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Some Difference Equations for Srivastava's λ-Generalized Hurwitz–Lerch Zeta Functions with Applications

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Abstract: In this article, we establish some new difference equations for the family of λ-generalized Hurwitz–Lerch zeta functions. These difference equations proved worthwhile to study these newly defined functions in terms of simpler functions. Several authors investigated such functions and their analytic properties, but no work has been reported for an estimation of their values. We perform some numerical computations to evaluate these functions for different values of the involved parameters. It is shown that the direct evaluation of involved integrals is not possible for the large values of parameter s; nevertheless, using our new difference equations, we can evaluate these functions for the large values of s. It is worth mentioning that for the small values of this parameter, our results are 100% accurate with the directly computed results using their integral representation. Difference equations so obtained are also useful for the computation of some new integrals of products of λ-generalized Hurwitz–Lerch zeta functions and verified to be consistent with the existing results. A derivative property of Mellin transforms proved fundamental to present this investigation.

Keywords: analytic number theory; λ-generalized Hurwitz–Lerch zeta functions; derivative properties; recurrence relations; integral representations; Mellin transform

1. Introduction

In this paper, we practice the customary symbolizations:

$$N := \{1,\ 2,\ \cdots\};\ N_0 := N \cup \{0\};\ Z^- := \{-1, -2,\ \cdots\};\ Z_0^- := Z^- \cup \{0\}, \tag{1}$$

where Z^- is the set of integers. The involved symbols R, R^+, and C represent the set of real, positive real, and complex numbers, consistently.

The Hurwitz–Lerch zeta function has always been a topic of motivation for several researchers due to its impact in analytic number theory and other applied sciences. Recently, Srivastava presented a considerably new universal family of Hurwitz–Lerch zeta functions defined by [1] (p. 1487, Equation (1.14)):

$$\begin{aligned}\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda) \\ = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp\left(-at-\frac{b}{t^\lambda}\right){}_p\Psi^*_q\left[\begin{matrix}(\lambda_1,\ \rho_1),\ldots,(\lambda_p,\ \rho_p) \\ (\mu_1,\sigma_1),\ldots,(\mu_p,\sigma_p)\end{matrix}\ ;ze^{-t}\right]dt; \\ (min[R(a),R(s)]>0;R(b)\geqq 0;\lambda\geqq 0)\end{aligned} \tag{2}$$

so that, evidently, one can get the subsequent connection with the extended Hurwitz–Lerch zeta functions $\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a)$ defined by the authors of [2] (p. 503, Equation (6.2)) (see also References [3,4]):

$$\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;0,\lambda)=\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a)=e^b\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;b,0). \tag{3}$$

In the above Equation (2), $\mathrm{p}^{\Psi^*}\mathrm{q}$ where ($\mathrm{p}, q\in\mathbb{N}_0$) is the standard Fox–Wright function defined by the authors of [4] (p. 2219, Equation (1)) (see also References [3] (p. 516, Equation (1)) and [2] (p. 493, Equation (2.1)):

$$\mathrm{p}^{\Psi^*}\mathrm{q}\left[\begin{array}{ll}(\lambda_1,\ \rho_1),\ldots, & (\lambda_p,\ \rho_p)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{array};z\right]=\sum_{\chi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}\frac{z^\chi}{\chi!}. \tag{4}$$

Pochhammer symbols $([\lambda_p])_{\rho_p n}:=[\lambda_1]_{\rho_p n}\cdots[\lambda_p]_{\rho_p n}$ symbolize the shifted factorial defined in terms of the basic Gamma function as follows:

$$(\lambda)_\rho=\frac{\Gamma(\lambda+\rho)}{\Gamma(\lambda)}=\begin{cases}1 & (\rho=0,\lambda\in\mathbb{C}\backslash\{0\})\\ \lambda(\lambda+1)\ldots(\lambda+\chi-1) & (\rho=\chi\in\mathbb{N};\lambda\in\mathbb{C}),\end{cases} \tag{5}$$

$$\Delta:=\sum_{j=1}^{q}\sigma_j-\sum_{j=1}^{p}\rho_j \text{ and } \nabla:=\left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right).\left(\prod_{j=1}^{q}\sigma_j^{\sigma_j}\right).$$

The series given by Equation (4) converges in the entire complex z-plane for $\Delta>-1$; and if $\Delta=0$, the series (Equation (4)) converges only for $|z|<\nabla$. For more detailed discussion of such functions, we refer the interested reader to also see References [5–9].

The analysis of Srivastava's λ-generalized Hurwitz–Lerch zeta functions and its different forms have attracted noteworthy concern, and many papers have subsequently appeared on this subject. Jankov et al. [10] and Srivastava et al. [3] discussed some inequalities for different cases of λ-generalized Hurwitz–Lerch zeta functions. Srivastava et al. [11] introduced a nonlinear operator related with the λ-generalized Hurwitz–Lerch zeta functions to analyze the inclusion properties of definite subclass of special type of meromorphic functions. Srivastava and Gaboury [12] deliberated on new expansion formulas for such functions (see, for details, References [13,14]; see also the further thoroughly associated studies cited in each of these publications). Luo and Raina [4] discussed some new inequalities involving Srivastava's λ-generalized Hurwitz–Lerch zeta functions and obtained the following series representation [4] (p. 2221, Equation (6)):

$$\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda)=\frac{1}{\lambda\Gamma(s)}\sum_{\chi=0}^{\infty}\frac{([\lambda_p])_{\rho_p\chi}}{([\mu_q])_{\sigma_q\chi}}Z^{\frac{s}{\lambda}}_{\frac{1}{\lambda}}(a+\chi)^\lambda b\frac{z^\chi}{\chi!(\chi+a)^s},$$
$$\Big(\lambda j\in R(j=1,..,p)\text{ and }\mu_j\in R\backslash Z-0\ (j=1,\ldots,q);\rho j>0(j,\ldots,p);\sigma j>0(j=1,\ldots,q);1+\Delta\geq 0\Big). \tag{6}$$

Srivastava beautifully described important results about the zeta and related functions in an expository article [15]. Choi et al. [16] further discussed these functions by introducing one more variable. Srivastava et al. [17] presented an innovative integral transform connected with the λ-extended Hurwitz–Lerch zeta function. More recently, Tassaddiq [18] obtained a new representation for this family of the λ-generalized Hurwitz–Lerch zeta functions in terms of complex delta functions such that the definition of these functions is formalized over the space of entire test functions denoted by Z. The author also listed and discussed all the possible special cases of Srivastava's λ-generalized Hurwitz–Lerch zeta functions [18] (p. 4) in the form of a table. For the purposes of our present investigation, this table is given on the next page. For any use of the special cases of the generalized Hurwitz–Lerch zeta functions, the reader is referred to this table. For more detailed study of zeta and related functions, we refer the interested reader to References [19–40] and further bibliography cited therein.

Table 1. Different special cases of λ-generalized Hurwitz–Lerch zeta functions [18].

$min[\Re(a),\Re(s)] > 0$; $\Re(b) \geqq 0$; $\lambda \geqq 0$; $\rho = \rho_1,\ldots,\rho_p$; $\sigma = \sigma_1,\ldots,\sigma_q$; $\lambda^* = \lambda_1,\ldots,\lambda_p$; $\mu = \mu_1,\ldots,\mu_q$		($p-1 = q = 0$; $\lambda_1 = \mu$; $\rho_1 = 1$)					($p-1 = q = 0$; $\lambda_1 = \mu$; $\rho_1 = 1$)	
			$\lambda = 1$	$\mu = 1$	$\lambda = \mu = 1$	$b = 0$	$b = 0$	$\mu = 1; b = 0$
λ-Generalized Hurwitz–Lerch Zeta Functions	$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z,s,a;b,\lambda)$ [1], (p. 1487), Equation (1.14)	$\Theta^{\lambda}_{\mu}(\mp z,s,a;b)$ [41], (p. 90), Equation (6) and [42]	$\Phi^{*}_{\mu}(\pm z,s,a,b)$	$\Phi_b(\pm z,s,a,\lambda)$	$\Phi_b(\pm z,s,a)$	$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z,s,a)$ ([1], p. 1486, Equation (1.11)) & [2]	$\Phi^{*}_{\mu}(\pm z,s,a)$ ([43], p. 100, Equation (1.5))	$\Phi(\pm z,s,a)$ ([44], p. 27, Equation (1.11))
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm e^{-x},s,a;b,\lambda)$ λ-Generalized Extended Fermi–Dirac and Extended Bose–Einstein Functions	$\Theta^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s,a;b,\lambda)$	$\Theta^{\lambda}_{\mu}(x,s,a;b)$	$\Theta^{*}_{\mu}(x,s,a,b)$	$\Theta_b(x,s,a;\lambda)$	$\Theta_b(x,s,a)$	$\Theta^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s,a)$	$\Theta^{*}_{\mu}(x,s,a)$ ([43], p. 12, Equation (45))	$\Theta_a(x;s)$ ([45], p. 9, Equation (3.14))
	$\Psi^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s,a;b,\lambda)$	$\Psi^{\lambda}_{\mu}(x,s,a;b)$	$\Psi^{*}_{\mu}(x,s,a,b)$	$\Psi_b(x,s,a,\lambda)$	$\Psi_b(x,s,a)$	$\Psi^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s,a)$	$\Psi^{*}_{\mu}(x,s,a)$ ([43], p. 12, Equation (45))	$\Psi_a(x;s)$ [45], p. 115, Equation (4.4)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z,s,1;b,\lambda)$ λ-Generalized Polylogarithm Functions	$\mathrm{Li}^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm z,s;b,\lambda)$	$\mathrm{Li}^{\lambda}_{\mu}(\mp z,s,a;b)$	$\mathrm{Li}^{*}_{\mu}(z,s,b)$	$\mathrm{Li}_b(z,s,\lambda)$	$\mathrm{Li}_b(z,s)$	$\mathrm{Li}^{(\rho;\sigma)}_{\lambda^*,\mu}(z,s)$	$\mathrm{Li}^{*}_{\mu}(z,s)$ ([43], p. 12, Equation (47))	$\mathrm{Li}_s(z)$ [44], (Chapter 1)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm e^{-x},s+1,1;b,\lambda)$ λ-Generalized Fermi–Dirac and Bose–Einstein Functions	$\mathrm{F}^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s;b,\lambda)$	$\mathrm{F}^{\lambda}_{\mu}(x,s,a;b)$	$\mathrm{F}^{*}_{\mu}(\mathrm{x},\mathrm{s},\mathrm{b})$	$\mathrm{F_b}(x,\mathrm{s},\lambda)$	$\mathrm{F_b}(x,\mathrm{s})$	$\mathrm{F}^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s)$	$\mathrm{F}^{*}_{\mu}(\mathrm{x},\mathrm{s})$ ([43], p. 12, Equation (47))	$\mathrm{F_s}(\mathrm{x})$ ([45], p. 109, Equation (1.12)
	$\mathrm{B}^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s;b,\lambda)$	$\mathrm{B}^{\lambda}_{\mu}(x,s,a;b)$	$B^{*}_{\mu}(x,s,b)$	$B_b(x,s,\lambda)$	$B_b(x,s)$	$\mathrm{B}^{(\rho;\sigma)}_{\lambda^*,\mu}(x,s)$	$\mathrm{B}^{*}_{\mu}(x,s)$ ([43], p. 12, Equation (45))	$\mathrm{B_s}(\mathrm{x})$ ([45], p. 109, Equation (1.12))
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm 1,s,a;b,\lambda)$ λ-Generalized Hurwitz zeta Functions	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s,a;b,\lambda)$	$\zeta^{\lambda}_{\mu}(s,a;b)$	$\zeta^{*}_{\mu}(s,a,b)$	$\zeta_b(s,a,\lambda)$	$\zeta_b(s,a)$ [46], p. 308	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s,a)$	$\zeta^{*}_{\mu}(s,a)$ [43]	$\zeta(s,a)$ [44], (Chapter 1)
$\Phi^{(\rho;\sigma)}_{\lambda^*,\mu}(\pm 1,s,1;b,\lambda)$ λ-Generalized Riemann Zeta Functions	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s)$	$\zeta^{\lambda}_{\mu}s;b)$	$\zeta^{*}_{\mu}(s,b)$	$\zeta_b(s,\lambda)$	$\zeta_b(s)$ [46], p. 308	$\zeta^{(\rho;\sigma)}_{\lambda^*,\mu}(s)$	$\zeta^{*}_{\mu}(s)$ [43]	$\zeta(s)$ [44], (Chapter 1)

In this research, our focus is to establish some new difference equations for the family of λ-generalized Hurwitz–Lerch zeta functions and its special cases by following the approach of Tassaddiq and Qadir [33]. From the above discussion and Table 1, we can notice that several authors presented and studied worthwhile generalizations of the Hurwitz–Lerch zeta functions. They obtained various analytic formulas, integral, and series representations. However, as we deeply study Riemann zeta functions, we know their values, their graphs, and several other important aspects. We could not develop this approach for these generalizations. Bayad and Chiki [43] obtained reduction and duality formulas of the generalized Hurwitz–Lerch zeta functions. Their results contain the earlier obtained results of Choi [47]. These reduction formulas were concerned with the reduction of one parameter that represent the generalized Hurwitz–Lerch zeta $\Phi_{\mu}^{*}(z,s,a)$ and Hurwitz zeta functions $\zeta_{\mu}^{*}(s,a)$ in terms of Hurwitz–Lerch zeta $\Phi(z,s,a)$ and Hurwitz zeta $\zeta(s,a)$ functions, respectively. The difference equations presented here have the advantage of reducing the generalized Hurwitz–Lerch zeta $\Phi_{\mu}^{*}(z,s,a)$ and the generalized Hurwitz zeta functions $\zeta_{\mu}^{*}(s,a)$ in terms of basic polylogarithm $\mathrm{Li}_s(z)$ and zeta functions $\zeta(s)$, respectively. That means we have reduced one more parameter and our results are simple enough to evaluate these functions for different values of the involved parameters. By following the approach developed in this paper, we can initiate a deeper analysis of these functions that will enhance their applications. The Riemann hypothesis is a well-known unsolved problem in analytic number theory [22]. It states that "all the non-trivial zeros of the zeta function exist on the real line $s = \frac{1}{2}$". These zeros seem to be complex conjugates and hence symmetric on this line. The integrals of the zeta function and its generalizations are vital in the study of Riemann hypothesis and for the investigation of zeta functions themselves. The study of distributions in statistical inference and reliability theory [1,48,49] also involves such integrals. Difference equations obtained in this investigation are worthwhile to evaluate integrals of products of the family of λ-generalized Hurwitz–Lerch zeta functions that are consistent with the existing results.

The plan of the paper as follows: We present some new difference equations involving the λ-generalized Hurwitz–Lerch zeta functions in Section 2 and obtain similar results for other related functions. We discuss some applications of these difference equations in Section 3 by evaluating some special cases of the function. Based upon the results of Section 2, we evaluate new integrals of products of these functions in Section 4. We conclude our results in the last Section 5 by highlighting some future directions of this work.

Throughout this investigation, conditions on the parameters will be considered standard as given in Equations (1)–(6) and Table 1 unless otherwise stated.

2. Results

New Difference Equation of the λ-Generalized Hurwitz–Lerch Zeta Functions

Theorem 1. *Prove that λ-Generalized Hurwitz–Lerch zeta functions satisfy the following relation:*

$$\begin{gathered}\Gamma(s)\Phi^{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,,\mu_q}_{(\lambda_1+\rho_1,\ldots,\lambda_p\rho_p,\mu_1+\sigma_1,\ldots,\mu_q+\sigma_q)}(z,s,a+1;b,\lambda)= \\ \frac{(\mu_1)_{\sigma_1}\cdots\cdots(\mu_q)_{\sigma_q}}{z.(\lambda_1)_{\rho_1}\cdots\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c} b\lambda\Gamma(s-\lambda-1)\Phi^{(\rho_1,\ldots,\rho_p\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,,\mu_q}(z,s-\lambda-1,a,b)+\Gamma(s) \\ \left[\Phi^{(\rho_1,\ldots,\rho_p\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,,\mu_q}(z,s-1,a;b,\lambda)-a\Phi^{(\rho_1,\ldots,\rho_p\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,,\mu_q}(z,s,a;b,\lambda)\right]\end{array}\right].\end{gathered} \tag{7}$$

Proof: Consider the function:

$$f(t)=\exp\left(-at-\frac{b}{t^{\lambda}}\right)\,p^{\Psi^{*}}q\left[\begin{array}{ccc}(\lambda_1,\ \rho_1),\ldots, & (\lambda_p,\ \rho_p) & ;ze^{-t} \\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q) & \end{array}\right] \tag{8}$$

and differentiate Equation (8) to get:

$$\begin{aligned}
&\frac{d}{dt}\left[\exp\left(-at-\frac{b}{t^{\lambda}}\right)p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1,\rho_1),\ldots, & \left(\lambda_p,\rho_p\right)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{array};ze^{-t}\right]\right]=\\
&-a.\exp\left(-at-\frac{b}{t^{\lambda}}\right)p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1,\rho_1),\ldots, & \left(\lambda_p,\rho_p\right)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{array};ze^{-t}\right]\\
&+b\lambda\frac{\exp\left(-at-\frac{b}{t^{\lambda}}\right)}{t^{\lambda+1}}p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1,\rho_1),\ldots, & \left(\lambda_p,\rho_p\right)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{array};ze^{-t}\right]\\
&-z\frac{(\lambda_1)_{\rho_1}\ldots(\lambda_p)_{\rho_p}}{(\mu_1)_{\sigma_1}\ldots(\mu_q)_q}\exp\left(-(a+1)t-\frac{b}{t^{\lambda}}\right)p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1+\rho_1,\rho_1),\ldots, & \left(\lambda_p+\rho_p,\rho_p\right)\\ (\mu_1+\sigma_1,\sigma_1),\ldots, & (\mu_q+\sigma_q,\sigma_q)\end{array};ze^{-t}\right]
\end{aligned}\tag{9}$$

so that:

$$\begin{aligned}
f'(t)=&\exp\left(-at-\frac{b}{t^{\lambda}}\right)p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1,\rho_1),\ldots, & \left(\lambda_p,\rho_p\right)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{array}ze^{-t}\right]\left[-a+\frac{b\lambda}{t^{\lambda+1}}\right]\\
&-z\frac{(\lambda_1)_{\rho_1}\ldots(\lambda_p)_{\rho_p}}{(\mu_1)_{\sigma_1}\ldots(\mu_q)_q}\exp\left(-(a+1)t-\frac{b}{t^{\lambda}}\right)p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1+\rho_1,\rho_1),\ldots, & \left(\lambda_p+\rho_p,\rho_p\right)\\ (\mu_1+\sigma_1,\sigma_1),\ldots, & (\mu_q+\sigma_q,\sigma_q)\end{array};ze^{-t}\right],
\end{aligned}\tag{10}$$

where we have used the usual differentiation and the derivative property, which is obtained on the same lines as given by Reference [1] (p. 1492, Equation (3.1)):

$$\begin{aligned}
&\frac{d}{dt}\left[p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1,\rho_1),\ldots, & \left(\lambda_p,\rho_p\right)\\ (\mu_1,\sigma_1),\ldots, & (\mu_q,\sigma_q)\end{array};ze^{-t}\right]\right]=\\
&-ze^{-t}\frac{(\lambda_1)_{\rho_1}\ldots(\lambda_p)_{\rho_p}}{(\mu_1)_{\sigma_1}\ldots(\mu_q)_q}p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1+\rho_1,\rho_1),\ldots, & \left(\lambda_p+\rho_p,\rho_p\right)\\ (\mu_1+\sigma_1,\sigma_1),\ldots, & (\mu_q+\sigma_q,\sigma_q)\end{array};ze^{-t}\right].
\end{aligned}\tag{11}$$

Taking Mellin transform on both sides of Equation (8) and using the defining integral representation as given in Equation (2), we can write:

$$\Gamma(s)\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda)=M\left[\exp\left(-at-\frac{b}{t^{\lambda}}\right)p^{\Psi^*}q\left[\begin{array}{ll}(\lambda_1,\ \rho_1),\ldots, & \left(\lambda_p,\ \rho_p\right)\\ (\mu_1,\sigma_1),\ldots, & (\mu_p,\sigma_p)\end{array}ze^{-t}\right];s\right].\tag{12}$$

Using the derivative property of Mellin transform given by, see [50] (Chapter 10):

$$M[\,u'(y);\ \tau]=\ -(\tau-1)M[u(y);\ \tau-1]\tag{13}$$

we obtain the following equation:

$$\begin{aligned}
&\Gamma(s)\left[a\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda)-\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s-1,a;b,\lambda)\right]=\\
&b\lambda\Gamma(s-\lambda-1)\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s-\lambda-1,a,b)\\
&-\frac{(\lambda_1)_{\rho_1}\ldots(\lambda_p)_{\rho_p}}{(\mu_1)_{\sigma_1}\ldots(\mu_q)_q}z\Gamma(s)\Phi^{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}_{(\lambda_1+\rho_1,\ldots,\lambda_p\rho_p,\mu_1+\sigma_1,\ldots,\mu_q+\sigma_q)}(z,s,a+1;b,\lambda)
\end{aligned}\tag{14}$$

which leads to:

$$\begin{aligned}
&\frac{(\lambda_1)_{\rho_1}\ldots(\lambda_p)_{\rho_p}}{(\mu_1)_{\sigma_1}\ldots(\mu_q)_q}z\Gamma(s)\Phi^{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}_{(\lambda_1+\rho_1,\ldots,\lambda_p\rho_p,\mu_1+\sigma_1,\ldots,\mu_q+\sigma_q)}(z,s,a+1;b,\lambda)=\\
&b\lambda\Gamma(s-\lambda-1)\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s-\lambda-1,a,b)\\
&+\Gamma(s)\left[\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s-1,a;b,\lambda)-a\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda)\right].
\end{aligned}\tag{15}$$

After some simple modifications, one can arrive at the required result of Equation (7). □

Remark 1. *We can obtain similar results for other related functions as listed in Table 1 by considering different parameter values in the resulting corollaries.*

Corollary 1. *λ-Generalized Extended Fermi–Dirac functions have the following representation:*

$$\Gamma(s)\Theta^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1+\rho_1,\dots,\lambda_p+\rho_p,\mu_1+\sigma_1,\dots,,\mu_q\sigma_q}(x,s,a+1;b,\lambda) = \frac{e^x\cdot(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c}\Gamma(s)\left[a\Theta^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s,a;b,\lambda)-\Theta^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-1,a;b,\lambda)\right]\\ -b\lambda\Gamma(s-\lambda-1)\Theta^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-\lambda-1,a,b,\lambda)\end{array}\right] \tag{16}$$

and λ-Generalized Extended Bose–Einstein functions have the following representation:

$$\begin{aligned}\Gamma(s)\Psi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1+\rho_1,\dots,\lambda_p+\rho_p,\mu_1+\sigma_1,\dots,,\mu_q+\sigma_q}(x,s,a+1;b,\lambda)\\ =\frac{e^x.(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\Big[b\lambda\Gamma(s-\lambda-1)\Psi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-\lambda-1,a,b,\lambda)\\ +\Gamma(s)\left[\Psi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-1,a;b,\lambda)-a\Psi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s,a;b,\lambda)\right]\Big].\end{aligned} \tag{17}$$

Proof. The results follow directly from Equation (7) upon replacing $\mathbf{z} \longrightarrow \pm\mathbf{e}^{-\mathbf{x}}$ and using the parallel case given in row 2 and column 2 of Table 1. □

Corollary 2. *λ-Generalized Fermi–Dirac functions have the following representation:*

$$\Gamma(s)\Theta^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1+\rho_1,\dots,\lambda_p+\rho_p,\mu_1+\sigma_1,\dots,,\mu_q+\sigma_q}(x,s,2;b,\lambda) = \frac{e^x.(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c}b\lambda\Gamma(s-\lambda-1)F^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-\lambda-1,b,\lambda)\\ +\Gamma(s)\left[F^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-1;b,\lambda)-F^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s;b,\lambda)\right]\end{array}\right] \tag{18}$$

and λ-Generalized Bose–Einstein functions have the following representation:

$$\begin{aligned}\Gamma(s)\Psi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1+\rho_1,\dots,\lambda_p+\rho_p,\mu_1+\sigma_1,\dots,,\mu_q+\sigma_q}(x,s,2;b,\lambda)\\ =\frac{e^x.(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\Big[b\lambda\Gamma(s-\lambda-1)B^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-\lambda-1,b,\lambda)\\ +\Gamma(s)\left[B^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s-1;b,\lambda)-B^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(x,s;b,\lambda)\right]\Big].\end{aligned} \tag{19}$$

Proof. The results follow directly from Equation (7) upon replacing $z \longrightarrow \pm e^{-x}; a \longrightarrow 1$ and taking the item from Table 1 corresponding to these parameter values. □

Corollary 3. *λ-Generalized Polylogarithm functions have the following representation:*

$$\Gamma(s)Li^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1+\rho_1,\dots,\lambda_p+\rho_p,\mu_1+\sigma_1,\dots,,\mu_q+\sigma_q}(z,s,2;b,\lambda) = \frac{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{z.(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c}b\lambda\Gamma(s-\lambda-1)Li^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(z,s-\lambda-1,b,\lambda)\\ +\Gamma(s)\left[Li^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(z,s-1;b,\lambda)-Li^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,,\mu_q}(z,s;b,\lambda)\right]\end{array}\right]. \tag{20}$$

Proof. The result follows directly from Equation (7) upon replacing $a \longrightarrow 1$ and considering the specific case of these parameter values from Table 1. □

Corollary 4. *λ-Generalized Hurwitz zeta functions have the following representation:*

$$\begin{aligned}\Gamma(s)\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1+\rho_1,\ldots,\lambda_p+\rho_p,\mu_1+\sigma_1,\ldots,\mu_q+\sigma_q}(s,a+1;b,\lambda)\\ = \frac{(\mu_1)_{\sigma_1}\cdots\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots\cdots(\lambda_p)_{\rho_p}}\Big[b\lambda\Gamma(s-\lambda-1)\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s-\lambda-1,a;b,\lambda)\\ +\Gamma(s)\Big[\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s-1,a;b,\lambda) - a\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s,a;b,\lambda)\Big]\Big].\end{aligned} \tag{21}$$

Proof. The result follows directly from Equation (7) upon replacing $z \longrightarrow 1$ and in view of the defined item from Table 1 dependable on these parameter values. □

Corollary 5. *λ-Generalized Riemann zeta functions have the following representation:*

$$\begin{aligned}&\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1+\rho_1,\ldots,\lambda_p+\rho_p,\mu_1+\sigma_1,\ldots,\mu_q+\sigma_q}(s,2;b,\lambda)\\ &= \frac{(\mu_1)_{\sigma_1}\cdots\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c} b\lambda\Gamma(s-\lambda-1)\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s-\lambda-1;b,\lambda)\\ +\Gamma(s)\left[\zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s-1;b,\lambda) - \zeta^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(s;b,\lambda)\right]\end{array}\right].\end{aligned} \tag{22}$$

Proof. The result follows directly from Equation (7) upon replacing $z \longrightarrow 1; a \longrightarrow 1$ and with reference to the definite element from Table 1 stable with these parameter values. □

Remark 2. *We can get similar representations for other special cases of these functions by considering different parameter variations in view of Table 1 column-wise.*

Note that by taking $b = 0$ in the above results, we can get the following formulae for unified extended Hurwitz–Lerch zeta functions $\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a;0,\lambda)$:

$$\begin{aligned}&\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1+\rho_1,\ldots,\lambda_p+\rho_p,\mu_1+\sigma_1,\ldots,\mu_q+\sigma_q}(z,s,a+1)\\ &= \frac{(\mu_1)_{\sigma_1}\cdots\cdots(\mu_q)_{\sigma_q}}{z.(\lambda_1)_{\rho_1}\cdots\cdots(\lambda_p)_{\rho_p}}\left[\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s-1,a) - a\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(z,s,a)\right].\end{aligned} \tag{23}$$

Next, by selecting $p-1 = q = 0; \lambda_1 = \mu \neq 0;\ 1 = \rho_1,$ in the above results, we can get the following result for unified Hurwitz–Lerch zeta functions $\Phi^*_\mu(z,s,a)$:

$$\int_0^\infty \frac{t^{s-1}e^{-(a+1)t}}{(1-ze^{-t})^{\mu+1}}dt = \Gamma(s)\Phi^*_{\mu+1}(z,s,a+1) = \frac{\Gamma(s)}{\mu z}\left[\Phi^*_\mu(z,s-1,a) - a\Phi^*_\mu(z,s,a)\right]. \tag{24}$$

Next, we note that by taking $\mu = 1$, we get for Hurwitz–Lerch zeta functions:

$$\int_0^\infty \frac{t^{s-1}e^{-(a+1)t}}{(1-ze^{-t})^2}dt = \Gamma(s)\Phi^*_2(z,s,a+1) = \frac{\Gamma(s)[\Phi(z,s-1,a) - a\Phi(z,s,a)]}{z}. \tag{25}$$

If we consider the same parameter values as above but with $b \neq 0$, then we can find the following new results for the extended Riemann and Hurwitz zeta functions:

$$\begin{aligned}\mu z\Gamma(s)\Theta^{\lambda}_{\mu+1}(z,s,a+1,b)&\\ &=b\lambda\Gamma(s-\lambda-1)\Theta^{\lambda}_{\mu}(z,s-\lambda-1,a,b)\\ &+\Gamma(s)\Big[\Theta^{\lambda}_{\mu}(z,s-1,a,b)-a\Theta^{\lambda}_{\mu}(z,s,a,b)\Big]\end{aligned} \tag{26}$$

$$\begin{aligned}\int_0^{\infty}\tfrac{t^{s-1}e^{-(a+1)t-\frac{b}{t}}}{(1-ze^{-t})^2}dt&=\Gamma(s)\Phi^*_2(z,s,a+1;b,1)\\ &=\tfrac{b\Gamma(s-2)}{z}\Phi_b(z,s-2,a)+\tfrac{\Gamma(s)}{z}[\Phi_b(z,s-1,a)-a\Phi_b(z,s,a)]\end{aligned} \tag{27}$$

$$\int_0^{\infty}\frac{t^{s-1}e^{-2t-\frac{b}{t}}}{(1-e^{-t})^2}dt=\Gamma(s)\zeta^*_2(s,2;b,1)=b\Gamma(s-2)\zeta_b(s-2)+\Gamma(s)[\zeta_b(s-1)-\zeta_b(s)]. \tag{28}$$

3. Some applications of the difference equation

In this section, we consider some interesting special cases of difference equations. On one side, these are useful to know the values of generalized Hurwitz zeta functions $\Phi^*_{\mu}(z,s,a)$ in terms of zeta functions, and on the other, they lead to the computation of some elementary integrals that are nontrivial to obtain for small values of $\mu=2,3,4,5$ and the large values of s.

Taking $\mu=2$ and $a\longrightarrow a+1$ in Equation (24), we get:

$$\int_0^{\infty}\tfrac{t^{s-1}e^{-(a+2)t}}{(1-ze^{-t})^3}dt=\Gamma(s)\Phi^*_3(z,s,a+2)=\tfrac{\Gamma(s)}{1.2.z}[\Phi^*_2(z,s-1,a+1)-(a+1)\Phi^*_2(z,s,a+1)]. \tag{29}$$

Next, making use of Equation (25) on the right-hand side of the above Equation (29) leads to the following form of $\Phi^*_3(z,s,a+2)$ in terms of the Hurwitz–Lerch zeta function:

$$\begin{aligned}\int_0^{\infty}\tfrac{t^{s-1}e^{-(a+2)t}}{(1-ze^{-t})^3}dt&=\tfrac{\Gamma(s)}{1.2.z^2}[\Phi(z,s-2,a)-a\Phi(z,s-1,a)-(a+1)\Phi(z,s-1,a)+a(a+1)\Phi(z,s,a)]\\ &=\tfrac{\Gamma(s)}{1.2.z^2}[\Phi(z,s-2,a)-(2a+1)\Phi(z,s-1,a)+a(a+1)\Phi(z,s,a)].\end{aligned} \tag{30}$$

Now we consider some interesting special cases of the above Equation (30).

For $z=1$, it leads to the following representation in terms of the Hurwitz zeta function:

$$\int_0^{\infty}\frac{t^{s-1}e^{-(a+2)t}}{(1-e^{-t})^3}dt=\frac{\Gamma(s)}{1.2}[\zeta(s-2,a)-(2a+1)\zeta(s-1,a)+a(a+1)\zeta(s,a)]; \tag{31}$$

$$\int_0^{\infty}\frac{t^{s-1}e^{-(a+2)t}}{(1-e^{-t})^3}dt=\frac{\Gamma(s)}{1.2}[\zeta(s-2,a)-(2a+1)\zeta(s-1,a)+a(a+1)\zeta(s,a)]. \tag{32}$$

For $a=1$, we get the following representation in terms of the polylogarithm function:

$$\int_0^{\infty}\frac{t^{s-1}e^{-3t}}{(1-ze^{-t})^3}dt=\frac{\Gamma(s)}{1.2.z^3}[Li_{s-2}(z)-3Li_{s-1}(z)+2Li_s(z)]. \tag{33}$$

For $a=1,\ z=1$, it leads to the following relation in terms of the zeta function:

$$\int_0^{\infty}\frac{t^{s-1}e^{-3t}}{(1-e^{-t})^3}dt=\frac{\Gamma(s)}{1.2}[2\zeta(s)+\zeta(s-2)-3\zeta(s-1)];(s\neq 1,2,3). \tag{34}$$

For $s=4$ in Equation (34), we get the following integral:

$$\int_0^{\infty}\frac{t^{3}e^{-3t}}{(1-e^{-t})^3}dt=\frac{\Gamma(4)}{1.2}[2\zeta(4)+\zeta(2)-3\zeta(3)]. \tag{35}$$

Similarly, by considering different values of s, we can produce the following Tables 2 and 3 of values. These computations show that Mathematica is unable to compute the involved integral on a commonly available computer for the large values of s, but it can be done using these new difference equations. For small values of s, our results are 100% accurate with the direct computed results

Table 2. Computation of $\int_0^\infty \frac{t^{s-1}e^{-3t}}{(1-e^{-t})^3}dt$.

s	Direct Evaluation by Mathematica	Using difference Equation (34)
4	0.610229	0.610229
30	4.29669×10^{16}	4.29669×10^{16}
40	1.67783×10^{27}	1.67783×10^{27}
45	8.998×10^{32}	8.998×10^{32}
46	1.3497×10^{34}	1.3497×10^{34}
48	3.24227×10^{36}	3.24227×10^{36}
48.5	1.29353×10^{37}	1.29353×10^{37}
48.9	3.92781×10^{37}	3.92781×10^{37}
49	5.18762×10^{37}	5.18762×10^{37}
52	2.40071×10^{41}	2.40071×10^{41}
56	2.426×10^{46}	2.426×10^{46}
160	Unable to compute	1.349×10^{206}
220	Unable to compute	1.121×10^{314}
400	Unable to compute	2.269×10^{675}

Putting $\mu = 3,\ a \longrightarrow a+2$ in Equation (24), we get:

$$\begin{aligned} &\int_0^\infty \frac{t^{s-1}e^{-(a+3)t}}{(1-ze^{-t})^{3+1}}dt = \Gamma(s)\Phi^*_{3+1}(z,s,a+3) \\ &= \frac{\Gamma(s)}{3.z}[\Phi^*_3(z,s-1,a+2) - (a+2)\Phi^*_3(z,s,a+2)]. \end{aligned} \tag{36}$$

Next, combining the above two results (Equations (30) and (36)), we get the following representation in terms of the Hurwitz–Lerch zeta function

$$\begin{aligned} &\int_0^\infty \frac{t^{s-1}e^{-(a+3)t}}{(1-ze^{-t})^4}dt \\ &= \frac{\Gamma(s)}{1.2.3.z^3}\left[\begin{array}{c} \Phi(z,s-3,a) - 3(a+1)\Phi(z,s-2,a) + \left(3a^2+6a+2\right)\Phi(z,s-1,a) \\ -a(a+1)(a+2)\Phi(z,s,a) \end{array}\right]. \end{aligned} \tag{37}$$

Some interesting special cases: For $z = 1$, it leads to the following representation in terms of the Hurwitz zeta function:

$$\begin{aligned} &\int_0^\infty \frac{t^{s-1}e^{-(a+3)t}}{(1-e^{-t})^4}dt = \Gamma(s)\zeta^*_4(s,a+3) \\ &= \frac{\Gamma(s)}{1.2.3}\left[\begin{array}{c} \zeta(s-3,a) - 3(a+1)\zeta(s-2,a) + \left(3a^2+6a+2\right)\zeta(s-1,a) \\ -a(a+1)(a+2)\zeta(s,a) \end{array}\right]. \end{aligned} \tag{38}$$

For $a = 1$, we get the following representation in terms of the polylogarithm function:

$$\begin{aligned} &\int_0^\infty \frac{t^{s-1}e^{-4t}}{(1-ze^{-t})^4}dt = \Gamma(s)\Phi^*_4(z,s,4) \\ &= \frac{\Gamma(s)}{1.2.3.z^4}[Li_{s-3}(z) - 6Li_{s-2}(z) + 11Li_{s-1}(z) - 6Li_s(z)]. \end{aligned} \tag{39}$$

For $a = 1,\ z = 1$, it leads to the following relation in terms of the zeta function:

$$\begin{gathered}\int_0^\infty \frac{t^{s-1}e^{-4t}}{(1-e^{-t})^4}dt = \Gamma(s)\zeta_4^*(s,4)\\ = \frac{\Gamma(s)}{1.2.3}\left[\ \zeta(s-3)-6\zeta(s-2)+11\zeta(s-1)-6\zeta(s)\ \right];\ (s \neq 1,2,3,4).\end{gathered} \tag{40}$$

For s = 5, we have:

$$\begin{gathered}\int_0^\infty \frac{t^{4}e^{-4t}}{(1-e^{-t})^4}dt = \Gamma(5)\zeta_4^*(4,4)\\ = \frac{\Gamma(5)}{1.2.3}\left[\ \zeta(2)-6\zeta(3)+11\zeta(4)-6\zeta(5)\ \right].\end{gathered} \tag{41}$$

Table 3. Computation of $\int_0^\infty \frac{t^{s-1}e^{-4t}}{(1-e^{-t})^4}dt$.

s	Direct Evaluation by Mathematica	By using difference Equation (40)
90	1.07719×10^{82}	1.07719×10^{82}
100	5.8077×10^{95}	5.8077×10^{95}
140	Unable to compute	1.78191×10^{156}
160	Unable to compute	1.37955×10^{186}

Similarly, putting $\mu = 4,\ a \longrightarrow a+3$ in Equation (24), we get:

$$\begin{gathered}\int_0^\infty \frac{t^{s-1}e^{-(a+4)t}}{(1-ze^{-t})^{4+1}}dt = \Gamma(s)\Phi_{4+1}^*(z,s,a+4)\\ = \frac{\Gamma(s)}{4.z}\left[\Phi_4^*(z,s-1,a+3)-(a+3)\Phi_4^*(z,s,a+3)\right].\end{gathered} \tag{42}$$

Next, combining the above two results (Equations (38) and (42)), we get:

$$\begin{gathered}\int_0^\infty \frac{t^{s-1}e^{-(a+4)t}}{(1-ze^{-t})^5}dt\\ = \frac{\Gamma(s)}{1.2.3.4.z^4}\left[\begin{array}{c}\Phi(z,s-3,a)+\left(6a^2+18a+11\right)\Phi(z,s-2,a)\\ -\left(4a^3+18a^2+22a+6\right)\Phi(z,s-2,a)+a(a+1)(a+2)(a+3)\Phi(z,s,a)\end{array}\right].\end{gathered} \tag{43}$$

Some interesting special cases:
For z = 1:

$$\begin{gathered}\int_0^\infty \frac{t^{s-1}e^{-(a+4)t}}{(1-e^{-t})^5}dt = \Gamma(s)\zeta_5^*(s,a+4)\\ = \frac{\Gamma(s)}{1.2.3.4}\left[\begin{array}{c}\zeta(s-4,a)-2(2a+3)\zeta(s-3,a)+\left(6a^2+18a+11\right)\zeta(s-2,a)\\ -\left(4a^3+18a^2+22a+6\right)\zeta(s-2,a)+a(a+1)(a+2)(a+3)\zeta(s,a)\end{array}\right].\end{gathered} \tag{44}$$

For a = 1:

$$\begin{gathered}\int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-ze^{-t})^5}dt = \Gamma(s)\Phi_5^*(z,s,5)\\ = \frac{\Gamma(s)}{1.2.3.4.z^5}\left[\begin{array}{c}Li_{s-4}(z)-10Li_{s-3}(z)+37Li_{s-2}(z)\\ -50Li_{s-2}(z)+24Li_s(z)\end{array}\right].\end{gathered} \tag{45}$$

For $a = 1,\ z = 1$:

$$\begin{gathered}\int_0^\infty \frac{t^{s-1}e^{-5t}}{(1-e^{-t})^5}dt = \Gamma(s)\zeta_5^*(s,5)\\ = \frac{\Gamma(s)}{1.2.3.4}\left[\begin{array}{c}\zeta(s-4)-10\zeta(s-3)+37\zeta(s-2)\\ -50\zeta(s-2)+24\zeta(s)\end{array}\right];(s \neq 1,2,3,4,5).\end{gathered} \tag{46}$$

Now put $s = 6$:

$$\begin{gathered}\int_0^\infty \frac{t^{5}e^{-5t}}{(1-e^{-t})^5}dt = \Gamma(6)\zeta_5^*(6,5)\\ = \frac{\Gamma(6)}{1.2.3.4}\left[\begin{array}{c}\zeta(2)-10\zeta(3)+37\zeta(4)\\ -50\zeta(5)+24\zeta(6)\end{array}\right].\end{gathered} \tag{47}$$

Continuing in this way, by putting $\mu = 5,\ a \longrightarrow a+4$ in Equation (24), we get:

$$\begin{aligned} &\int_0^\infty \frac{t^{s-1}e^{-(a+5)t}}{(1-ze^{-t})^{5+1}}dt = \Gamma(s)\Phi^*_{5+1}(z,s,a+5) \\ &= \frac{\Gamma(s)}{5.z}[\Phi^*_5(z,s-1,a+4) - (a+4)\Phi^*_5(z,s,a+4)]. \end{aligned} \tag{48}$$

Next, combining the above two results of Equations (43) and (48), we can get $\Phi^*_6(z,s,a+4)$ in terms of Hurwitz–Lerch zeta functions. Similarly, for nonzero values of z, for example, $z = 0.3; a = 1$; $\mu = 3$ in Equation (33) we have:

$$\int_0^\infty \frac{\left(e^{-3\ t}t^4\right)}{\left(1-\frac{3\ e^{-t}}{10}\right)^3}dt = \ 0.125061 \tag{49}$$

$$\frac{\Gamma(5)(Li_3(0.3) -\ 3\ Li_4(0.3) +\ 2\ Li_5(0.3))}{2\ (0.3)^3} = \ 0.125061 \tag{50}$$

and so on and so forth.

4. Integrals of products of the family of λ-Generalized Hurwitz–Lerch zeta functions

By means of the basic Parseval's identity of Mellin transform [50] (Chapter 10) and difference equations obtained in Section 2, we can get the following integral formulae in view of Equation (2) and column 3 of Table 1. For example, for the generalized Hurwitz–Lerch zeta functions $\Theta^\lambda_\mu(z,s,a,b)$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Theta^\lambda_\mu(z,s,a,b)\Theta^\lambda_\delta(z,w-s,a,b) = \int_0^\infty \frac{t^{w-1}e^{-2at-\frac{2b}{t^\lambda}}}{(1-ze^{-t})^{\mu+\delta}}dt \tag{51}$$

that leads to the following by simply replacing $\mu \longrightarrow \mu+\delta-1$ in Equation (26):

$$\Gamma(\omega)\Theta^\lambda_{\mu+\delta}(z,\omega,2a,2b) = \frac{1}{z(\mu-1+\delta)}\begin{bmatrix} 2b\lambda\Gamma(\omega-\lambda-1)\Theta^\lambda_{\mu-1+\delta}(z,\omega-\lambda-1,2a-1,2b) \\ +\Gamma(\omega)\Theta^\lambda_{\mu-1+\delta}(z,\omega-1,2a-1,2b) \\ -(2a-1)\Gamma(\omega)\Theta^\lambda_{\mu-1+\delta}(z,w,2a-1,2b) \end{bmatrix}. \tag{52}$$

Therefore, we get the following new integral formulae for other related cases given in column 3 of Table 1 and Equations (51) and (52):

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Theta^\lambda_\mu(z,s,a,b)\Theta^\lambda_\delta(z,w-s,a,b) = \int_0^\infty \frac{t^{w-1}e^{-2at-\frac{2b}{t^\lambda}}}{(1+e^{-x}e^{-t})^{\mu+\delta}}dt \\ &= \frac{e^x}{\mu-1+\delta}\begin{bmatrix} \Gamma(\omega)\Big[(2a-1)\Theta^\lambda_{\mu-1+\delta}(x,w,2a-1,2b) - \Theta^\lambda_{\mu-1+\delta}(x,\omega-1,2a-1,2b)\Big] \\ -2b\lambda\Gamma(\omega-\lambda-1)\Theta^\lambda_{\mu-1+\delta}(x,\omega-\lambda-1,2a-1,\ 2b) \end{bmatrix} \end{aligned} \tag{53}$$

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)F^\lambda_\mu(x,s,b)F^\lambda_\delta(x,w-s,b) = \int_0^\infty \frac{t^{w-1}e^{-2t-\frac{2b}{t^\lambda}}}{(1+e^{-x}e^{-t})^{\mu+\delta}}dt \\ &= \frac{e^x}{\mu-1+\delta}\begin{bmatrix} \Gamma(\omega)\Big[F^\lambda_{\mu-1+\delta}(x,w,2b) - F^\lambda_{\mu-1+\delta}(x,\omega-1,2b)\Big] \\ -2b\lambda\Gamma(\omega-\lambda-1)F^\lambda_{\mu-1+\delta}(x,\omega-\lambda-1,\ 2b) \end{bmatrix} \end{aligned} \tag{54}$$

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Psi^\lambda_\mu(z,s,a,b)\Psi^\lambda_\delta(z,w-s,a,b) = \int_0^\infty \frac{t^{w-1}e^{-2at-\frac{2b}{t^\lambda}}}{(1-e^{-x}e^{-t})^{\mu+\delta}}dt \\ &= \frac{e^x}{\mu-1+\delta}\begin{bmatrix} 2b\lambda\Gamma(\omega-\lambda-1)\Psi^\lambda_{\mu-1+\delta}(x,\omega-\lambda-1,2a-1,\ 2b) \\ +\Gamma(\omega)\Big[\Psi^\lambda_{\mu-1+\delta}(x,\omega-1,2a-1,2b)\Big] - (2a-1)\Psi^\lambda_{\mu-1+\delta}(x,w,2a-1,2b) \end{bmatrix} \end{aligned} \tag{55}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)B_{\mu}^{\lambda}(x,s,b)B_{\delta}^{\lambda}(x,w-s,b)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t-\frac{2b}{t^{\lambda}}}}{(1-e^{-x}e^{-t})^{\mu+\delta}}dt\\&=\tfrac{e^x}{\mu-1+\delta}\left[\begin{array}{c}2b\lambda\Gamma(\omega-\lambda-1)B_{\mu-1+\delta}^{\lambda}(x,\omega-\lambda-1,\ 2b)\\+\Gamma(\omega)\left[B_{\mu-1+\delta}^{\lambda}(x,\omega-1,2b)-B_{\mu-1+\delta}^{\lambda}(x,w,2b)\right]\end{array}\right]\end{aligned} \tag{56}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)Li_{\mu}^{\lambda}(z,s,b)Li_{\delta}^{\lambda}(z,w-s,b)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t-\frac{2b}{t^{\lambda}}}}{(1-ze^{-t})^{\mu+\delta}}dt\\&=\tfrac{1}{(\mu-1+\delta)z}\left[\begin{array}{c}2b\lambda\Gamma(\omega-\lambda-1)Li_{\mu-1+\delta}^{\lambda}(x,\omega-\lambda-1,\ 2b)\\+\Gamma(\omega)\left[Li_{\mu-1+\delta}^{\lambda}(x,\omega-1,2b)-Li_{\mu-1+\delta}^{\lambda}(x,w,2b)\right]\end{array}\right]\end{aligned} \tag{57}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\zeta_{\mu}^{\lambda}(s,a;b)\zeta_{\delta}^{\lambda}(w-s,a;b)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2at-\frac{2b}{t^{\lambda}}}}{(1-e^{-t})^{\mu+\delta}}dt\\&=\tfrac{1}{(\mu-1+\delta)}\left[\begin{array}{c}2b\lambda\Gamma(\omega-\lambda-1)\zeta_{\mu-1+\delta}^{\lambda}(\omega-\lambda-1,2a-1,\ 2b)\\+\Gamma(\omega)\left[\zeta_{\mu-1+\delta}^{\lambda}(w-1,2a-1,2b)-(2a-1)\zeta_{\mu-1+\delta}^{\lambda}(w,2a-1,2b)\right]\end{array}\right]\end{aligned} \tag{58}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\zeta_{\mu}^{\lambda}(s;b)\zeta_{\delta}^{\lambda}(w-s;b)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t-\frac{2b}{t^{\lambda}}}}{(1-e^{-t})^{\mu+\delta}}dt\\&=\tfrac{1}{(\mu-1+\delta)}\left[\begin{array}{c}2b\lambda\Gamma(\omega-\lambda-1)\zeta_{\mu-1+\delta}^{\lambda}(\omega-\lambda-1,2b)\\+\Gamma(\omega)\left[\zeta_{\mu-1+\delta}^{\lambda}(w-1,2b)-\zeta_{\mu-1+\delta}^{\lambda}(w,2b)\right]\end{array}\right].\end{aligned} \tag{59}$$

Next, for $b=0$, we can get the following new formulae in view of column 8 of Table 1 and Equation (53):

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Phi_{\mu}^{*}(z,s,a)\ \Phi_{\delta}^{*}(z,w-s,a)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2at}}{(1-ze^{-t})^{\mu+\delta}}dt\\&=\tfrac{\Gamma(\omega)}{(\mu-1+\delta)z}\left[\Phi_{\mu-1+\delta}^{*}\ (z,\omega-1,2a-1)-(2a-1)\Phi_{\mu-1+\delta}^{*}(z,w,2a-1)\right]\end{aligned} \tag{60}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Theta_{\mu}^{*}(x,s,a)\Theta_{\delta}^{*}(x,w-s,a)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2at}}{(1+e^{-x}e^{-t})^{\mu+\delta}}dt\\&=\tfrac{e^x\Gamma(\omega)}{\mu-1+\delta}\left[(2a-1)\Theta_{\mu-1+\delta}^{*}(x,w,2a-1)-\Theta_{\mu-1+\delta}^{*}(x,\omega-1,2a-1)\right]\end{aligned} \tag{61}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)F_{\mu}^{*}(x,s)F_{\delta}^{*}(x,w-s)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t}}{(1+e^{-x}e^{-t})^{\mu+\delta}}dt\\&=\tfrac{e^x\Gamma(\omega)}{\mu-1+\delta}\left[F_{\mu-1+\delta}^{*}(x,w)-F_{\mu-1+\delta}^{*}(x,\omega-1)\right]\end{aligned} \tag{62}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Psi_{\mu}^{*}(x,s,a)\Psi_{\delta}^{*}(x,w-s,a)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2at}}{(1-e^{-x}e^{-t})^{\mu+\delta}}dt\\&=\tfrac{e^x\Gamma(\omega)}{\mu-1+\delta}\left[\Psi_{\mu-1+\delta}^{*}(x,\omega-1,2a-1)-(2a-1)\Psi_{\mu-1+\delta}^{*}(x,w,2a-1)\right]\end{aligned} \tag{63}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)B_{\mu}^{*}(x,s)B_{\delta}^{*}(x,w-s)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t}}{(1-e^{-x}e^{-t})^{\mu+\delta}}dt\\&=\tfrac{e^x\Gamma(\omega)}{\mu-1+\delta}\left[B_{\mu-1+\delta}^{*}(x,\omega-1)-B_{\mu-1+\delta}^{*}(x,w)\right]\end{aligned} \tag{64}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)Li_{\mu}^{*}(z,s)\ Li_{\delta}^{*}(z,w-s)=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t}}{(1-ze^{-t})^{\mu+\delta}}dt\\&=\tfrac{\Gamma(\omega)}{(\mu-1+\delta)z}\left[Li_{\mu-1+\delta}^{*}\ (z,\omega-1)-Li_{\mu-1+\delta}^{*}(z,w)\right]\end{aligned} \tag{65}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\zeta_{\mu}^{*}(s,a)\ \zeta_{\delta}^{*}(w-s,a)ds=\int_0^{\infty}\tfrac{t^{w-1}e^{-2at}}{(1-e^{-t})^{\mu+\delta}}dt\\&=\tfrac{\Gamma(\omega)}{(\mu-1+\delta)}\left[\zeta_{\mu-1+\delta}^{*}\ (\omega-1,a)-(2a-1)\zeta_{\mu-1+\delta}^{*}(w,a)\right]\end{aligned} \tag{66}$$

$$\begin{aligned}&\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\zeta_{\mu}^{*}(s)\ \zeta_{\delta}^{*}(w-s)ds=\int_0^{\infty}\tfrac{t^{w-1}e^{-2t}}{(1-e^{-t})^{\mu+\delta}}dt\\&=\tfrac{\Gamma(\omega)}{(\mu-1+\delta)}\left[\zeta_{\mu-1+\delta}^{*}\ (\omega-1)-\zeta_{\mu-1+\delta}^{*}(w)\right].\end{aligned} \tag{67}$$

Next, if we consider $\delta = \mu = \lambda = 1; b \neq 0$ in Equation (53), we can obtain the following new integral formulae in view of column 6 of Table 1:

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\Phi_b(z,s,a)\ \Phi_b(z,w-s,a)ds &= \int_0^\infty \frac{t^{w-1}e^{-2at-\frac{2b}{t}}}{(1-ze^{-t})^2}dt \\ &= \frac{2b\Gamma(\omega-2)\Phi_{2b}(z,\omega-2,2a-1)+\Gamma(\omega)[\Phi_{2b}(z,\omega-1,2a-1)-(2a-1)\Phi_{2b}(z,w,2a-1)]}{z}\end{aligned} \tag{68}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\mathrm{Li}_b(z,s)\ Li_b(z,w-s)ds &= \int_0^\infty \frac{t^{w-1}e^{-2t-\frac{2b}{t}}}{(1-ze^{-t})^2}dt \\ &= \frac{2b\Gamma(\omega-2)\mathrm{Li}_{2b}(z,\omega-2)+\Gamma(\omega)[\mathrm{Li}_{2b}(z,\omega-1)-\mathrm{Li}_{2b}(z,w)]}{z}\end{aligned} \tag{69}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\zeta_b(s,a)\ \zeta_b(w-s,a)ds &= \int_0^\infty \frac{t^{w-1}e^{-2at-\frac{2b}{t}}}{(1-e^{-t})^2}dt \\ = 2b\Gamma(\omega-2)\zeta_{2b}(\omega-2,2a-1) &+ \Gamma(\omega)[\zeta_{2b}(\omega-1,2a-1)-(2a-1)\zeta_{2b}(w,2a-1)]\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(w-s)\zeta_b(s)\ \zeta_b(w-s)ds &= \int_0^\infty \frac{t^{w-1}e^{-2t-\frac{2b}{t}}}{(1-e^{-t})^2}dt \\ &= 2b\Gamma(\omega-2)\zeta_{2b}(\omega-2)+\Gamma(\omega)[\zeta_{2b}(\omega-1)-\zeta_{2b}(w)]\end{aligned}. \tag{71}$$

Similarly, by considering the different parameter values consistent with the results obtained in Section 2, one can obtain more integral formulae for the family of zeta and associated functions. One model is the following by means of Theorem 1:

$$\begin{aligned}&\frac{1}{2\pi \mathrm{i}}\int_{\mathrm{c-i\infty}}^{\mathrm{c+i\infty}}\Gamma(\mathrm{s})\Gamma(\mathrm{w-s})\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,s,a,b};\lambda)\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,w-s,a,b};\lambda)\mathrm{ds} \\ &= \frac{(\mu_1)_{\sigma_1}\cdots\cdots(\mu_q)_{\sigma_q}}{\mathrm{z}(\lambda_1)_{\rho_1}\cdots\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c} 2\mathrm{b}\lambda\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}\Gamma(\mathrm{w}-\lambda-1)(\mathrm{z,w}-\lambda-1,2\mathrm{a}-1,2\mathrm{b}) \\ +\Gamma(\omega)\left[\begin{array}{c}\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,w}-1,2\mathrm{a}-1;2\mathrm{b},\lambda)- \\ (2\mathrm{a}-1)\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,w},2\mathrm{a}-1;2\mathrm{b},\lambda)\end{array}\right]\end{array}\right]\end{aligned} \tag{72}$$

and for $b = 0$, it leads to:

$$\begin{aligned}&\frac{1}{2\pi \mathrm{i}}\int_{\mathrm{c-i\infty}}^{\mathrm{c+i\infty}}\Gamma(\mathrm{s})\Gamma(\mathrm{w-s})\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,s,a})\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,w-s,a})\mathrm{ds} \\ &= \frac{\Gamma(\omega)(\mu_1)_{\sigma_1}\cdots\cdots(\mu_q)_{\sigma_q}}{\mathrm{z}(\lambda_1)_{\rho_1}\cdots\cdots(\lambda_p)_{\rho_p}}\left[\begin{array}{c}\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,w}-1,2\mathrm{a}-1;2\mathrm{b},\lambda)- \\ (2\mathrm{a}-1)\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}(\mathrm{z,w},2\mathrm{a}-1;2\mathrm{b},\lambda)\end{array}\right].\end{aligned} \tag{73}$$

5. Discussion and Future Directions

In this study, we obtained some recurrence relations for the newly defined family of the λ-generalized Hurwitz–Lerch zeta functions using the familiar Mellin transforms. These relations proved valuable to acquire new integral formulae involving the family of zeta functions. The outcomes were also confirmed with the previous obtained results as special cases. It is remarkable that the recurrence relations obtained in this research work are worthwhile to achieve simple relations such as Equations (34) and (40) that express special cases of λ-generalized Hurwitz–Lerch zeta functions in terms of Riemann zeta functions, so that we can evaluate the values of these functions. By following the method, we can obtain significant new results by considering the further specific values of the involved parameters. This is useful for the further analysis of these functions by plotting the graphs and deriving different series and asymptotic representations, etc. This work is in progress and would be a part of some future research.

λ-generalized Hurwitz–Lerch zeta functions analytically generalize the functions of the zeta family and offer consideration for some further presumable new members of this family that are not discussed in the literature. This aspect is most suitable for attaining new consequences from one key result. Our foremost results produce simultaneously important new results for a class of well-studied functions by applying the new difference equations. The Bose–Einstein and Fermi–Dirac functions are of fundamental importance in quantum statistics that contracts by means of two specific categories of spin symmetry, that is, fermions and bosons. Fitting together these functions here with the λ-generalized Hurwitz–Lerch zeta functions yields substantial new identities for them that provides clues regarding the forthcoming applications of these difference equations in quantum physics and associated fields. This practice to acquire the outcomes by making use of new difference equations explores the required simplicity that always inspires hope. We have discussed here the direct consequences of our results. It is remarked that the method established in this research is in fact noteworthy for the analysis and study of these higher transcendental functions.

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