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A Note on the Truncated-Exponential Based Apostol-Type Polynomials

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Abstract: In this paper, we propose to investigate the truncated-exponential-based Apostol-type polynomials and derive their various properties. In particular, we establish the operational correspondence between this new family of polynomials and the familiar Apostol-type polynomials. We also obtain some implicit summation formulas and symmetric identities by using their generating functions. The results, which we have derived here, provide generalizations of the corresponding known formulas including identities involving generalized Hermite-Bernoulli polynomials.

Keywords: truncated-exponential polynomials; monomiality principle; generating functions; Apostol-type polynomials and Apostol-type numbers; Bernoulli, Euler and Genocchi polynomials; Bernoulli, Euler, and Genocchi numbers; operational methods; summation formulas; symmetric identities

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1. Introduction

Operational techniques involving differential operators, which is a consequence of the monomiality principle, provide efficient tools in the theory of conventional polynomial systems and their various generalizations. Steffensen [1] suggested the concept of poweroid, which happens to be behind the idea of monomiality. The principle of monomiality was subsequently reformulated and developed by Dattoli [2]. The strategy underlining this viewpoint is apparently simple, but the outcomes are remarkably deep.

In the theory of the monomiality principle, a polynomial set $p_n(x)$ $(n \in \mathbb{N};\ x \in \mathbb{C})$ is quasi-monomial if there exist two operators $\widehat{M}$ and $\widehat{P}$, which are named the *multiplicative* and the *derivative* operators, respectively, are defined as follows:

$$\widehat{M}\{p_n(x)\} = p_{n+1}(x) \qquad \text{and} \qquad \widehat{P}\{p_n(x)\} = np_{n-1}(x),$$

together with the initial condition given by

$$p_0(x) = 1. \tag{1}$$

The operators $\widehat{M}$ and $\widehat{P}$ satisfy the following commutation relation:

$$[\widehat{M}, \widehat{P}] = \widehat{1}. \tag{2}$$

Thus, clearly, these operators display a Weyl group structure.

The properties of the polynomials $p_n(x)$ can be deduced from those of the operators $\widehat{M}$ and $\widehat{P}$. If $\widehat{M}$ and $\widehat{P}$ possess a differential character, then the polynomials $p_n(x)$ satisfy the following differential equation:

$$\widehat{M}\widehat{P}\{p_n(x)\} = np_n(x). \tag{3}$$

The polynomial family $p_n(x)$ can be explicitly constructed through the action of $\widehat{M^n}$ on $p_0(x)$ as follows:

$$p_n(x) = \widehat{M}^n\{p_0(x)\}. \tag{4}$$

Just as in (1), we shall always assume that $p_0(x) = 1$. In view of the above identity (4), the exponential generating function of $p_n(x)$ can be written in the form:

$$\exp(t\widehat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x)\, \frac{t^n}{n!} \quad (|t| < \infty)\,. \tag{5}$$

We now introduce the truncated-exponential polynomials $e_n(x)$ (see [3]) defined by the following series:

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \tag{6}$$

that is, by the first $n+1$ terms of the Taylor-Maclaurin series for the exponential function e^x. These truncated-exponential polynomials play an important rôle in many problems in optics and quantum mechanics. However, their properties are apparently as widespread as they should be. The truncated-exponential polynomials $e_n(x)$ have been used to evaluate several overlapping integrals associated with the optical mode evolution or for characterizing the structure of the flattened beams. Their usefulness has led to the possibility of appropriately extending their definition. Actually, Dattoli et al. [4] systematically studied the properties of these polynomials.

The definition (6) does lead us to most (if not all) of the properties of the polynomials $e_n(x)$. We note the following representation:

$$e_n(x) = \frac{1}{n!} \int_0^{\infty} e^{-\xi}\, (x+\xi)^n \, \mathrm{d}\xi, \tag{7}$$

which follows readily from the classical gamma-function representation (see, for details, [3]). Consequently, we have the following generating function for the truncated-exponential polynomials $e_n(x)$ (see [4]):

$$\frac{e^{xt}}{1-t} = \sum_{n=0}^{\infty} e_n(x)\, t^n. \tag{8}$$

The definition (6) of $e_n(x)$ can thus be extended to a family of potentially useful truncated-exponential polynomials as follows (see [4]):

$$[2]e_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-2k}}{(n-2k)!}, \tag{9}$$

which obviously possesses a generating function in the form (see [4]):

$$\frac{e^{xt}}{1-t^2} = \sum_{n=0}^{\infty} [2]e_n(x) t^n. \tag{10}$$

We also recall the higher-order truncated-exponential polynomials $[r]e_n(x)$, which are defined by the following series (see [4]):

$$[r]e_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-rk}}{(n-rk)!} \tag{11}$$

and specified by the following generating function (see [4]):

$$\frac{e^{xt}}{1-t^r} = \sum_{n=0}^{\infty} [r]e_n(x)t^n. \tag{12}$$

The special two-variable case of the polynomials in (11) (that is, the case when $r = 2$) are important for applications. Moreover, these polynomials help us derive several potentially useful identities in a simple way and in investigating other novel families of polynomial systems. Actually, Equation (12) enables us to give a new family of polynomials as has been given in Theorem 1.

A 2-variable extension of the truncated-exponential polynomials is given by (see [4])

$$[2]e_n(x,y) = \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{(n-2k)!} \tag{13}$$

and possesses the following generating function (see [4]):

$$\frac{e^{xt}}{1-yt^2} = \sum_{n=0}^{\infty} [2]e_n(x,y)t^n. \tag{14}$$

With a view to introducing a mixed family of polynomials related to the familiar Sheffer sequence, we first consider the 2-variable truncated-exponential polynomials (2VTEP) $e_n^{(r)}(x,y)$ of order r, which are expressed explicitly by (see [5])

$$e_n^{(r)}(x,y) = \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-rk}}{(n-rk)!} \tag{15}$$

and which are generated by

$$\frac{e^{xt}}{1-yt^r} = \sum_{n=0}^{\infty} e_n^{(r)}(x,y)\,\frac{t^n}{n!}. \tag{16}$$

From (8), (10), (12), (14) and (16), we can deduce several special cases of the 2VTEP $e_n^{(r)}(x,y)$, For example, we have

$$e_n^{(2)}(x,y) = [2]e_n(x,y) \quad e_n^{(1)}(x,1) = [r]e_n(x) \quad e_n^{(2)}(x,1) = [2]e_n(x) \qquad \text{and} \qquad e_n^{(1)}(x,1) = e_n(x). \tag{17}$$

As it is shown in [6,7], the 2VTEP $e_n^{(r)}(x,y)$ are quasi-monomial (see also [1,2]) with respect to multiplicative and derivative operators given by

$$\widehat{M}_{e^{(r)}} = (x + ry\partial_y y\partial_x^{r-1}) \tag{18}$$

and

$$\widehat{P}_{e^{(r)}} = \partial_x, \tag{19}$$

where

$$\partial_x = \frac{\partial}{\partial x} \qquad \text{and} \qquad \partial_y = \frac{\partial}{\partial y}.$$

Thus, if we apply the monomiality principle as well as the Equations (18) and (19), we have

$$\widehat{M}_{e^{(r)}}\{e_n^{(r)}(x,y)\} = e_{n+1}^{(r)}(x,y) \tag{20}$$

and

$$\widehat{P}_{e^{(r)}}\{e_n^{(r)}(x,y)\} = ne_{n-1}^{(r)}(x,y), \tag{21}$$

respectively.

The 2VTEP $e_n^{(r)}(x,y)$ are quasi-monomial, so their properties can be derived from those of the multiplicative and derivative operators $\widehat{M}_{e^{(r)}}$ and $\widehat{P}_{e^{(r)}}$, respectively. We thus find that

$$\widehat{M}_{e^{(r)}}\widehat{P}_{e^{(r)}}\{e_n^{(r)}(x,y)\} = ne_n^{(r)}(x,y), \tag{22}$$

which satisfies a differential equation for $e_n^{(r)}(x,y)$ as follows:

$$(r\partial_x + ry\partial_y y\partial_x^r - n)e_n^{(r)}(x,y) = 0. \tag{23}$$

Again, since $e_0^{(r)}(x,y) = 1$, the 2VTEP $e_n^{(r)}(x,y)$ can be explicitly constructed as follows:

$$e_n^{(r)}(x,y) = \widehat{M}^n_{e^{(r)}}\{e_0^{(r)}(x,y)\} = \widehat{M}^n_{e^{(r)}}\{1\}. \tag{24}$$

Equation (24) yields the following generating function of the 2VTEP $e_n^{(r)}(x,y)$:

$$\exp(\widehat{M}_{e^{(r)}}t)\{1\} = \sum_{n=0}^{\infty} e_n^{(r)}(x,y)\frac{t^n}{n!} \quad (|t| < \infty). \tag{25}$$

We can easily verify the following relation between $\widehat{M}_{e^{(r)}}$ and $\widehat{P}_{e^{(r)}}$:

$$[\widehat{P}_{e^{(r)}}, \widehat{M}_{e^{(r)}}] = \widehat{1}. \tag{26}$$

Denoting the classical Bernoulli, Euler and Genocchi polynomials by $B_n(x)$, $E_n(x)$ and $G_n(x)$, respectively, we now recall their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of order α, which are generated by (see, for details, [8–14]; see also [15] as well as the references cited therein):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \qquad (|t| < 2\pi;\ 1^{\alpha} := 1), \tag{27}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \qquad (|t| < \pi;\ 1^{\alpha} := 1) \tag{28}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < \pi;\ \alpha \in \mathbb{N}_0). \tag{29}$$

Obviously, we have

$$B_n^{(1)}(x) =: B_n(x), \quad E_n^{(1)}(x) =: E_n(x) \quad \text{and} \quad G_n^{(1)}(x) =: G_n(x). \tag{30}$$

It is also known that

$$B_n^{(1)}(0) =: B_n, \quad E_n^{(1)}(0) =: E_n \quad \text{and} \quad G_n^{(1)}(0) =: G_n \tag{31}$$

for the Bernoulli, Euler, and Genocchi numbers B_n, E_n and G_n, respectively.

The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order α was introduced by Luo and Srivastava (see [16,17]). Subsequently, the Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ and the Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order α were analogously studied by Luo (see [18–20]; see also [21–27]).

Definition 1. *The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α are defined by*

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\,\frac{t^n}{n!} \tag{32}$$
$$\left(\,|t| < 2\pi \text{ when } \lambda = 1;\ |t| < |\log\lambda| \text{ when } \lambda \neq 1;\ 1^{\alpha} := 1\right)$$

with

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x;1) \qquad \text{and} \qquad B_n^{(\alpha)}(\lambda) = B_n^{(\alpha)}(0;\lambda), \tag{33}$$

where $B_n^{(\alpha)}(\lambda)$ denotes the Apostol-Bernoulli numbers of order α.

Definition 2. *The Apostol-Euler polynomials $E_n^{(\alpha)}(x)$ of order α are defined by*

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\,\frac{t^n}{n!} \tag{34}$$
$$\left(|t| < \pi \text{ when } \lambda = 1;\ |t| < |\log(-\lambda)| < \pi \text{ when } \lambda \neq 1;\ 1^{\alpha} := 1\right)$$

with

$$E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x;1) \quad \text{and} \quad E_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0;\lambda), \tag{35}$$

where $E_n^{(\alpha)}(\lambda)$ denotes the Apostol-Euler numbers of order α.

Definition 3. *The Apostol-Genocchi polynomials $G_n^{(\alpha)}(x)$ of order α are defined by*

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\,\frac{t^n}{n!} \tag{36}$$
$$\left(|t| < \pi \text{ when } \lambda = 1;\ |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1;\ 1^{\alpha} := 1\right) \tag{37}$$

with

$$G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x;1) \qquad \text{and} \qquad G_n^{(\alpha)}(\lambda) = G_n^{(\alpha)}(0;\lambda), \tag{38}$$

where $G_n^{(\alpha)}(\lambda)$ denotes the Apostol-Genocchi numbers of order α.

Remark 1. *Whenever $\lambda = 1$ in (32) and $\lambda = -1$ in (36), the order α of the Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ and the order α of the Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ should obviously be constrained to take on nonnegative integer values (see, for details, [14]). A similar remark would apply also to the order α in all other analogous situations considered in this paper.*

Among other authors, Özden (see [28,29]), Özden et al. ([30]) and Özarslan (see [31,32]) introduced and studied the unification of the above-defined Apostol-type polynomials. In particular, Özden ([29]) defined the unified polynomials $Y_{n,\beta}^{(\alpha)}(x;k,a,b)$ of higher order by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x;k,a,b)\,\frac{t^n}{n!} \tag{39}$$

$$\left(|t| < 2\pi \text{ when } \beta = a;\; |t| < \left|b\log\left(\frac{\beta}{a}\right)\right| \text{ when } \beta \neq a;\; 1^{\alpha} := 1;\; k \in \mathbb{N}_0;\; a,b \in \mathbb{R}\setminus\{0\};\; \alpha,\beta \in \mathbb{C}\right).$$

By putting $x = 0$ in (39), we can readily obtain the corresponding unification $Y_{n,\beta}^{(\alpha)}(k,a,b)$ of the Apostol-type polynomials, which is generated by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(k,a,b)\,\frac{t^n}{n!}. \tag{40}$$

In fact, from Equations (32), (34), (36) and (39), we have

$$Y_{n,\lambda}^{(\alpha)}(x;1,1,1) = B_n^{(\alpha)}(x;\lambda), \tag{41}$$

$$Y_{n,\lambda}^{(\alpha)}(x;0,-1,1) = E_n^{(\alpha)}(x;\lambda) \tag{42}$$

and

$$Y_{n,\lambda}^{(\alpha)}\left(x;1,-\frac{1}{2},1\right) = G_n^{(\alpha)}(x;\lambda). \tag{43}$$

Definition 4. *For an arbitrary real or complex parameter λ, the number $S_k(n,\lambda)$ is given by Zhang and Yang (see [19])*

$$\sum_{k=0}^{\infty} S_k(n,\lambda)\,\frac{t^k}{k!} = \frac{\lambda e^{(n+1)t} - 1}{\lambda e^t - 1}, \tag{44}$$

which, for $\lambda = 1$, yields

$$S_k(n,1) =: S_k(n).$$

Our main objective in this article is to first appropriately combine the 2-variable truncated-exponential polynomials and the Apostol-type polynomials by means of operational techniques. This leads us to the truncated-exponential-based Apostol-type polynomials. By framing these polynomials within the context of the monomiality principle, we then establish their potentially useful properties. We also derive some other properties and investigate several implicit summation formulas for this general family of polynomials by making use of several different analytical techniques on their generating functions. We choose to point out some relevant connections between the truncated-exponential polynomials and the Apostol-type polynomials and thereby derive extensions of several symmetric identities.

2. Two-Variable Truncated-Exponential-Based Apostol-Type Polynomials

We now start with the following theorem arising from the generating functions for the truncated-exponential-based Apostol-type polynomials (TEATP), which are denoted by ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$.

Theorem 1. *The generating function for the 2-variable truncated-exponential-based Apostol-type polynomials ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ is given by*

$$\sum_{n=0}^{\infty} \left({}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\right)\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^r}\right). \tag{45}$$

Proof. Replacing x in the left-hand side and the right-hand side of (39) by the multiplicative operator $\widehat{M}_e^{(r)}$ of the 2VTEATP ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$, we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha \exp(\widehat{M}_e^{(r)}t)\{1\} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(\widehat{M}_e^{(r)};k,a,b)\ \frac{t^n}{n!} \qquad \left(|t| < \left|b\ \log\left(\frac{\beta}{a}\right)\right|\right). \tag{46}$$

Using Equation (25) in the left-hand side and Equation (18) in the right-hand side of Equation (46), we see that

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha \sum_{n=0}^{\infty} e_n^{(r)}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}\left(x+\frac{\phi'(y,\partial_x)}{\phi(y,\partial_x)};k,a,b\right)\ \frac{t^n}{n!}. \tag{47}$$

Now, using Equation (16) in the left-hand side and denoting the resulting 2-variable truncated-exponential-based Apostol-type polynomials (2VTEATP) in the right-hand side by ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$, we have

$${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) = Y_{n,\beta}^{(\alpha)}(\widehat{M}_e^{(r)};k,a,b) = Y_{n,\beta}^{(\alpha)}\left(x+\frac{\phi'(y,\partial_x)}{\phi(y,\partial_x)};k,a,b\right), \tag{48}$$

which yields the assertion (45) of Theorem 1. □

Remark 2. *Equation (48) gives the operational representation involving the unified Apostol-type polynomials* $Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ *and 2VTEATP* ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$.

To frame the 2VTEATP ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ within the context of monomiality principle, we state the following result.

Theorem 2. *The 2VTEATP* ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ *are quasi-monomial with respect to the following multiplicative and derivative operators*:

$$\widehat{M}_{e^{(r)}Y} = x + ry\partial_y y\partial_x^{r-1} + \frac{\alpha k(\beta^b e^t - a^b) - \alpha\beta^b\partial_x e^{\partial_x}}{\partial_x(\beta^b e^t - a^b)} \tag{49}$$

and

$$\widehat{P}_{e^{(r)}Y} = \partial_x. \tag{50}$$

Proof. Let us consider the following expression:

$$\partial_x\left\{e^{xt}\frac{1}{1-yt^r}\right\} = t\{e^{xt}\frac{1}{1-yt^r}\right\}. \tag{51}$$

Differentiating both sides of Equation (45) partially with respect to t, we see that

$$\begin{aligned}\left(x + ry\partial_y y\partial_x^{r-1} + \frac{\alpha k(\beta^b e^t - a^b) - \alpha\beta^b te^t}{t(\beta^b e^t - a^b)}\right)\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha \frac{e^{xt}}{1-yt^r}\\ = \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n+1,\beta}^{(\alpha)}(x,y;k,a,b)\ \frac{t^n}{n!}.\end{aligned} \tag{52}$$

Since

$$\phi(y,t) = \frac{1}{1-yt^r}$$

is an invertible series of t, therefore,

$$\frac{\phi^{'}(y,\partial_x)}{\phi(y,\partial_x)}$$

possesses a power-series expansion in t. Thus, using (51), Equation (52) becomes

$$\begin{aligned}&\left(x + ry\partial_y y\partial_x^{r-1} + \frac{\alpha k(\beta^b e^{\partial_x} - a^b) - \alpha\beta^b\partial_x e^{\partial_x}}{\partial_x(\beta^b e^t - a^b)}\right)\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha}\frac{e^{xt}}{1-yt^r}\\ &= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n+1,\beta}^{(\alpha)}(x,y;k,a,b)\,\frac{t^n}{n!}.\end{aligned} \tag{53}$$

Again, by using the generating function (45) in left-hand side of Equation (53) and rearranging the resulting summation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(x + ry\partial_y y\partial_x^{r-1} + \frac{\alpha k(\beta^b e^{\partial_x} - a^b) - \alpha\beta^b\partial_x e^{\partial_x}}{\partial_x(\beta^b e^t - a^b)}\right)\left\{{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\right\}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n+1,\beta}^{(\alpha)}(x,y;k,a,b)\,\frac{t^n}{n!}.\end{aligned} \tag{54}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in the Equation (54), we get

$$\begin{aligned}&\left(x + ry\partial_y y\partial_x^{r-1} + \frac{\alpha k(\beta^b e^{\partial_x} - a^b) - \alpha\beta^b\partial_x e^{\partial_x}}{\partial_x(\beta^b e^t - a^b)}\right)\left\{{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\right\}\\ &= {}_{e^{(r)}}Y_{n+1,\beta}^{(\alpha)}(x,y;k,a,b),\end{aligned} \tag{55}$$

which, in view of the monomiality principle exhibited in Equation (20) for ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$, yields the assertion (49) of Theorem 2.

We now prove the assertion (50) of Theorem 2. For this purpose, we start with the following identity arising from Equations (45) and (51):

$$\partial_x\left\{\sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_{e^{(r)}}Y_{n-1,\beta}^{(\alpha)}(x,y;k,a,b)\,\frac{t^n}{(n-1)!}. \tag{56}$$

Rearranging the summation in the left-hand side of Equation (56), and then equating the coefficients of the same powers of t in both sides of the resulting equation, we find that

$$\partial_x\left\{{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\right\} = {}_{e^{(r)}}Y_{n-1,\beta}^{(\alpha)}(x,y;k,a,b) \qquad (n\in\mathbb{N}), \tag{57}$$

which, in view of the monomiality principle exhibited in Equation (21) for ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b))$, yields the assertion (50) of Theorem 2. Our demonstration of Theorem 2 is thus completed. □

We note that the properties of quasi-monomials can be derived by means of the actions of the multiplicative and derivative operators. We derive the differential equation for the 2VTEATP ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ in the following theorem.

Theorem 3. *The 2VTEATP* ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ *satisfies the following differential equation*:

$$\left(x\partial_x + ry\partial_y y\partial_x^{r} + \frac{\alpha k(\beta^b e^t - a^b) - \alpha\beta^b\partial_x e^{\partial_x}}{(\beta^b e^t - a^b)} - n\right)\left\{{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\right\} = 0, \tag{58}$$

Proof. Theorem 3 can be easily proved by combining (49) and (50) with the monomiality principle exhibited in (22). □

Remark 3. *When $r = 2$, the 2VTEP $e^{(r)}(x,y)$ of order r reduces to the 2VTEP ${}_{[2]}e_n(x,y)$. Therefore, if we set $r = 2$ in Equation (45), we get the following generating function for the 2-variable truncated-exponential Apostol-type polynomials (2VTEATP) ${}_{[2]e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)$:*

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^2}\right) = \sum_{n=0}^{\infty} {}_{[2]e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)\,\frac{t^n}{n!}. \tag{59}$$

The series definition and other results for the 2VTEATP ${}_{[2]e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)$ can be obtained by taking $r = 2$ in Theorems 1 and 2. Table 1 shown the special cases of the 2VTEATP ${}_{\cdot e^{(r)}}Y_n(x,y;k,a,b)$.

Remark 4. *For the case $y = 1$, the polynomials ${}_{[2]}e_n(x,1)$ reduce to the truncated-exponential polynomials ${}_{[2]}e_n(x)$. Therefore, by taking $y = 1$ in Equation (59), we get the following generating function for the truncated-exponential Apostol-type polynomials (TEATP) ${}_{[2]e^{(r)}}Y^{(\alpha)}_{n,\beta}(x;k,a,b)$:*

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\left(\frac{1}{1-t^2}\right) = \sum_{n=0}^{\infty} {}_{[2]e^{(r)}}Y^{(\alpha)}_{n,\beta}(x;k,a,b)\,\frac{t^n}{n!}. \tag{60}$$

Table 1. Some special cases of the 2VTEATP ${}_{\cdot e^{(r)}}Y_n(x,y;k,a,b)$.

S. No.	Values of the Parameter	Relation between the 2VTEATP ${}_{e^{(r)}}Y_n(x,y;k,a,b)$ and Its Special Case	Name of the Resultant Special Polynomials	Generating Functions and the Resultant of Special Polynomials
I.	$k=a=b=1, \beta=\lambda$	${}_{e^{(r)}}Y_n(x,y;1,1,\lambda) = {}_{e^{(r)}}B^{(\alpha)}_n(x,y;\lambda)$	2-variable truncated-exponential-based Apostol-Bernoulli polynomial	$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^r}\right) = \sum_{n=0}^{\infty} {}_{e^{(r)}}B^{(\alpha)}_n(x,y;\lambda)\,\frac{t^n}{n!}$
II.	$k+1=-a=b=1, \beta=\lambda$	${}_{e^{(r)}}Y_n(x,y;0,-1,1,\lambda) = {}_{e^{(r)}}E^{(\alpha)}_n(x,y;\lambda)$	2-variable truncated-exponential-based Apostol-Euler polynomial	$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^r}\right) = \sum_{n=0}^{\infty} {}_{e^{(r)}}E^{(\alpha)}_n(x,y;\lambda)\,\frac{t^n}{n!}$
III.	$k=-2a=b=1, 2\beta=\lambda$	${}_{e^{(r)}}Y_n(x,y;1,-\frac{1}{2},1,\lambda) = {}_{e^{(r)}}G^{(\alpha)}_n(x,y;\lambda)$	2-variable truncated-exponential-based Apostol-Genocchi polynomial	$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^r}\right) = \sum_{n=0}^{\infty} {}_{e^{(r)}}G^{(\alpha)}_n(x,y;\lambda)\,\frac{t^n}{n!}$

In the case when $\lambda = 1$, the results obtained above for the 2VTEABP ${}_{e^{(r)}}B^{(\alpha)}_n(x,y;\lambda)$, 2VTEAEP ${}_{e^{(r)}}E^{(\alpha)}_n(x,y;\lambda)$ and 2VTEAGP ${}_{e^{(r)}}G^{(\alpha)}_n(x,y;\lambda)$ give the corresponding results for the 2-variable truncated-exponential Bernoulli polynomials (2VTEBP) (of order α) ${}_{e^{(r)}}B^{(\alpha)}_n(x,y)$, 2-variable truncated-exponential Euler polynomials (2VTEBP) (of order α) ${}_{e^{(r)}}E^{(\alpha)}_n(x,y)$ and 2-variable truncated-exponential Genocchi polynomials (2VTGBP) (of order α) ${}_{e^{(r)}}G^{(\alpha)}_n(x,y)$ [6]. Again for $\alpha = 1$, we get the corresponding results for the 2-variable truncated-exponential Bernoulli polynomials (2VTEBP) ${}_{e^{(r)}}B_n(x,y)$, 2-variable truncated-exponential Euler polynomials (2VTEEP) ${}_{e^{(r)}}E_n(x,y)$ and 2-variable truncated-exponential Genocchi polynomials (2VTEGP) ${}_{e^{(r)}}G_n(x,y)$.

3. Implicit Formulas Involving the 2-Variable Truncated-Exponential Based Apostol-Type Polynomials

In this section, we employ the definition of the 2-variable truncated-exponential-based Apostol-type polynomials ${}_{e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)$ that help in proving the generalizations of the previous works of Khan et al. [33] and Pathan and Khan (see [34–36]). For the derivation of implicit formulas involving the 2-variable truncated-exponential-based Apostol-type polynomials ${}_{e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)$, the same considerations as developed for the ordinary Hermite and related polynomials in the works by Khan et al. [33] and Pathan et al. (see [34–36]) apply as well. We first prove the following results involving the 2-variable truncated-exponential-based Apostol-type polynomials ${}_{e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)$.

Theorem 4. *The following implicit summation formulas for the 2-variable truncated-exponential-based Apostol-type polynomials* ${}_{e^{(r)}}Y^{(\alpha)}_{n,\beta}(x,y;k,a,b)$ *holds true:*

$$ {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(z,y;k,a,b) = \sum_{n=0}^{q}\sum_{p=0}^{l}\binom{q}{n}\binom{l}{p}(z-x)^{n+p}\ {}_{e^{(r)}}Y^{(\alpha)}_{q+l-n-p,\beta}(x,y;k,a,b). \tag{61} $$

Proof. We replace t by $t+u$ and rewrite (45) as follows:

$$ \left(\frac{2^{1-k}(t+u)^k}{\beta^b e^{t+u}-a^b}\right)^{\alpha}\left(\frac{1}{1-y(t+u)^r}\right) = e^{-x(t+u)}\sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(x,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!}. \tag{62} $$

Replacing x by z in the Equation (62) and equating the resulting equation to the above equation, we get

$$ e^{(z-x)(t+u)}\sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(x,y;k,a,b)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{n,\beta}(z,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!}. \tag{63} $$

Upon expanding the exponential function (63), we get

$$ \sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(x,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(z,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!}, \tag{64} $$

which, by appealing to the following series manipulation formula:

$$ \sum_{N=0}^{\infty} f(N)\,\frac{(x+y)^N}{N!} = \sum_{m,n=0}^{\infty} f(m+n)\,\frac{x^m}{m!}\,\frac{y^n}{n!} \tag{65} $$

in the left-hand side of (64), becomes

$$ \sum_{n,p=0}^{\infty}\frac{(z-x)^{n+p}t^n u^p}{n!\ p!}\sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(x,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(z,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!}. \tag{66} $$

Now, replacing q by $q-n$ and l by $l-p$, and using a lemma in [37] in the left-hand side of (66), we get

$$ \begin{aligned} &\sum_{q,l=0}^{\infty}\sum_{n=0}^{q}\sum_{p=0}^{l}\frac{(z-x)^{n+p}}{n!\ p!}\ {}_{e^{(r)}}Y^{(\alpha)}_{q+l-n-p,\beta}(x,y;k,a,b)\ \frac{t^q}{(q-n)!}\ \frac{u^l}{(l-p)!} \\ &\qquad = \sum_{q,l=0}^{\infty} {}_{e^{(r)}}Y^{(\alpha)}_{q+l,\beta}(z,y;k,a,b)\,\frac{t^q}{q!}\,\frac{u^l}{l!}. \end{aligned} \tag{67} $$

Finally, on equating the coefficients of the like powers of t and u in the equation (67), we get the required result (61) asserted by Theorem 4. □

If we set

$$ k=a=b=1 \qquad \text{and} \qquad \beta=\lambda $$

in Theorem 4, we get the following corollary.

Corollary 1. *The following implicit summation formula for the truncated-exponential-based Bernoulli polynomials* ${}_{e^{(r)}}B_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}B_{q+l}^{(\alpha)}(z,y;\lambda) = \sum_{n=0}^{q}\sum_{p=0}^{l}\binom{q}{n}\binom{l}{p}(z-x)^{n+p}\ {}_{e^{(r)}}B_{q+l-p-n}^{(\alpha)}(x,y;\lambda). \tag{68} $$

For

$$ k+1=-a=b=1 \quad \text{and} \quad \beta=\lambda $$

in Theorem 4, we get the following corollary.

Corollary 2. *The following implicit summation formula for the truncated-exponential-based Euler polynomials* ${}_{e^{(r)}}E_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}E_{q+l}^{(\alpha)}(z,y;\lambda) = \sum_{n=0}^{q}\sum_{p=0}^{l}\binom{q}{n}\binom{l}{p}(z-x)^{n+p}{}_{e^{(r)}}E_{q+l-p-n}^{(\alpha)}(x,y;\lambda). \tag{69} $$

Letting

$$ k=-2a=b=1 \quad \text{and} \quad 2\beta=\lambda $$

in Theorem 4, we get the following corollary.

Corollary 3. *The following implicit summation formulas for the truncated-exponential-based Genocchi polynomials* ${}_{e^{(r)}}G_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}G_{q+l}^{(\alpha)}(z,y;\lambda) = \sum_{n=0}^{q}\sum_{p=0}^{l}\binom{q}{n}\binom{l}{p}(z-x)^{n+p}{}_{e^{(r)}}G_{q+l-p-n}^{(\alpha)}(x,y;\lambda). \tag{70} $$

Theorem 5. *The following implicit summation formula involving the* 2*-variable truncated-exponential-based Apostol-type polynomials* ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ *holds true*:

$$ {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) = \sum_{s=0}^{n}\binom{n}{s}Y_{n-s,\beta}^{(\alpha)}(k,a,b)e_s^{(r)}(x,y). \tag{71} $$

Proof. By the definition (45), we have

$$ \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^r}\right) = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(k,a,b)\,\frac{t^n}{n!}\sum_{s=0}^{\infty} e_s^{(r)}(x,y)\,\frac{t^s}{s!}. \tag{72} $$

Now, replacing n by $n-s$ in the right-hand side of the Equation (72) and comparing the coefficients of t, we get the result (71) asserted by Theorem 5. □

If we set

$$ k=a=b=1 \quad \text{and} \quad \beta=\lambda $$

in Theorem 5, we get the following corollary.

Corollary 4. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Bernoulli polynomials* ${}_{e^{(r)}}B_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}B_n^{(\alpha)}(x+z,y+u;\lambda) = \sum_{s=0}^{n}\binom{n}{s}B_{n-s}^{(\alpha)}(\lambda)e_s^{(r)}(x,y). \tag{73} $$

For
$$k + 1 = -a = b = 1 \qquad \text{and} \qquad \beta = \lambda$$
in Theorem 5, we get the following corollary.

Corollary 5. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Euler polynomials* ${}_{e^{(r)}}E_n^{(\alpha)}(x, y; \lambda)$ *holds true*:

$$ {}_{e^{(r)}}E_n^{(\alpha)}(x + z, y + u; \lambda) = \sum_{s=0}^{n} \binom{n}{s} E_{n-s}^{(\alpha)}(\lambda) e_s^{(r)}(x, y). \tag{74}$$

Letting
$$k = -2a = b = 1 \qquad \text{and} \qquad 2\beta = \lambda$$
in Theorem 5, we get the following corollary.

Corollary 6. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Genocchi polynomials* ${}_{e^{(r)}}G_n^{(\alpha)}(x, y; \lambda)$ *holds true*:

$$ {}_{e^{(r)}}G_n^{(\alpha)}(x + z, y + u; \lambda) = \sum_{s=0}^{n} \binom{n}{s} G_{n-s}^{(\alpha)}(\lambda) e_s^{(r)}(x, y). \tag{75}$$

Theorem 6. *The following implicit summation formula involving the* 2*-variable truncated-exponential-based Apostol-type polynomials* ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x, y; k, a, b)$ *holds true*:

$$ {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x + z, y; k, a, b) = \sum_{s=0}^{n} \binom{n}{s} {}_{e^{(r)}}Y_{n-s,\beta}^{(\alpha)}(x, y; k, a, b) z^s. \tag{76}$$

Proof. We first replace x by $x + z$ in (45). Then, by using (16), we rewrite the generating function (45) as follows:

$$\begin{aligned} \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x+z)t} \left(\frac{1}{1 - yt^r}\right) &= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x, y; k, a, b) \frac{t^n}{n!} \sum_{s=0}^{\infty} \frac{(zt)^s}{s!} \\ &= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x + z, y; k, a, b) \frac{t^n}{n!}. \end{aligned} \tag{77}$$

Furthermore, upon replacing n by $n - s$ in *l.h.s* and comparing the coefficients of t^n, we complete the proof of Theorem 6. □

For
$$k = a = b = 1 \qquad \text{and} \qquad \beta = \lambda$$
in Theorem 6, we get the following corollary.

Corollary 7. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Bernoulli polynomials* ${}_{e^{(r)}}B_n^{(\alpha)}(x, y; \lambda)$ *holds true*:

$$ {}_{e^{(r)}}B_n^{(\alpha)}(x + z, y + u; \lambda) = \sum_{s=0}^{n} \binom{n}{s} {}_{e^{(r)}}B_{n-s}^{(\alpha)}(x, y; \lambda) H_s(z, u). \tag{78}$$

Upon setting
$$k + 1 = -a = b = 1 \qquad \text{and} \qquad \beta = \lambda$$
in Theorem 6, we get the following corollary.

Corollary 8. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Euler polynomials* ${}_{e^{(r)}}E_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}E_n^{(\alpha)}(x+z,y+u;\lambda) = \sum_{s=0}^{n} \binom{n}{s} {}_{e^{(r)}}E_{n-s}^{(\alpha)}(x,y;\lambda) H_s(z,u). \tag{79} $$

Letting

$$ k = -2a = b = 1 \qquad \text{and} \qquad 2\beta = \lambda $$

in Theorem 6, we get the following corollary.

Corollary 9. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Genocchi polynomials* ${}_{e^{(r)}}G_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}G_n^{(\alpha)}(x+z,y+u;\lambda) = \sum_{s=0}^{n} \binom{n}{s} {}_{e^{(r)}}G_{n-s}^{(\alpha)}(x,y;\lambda) H_s(z,u). \tag{80} $$

Theorem 7. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Apostol-type polynomials* ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ *holds true*:

$$ {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) = \sum_{r=0}^{n} \binom{n}{r} Y_{n-r,\beta}^{(\alpha)}(x-z;k,a,b) e^{(r)}(z,y). \tag{81} $$

Proof. Let us rewrite Equation (45) as follows:

$$ \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{(x-z+z)t} \left(\frac{1}{1-yt^r}\right) = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x-z;k,a,b)\frac{t^n}{n!} \sum_{r=0}^{\infty} e^{(r)}(z,y)\frac{t^r}{r!}. \tag{82} $$

Replacing n by $n-r$ and using (45), and then equating the coefficients of the of t^n, we complete the proof of Theorem 7. □

For

$$ k = a = b = 1 \qquad \text{and} \qquad \beta = \lambda $$

in Theorem 7, we get the following corollary.

Corollary 10. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Apostol-type Bernoulli polynomials* ${}_{e^{(r)}}B_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}B_n^{(\alpha)}(x,y;\lambda) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(\alpha)}(x-z;\lambda) e^{(r)}(z,y). \tag{83} $$

Letting

$$ k+1 = -a = b = 1 \qquad \text{and} \qquad \beta = \lambda $$

in Theorem 7, we get the following corollary.

Corollary 11. *The following implicit summation formula for the* 2*-variable truncated-exponential-based Apostol-type Euler polynomials* ${}_{e^{(r)}}E_n^{(\alpha)}(x,y;\lambda)$ *holds true*:

$$ {}_{e^{(r)}}E_n^{(\alpha)}(x,y;\lambda) = \sum_{r=0}^{n} \binom{n}{r} E_{n-r}^{(\alpha)}(x-z;\lambda) e^{(r)}(z,y). \tag{84} $$

If we set

$$k = -2a = b = 1 \qquad \text{and} \qquad 2\beta = \lambda$$

in Theorem 7, we get the following corollary.

Corollary 12. *The following implicit summation formula for the 2-variable truncated-exponential-based Apostol-type Genocchi polynomials ${}_{e^{(r)}}G_n^{(\alpha)}(x,y;\lambda)$ holds true:*

$$ {}_{e^{(r)}}G_n^{(\alpha)}(x,y;\lambda) = \sum_{r=0}^{n} \binom{n}{r} G_{n-r}^{(\alpha)}(x-z;\lambda) e^{(r)}(z,y). \tag{85}$$

Theorem 8. *The following implicit summation formula for the 2-variable truncated-exponential-based Apostol-type polynomials ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ holds true:*

$$ {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x+1,y;k,a,b) = \sum_{m=0}^{n} \binom{n}{m} {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}(x,y;k,a,b). \tag{86}$$

Proof. Using the generating function (45), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left({}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x+1,y;k,a,b) - {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) \right) \frac{t^n}{n!} \\
&= \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b} \right)^{\alpha} \left(\frac{1}{1-yt^r} \right) (e^t - 1) \\
&= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) \frac{t^n}{n!} \left(\sum_{r=0}^{\infty} \frac{t^m}{m!} - 1 \right) \\
&= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) \frac{t^n}{n!} \sum_{r=0}^{\infty} \frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}(x,y;k,a,b) - {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b) \right] \frac{t^n}{n!}.
\end{aligned}$$

which, upon equating the coefficients of t^n, yields the assertion (86) of Theorem 8. □

Remark 5. *Several corollaries and consequences of Theorem 11 can be deduced by using many of the aforementioned specializations of the various parameters involved in Theorem 8.*

4. General Symmetry Identities

In this section, we give general symmetry identities for the 2-variable truncated-exponential-based Apostol-type polynomials ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ by applying the generating functions (39) and (45). The results extend some known identities of Özarslan (see [31,32]), Khan [38], and Pathan and Khan (see [34–36]).

Theorem 9. *Let $\alpha, k \in \mathbb{N}_0$, $a, b \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{C}$, $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then the following symmetry identity holds true:*

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} d^m c^{n-m} \; {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}(dx, d^r y;k,a,b) \; {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(cX, c^r Y;k,a,b) \\
&= \sum_{m=0}^{n} \binom{n}{m} c^m d^{n-m} \; {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}(cx, c^r y;k,a,b) {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(dX, d^r Y;k,a,b).
\end{aligned} \tag{87}$$

Proof. Let us first consider the following expression:

$$g(t) = \left(\frac{c^k d^k 2^{2(1-k)} t^{2k}}{(\beta^b e^{ct} - a^b)(\beta^b e^{dt} - a^b)} \right)^{\alpha} e^{cdxt} \left(\frac{1}{1 - y(cdt)^r} \right) e^{cdXt} \left(\frac{1}{1 - Y(cdt)^r} \right),$$

which shows that the function $g(t)$ is symmetric in the parameters a and b. Then, by expanding $g(t)$ into series in two different ways, we get

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(dx, d^r y; k, a, b) \frac{(ct)^n}{n!} \sum_{m=0}^{\infty} {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(cX, c^r Y; k, a, b) \frac{(dt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} d^m c^{n-m} {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}(dx, d^r y; k, a, b) {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(cX, c^r Y; k, a, b) t^n \end{aligned} \tag{88}$$

and

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(cx, c^r y; k, a, b) \frac{(dt)^n}{n!} \sum_{m=0}^{\infty} {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(dX, d^r Y; k, a, b) \frac{(ct)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} c^m d^{n-m} {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}(cx, c^r y; k, a, b) {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(dX, d^r Y; k, a, b) t^n. \end{aligned} \tag{89}$$

Comparing the coefficients of t^n on the right-hand sides of Equations (88) and (89), we arrive at the desired result (87). □

For

$$k = a = b = 1 \quad \text{and} \quad \beta = \lambda$$

in Theorem 9, we get the following corollary.

Corollary 13. *For all $c, d, r \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the following symmetry identity for the 2-variable truncated-exponential-based Apostol-type Bernoulli polynomials holds true*:

$$\begin{aligned} &\sum_{m=0}^{n} \binom{n}{m} d^m c^{n-m} {}_{e^{(r)}}B_{n-m}^{(\alpha)}(dx, d^r y; \lambda) {}_{e^{(r)}}B_{m}^{(\alpha)}(cX, c^r Y; \lambda) \\ &\quad = \sum_{m=0}^{n} \binom{n}{m} c^m d^{n-m} {}_{e^{(r)}}B_{n-m}^{(\alpha)}(cx, c^r y; \lambda) {}_{e^{(r)}}B_{m}^{(\alpha)}(dX, d^r Y; \lambda). \end{aligned} \tag{90}$$

Putting

$$k + 1 = -a = b = 1 \quad \text{and} \quad \beta = \lambda$$

in Theorem 9, we get the following corollary.

Corollary 14. *For all $r \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the following symmetry identity for the 2-variable truncated-exponential-based Apostol-type Euler polynomials holds true*:

$$\begin{aligned} &\sum_{m=0}^{n} \binom{n}{m} d^m c^{n-m} {}_{e^{(r)}}E_{n-m}^{(\alpha)}(dx, d^r y; \lambda) {}_{e^{(r)}}E_{m}^{(\alpha)}(cX, c^r Y; \lambda) \\ &\quad = \sum_{m=0}^{n} \binom{n}{m} c^m d^{n-m} {}_{e^{(r)}}E_{n-m}^{(\alpha)}(cx, c^r y; \lambda) {}_{e^{(r)}}E_{m}^{(\alpha)}(dX, d^r Y; \lambda). \end{aligned} \tag{91}$$

If we set

$$k = -2a = b = 1 \quad \text{and} \quad 2\beta = \lambda$$

in Theorem 9, we get the following corollary.

Corollary 15. *For all $r \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the following symmetry identity for the 2-variable truncated-exponential-based Apostol-type Genocchi polynomials holds true:*

$$\sum_{m=0}^{n} \binom{n}{m} d^m c^{n-m} \;_{e^{(r)}}G_{n-m}^{(\alpha)}(dx, d^r y; \lambda) \;_{e^{(r)}}G_m^{(\alpha)}(cX, c^r Y; \lambda)$$
$$= \sum_{m=0}^{n} \binom{n}{m} c^m d^{n-m} \;_{e^{(r)}}G_{n-m}^{(\alpha)}(cx, c^r y; \lambda) \;_{e^{(r)}}G_m^{(\alpha)}(dX, d^r Y; \lambda). \tag{92}$$

Theorem 10. *Let $\alpha, k \in \mathbb{N}_0$, $a, b \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{C}$, $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then the following symmetry identity holds true:*

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m \;_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(dx + \frac{d}{c}i + j, d^r y; k, a, b\right) \;_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(cX, c^r Y; k, a, b)$$
$$= \sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} d^{n-m} c^m \;_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(cx + \frac{c}{d}i + j, c^r y; k, a, b\right) \;_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(dX, d^r Y; k, a, b). \tag{93}$$

Proof. Let us first consider the following application:

$$g(t) = \left(\frac{c^k d^k 2^{2(1-k)} t^{2k}}{(\beta^b e^{ct} - a^b)(\beta^b e^{dt} - a^b)}\right)^{\alpha} e^{cdxt} \left(\frac{1}{1 - y(cdt)^r}\right) \frac{(e^{cdt} - 1)^2}{(e^{ct} - 1)(e^{dt} - 1)} e^{cdXt} \left(\frac{1}{1 - Y(cdt)^r}\right)$$
$$= \left(\frac{2^{(1-k)} c^k t^k}{\beta^b e^{ct} - a^b}\right)^{\alpha} e^{cdxt} \left(\frac{1}{1 - y(cdt)^r}\right) \left(\frac{e^{cdt} - 1}{e^{ct} - 1}\right) \left(\frac{2^{(1-k)} d^k t^k}{\beta^b e^{dt} - a^b}\right)^{\alpha}$$
$$\cdot e^{cdXt} \left(\frac{1}{1 - Y(cdt)^r}\right) \left(\frac{1}{e^{cdt} - 1} e^{dt} - 1\right)$$
$$= \left(\frac{2^{(1-k)} c^k t^k}{(\beta^b e^{ct} - a^b}\right)^{\alpha} e^{cdxt} \left(\frac{1}{1 - y(cdt)^r}\right) \sum_{i=0}^{c-1} e^{dti} \left(\frac{2^{(1-k)} d^k t^k}{\beta^b e^{dt} - a^b}\right)^{\alpha}$$
$$\cdot e^{cdXt} \left(\frac{1}{1 - Y(cdt)^r}\right) e^{cdyt} \sum_{j=0}^{d-1} e^{ctj}$$
$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-m} d^m \;_{e^{(r)}} \right.$$
$$\left. \cdot Y_{n-m,\beta}^{(\alpha)}\left(dx + \frac{d}{c}i + j, d^r y; k, a, b\right) \;_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(cX, c^r Y; k, a, b)\right] t^n. \tag{94}$$

On the other hand, we have

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} d^{n-m} c^m \right.$$
$$\left. \cdot \;_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(cx + \frac{c}{d}i + j, c^r y; k, a, b\right) \;_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(dX, d^r Y; k, a, b)\right) t^n. \tag{95}$$

By comparing the coefficients of t^n on the right-hand sides of (94) and (95), we arrive at the desired result (93) asserted by Theorem 10. □

Remark 6. *Several corollaries and consequences of Theorem 11 can be derived by making use of many of the aforementioned specializations of the various parameters involved in Theorem 10.*

Theorem 11. *For each pair of integers a and b and all integers $n \in \mathbb{N}_0$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{c-1}\sum_{j=0}^{d-1}c^{n-m}d^{m}\ {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(dx+\frac{d}{c}\,i,d^{r}y;k,a,b\right)\ {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(cX+\frac{c}{d}\,j,c^{r}Y;k,a,b)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{d-1}\sum_{j=0}^{c-1}d^{n-m}c^{m}\ {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(cx+\frac{c}{d}\,i,c^{r}y;k,a,b\right)$$
$$\cdot\ {}_{e^{(r)}}Y_{m,\beta}^{(\alpha)}(dX+\frac{d}{c}\,j,d^{r}Y;k,a,b). \tag{96}$$

Proof. The proof of Theorem 11 is analogous to that of Theorem 10, so we omit the details involved in the proof of Theorem 11. □

Remark 7. *Several corollaries and consequences of Theorem 11 can be derived by applying many of the aforementioned specializations of the various parameters involved in Theorem 11.*

We conclude our present investigation by proving the following symmetric identity involving the number $S_k(n,\lambda)$, which is defined by (44).

Theorem 12. *For all positive integers a and b, and for $n \in \mathbb{N}_0$, the following symmetric identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}c^{n-m}d^{m}\ {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(dx,d^{r}y;k,a,b\right)\sum_{i=0}^{m}\binom{m}{i}S_i\left(c-1;\left(\frac{\beta}{a}\right)^{b}\right)\ {}_{e^{(r)}}Y_{m-i,\beta}^{(\alpha)}(cX,c^{r}Y;k,a,b)$$
$$=\sum_{m=0}^{n}\binom{n}{m}c^{m}d^{n-m}\ {}_{e^{(r)}}Y_{n-m,\beta}^{(\alpha)}\left(cx,c^{r}y;k,a,b\right)\sum_{i=0}^{m}\binom{m}{i}S_i\left(d-1;\left(\frac{\beta}{a}\right)^{b}\right)$$
$$\cdot\ {}_{e^{(r)}}Y_{m-i,\beta}^{(\alpha)}(dX,d^{r}Y;k,a,b). \tag{97}$$

Proof. We first consider the function $g(t)$ given by

$$g(t)=\frac{(2^{2(1-k)}c^{k}d^{k}t^{2k})^{\alpha}(\beta^{b}e^{cdt}-a^{b})}{(\beta^{b}e^{ct}-a^{b})^{\alpha}(\beta^{b}e^{dt}-a^{b})^{\alpha+1}}\ e^{cdxt}\left(\frac{1}{1-y(cdt)^{r}}\right)e^{cdXt}\left(\frac{1}{1-Y(cdt)^{r}}\right)$$
$$=\left(\frac{2^{(1-k)}c^{k}t^{k}}{\beta^{b}e^{ct}-a^{b}}\right)^{\alpha}e^{cdxt}\left(\frac{1}{1-y(cdt)^{r}}\right)\left(\frac{\beta^{b}e^{cdt}-a^{b}}{\beta^{b}e^{dt}-a^{b}}\right)\left(\frac{2^{(1-k)}d^{k}t^{k}}{\beta^{b}e^{dt}-a^{b}}\right)^{\alpha}e^{cdXt}\left(\frac{1}{1-Y(cdt)^{r}}\right)$$
$$=\left(\sum_{n=0}^{\infty}{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}\left(dx,d^{r}y;k,a,b\right)\frac{(ct)^{n}}{n!}\right)\left[\sum_{n=0}^{\infty}S_n\left(c-1;\left(\frac{\beta}{a}\right)^{b}\right)\frac{(dt)^{n}}{n!}\right]$$
$$\cdot\left(\sum_{n=0}^{\infty}{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(cX,c^{r}Y;k,a,b)\frac{(dt)^{n}}{n!}\right).$$

Using similar arguments as above, we get

$$g(t)=\left(\sum_{n=0}^{\infty}{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}\left(cx,c^{r}y;k,a,b\right)\frac{(dt)^{n}}{n!}\right)\left[\sum_{n=0}^{\infty}S_n\left(d-1;\left(\frac{\beta}{a}\right)^{b}\right)\frac{(ct)^{n}}{n!}\right]$$
$$\cdot\left(\sum_{n=0}^{\infty}{}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(dX,d^{r}Y;k,a,b)\frac{(ct)^{n}}{n!}\right). \tag{98}$$

Finally, after a suitable manipulation with the summation index in (98) followed by a comparison of the coefficients of t^n, the proof of Theorem 12 is completed. □

5. Conclusions

Özden ([29]) defined the unified polynomials $Y_{n,\beta}^{(\alpha)}(x;k,a,b)$ of order α by means of the following generating function (see also Remark 1 above):

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(\alpha)}(x;k,a,b)\,\frac{t^n}{n!}$$

$$\left(|t| < 2\pi \text{ when } \beta = a;\ |t| < \left|b\log\left(\frac{\beta}{a}\right)\right| \text{ when } \beta \neq a;\ 1^{\alpha} := 1;\ k \in \mathbb{N}_0;\ a,b \in \mathbb{R}\setminus\{0\};\ \alpha,\beta \in \mathbb{C}\right).$$

Basing our investigation upon this generating function, we have introduced generating function for the 2-variable truncated-exponential-based Apostol-type polynomials denoted by ${}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ as follows:

$$\sum_{n=0}^{\infty} {}_{e^{(r)}}Y_{n,\beta}^{(\alpha)}(x,y;k,a,b)\,\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}\left(\frac{1}{1-yt^r}\right),$$

which we have found to be instrumental in deriving quasi-monomiality with respect to the following multiplicative and derivative operators:

$$\widehat{M}_{e^{(r)}Y} = x + ry\partial_y y\partial_x^{r-1} + \frac{\alpha k(\beta^b e^t - a^b) - \alpha\beta^b\partial_x e^{\partial_x}}{\partial_x(\beta^b e^t - a^b)}$$

and

$$\widehat{P}_{e^{(r)}Y} = \partial_x.$$

We have also presented a further investigation to obtain some implicit summation formulas and symmetric identities by means of their generating functions.

In our next investigation, we propose to study an appropriate combination of the operational approach with that involving integral transforms with a view to studying integral representations related to the truncated-exponential-based Apostol-type polynomials which we have introduced and studied in this article.

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