl.* **2001**, *259*, 566–581. [[CrossRef](#)]
24. Ganigi, M.R.; Uralegaddi, B.A. New criteria for meromorphic univalent functions. *Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie Nouvelle Série* **1989**, *33*, 9–13.
25. Yang, D. On a class of meromorphic starlike multivalent functions. *Bull. Inst. Math. Acad. Sin.* **1996**, *24*, 151–157.
26. Challab, K.; Darus, M.; Ghanim, F. On a certain subclass of meromorphic functions defined by a new linear differential operator. *J. Math. Fund. Sci.* **2017**, *49*, 269–282. [[CrossRef](#)]

27. Elrifai, E.A.; Darwish, H.E.; Ahmed, A.R. On certain subclasses of meromorphic functions associated with certain differential operators. *Appl. Math. Lett.* **2012**, *25*, 952–958. [[CrossRef](#)]
28. Lashin, A.Y. On certain subclasses of meromorphic functions associated with certain integral operators. *Comput. Math. Appl.* **2010**, *59*, 524–531 [[CrossRef](#)]



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).