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# Product Operations on $q$-Rung Orthopair Fuzzy Graphs

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**Abstract:** The $q$-rung orthopair fuzzy graph is an extension of intuitionistic fuzzy graph and Pythagorean fuzzy graph. In this paper, the degree and total degree of a vertex in $q$-rung orthopair fuzzy graphs are firstly defined. Then, some product operations on $q$-rung orthopair fuzzy graphs, including direct product, Cartesian product, semi-strong product, strong product, and lexicographic product, are defined. Furthermore, some theorems about the degree and total degree under these product operations are put forward and elaborated with several examples. In particular, these theorems improve the similar results in single-valued neutrosophic graphs and Pythagorean fuzzy graphs.

**Keywords:** $q$-rung orthopair fuzzy graph; product operations; $q$-rung orthopair fuzzy sets; total degree

## 1. Introduction

In 2017, Yager proposed the concept of $q$-rung orthopair fuzzy sets ($q$-ROFSs) [1], which is a generalization of intuitionistic fuzzy sets (IFSs) [2] and Pythagorean fuzzy sets (PFSs) [3,4]. The $q$-ROFSs are fuzzy sets in which the membership grades of an element $x$ are pairs of values in the unit interval, $<\mu_A(x), \nu_A(x)>$, one of which indicates membership degree in the fuzzy set and the other nonmembership degree [1]. For the $q$-ROFSs, the membership grades need to satisfy the following conditions: $(\mu_A(x))^q + (\nu_A(x))^q \leq 1$, $\mu_A(x) \in [0,1]$, $\nu_A(x) \in [0,1]$ and $q \geq 1$, where the parameter $q$ determines the range of information expression. As $q$ increases, the range of information expression increases. As we all known, IFSs require the condition $\mu_A(x) + \nu_A(x) \leq 1$ and PFSs require the condition $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$. It is obvious to observe that $q$-ROFSs further diminish the restriction of IFSs and PFSs on membership grades. Therefore, compared with IFSs and PFSs, $q$-ROFSs provide decision-makers more elasticity to voice opinions with respect to membership grades of an element. Recently, the $q$-ROFSs have become a hotspot research topic and attracted broad attention [5–17].

Graph is a convenient tool to describe the decision-making problems diagrammatically [18]. By using this tool, the decision-making objects and their relationships are represented by vertex and edge. With different representations of decision-making information, many different types of graphs have been proposed, such as fuzzy graph [19], intuitionistic fuzzy graph (IFG) [20], single-valued neutrosophic graph (SVNG) [21], intuitionistic fuzzy soft graph [22], rough fuzzy graph [23], Pythagorean fuzzy graph (PFG) [24]. In consideration of the superiority of $q$-ROFSs, Habib et al. [25] proposed the concept of $q$-rung orthopair fuzzy graph ($q$-ROFG) based on the $q$-ROFSs in 2019. The $q$-ROFG is an extension of IFG [20] and PFG [24]. Compared with IFG and PFG, $q$-ROFG has a more powerful ability to model uncertainty in decision-making problems.

Product operations on graphs are highly important part in graph theory [26]. Many scholars have discussed product operations on different graphs. Mordeson and Peng [27–30] defined some product operations on fuzzy graphs. Later, using these operations, the degree of the vertices is obtained from two fuzzy graphs in [31,32]. Gong and Wang [33] defined some product operations on fuzzy hypergraphs. Sahoo and Pal [34] presented some product operations on IFGs and calculated the degree of a vertex in IFGs. Rashmanlou et al. [35] proposed product operations on interval-valued fuzzy graphs and study about the degree of a vertex in interval-valued fuzzy graphs. Naz et al. [21] discussed some product operations of SVNGs and applied SVNGs to multi-criteria decision-making. More recently, Akram et al. [24] investigated some product operations of PFGs and the degree and total degree of a vertex in PFGs. However, the product operations on $q$-ROFGs have not been researched yet, so we will pay our attention to this subject in this paper. Moreover, we have found that in SVNGs and PFGs, the results about the degree and total degree under some product operations fail to work in some cases. To improve these results, we introduced the number of adjacent vertices and obtained some more general theorems.

The reminder of this paper is organized as follows. Some notions of $q$-ROFSs and $q$-ROFGs are reviewed in Section 2. The degree and total degree of a vertex in a $q$-ROFG are defined in Section 3. Some product operations on $q$-ROFGs, such as direct product, Cartesian product, semi-strong product, strong product and lexicographic product, are defined, and the theorems about the degree and total degree under the defined product operations are obtained in Section 4. Some conclusions are given in Section 5.

## 2. Preliminaries

In this section, we review some definitions that are necessary.

### 2.1. Graph Theory

**Definition 1** ([19])**.** *A graph is a pair of sets $G = (V, E)$, satisfying $E(G) \subseteq V \times V$. The elements of $V(G)$ and $E(G)$ are the vertices and edges of the graph $G$, respectively. The standard products of graphs: direct product, Cartesian product, semi-strong product, strong product and lexicographic product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ will be denoted by $G_1 \times G_2$, $G_1 \square G_2$, $G_1 \bullet G_2$, $G1 \boxtimes G_2$ and $G_1[G_2]$, respectively. Let $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$. Then*

$$E(G_1 \times G_2) = \{(x_1, x_2), (y_1, y_2) \mid x_1 y_1 \in E_1 \text{ and } x_2 y_2 \in E_2\},$$
$$E(G_1 \square G_2) = \{(x_1, x_2), (y_1, y_2) \mid x_1 = y_1 \text{ and } x_2 y_2 \in E_2, \text{ or } x_1 y_1 \in E_1 \text{ and } x_2 = y_2\},$$
$$E(G_1 \bullet G_2) = \{(x_1, x_2), (y_1, y_2) \mid x_1 = y_1 \text{ and } x_2 y_2 \in E_2, \text{ or } x_1 y_1 \in E_1 \text{ and } x_2 y_2 \in E_2\},$$
$$E(G_1 \boxtimes G_2) = E(G_1 \square G_2) \cup E(G_1 \times G_2),$$
$$E(G_1[G_2]) = \{(x_1, x_2), (y_1, y_2) \mid x_1 y_1 \in E_1, \text{ or } x_1 = y_1 \text{ and } x_2 y_2 \in E_2\}.$$

**Definition 2** ([19])**.** *A fuzzy subset $\xi$ of a set $V$ is a function $\xi : V \to [0, 1]$. A fuzzy relation on a set $V$ is a mapping $\eta : V \times V \to [0, 1]$ such that $\eta(x, y) \leq \xi(x) \wedge \xi(y)$ for all $x, y \in V$. A fuzzy graph is a pair $G = (\xi, \eta)$, where $\xi$ is a fuzzy subset of a set $V$ and $\eta$ is a fuzzy relation on $\xi$.*

### 2.2. q-Rung Othopair Fuzzy Set

**Definition 3** ([1])**.** *Let $X$ be a universe of discourse, a q-ROFS $\mathcal{A}$ defined on $X$ is given by*

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle \,|x \in X\}$$

*where $\mu_{\mathcal{A}}(x) \in [0, 1]$ and $\nu_{\mathcal{A}}(x) \in [0, 1]$ respectively represent the membership and nonmembership degrees of the element $x$ to the set $\mathcal{A}$ satisfying $\mu^q_{\mathcal{A}}(x) + \nu^q_{\mathcal{A}}(x) \leq 1, (q \geq 1)$. The indeterminacy degree of the element $x$ to the set $\mathcal{A}$ is $\pi_{\mathcal{A}}(x)^q = (\mu_{\mathcal{A}}(x)^q + \nu_{\mathcal{A}}(x)^q - \mu_{\mathcal{A}}(x)^q \nu_{\mathcal{A}}(x)^q)^{1/q}$. For convenience, the pair $(\mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x))$ is called a q-rung orthopair fuzzy number (q-ROFN) [8].*

### 2.3. q-Rung Orthopair Fuzzy Graph

**Definition 4** ([25]). *A q-ROFS $\mathcal{Q}$ on $X \times X$ is said to be a q-rung orthopair fuzzy relation (q-ROFR) on X, denoted by*

$$\mathcal{Q} = \{\langle xy, \mu_{\mathcal{Q}}(xy), \nu_{\mathcal{Q}}(xy)\rangle | xy \in X \times X\},$$

*where $\mu_{\mathcal{Q}} : X \times X \to [0,1]$ and $\nu_{\mathcal{Q}} : X \times X \to [0,1]$ represent the membership and nonmembership function of $\mathcal{Q}$, respectively, such that $0 \le {\mu_{\mathcal{Q}}}^q(xy) + {\nu_{\mathcal{Q}}}^q(xy) \le 1$ for all $xy \in X \times X$ and $q \ge 1$. The proposed concept of q-ROFG is a generalization of IFG [20] and PFG [24].*

**Definition 5** ([25]). *A q-ROFG on a non-empty set X is a pair $\mathcal{G} = (\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ is a q-ROFS on X and $\mathcal{Q}$ is a q-ROFR on X such that*

$$\mu_{\mathcal{Q}}(xy) \le \min\{\mu_{\mathcal{P}}(x), \mu_{\mathcal{P}}(y)\}, \nu_{\mathcal{Q}}(xy) \ge \max\{\nu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(y)\}$$

*and $0 \le {\mu_{\mathcal{Q}}}^q(xy) + {\nu_{\mathcal{Q}}}^q(xy) \le 1$ for all $x, y \in X$ and $q \ge 1$. We call $\mathcal{P}$ and $\mathcal{Q}$ the q-rung orthopair fuzzy vertex set and the q-rung orthopair fuzzy edge set of $\mathcal{G}$, respectively.*

## 3. The Degree and Total Degree

In this section, the degree and total degree of a vertex in a $q$-ROFG are defined.

**Definition 6.** *The degree and total degree of a vertex $x \in V$ in a q-ROFG $\mathcal{G}$ are defined as $d_{\mathcal{G}}(x) = (d_\mu(x), d_\nu(x))$ and $td_{\mathcal{G}}(x) = (td_\mu(x), td_\nu(x))$, respectively, where*

$$d_\mu(x) = \sum_{x,y \neq x \in V} \mu_{\mathcal{Q}}(xy),\ d_\nu(x) = \sum_{x,y \neq x \in V} \nu_{\mathcal{Q}}(xy),$$
$$td_\mu(x) = \sum_{x,y \neq x \in V} \mu_{\mathcal{Q}}(xy) + \mu_{\mathcal{P}}(x),\ td_\nu(x) = \sum_{x,y \neq x \in V} \nu_{\mathcal{Q}}(xy) + \nu_{\mathcal{P}}(x).$$

**Example 1.** *Considering a road network problem, there are four locations $l, m, n, o$, assume that locations are performed by vertices, roads by edges, and the traffic congestion between adjacent locations is subjectively evaluated by decision-maker. The road network can be performed as a q-ROFG $\mathcal{G} = (\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ and $\mathcal{Q}$ respectively represent a q-ROFS of locations (vertices) and a q-ROFS of roads (edges). The traffic congestion of locations and roads are respectively denoted as $(\mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x))$ and $(\mu_{\mathcal{Q}}(x), \nu_{\mathcal{Q}}(x))$, see Figure 1. For example, $\frac{l}{(0.6,0.5)}$ means that the congestion degree of location l is 0.6 and the non-congestion degree of location l is 0.5. $\frac{lm}{(0.5,0.9)}$ means that the congestion degree of road lm is 0.5 and the non-congestion degree of road lm is 0.9.*

$$\mathcal{P} = \left(\frac{l}{(0.6, 0.5)}, \frac{m}{(0.7, 0.9)}, \frac{n}{(0.3, 0.2)}, \frac{o}{(0.5, 0.1)}\right),$$
$$\mathcal{Q} = \left(\frac{lm}{(0.5, 0.9)}, \frac{mn}{(0.1, 0.9)}, \frac{no}{(0.2, 0.5)}\right).$$

To obtain more traffic congestion information of the road network, the degree and total degree of each location are calculated. By Definition 6, $d_{\mathcal{G}}(m) = (d_\mu(m), d_\nu(m))$. Since $d_\mu(x) = \sum_{x,y \neq x \in V} \mu_{\mathcal{Q}}(xy)$ *and* $d_\nu(x) = \sum_{x,y \neq x \in V} \nu_{\mathcal{Q}}(xy)$, we can get $d_{\mathcal{G}}(m) = (\mu_{\mathcal{Q}}(lm) + \mu_{\mathcal{Q}}(mn), \nu_{\mathcal{Q}}(lm) + \nu_{\mathcal{Q}}(mn)) = (0.5 + 0.1, 0.9 + 0.9) = (0.6, 1.8)$. The degree of the location $m$ represents the sum of congestion grades between $m$ and other neighbor locations. By Definition 6, $td_{\mathcal{G}}(m) = (td_\mu(m), td_\nu(m))$. Since $td_\mu(x) = \sum_{x,y \neq x \in V} \mu_{\mathcal{Q}}(xy) + \mu_{\mathcal{P}}(x)$ *and* $td_\nu(x) = \sum_{x,y \neq x \in V} \nu_{\mathcal{Q}}(xy) + \nu_{\mathcal{P}}(x)$, so we can get $td_{\mathcal{G}}(m) = (\mu_{\mathcal{Q}}(lm) + \mu_{\mathcal{Q}}(mn) + \mu_{\mathcal{P}}(m), \nu_{\mathcal{Q}}(lm) + \nu_{\mathcal{Q}}(mn) + \nu_{\mathcal{P}}(m)) = (0.5 + 0.1 + 0.7, 0.9 + 0.9 + 0.9) = (1.3, 2.7)$. The total degree of the location $m$ represents the sum of total congestion grades of the location $m$ in road network. Similarly, we can obtain $d_{\mathcal{G}}(l)$ = (0.5, 0.9), $td_{\mathcal{G}}(l)$ = (1.1, 1.4), $d_{\mathcal{G}}(n)$ = (0.3, 1.4), $td_{\mathcal{G}}(n)$ = (0.6, 1.6), $d_{\mathcal{G}}(o)$ = (0.2, 0.5) and $td_{\mathcal{G}}(o)$ = (0.7, 0.6).

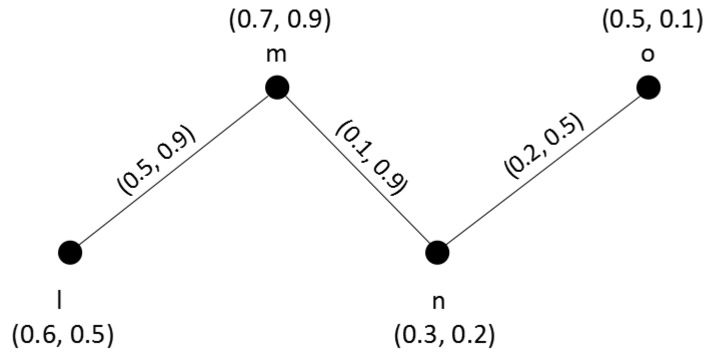


**Figure 1.** A road network using $q$-rung orthopair fuzzy graph ($q$-ROFG) with $q$ = 4.

## 4. Some Product Operations on $q$-Rung Orthopair Fuzzy Graphs

In this section, product operations on $q$-ROFGs, including direct product, Cartesian product, semi-strong product, strong product and lexicographic product, are analyzed.

**Definition 7.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. The direct product of $\mathcal{G}_1$ and $\mathcal{G}_2$ is denoted by $\mathcal{G}_1 \times \mathcal{G}_2 = (\mathcal{P}_1 \times \mathcal{P}_2, \mathcal{Q}_1 \times \mathcal{Q}_2)$ and defined as:*

*(i)* $\begin{cases} (\mu_{\mathcal{P}_1} \times \mu_{\mathcal{P}_2})(x_1, x_2) = \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2) \\ (\nu_{\mathcal{P}_1} \times \nu_{\mathcal{P}_2})(x_1, x_2) = \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2) \; for\ all\ (x_1, x_2) \in V_1 \times V_2, \end{cases}$

*(ii)* $\begin{cases} (\mu_{\mathcal{Q}_1} \times \mu_{\mathcal{Q}_2})(x_1, x_2)(y_1, y_2) = \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\ (\nu_{\mathcal{Q}_1} \times \nu_{\mathcal{Q}_2})(x_1, x_2)(y_1, y_2) = \nu_{\mathcal{Q}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \; for\ all\ x_1y_1 \in E_1,\ for\ all\ x_2y_2 \in E_2. \end{cases}$

**Remark 1.** *The direct product of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be understood that the edges of $\mathcal{G}_1$ combine with the each edge of $\mathcal{G}_2$ to form a new graph $\mathcal{G}_1 \times \mathcal{G}_2$.*

**Proposition 1.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the q-ROFGs of the graphs $G_1$ and $G_2$ respectively. The direct product $\mathcal{G}_1 \times \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is a q-ROFG.*

**Definition 8.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. Then, for any vertex, $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned} (d_\mu)_{\mathcal{G}_1 \times \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \times E_2} (\mu_{\mathcal{Q}_1} \times \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2), \\ (d_\nu)_{\mathcal{G}_1 \times \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \times E_2} (\nu_{\mathcal{Q}_1} \times \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \nu_{\mathcal{Q}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2). \end{aligned}$$

**Theorem 1.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. If $\mu_{\mathcal{Q}_2} \geq \mu_{\mathcal{Q}_1}, \nu_{\mathcal{Q}_2} \leq \nu_{\mathcal{Q}_1}$, then $d_{\mathcal{G}_1 \times \mathcal{G}_2}(x_1, x_2) = |c(x_2)|\, d_{\mathcal{G}_1}(x_1)$, where $|c(x_2)| = \sum_{x_2y_2 \in E_2} 1$, represents the number of points adjacent to $x_2$ in $\mathcal{G}_2$ and if $\mu_{\mathcal{Q}_1} \geq \mu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1} \leq \nu_{\mathcal{Q}_2}$, then $d_{\mathcal{G}_1 \times \mathcal{G}_2}(x_1, x_2) = |c(x_1)|\, d_{\mathcal{G}_2}(x_2)$ for all $(x_1, x_2) \in V_1 \times V_2$, where $|c(x_1)| = \sum_{x_1y_1 \in E_1} 1$ represents the number of points adjacent to $x_1$ in $\mathcal{G}_1$.*

**Proof.** By definition of degree of a vertex in $\mathcal{G}_1 \times \mathcal{G}_2$, we have

$$\begin{aligned}
(d_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (\mu_{\mathcal{Q}_1}\times\mu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) \\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \mu_{\mathcal{Q}_1}(x_1y_1)\wedge\mu_{\mathcal{Q}_2}(x_2y_2) \\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \mu_{\mathcal{Q}_1}(x_1y_1)\ (since\ \mu_{\mathcal{Q}_2}\geq\mu_{\mathcal{Q}_1}) \\
&= \sum_{x_2y_2\in E_2} 1\times \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= |c(x_2)| \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= |c(x_2)|\,(d_\mu)_{\mathcal{G}_1}(x_1), \\
(d_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (\nu_{\mathcal{Q}_1}\times\nu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) \\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \nu_{\mathcal{Q}_1}(x_1y_1)\vee\nu_{\mathcal{Q}_2}(x_2y_2) \\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \nu_{\mathcal{Q}_1}(x_1y_1)\ (since\ \nu_{\mathcal{Q}_2}\leq\nu_{\mathcal{Q}_1}) \\
&= \sum_{x_2y_2\in E_2} 1\times \sum_{x_1y_1\in E_1} \nu_{\mathcal{Q}_1}(x_1y_1) \\
&= |c(x_2)| \sum_{x_1y_1\in E_1} \nu_{\mathcal{Q}_1}(x_1y_1) \\
&= |c(x_2)|\,(d_\nu)_{\mathcal{G}_1}(x_1).
\end{aligned}$$

Hence, $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = |c(x_2)|\, d_{\mathcal{G}_1}(x_1)$. Likewise, it is easy to show that if $\mu_{\mathcal{Q}_1}\geq\mu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1}\leq\nu_{\mathcal{Q}_2}$, then $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = |c(x_1)|\, d_{\mathcal{G}_2}(x_2)$. □

**Remark 2.** *In the SVNGs [21] and PFGs [24], If $\mu_{\mathcal{Q}_2}\geq\mu_{\mathcal{Q}_1}, \nu_{\mathcal{Q}_2}\leq\nu_{\mathcal{Q}_1}$, then $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = d_{\mathcal{G}_1}(x_1)$. If $\mu_{\mathcal{Q}_1}\geq\mu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1}\leq\nu_{\mathcal{Q}_2}$, then $d_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = d_{\mathcal{G}_2}(x_2)$ (cf. Theorem 3.4 in [21] and Theorem 1 in [24]). It is obvious that they do not consider the effect of $|c(x_2)|$ or $|c(x_1)|$ on the degree under direct product.*

**Definition 9.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1,x_2)\in V_1\times V_2$,*

$$\begin{aligned}
(td_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (\mu_{\mathcal{Q}_1}\times\mu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) + (\mu_{\mathcal{P}_1}\times\mu_{\mathcal{P}_2})(x_1,x_2) \\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \mu_{\mathcal{Q}_1}(x_1y_1)\wedge\mu_{\mathcal{Q}_2}(x_2y_2) + \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{P}_2}(x_2), \\
(td_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\times E_2} (\nu_{\mathcal{Q}_1}\times\nu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) + (\nu_{\mathcal{P}_1}\times\nu_{\mathcal{P}_2})(x_1,x_2) \\
&= \sum_{x_1y_1\in E_1, x_2y_2\in E_2} \nu_{\mathcal{Q}_1}(x_1y_1)\vee\nu_{\mathcal{Q}_2}(x_2y_2) + \nu_{\mathcal{P}_1}(x_1)\vee\nu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

**Theorem 2.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any $(x_1,x_2)\in V_1\times V_2$, if*

*(1)* $\mu_{\mathcal{Q}_2}\geq\mu_{\mathcal{Q}_1}$, *then* $(td_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = |c(x_2)|\,(d_\mu)_{\mathcal{G}_1}(x_1) + \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{P}_2}(x_2)$;
*(2)* $\nu_{\mathcal{Q}_2}\leq\nu_{\mathcal{Q}_1}$, *then* $(td_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = |c(x_2)|\,(d_\nu)_{\mathcal{G}_1}(x_1) + \nu_{\mathcal{P}_1}(x_1)\vee\nu_{\mathcal{P}_2}(x_2)$;
*(3)* $\mu_{\mathcal{Q}_1}\geq\mu_{\mathcal{Q}_2}$, *then* $(td_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = |c(x_1)|\,(d_\mu)_{\mathcal{G}_2}(x_2) + \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{P}_2}(x_2)$;
*(4)* $\nu_{\mathcal{Q}_2}\leq\nu_{\mathcal{Q}_1}$, *then* $(td_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = |c(x_1)|\,(d_\nu)_{\mathcal{G}_2}(x_2) + \nu_{\mathcal{P}_1}(x_1)\vee\nu_{\mathcal{P}_2}(x_2)$.

*In the above equalities, $|c(x_2)|$ represents the number of points adjacent to $x_2$ in $\mathcal{G}_2$ and $|c(x_1)|$ represents the number of points adjacent to $x_1$ in $\mathcal{G}_1$.*

**Proof.** The proof can be obtained by Definition 9 and Theorem 1. □

**Remark 3.** *In the PFGs [24], if*

*(1)* $\mu_{\mathcal{Q}_2} \geq \mu_{\mathcal{Q}_1}$, *then* $(td_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = (d_\mu)_{\mathcal{G}_1}(x_1) + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2)$;
*(2)* $\nu_{\mathcal{Q}_2} \leq \nu_{\mathcal{Q}_1}$, *then* $(td_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = (d_\nu)_{\mathcal{G}_1}(x_1) + \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2)$;
*(3)* $\mu_{\mathcal{Q}_1} \geq \mu_{\mathcal{Q}_2}$, *then* $(td_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = (d_\mu)_{\mathcal{G}_2}(x_2) + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2)$;
*(4)* $\nu_{\mathcal{Q}_1} \leq \nu_{\mathcal{Q}_2}$, *then* $(td_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(x_1,x_2) = (d_\nu)_{\mathcal{G}_2}(x_2) + \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2)$ *(cf. Theorem 2 in [24]).*

*It is obvious that they do not consider the effect of $|c(x_2)|$ or $|c(x_1)|$ on the total degree under direct product.*

**Example 2.** *Consider two q-ROFGs $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ on $V_1 = \{l, m\}$ and $V_2 = \{n, p, s\}$, respectively, as shown in Figure 2. Their direct product $\mathcal{G}_1 \times \mathcal{G}_2$ is shown in Figure 3.*

*Since $\mu_{\mathcal{Q}_2} \geq \mu_{\mathcal{Q}_1}$, $\nu_{\mathcal{Q}_2} \leq \nu_{\mathcal{Q}_1}$, by Theorem 1, we have*

$$(d_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(l,p) = |c(p)|\,(d_\mu)_{\mathcal{G}_1}(l) = |\{n,s\}|\,(d_\mu)_{\mathcal{G}_1}(l) = 2\times 0.1 = 0.2,$$
$$(d_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(l,p) = |c(p)|\,(d_\nu)_{\mathcal{G}_1}(l) = |\{n,s\}|\,(d_\nu)_{\mathcal{G}_1}(l) = 2\times 0.8 = 1.6.$$

*Therefore, $(d)_{\mathcal{G}_1\times\mathcal{G}_2}(l,p)$=(0.2, 1.6). In addition, by Theorem 2, we have*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1\times\mathcal{G}_2}(l,p) &= |c(p)|\,(d_\mu)_{\mathcal{G}_1}(l) + \mu_{\mathcal{P}_1}(l)\wedge\mu_{\mathcal{P}_2}(p) = |\{n,s\}|\,(d_\mu)_{\mathcal{G}_1}(l) + \mu_{\mathcal{P}_1}(l)\wedge\mu_{\mathcal{P}_2}(p)\\ &= 2\times 0.1 + 0.9\wedge 0.9 = 1.1,\\ (td_\nu)_{\mathcal{G}_1\times\mathcal{G}_2}(l,p) &= |c(p)|\,(d_\nu)_{\mathcal{G}_1}(l) + \nu_{\mathcal{P}_1}(l)\vee\nu_{\mathcal{P}_2}(p) = |\{n,s\}|\,(d_\nu)_{\mathcal{G}_1}(l) + \nu_{\mathcal{P}_1}(l)\vee\nu_{\mathcal{P}_2}(p)\\ &= 2\times 0.8 + 0.6\vee 0.5 = 2.2.\end{aligned}$$

*Therefore, $(td)_{\mathcal{G}_1\times\mathcal{G}_2}(l,p)$=(1.1, 2.2). Likewise, we can get the degree and total degree of each vertex in $\mathcal{G}_1 \times \mathcal{G}_2$.*

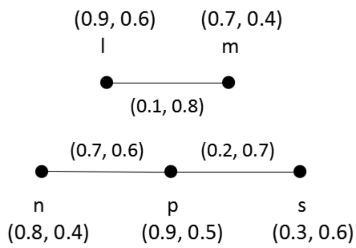


**Figure 2.** Two $q$-ROFGs with $q$ = 3.

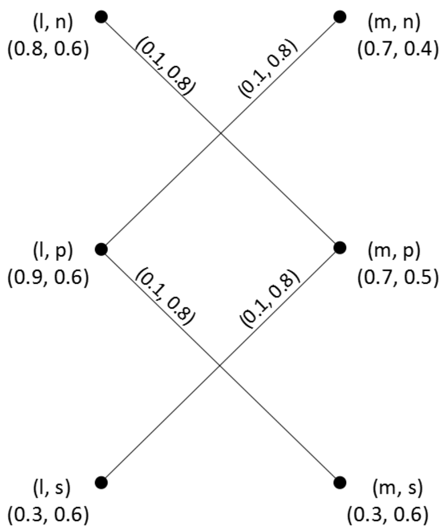


**Figure 3.** Direct product of two $q$-ROFGs.

**Remark 4.** *Klement and Mesiar [36] show that results concerning various fuzzy structures actually follow from results of ordinary fuzzy structures. These results include those from PFSs, IFSs, and many others. Although PFSs and q-rung orthopair fuzzy sets are isomorphism, Theorem 1 and Theorem 2 in this paper cannot be obtained from the results of PFGs. In the PFGs [24], they do not consider the effect of* $|c(x_2)| = \sum_{x_2y_2 \in E_2} 1$ *and their results fail to work in Example 2. For example, when using theorem 1 in PFGs [24], we can get*

$$(d_\mu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) = (d_\mu)_{\mathcal{G}_1}(l) = 0.1$$
$$(d_\nu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) = (d_\nu)_{\mathcal{G}_1}(l) = 0.8.$$

*However,* $(d_\mu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) = 0.2 \neq (d_\mu)_{\mathcal{G}_1}(l) = 0.1$ *and* $(d_\nu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) = 1.6 \neq (d_\nu)_{\mathcal{G}_1}(l) = 0.8$. *When using theorem 2 in PFGs [24], we can get*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) &= (d_\mu)_{\mathcal{G}_1}(l) + \mu_{\mathcal{P}_1}(l) \wedge \mu_{\mathcal{P}_2}(p) = (d_\mu)_{\mathcal{G}_1}(l) + \mu_{\mathcal{P}_1}(l) \wedge \mu_{\mathcal{P}_2}(p) \\ &= 0.1 + 0.9 \wedge 0.9 = 1.0, \\ (td_\nu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) &= (d_\nu)_{\mathcal{G}_1}(l) + \nu_{\mathcal{P}_1}(l) \vee \nu_{\mathcal{P}_2}(p) = (d_\nu)_{\mathcal{G}_1}(l) + \nu_{\mathcal{P}_1}(l) \vee \nu_{\mathcal{P}_2}(p) \\ &= 0.8 + 0.6 \vee 0.5 = 1.4.\end{aligned}$$

*However,* $(td_\mu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) = 1.1 \neq 1.0$ *and* $(td_\nu)_{\mathcal{G}_1 \times \mathcal{G}_2}(l,p) = 2.2 \neq 1.4$.

**Definition 10.** *Let* $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ *and* $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ *be two q-ROFGs of* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$, *respectively. The Cartesian product of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *is denoted by* $\mathcal{G}_1 \Box \mathcal{G}_2 = (\mathcal{P}_1 \Box \mathcal{P}_2, \mathcal{Q}_1 \Box \mathcal{Q}_2)$ *and defined as:*

*(i)* $$\begin{cases} (\mu_{\mathcal{P}_1} \Box \mu_{\mathcal{P}_2})(x_1, x_2) = \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2) \\ (\nu_{\mathcal{P}_1} \Box \nu_{\mathcal{P}_2})(x_1, x_2) = \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2) \ for\ all\ (x_1, x_2) \in V_1 \times V_2, \end{cases}$$

*(ii)* $$\begin{cases} (\mu_{\mathcal{Q}_1} \Box \mu_{\mathcal{Q}_2})(x, x_2)(x, y_2) = \mu_{\mathcal{P}_1}(x) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\ (\nu_{\mathcal{Q}_1} \Box \nu_{\mathcal{Q}_2})(x, x_2)(x, y_2) = \nu_{\mathcal{P}_1}(x) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \ for\ all\ x \in V_1,\ for\ all\ x_2y_2 \in E_2, \end{cases}$$

*(iii)* $$\begin{cases} (\mu_{\mathcal{Q}_1} \Box \mu_{\mathcal{Q}_2})(x_1, z)(y_1, z) = \mu_{\mathcal{P}_1}(x_1y_1) \wedge \mu_{\mathcal{P}_2}(z) \\ (\nu_{\mathcal{Q}_1} \Box \nu_{\mathcal{Q}_2})(x_1, z)(y_1, z) = \nu_{\mathcal{P}_1}(x_1x_2) \vee \nu_{\mathcal{Q}_2}(z) \ for\ all\ z \in V_2,\ for\ all\ x_1y_1 \in E_1. \end{cases}$$

**Remark 5.** *The Cartesian product of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *can be understood that the vertices of* $\mathcal{G}_1$ *combine with the each edge of* $\mathcal{G}_2$ *and the vertices of* $\mathcal{G}_2$ *combine with the each edge of* $\mathcal{G}_1$ *to form a new graph* $\mathcal{G}_1 \Box \mathcal{G}_2$.

**Proposition 2.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the q-ROFGs of the graphs $G_1$ and $G_2$, respectively. The Cartesian product $\mathcal{G}_1 \square \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_1$ is a q-ROFG.*

**Definition 11.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned}
(d_\mu)_{\mathcal{G}_1 \square \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \square E_2} (\mu_{\mathcal{Q}_1} \square \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1 y_1), \\
(d_\nu)_{\mathcal{G}_1 \square \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \square E_2} (\nu_{\mathcal{Q}_1} \square \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1 y_1).
\end{aligned}$$

**Theorem 3.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$ and $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$. Then $d_{\mathcal{G}_1 \square \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$ for any $(x_1, x_2) \in V_1 \times V_2$.*

**Proof.** By definition of degree of a vertex in $\mathcal{G}_1 \square \mathcal{G}_2$, we have

$$\begin{aligned}
(d_\mu)_{\mathcal{G}_1 \square \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \square E_2} (\mu_{\mathcal{Q}_1} \square \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad (\textit{By using } \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2} \textit{ and } \mu_{\mathcal{P}_1} \leq \mu_{\mathcal{Q}_2}) \\
&= \sum_{x_1 = y_1} 1 \times \sum_{x_2 y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2} 1 \times \sum_{x_1 y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&= \sum_{x_2 y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_1 y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&= (d_\mu)_{\mathcal{G}_1}(x_1) + (d_\mu)_{\mathcal{G}_2}(x_2), \\
(d_\nu)_{\mathcal{G}_1 \square \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \square E_2} (\nu_{\mathcal{Q}_1} \square \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1 y_1) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \nu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \nu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad (\textit{By using } \nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2} \textit{ and } \nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}) \\
&= \sum_{x_1 = y_1} 1 \times \sum_{x_2 y_2 \in E_2} \nu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2} 1 \times \sum_{x_1 y_1 \in E_1} \nu_{\mathcal{Q}_1}(x_1 y_1) \\
&= \sum_{x_2 y_2 \in E_2} \nu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_1 y_1 \in E_1} \nu_{\mathcal{Q}_1}(x_1 y_1) \\
&= (d_\nu)_{\mathcal{G}_1}(x_1) + (d_\nu)_{\mathcal{G}_2}(x_2).
\end{aligned}$$

Hence, $d_{\mathcal{G}_1 \square \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$. $\square$

**Definition 12.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned}
(td_\mu)_{\mathcal{G}_1\square\mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\square E_2} (\mu_{\mathcal{Q}_1}\square\mu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) + (\mu_{\mathcal{P}_1}\square\mu_{\mathcal{P}_2})(x_1,x_2) \\
&= \sum_{x_1=y_1,x_2y_2\in E_2} \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2,x_1y_1\in E_1} \mu_{\mathcal{P}_2}(x_2)\wedge\mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{P}_2}(x_2), \\
(td_\nu)_{\mathcal{G}_1\square\mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\square E_2} (\nu_{\mathcal{Q}_1}\square\nu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) + (\nu_{\mathcal{P}_1}\square\nu_{\mathcal{P}_2})(x_1,x_2) \\
&= \sum_{x_1=y_1,x_2y_2\in E_2} \nu_{\mathcal{P}_1}(x_1)\wedge\nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2,x_1y_1\in E_1} \nu_{\mathcal{P}_2}(x_2)\wedge\nu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \nu_{\mathcal{P}_1}(x_1)\vee\nu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

**Theorem 4.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any $(x_1, x_2) \in V_1 \times V_2$,*

*(1) If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$ and $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$, then*

$$(td_\mu)_{\mathcal{G}_1\square\mathcal{G}_2}(x_1,x_2) = (td_\mu)_{\mathcal{G}_1}(x_1) + (td_\mu)_{\mathcal{G}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1)\vee\mu_{\mathcal{P}_2}(x_2);$$

*(2) If $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$ and $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$, then*

$$(td_\nu)_{\mathcal{G}_1\square\mathcal{G}_2}(x_1,x_2) = (td_\nu)_{\mathcal{G}_1}(x_1) + (td_\nu)_{\mathcal{G}_2}(x_2) - \nu_{\mathcal{P}_1}(x_1)\wedge\nu_{\mathcal{P}_2}(x_2).$$

**Proof.** By definition of total degree of a vertex in $\mathcal{G}_1\square\mathcal{G}_2$,

$$\begin{aligned}
(1)\quad (td_\mu)_{\mathcal{G}_1\square\mathcal{G}_2}(x_1,x_2) &= \sum_{x_1=y_1,x_2y_2\in E_2} \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2,x_1y_1\in E_1} \mu_{\mathcal{P}_2}(x_2)\wedge\mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \mu_{\mathcal{P}_1}(x_1)\wedge\mu_{\mathcal{P}_2}(x_2) \\
&= \sum_{x_1=y_1} 1\times\sum_{x_2y_2\in E_2}\mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2} 1\times\sum_{x_1y_1\in E_1}\mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) \\
&\quad + \mu_{\mathcal{P}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1)\vee\mu_{\mathcal{P}_2}(x_2)\ \ (\textit{since } \mu_{\mathcal{P}_1}\geq\mu_{\mathcal{Q}_2},\ \mu_{\mathcal{P}_2}\geq\mu_{\mathcal{Q}_1}) \\
&= \sum_{x_2y_2\in E_2}\mu_{\mathcal{Q}_2}(x_2y_2) + \mu_{\mathcal{P}_2}(x_2) + \sum_{x_1y_1\in E_1}\mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) \\
&\quad - \mu_{\mathcal{P}_1}(x_1)\vee\mu_{\mathcal{P}_2}(x_2) \\
&= (td_\mu)_{\mathcal{G}_1}(x_1) + (td_\mu)_{\mathcal{G}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1)\vee\mu_{\mathcal{P}_2}(x_2),
\end{aligned}$$

$$\begin{aligned}
(2)\quad (td_\nu)_{\mathcal{G}_1\square\mathcal{G}_2}(x_1,x_2) &= \sum_{x_1=y_1,x_2y_2\in E_2} \nu_{\mathcal{P}_1}(x_1)\vee\nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2,x_1y_1\in E_1} \nu_{\mathcal{P}_2}(x_2)\vee\nu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \nu_{\mathcal{P}_1}(x_1)\vee\nu_{\mathcal{P}_2}(x_2) \\
&= \sum_{x_1=y_1} 1\times\sum_{x_2y_2\in E_2}\nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2} 1\times\sum_{x_1y_1\in E_1}\nu_{\mathcal{Q}_1}(x_1y_1) + \nu_{\mathcal{P}_1}(x_1) \\
&\quad + \nu_{\mathcal{P}_2}(x_2) - \nu_{\mathcal{P}_1}(x_1)\wedge\nu_{\mathcal{P}_2}(x_2)\ \ (\textit{since } \nu_{\mathcal{P}_1}\leq\nu_{\mathcal{Q}_2},\ \nu_{\mathcal{P}_2}\leq\nu_{\mathcal{Q}_1}) \\
&= (td_\nu)_{\mathcal{G}_1}(x_1) + (td_\nu)_{\mathcal{G}_2}(x_2) - \nu_{\mathcal{P}_1}(x_1)\wedge\nu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

□

**Example 3.** *Consider two q-ROFGs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 2, where $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$, $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$ and $\nu_{\mathcal{P}_1} \leq \mu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$. Their Cartesian product $\mathcal{G}_1\square\mathcal{G}_2$ is shown in Figure 4.*

*By Theorem 3, we have*

$$(d_\mu)_{\mathcal{G}_1\Box\mathcal{G}_2}(l,p) = (d_\mu)_{\mathcal{G}_1}(l) + (d_\mu)_{\mathcal{G}_2}(p) = 0.1 + 0.7 + 0.2 = 1.0,$$
$$(d_\nu)_{\mathcal{G}_1\Box\mathcal{G}_2}(l,p) = (d_\nu)_{\mathcal{G}_1}(l) + (d_\nu)_{\mathcal{G}_2}(p) = 0.8 + 0.6 + 0.7 = 2.1.$$

*Therefore,* $d_{\mathcal{G}_1\Box\mathcal{G}_2}(l,p) = (1.0, 2.1)$. *In addition, by Theorem 4, we can get*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1\Box\mathcal{G}_2}(l,p) &= (td_\mu)_{\mathcal{G}_1}(l) + (td_\mu)_{\mathcal{G}_2}(p) - \mu_{\mathcal{P}_1}(l) \vee \mu_{\mathcal{P}_2}(p)\\ &= 0.9 + 0.1 + 0.7 + 0.2 + 0.9 - 0.9 \vee 0.9 = 1.9,\\ (td_\nu)_{\mathcal{G}_1\Box\mathcal{G}_2}(l,p) &= (td_\nu)_{\mathcal{G}_1}(l) + (td_\nu)_{\mathcal{G}_2}(p) - \nu_{\mathcal{P}_1}(l) \wedge \nu_{\mathcal{P}_2}(p)\\ &= 0.8 + 0.6 + 0.6 + 0.7 + 0.5 - 0.6 \wedge 0.5 = 2.7.\end{aligned}$$

*Therefore,* $td_{\mathcal{G}_1\Box\mathcal{G}_2}(l,p) = (1.9, 2.7)$. *Likewise, we can get the degree and total degree of each vertex in* $\mathcal{G}_1\Box\mathcal{G}_2$.

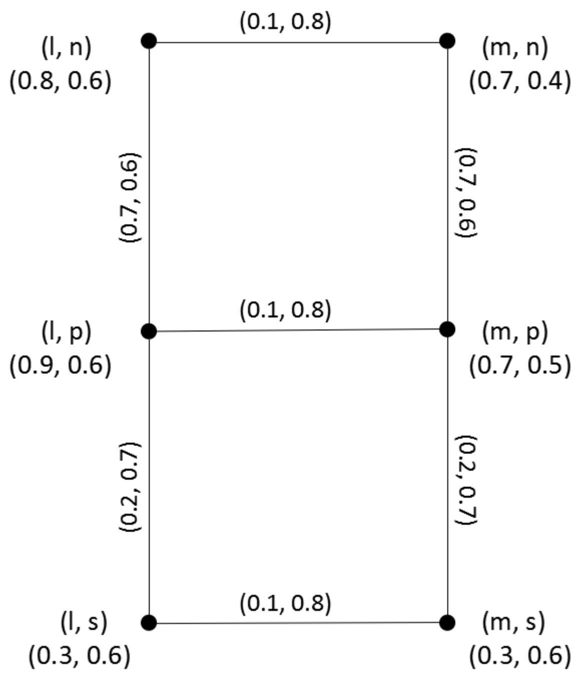


**Figure 4.** Cartesian product of two $q$-ROFGs.

**Definition 13.** *Let* $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ *and* $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ *be two q-ROFGs of the graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$*, respectively. The semi-strong product of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$*, denoted by* $\mathcal{G}_1 \bullet \mathcal{G}_2 = (\mathcal{P}_1 \bullet \mathcal{P}_2, \mathcal{Q}_1 \bullet \mathcal{Q}_2)$*, is defined as:*

*(i)* $$\begin{cases}(\mu_{\mathcal{P}_1} \bullet \mu_{\mathcal{P}_2})(x_1,x_2) = \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2)\\ (\nu_{\mathcal{P}_1} \bullet \nu_{\mathcal{P}_2})(x_1,x_2) = \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2) \text{ for all } (x_1,x_2) \in V_1 \times V_2,\end{cases}$$

*(ii)* $$\begin{cases}(\mu_{\mathcal{Q}_1} \bullet \mu_{\mathcal{Q}_2})(x,x_2)(x,y_2) = \mu_{\mathcal{P}_1}(x) \wedge \mu_{\mathcal{Q}_2}(x_2y_2),\\ (\nu_{\mathcal{Q}_1} \bullet \nu_{\mathcal{Q}_2})(x,x_2)(x,y_2) = \nu_{\mathcal{P}_1}(x) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \text{ for all } x \in V_1, \text{ for all } x_2y_2 \in E_2,\end{cases}$$

*(iii)* $$\begin{cases}(\mu_{\mathcal{Q}_1} \bullet \mu_{\mathcal{Q}_2})(x_1,x_2)(y_1,y_2) = \mu_{\mathcal{P}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2),\\ (\nu_{\mathcal{Q}_1} \bullet \nu_{\mathcal{Q}_2})(x_1,x_2)(y_1,y_2) = \nu_{\mathcal{P}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \text{ for all } x_1y_1 \in E_1, \text{ for all } x_2y_2 \in E_2.\end{cases}$$

**Remark 6.** *The semi-strong product of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *can be understood that the vertices of* $\mathcal{G}_1$ *combine with the each edge of* $\mathcal{G}_2$ *and the edges of* $\mathcal{G}_1$ *combine with the each edge of* $\mathcal{G}_2$ *to form a new graph* $\mathcal{G}_1 \bullet \mathcal{G}_2$.

**Proposition 3.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs of the graphs $G_1$ and $G_2$, respectively. The semi-strong product $\mathcal{G}_1 \bullet \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is a q-ROFG.*

**Definition 14.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned}(d_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \bullet E_2} (\mu_{\mathcal{Q}_1} \bullet \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2), \\ (d_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \bullet E_2} (\nu_{\mathcal{Q}_1} \bullet \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \nu_{\mathcal{Q}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2).\end{aligned}$$

**Theorem 5.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$ and $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$. Then $(d)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = |c(x_2)|\, d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$ for any $(x_1, x_2) \in V_1 \times V_2$, where $|c(x_2)|$ represents the number of points adjacent to $x_2$ in $\mathcal{G}_2$.*

**Proof.** By definition of degree of a vertex in $\mathcal{G}_1 \bullet \mathcal{G}_2$, we have

$$\begin{aligned}(d_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \bullet E_2} (\mu_{\mathcal{Q}_1} \bullet \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \\ &\quad (\textit{Since } \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2} \textit{ and } \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}) \\ &= \sum_{x_1 = y_1} 1 \times \sum_{x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2y_2 \in E_2} 1 \times \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\ &= \sum_{x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + |c(x_2)| \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\ &= (d_\mu)_{\mathcal{G}_2}(x_2) + |c(x_2)|\, (d_\mu)_{\mathcal{G}_1}(x_1).\end{aligned}$$

Analogously, it is easy to show that $(d_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = |c(x_2)|\, d_\nu \mathcal{G}_1(x_1) + d_\nu \mathcal{G}_2(x_2)$. Hence, $d_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = |c(x_2)|\, d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$. □

**Remark 7.** *In the SVNGs [21] and PFGs [24], if $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$ and $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$, then $(d)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$ (cf. Theorem 3.14 in [21] and Theorem 5 in [24]). It is obvious that they do not consider the effect of $|c(x_2)|$ on the degree under semi-strong product.*

**Definition 15.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$
\begin{aligned}
(td_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \bullet E_2} (\mu_{\mathcal{Q}_1} \bullet \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) + (\mu_{\mathcal{P}_1} \bullet \mu_{\mathcal{P}_2})(x_1, x_2) \\
&= \sum_{x_1=y_1, x_2y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\
&\quad + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2), \\
(td_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \bullet E_2} (\nu_{\mathcal{Q}_1} \bullet \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) + (\nu_{\mathcal{P}_1} \bullet \nu_{\mathcal{P}_2})(x_1, x_2) \\
&= \sum_{x_1=y_1, x_2y_2 \in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \nu_{\mathcal{Q}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \\
&\quad + \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2).
\end{aligned}
$$

**Theorem 6.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For all $(x_1, x_2) \in V_1 \times V_2$,*
*(1) If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, then*

$$(td_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = (|c(x_2)|)(td_\mu)_{\mathcal{G}_1}(x_1) + (td_\mu)_{\mathcal{G}_2}(x_2) + (1 - |c(x_2)|)\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2);$$

*(2) If $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$, then*

$$(td_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = (|c(x_2)|)(td_\nu)_{\mathcal{G}_1}(x_1) + (td_\nu)_{\mathcal{G}_2}(x_2) + (1 - |c(x_2)|)\nu_{\mathcal{P}_1}(x_1) - \nu_{\mathcal{P}_1}(x_1) \wedge \nu_{\mathcal{P}_2}(x_2).$$

*In the above equalities, $|c(x_2)|$ represents the number of points adjacent to $x_2$ in $\mathcal{G}_2$.*

**Proof.** By definition 6 of total degree of a vertex in $\mathcal{G}_1 \bullet \mathcal{G}_2$,

$$
\begin{aligned}
(1)(td_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) &= \sum_{x_1=y_1, x_2y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\
&\quad + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2) \\
&= \sum_{x_1=y_1} 1 \times \sum_{x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2y_2 \in E_2} 1 \times \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) \\
&\quad + \mu_{\mathcal{P}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2) \\
&\quad (\mathit{Since}\ \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2},\ \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}) \\
&= \sum_{x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + |c(x_2)| \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) \\
&\quad + \mu_{\mathcal{P}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2) \\
&= (|c(x_2)|)(td_\mu)_{\mathcal{G}_1}(x_1) + (td_\mu)_{\mathcal{G}_2}(x_2) + (1 - |c(x_2)|)\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2).
\end{aligned}
$$

Analogously, we can prove (2). □

**Remark 8.** *In the PFGs [24], if $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, then*

*$(td_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = (td_\mu)_{\mathcal{G}_1}(x_1) + (td_\mu)_{\mathcal{G}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2)$;*

*If $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$, then*

*$(td_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(x_1, x_2) = (td_\nu)_{\mathcal{G}_1}(x_1) + (td_\nu)_{\mathcal{G}_2}(x_2) - \nu_{\mathcal{P}_1}(x_1) \wedge \nu_{\mathcal{P}_2}(x_2)$ (cf. Theorem 6 in [24]).*

*It is obvious that they do not consider the effect of $|c(x_2)|$, $(1 - |c(x_2)|)\mu_{\mathcal{P}_1}(x_1)$ and $(1 - |c(x_2)|)\nu_{\mathcal{P}_1}(x_1)$ on the total degree under semi-strong product.*

**Example 4.** *Consider two q-ROFGs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 2, where $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$, $\mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$, $\nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$. Their semi-strong product $\mathcal{G}_1 \bullet \mathcal{G}_2$ is shown in Figure 5.*

*By Theorem 5, we can get*

$$\begin{aligned}(d_\mu)_{\mathcal{G}_1\bullet\mathcal{G}_2}(l,p) &= |c(p)|\,(d_\mu)_{\mathcal{G}_1}(l) + (d_\mu)_{\mathcal{G}_2}(p) = |\{n,s\}|\,(d_\mu)_{\mathcal{G}_1}(l) + (d_\mu)_{\mathcal{G}_2}(p)\\ &= 2\times 0.1 + 0.7 + 0.2 = 1.1,\\ (d_\nu)_{\mathcal{G}_1\bullet\mathcal{G}_2}(l,p) &= |c(p)|\,(d_\nu)_{\mathcal{G}_1}(l) + (d_\nu)_{\mathcal{G}_2}(p) = |\{n,s\}|\,(d_\nu)_{\mathcal{G}_1}(l) + (d_\nu)_{\mathcal{G}_2}(p)\\ &= 2\times 0.8 + 0.6 + 0.7 = 2.9.\end{aligned}$$

*Therefore, $d_{\mathcal{G}_1\bullet\mathcal{G}_2}(l,p) = (1.1, 2.9)$. In addition, by Theorem 6, we have*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1\bullet\mathcal{G}_2}(l,p) &= (|c(p)|)(td_\mu)_{\mathcal{G}_1}(l) + (td_\mu)_{\mathcal{G}_2}(p) + (1-|c(p)|)\mu_{\mathcal{P}_1}(l) - \mu_{\mathcal{P}_1}(l)\vee\mu_{\mathcal{P}_2}(p)\\ &= |\{n,s\}|\,(td_\mu)_{\mathcal{G}_1}(l) + (td_\mu)_{\mathcal{G}_2}(p) + (1-|c(p)|)\mu_{\mathcal{P}_1}(l) - \mu_{\mathcal{P}_1}(l)\vee\mu_{\mathcal{P}_2}(p)\\ &= 2\times(0.1+0.9) + 0.7 + 0.2 + 0.9 + (1-2)\times 0.9 - 0.9\vee 0.9 = 2.0,\\ (td_\nu)_{\mathcal{G}_1\bullet\mathcal{G}_2}(l,p) &= (|c(p)|)(td_\nu)_{\mathcal{G}_1}(l) + (td_\nu)_{\mathcal{G}_2}(p) + (1-|c(p)|)\nu_{\mathcal{P}_1}(l) - \nu_{\mathcal{P}_1}(l)\wedge\nu_{\mathcal{P}_2}(p)\\ &= |\{n,s\}|\,(td_\nu)_{\mathcal{G}_1}(l) + (td_\nu)_{\mathcal{G}_2}(p) + (1-|c(p)|)\nu_{\mathcal{P}_1}(l) - \nu_{\mathcal{P}_1}(l)\wedge\nu_{\mathcal{P}_2}(p)\\ &= 2\times(0.8+0.6) + 0.6 + 0.7 + 0.5 + (1-2)\times 0.6 - 0.6\wedge 0.5 = 3.5.\end{aligned}$$

*Therefore, $td_{\mathcal{G}_1\bullet\mathcal{G}_2}(m,p) = (2.0, 3.5)$. Likewise, we can get the degree and total degree of each vertex in $\mathcal{G}_1 \bullet \mathcal{G}_2$.*

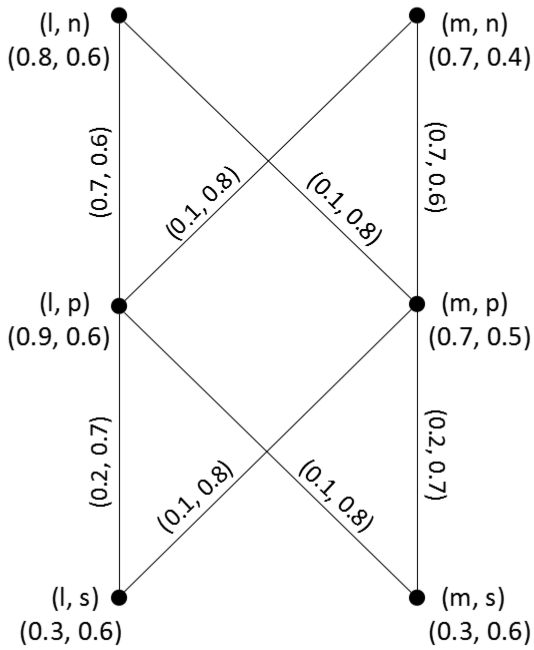


**Figure 5.** Semi-strong product of two $q$-ROFGs.

**Remark 9.** *In the PFGs [24], they do not consider the effect of* $|c(x_2)| = \sum_{x_2y_2 \in E_2} 1$ *and their results fail to work in Example 4. For example, when using theorem 5 in PFGs [24], we can get*

$$\begin{aligned}(d_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) &= (d_\mu)_{\mathcal{G}_1}(l) + (d_\mu)_{\mathcal{G}_2}(p) = (d_\mu)_{\mathcal{G}_1}(l) + (d_\mu)_{\mathcal{G}_2}(p) \\ &= 0.1 + 0.7 + 0.2 = 1.0, \\ (d_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) &= (d_\nu)_{\mathcal{G}_1}(l) + (d_\nu)_{\mathcal{G}_2}(p) = (d_\nu)_{\mathcal{G}_1}(l) + (d_\nu)_{\mathcal{G}_2}(p) \\ &= 0.8 + 0.6 + 0.7 = 2.1.\end{aligned}$$

*However,* $(d_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) = 1.1 \neq 1.0$ *and* $(d_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) = 2.9 \neq 2.1$.

*When using Theorem 6 in PFGs [24], we can get*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) &= (td_\mu)_{\mathcal{G}_1}(l) + (td_\mu)_{\mathcal{G}_2}(p) - \mu_{\mathcal{P}_1}(l) \vee \mu_{\mathcal{P}_2}(p) \\ &= (0.1 + 0.9) + 0.7 + 0.2 + 0.9 - 0.9 \vee 0.9 = 1.9, \\ (td_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) &= (td_\nu)_{\mathcal{G}_1}(l) + (td_\nu)_{\mathcal{G}_2}(p) - \nu_{\mathcal{P}_1}(l) \wedge \nu_{\mathcal{P}_2}(p) \\ &= (0.8 + 0.6) + 0.6 + 0.7 + 0.5 - 0.6 \wedge 0.5 = 2.7.\end{aligned}$$

*However,* $(td_\mu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) = 2.0 \neq 1.9$ *and* $(td_\nu)_{\mathcal{G}_1 \bullet \mathcal{G}_2}(l,p) = 3.5 \neq 2.7$.

**Definition 16.** *Let* $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ *and* $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ *be two q-ROFGs of the* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$*, respectively. The strong product of these two q-ROFGs is denoted by* $\mathcal{G}_1 \boxtimes \mathcal{G}_2 = (\mathcal{P}_1 \boxtimes \mathcal{P}_2, \mathcal{Q}_1 \boxtimes \mathcal{Q}_2)$ *and defined as:*

$$\text{(i)} \quad \begin{cases} (\mu_{\mathcal{P}_1} \boxtimes \mu_{\mathcal{P}_2})(x_1,x_2) = \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2) \\ (\nu_{\mathcal{P}_1} \boxtimes \nu_{\mathcal{P}_2})(x_1,x_2) = \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2) \; for \; all \; (x_1,x_2) \in V_1 \times V_2, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} (\mu_{\mathcal{Q}_1} \boxtimes \mu_{\mathcal{Q}_2})(x,x_2)(x,y_2) = \mu_{\mathcal{P}_1}(x) \wedge \mu_{\mathcal{Q}_2}(x_2y_2), \\ (\nu_{\mathcal{Q}_1} \boxtimes \nu_{\mathcal{Q}_2})(x,x_2)(x,y_2) = \nu_{\mathcal{P}_1}(x) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \; for \; all \; x \in V_1, \; for \; all \; x_2y_2 \in E_2, \end{cases}$$

$$\text{(iii)} \quad \begin{cases} (\mu_{\mathcal{Q}_1} \boxtimes \mu_{\mathcal{Q}_2})(x_1,z)(y_1,z) = \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{P}_2}(z) \\ (\nu_{\mathcal{Q}_1} \boxtimes \nu_{\mathcal{Q}_2})(x_1,z)(y_1,z) = \nu_{\mathcal{Q}_1}(x_1x_2) \vee \nu_{\mathcal{P}_2}(z) \; for \; all \; z \in V_2, \; for \; all \; x_1y_1 \in E_1, \end{cases}$$

$$\text{(iv)} \quad \begin{cases} (\mu_{\mathcal{Q}_1} \boxtimes \mu_{\mathcal{Q}_2})(x_1,x_2)(y_1,y_2) = \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\ (\nu_{\mathcal{Q}_1} \boxtimes \nu_{\mathcal{Q}_2})(x_1,x_2)(y_1,y_2) = \nu_{\mathcal{Q}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \; for \; all \; x_1y_1 \in E_1, \; for \; all \; x_2y_2 \in E_2. \end{cases}$$

**Remark 10.** *The strong product of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *can be understood that the vertices of* $\mathcal{G}_1$ *combine with the each edge of* $\mathcal{G}_2$*, the vertices of* $\mathcal{G}_2$ *combine with the each edge of* $\mathcal{G}_1$ *and the edges of* $\mathcal{G}_1$ *combine with the each edge of* $\mathcal{G}_2$ *to form a new graph* $\mathcal{G}_1 \boxtimes \mathcal{G}_2$.

**Proposition 4.** *Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be the q-ROFGs of the graphs* $G_1$ *and* $G_2$*, respectively. The strong product* $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ *of* $\mathcal{G}_1$ *and* $\mathcal{G}_1$ *is a q-ROFG.*

**Definition 17.** *Let* $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ *and* $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ *be two q-ROFGs. For any vertex* $(x_1,x_2) \in V_1 \times V_2$,

$$\begin{aligned}(d_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1,x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (\mu_{\mathcal{Q}_1} \boxtimes \mu_{\mathcal{Q}_2})((x_1,x_2)(y_1,y_2)) \\ &= \sum_{x_1 = y_1, x_2y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2 = y_2, x_1y_1 \in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\ &\quad + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2),\end{aligned}$$

$$
\begin{aligned}
(d_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (\nu_{\mathcal{Q}_1} \boxtimes \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2y_2 \in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2 = y_2, x_1y_1 \in E_1} \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \nu_{\mathcal{Q}_1}(x_1y_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2).
\end{aligned}
$$

**Theorem 7.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$, $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$, $\mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$, $\nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$. Then, for all $(x_1, x_2) \in V_1 \boxtimes V_2$, $d_{\mathcal{G}_1 \times \mathcal{G}_2}(x_1, x_2) = (1 + |c(x_2)|)\, d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$, where $|c(x_2)|$ represents the number of points adjacent to $x_2$ in $\mathcal{G}_2$.*

**Proof.** By definition of degree of a vertex in $\mathcal{G}_1 \boxtimes \mathcal{G}_2$, we have

$$
\begin{aligned}
(d_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (\mu_{\mathcal{Q}_1} \boxtimes \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1 = y_1, x_2y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2 = y_2, x_1y_1 \in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\
&= \sum_{x_1 = y_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2 = y_2, x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) + \sum_{x_1y_1 \in E_1, x_2y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad (\textit{Since } \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2},\ \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1} \textit{ and } \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}) \\
&= \sum_{x_1 = y_1} 1 \times \sum_{x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2 = y_2} 1 \times \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2y_2 \in E_2} 1 \times \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= \sum_{x_2y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) + |c(x_2)| \sum_{x_1y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= (d_\mu)_{\mathcal{G}_2}(x_2) + (d_\mu)_{\mathcal{G}_1}(x_1) + |c(x_2)|\, (d_\mu)_{\mathcal{G}_1}(x_1) \\
&= (d_\mu)_{\mathcal{G}_2}(x_2) + (1 + |c(x_2)|)\, (d_\mu)_{\mathcal{G}_1}(x_1).
\end{aligned}
$$

Analogously, it is easy to show that $(d_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = (1 + |c(x_2)|)\, (d_\nu)_{\mathcal{G}_1}(x_1) + (d_\nu)_{\mathcal{G}_2}(x_2)$. Hence, $d_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = (1 + |c(x_2)|)\, d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$. □

**Remark 11.** *In the SVNGs [21] and PFGs [24], If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$, $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$, $\mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$, $\nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$, then $d_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = |V_2|\, d_{\mathcal{G}_1}(x_1) + d_{\mathcal{G}_2}(x_2)$, where $|V_2|$ represents the number of vertices in $\mathcal{G}_2$ (cf. Theorem 3.19 in [21] and Theorem 7 in [24]). It is obvious that they do not consider the effect of $|c(x_2)|$ on the degree under strong product.*

**Definition 18.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned}
(td_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (\mu_{\mathcal{Q}_1} \boxtimes \mu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) + (\mu_{\mathcal{P}_1} \boxtimes \mu_{\mathcal{P}_2})(x_1, x_2) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad + \sum_{x_1 y_1 \in E_1, x_2 y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_1 y_1) \wedge \mu_{\mathcal{Q}_2}(x_2 y_2) + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2), \\
(td_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2) \in E_1 \boxtimes E_2} (\nu_{\mathcal{Q}_1} \boxtimes \nu_{\mathcal{Q}_2})((x_1, x_2)(y_1, y_2)) + (\nu_{\mathcal{P}_1} \boxtimes \nu_{\mathcal{P}_2})(x_1, x_2) \\
&= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad + \sum_{x_1 y_1 \in E_1, x_2 y_2 \in E_2} \nu_{\mathcal{Q}_1}(x_1 y_1) \vee \nu_{\mathcal{Q}_2}(x_2 y_2) + \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

**Theorem 8.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any $(x_1, x_2) \in V_1 \times V_2$,*

*(1) If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}, \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, then*

$$(td_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = (td_\mu)_{\mathcal{G}_2}(x_2) + (1 + |c(x_2)|)\,(td_\mu)_{\mathcal{G}_1}(x_1) - |c(x_2)|\,\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2);$$

*(2) If $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}, \mu_{\mathcal{Q}_1} \geq \mu_{\mathcal{Q}_2}$, then*

$$(td_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = (td_\nu)_{\mathcal{G}_2}(x_2) + (1 + |c(x_2)|)\,(td_\nu)_{\mathcal{G}_1}(x_1) - |c(x_2)|\,\nu_{\mathcal{P}_1}(x_1) - \nu_{\mathcal{P}_1}(x_1) \wedge \nu_{\mathcal{P}_2}(x_2).$$

*In the above equalities, $|c(x_2)|$ represents the number of points adjacent to $x_2$ in $\mathcal{G}_2$.*

**Proof.** For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned}
(1)\quad (td_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) &= \sum_{x_1 = y_1, x_2 y_2 \in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2, x_1 y_1 \in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad + \sum_{x_1 y_1 \in E_1, x_2 y_2 \in E_2} \mu_{\mathcal{Q}_1}(x_1 y_1) \wedge \mu_{\mathcal{Q}_2}(x_2 y_2) + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2), \\
&= \sum_{x_1 = y_1} 1 \times \sum_{x_2 y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2 y_2) + \sum_{x_2 = y_2} 1 \times \sum_{x_1 y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad + \sum_{x_2 y_2 \in E_2} 1 \times \sum_{x_1 y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1 y_1) \\
&\quad + \mu_{\mathcal{P}_1}(x_1) + \mu_{\mathcal{P}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2) \ (\text{since } \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2},\ \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1},\ \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}) \\
&= \sum_{x_2 y_2 \in E_2} \mu_{\mathcal{Q}_2}(x_2 y_2) + \mu_{\mathcal{P}_2}(x_2) + (1 + |c(x_2)|) \sum_{x_1 y_1 \in E_1} \mu_{\mathcal{Q}_1}(x_1 y_1) + (1 + |c(x_2)|)\,\mu_{\mathcal{P}_1}(x_1) \\
&\quad - ((1 + |c(x_2)|) - 1)\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2) \\
&= (td_\mu)_{\mathcal{G}_2}(x_2) + (1 + |c(x_2)|)\,(td_\mu)_{\mathcal{G}_1}(x_1) \\
&\quad - ((1 + |c(x_2)|) - 1)\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2) \\
&= (td_\mu)_{\mathcal{G}_2}(x_2) + (1 + |c(x_2)|)\,(td_\mu)_{\mathcal{G}_1}(x_1) - |c(x_2)|\,\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

Analogously, we can prove (2). □

**Remark 12.** *In the PFGs [24], if $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}, \mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, then*
*$(td_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = (td_\mu)_{\mathcal{G}_2}(x_2) + |V_2|\,(td_\mu)_{\mathcal{G}_1}(x_1) - (|V_2| - 1)\,\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2)$;*
*If $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}, \mu_{\mathcal{Q}_1} \geq \mu_{\mathcal{Q}_2}$, then*

$(td_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(x_1, x_2) = (td_\nu)_{\mathcal{G}_2}(x_2) + |V_2|\,(td_\nu)_{\mathcal{G}_1}(x_1) - (|V_2|-1)\,\nu_{\mathcal{P}_1}(x_1) - \nu_{\mathcal{P}_1}(x_1) \wedge \nu_{\mathcal{P}_2}(x_2)$ *(cf. Theorem 8 in [24]).*

*It is obvious that they do not consider the effect of $|c(x_2)|$ on the total degree under strong product.*

**Example 5.** *Consider two q-ROFGs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 2, where $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$, $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$, $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$, $\mu_{\mathcal{Q}_1} \leq \mu_{\mathcal{Q}_2}$, $\nu_{\mathcal{Q}_1} \geq \nu_{\mathcal{Q}_2}$ and their strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is shown in Figure 6.*

*By Theorem 7, we have*

$$
\begin{aligned}
(d_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) &= (d_\mu)_{\mathcal{G}_2}(p) + (1+|c(p)|)\,(d_\mu)_{\mathcal{G}_1}(l) \\
&= (d_\mu)_{\mathcal{G}_2}(p) + (1+|\{n,s\}|)\,(d_\mu)_{\mathcal{G}_1}(l) \\
&= 0.7 + 0.2 + (1+2) \times 0.1 = 1.2, \\
(d_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) &= (d_\nu)_{\mathcal{G}_2}(p) + (1+|c(p)|)\,(d_\nu)_{\mathcal{G}_1}(l) \\
&= (d_\nu)_{\mathcal{G}_2}(p) + (1+|\{n,s\}|)\,(d_\nu)_{\mathcal{G}_1}(l) \\
&= 0.6 + 0.7 + (1+2) \times 0.8 = 3.7.
\end{aligned}
$$

*Therefore, $d_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) = (1.2, 3.7)$. In addition, by Theorem 8, we have*

$$
\begin{aligned}
(td_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) &= (td_\mu)_{\mathcal{G}_2}(p) + (1+|c(p)|)\,(td_\mu)_{\mathcal{G}_1}(l) - |c(p)|\,\mu_{\mathcal{P}_1}(l) - \mu_{\mathcal{P}_1}(l) \vee \mu_{\mathcal{P}_2}(p) \\
&= (td_\mu)_{\mathcal{G}_2}(p) + (1+|\{n,s\}|)\,(td_\mu)_{\mathcal{G}_1}(l) - |\{n,s\}|\,\mu_{\mathcal{P}_1}(l) - \mu_{\mathcal{P}_1}(l) \vee \mu_{\mathcal{P}_2}(p) \\
&= 0.7 + 0.2 + 0.9 + (1+2) \times (0.1+0.9) - 2 \times 0.9 - 0.9 \vee 0.9 = 2.1, \\
(td_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) &= (td_\nu)_{\mathcal{G}_2}(p) + (1+|c(p)|)\,(td_\nu)_{\mathcal{G}_1}(l) - |c(p)|\,\nu_{\mathcal{P}_1}(l) - \nu_{\mathcal{P}_1}(l) \wedge \nu_{\mathcal{P}_2}(p) \\
&= (td_\nu)_{\mathcal{G}_2}(p) + (1+|\{n,s\}|)\,(td_\nu)_{\mathcal{G}_1}(l) - |\{n,s\}|\,\nu_{\mathcal{P}_1}(l) - \nu_{\mathcal{P}_1}(l) \wedge \nu_{\mathcal{P}_2}(p) \\
&= 0.6 + 0.7 + 0.5 + (1+2) \times (0.8+0.6) - 2 \times 0.6 - 0.6 \wedge 0.5 = 4.3.
\end{aligned}
$$

*Therefore, $td_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) = (2.1, 4.3)$. Likewise, we can find the degree and total degree of each vertex in $\mathcal{G}_1 \boxtimes \mathcal{G}_2$.*

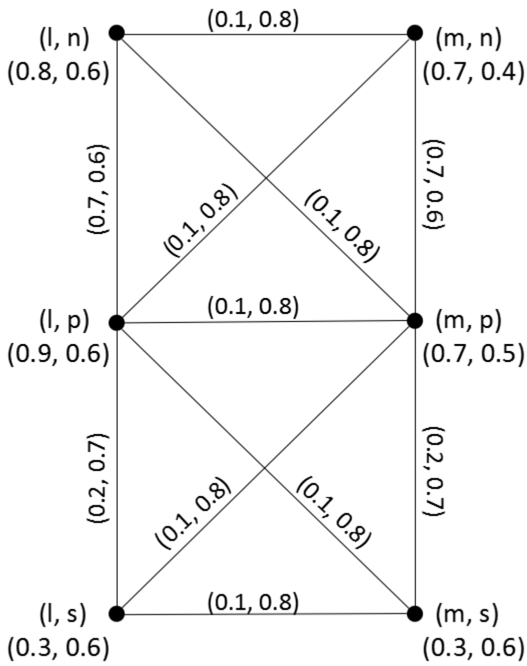


**Figure 6.** Strong product of two $q$-ROFGs.

**Remark 13.** *In the PFGs [24], they do not consider the effect of* $|c(x_2)| = \sum\limits_{x_2y_2 \in E_2} 1$. *For example, when using theorem 7 in PFGs [24], we can get*

$$(d_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) = (d_\mu)_{\mathcal{G}_2}(p) + p_2 (d_\mu)_{\mathcal{G}_1}(l) = 0.7 + 0.2 + 3 \times 0.1 = 1.2,$$
$$(d_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) = (d_\nu)_{\mathcal{G}_2}(p) + p_2 (d_\nu)_{\mathcal{G}_1}(l) = 0.6 + 0.7 + 3 \times 0.8 = 3.7.$$

*When using theorem 8 in PFGs [24], we can get*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) &= (td_\mu)_{\mathcal{G}_2}(p) + (p_2)\,(td_\mu)_{\mathcal{G}_1}(l) - (p_2 - 1)\,\mu_{\mathcal{P}_1}(l) - \mu_{\mathcal{P}_1}(l) \vee \mu_{\mathcal{P}_2}(p)\\ &= 0.7 + 0.2 + 0.9 + 3 \times (0.1 + 0.9) - (3-1) \times 0.9 - 0.9 \vee 0.9 = 2.1,\\ (td_\nu)_{\mathcal{G}_1 \boxtimes \mathcal{G}_2}(l,p) &= (td_\nu)_{\mathcal{G}_2}(p) + (p_2)\,(td_\nu)_{\mathcal{G}_1}(l) - (p_2 - 1)\,\nu_{\mathcal{P}_1}(l) - \nu_{\mathcal{P}_1}(l) \wedge \nu_{\mathcal{P}_2}(p)\\ &= 0.6 + 0.7 + 0.5 + 3 \times (0.8 + 0.6) - (3-1) \times 0.6 - 0.6 \wedge 0.5 = 4.3.\end{aligned}$$

*Although they get the same values as the Example 5, but the variable means different things.* $p_2$ *is represented by number of points in* $\mathcal{G}_2$. *Actually,* $p_2$ *should be replaced by* $1 + |c(x_2)|$ *in Example 5.*

**Definition 19.** *Let* $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ *and* $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ *be two q-ROFGs of the* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$, *respectively. The lexicographic product of these two q-ROFGs is denoted by* $\mathcal{G}_1[\mathcal{G}_2] = (\mathcal{P}_1[\mathcal{P}_2], \mathcal{Q}_1[\mathcal{Q}_2])$ *and defined as follows:*

$$\text{(i)} \quad \begin{cases} (\mu_{\mathcal{P}_1}[\mu_{\mathcal{P}_2}])(x_1,x_2) = \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2) \\ (\nu_{\mathcal{P}_1}[\nu_{\mathcal{P}_2}])(x_1,x_2) = \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2) \; for\ all\ (x_1,x_2) \in V_1 \times V_2, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} (\mu_{\mathcal{Q}_1}[\mu_{\mathcal{Q}_2}])(x,x_2)(x,y_2) = \mu_{\mathcal{P}_1}(x) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) \\ (\nu_{\mathcal{Q}_1}[\nu_{\mathcal{Q}_2}])(x,x_2)(x,y_2) = \nu_{\mathcal{P}_1}(x) \vee \nu_{\mathcal{Q}_2}(x_2y_2) \; for\ all\ x \in V_1,\ for\ all\ x_2y_2 \in E_2, \end{cases}$$

$$\text{(iii)} \quad \begin{cases} (\mu_{\mathcal{Q}_1}[\mu_{\mathcal{Q}_2}])(x_1,z)(y_1,z) = \mu_{\mathcal{Q}_1}(x_1y_1) \wedge \mu_{\mathcal{P}_2}(z) \\ (\nu_{\mathcal{Q}_1}[\nu_{\mathcal{Q}_2}])(x_1,z)(y_1,z) = \nu_{\mathcal{Q}_1}(x_1x_2) \vee \nu_{\mathcal{P}_2}(z) \; for\ all\ z \in V_2,\ for\ all\ x_1y_1 \in E_1, \end{cases}$$

$$\text{(iv)} \quad \begin{cases} (\mu_{\mathcal{Q}_1}[\mu_{\mathcal{Q}_2}])(x_1,x_2)(y_1,y_2) = \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{P}_2}(y_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\ (\nu_{\mathcal{Q}_1}[\nu_{\mathcal{Q}_2}])(x_1,x_2)(y_1,y_2) = \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{P}_2}(y_2) \vee \nu_{\mathcal{Q}_1}(x_1y_1) \; for\ all\ x_1y_1 \in E_1, x_2 \neq y_2. \end{cases}$$

**Remark 14.** *The lexicographic product of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *can be understood that the vertices of* $\mathcal{G}_1$ *combine with the each edge of* $\mathcal{G}_2$, *the vertices of* $\mathcal{G}_2$ *combine with the each edge of* $\mathcal{G}_1$ *and the edges of* $\mathcal{G}_1$ *combine with the two different vertices of* $\mathcal{G}_2$ *to form a new graph* $\mathcal{G}_1[\mathcal{G}_2]$.

**Proposition 5.** *The lexicographic product* $\mathcal{G}_1[\mathcal{G}_2]$ *of two q-ROFGs of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *is a q-ROFG.*

**Definition 20.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned}
(d_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1[E_2]} (\mu_{\mathcal{Q}_1}[\mu_{\mathcal{Q}_2}])((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(y_2) \wedge \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1), \\
(d_\nu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1[E_2]} (\nu_{\mathcal{Q}_1}[\nu_{\mathcal{Q}_2}])((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2, x_1y_1\in E_1} \nu_{\mathcal{P}_2}(y_2) \vee \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1y_1).
\end{aligned}$$

**Theorem 9.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}, \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$ and $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}, \nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$. Then, $d_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) = (d_\mu)_{\mathcal{G}_2}(x_2) + |V_2|\,(d_\mu)_{\mathcal{G}_1}(x_1)$, for any $(x_1, x_2) \in V_1 \times V_2$, where $|V_2|$ represents the number of vertices in $\mathcal{G}_2$.*

**Proof.** For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned}
(d_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1[E_2]} (\mu_{\mathcal{Q}_1}[\mu_{\mathcal{Q}_2}])((x_1, x_2)(y_1, y_2)) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(y_2) \wedge \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) + \sum_{x_2\neq y_2, x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad (\textit{Since } \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2} \textit{ and } \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}) \\
&= \sum_{x_1=y_1} 1 \times \sum_{x_2y_2\in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2} 1 \times \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2} 1 \times \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= \sum_{x_2y_2\in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \left(\sum_{x_2=y_2} 1 + \sum_{x_2\neq y_2} 1\right) \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&= (d_\mu)_{\mathcal{G}_2}(x_2) + |V_2|\,(d_\mu)_{\mathcal{G}_1}(x_1).
\end{aligned}$$

Analogously, we can show that $(d_\nu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) = (d_\nu)_{\mathcal{G}_2}(x_2) + |V_2|\,(d_\nu)_{\mathcal{G}_1}(x_1)$. Hence, $(d)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) = d_{\mathcal{G}_2}(x_2) + |V_2|\,d_{\mathcal{G}_1}(x_1)$. □

**Definition 21.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any vertex $(x_1, x_2) \in V_1 \times V_2$,*

$$\begin{aligned}
(td_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1[E_2]} (\mu_{\mathcal{Q}_1}[\mu_{\mathcal{Q}_2}])((x_1, x_2)(y_1, y_2)) + (\mu_{\mathcal{P}_1}[\mu_{\mathcal{P}_2}])(x_1, x_2) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(y_2) \wedge \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2), \\
(td_\nu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) &= \sum_{(x_1,x_2)(y_1,y_2)\in E_1\circ E_2} (\nu_{\mathcal{Q}_1}[\nu_{\mathcal{Q}_2}])((x_1, x_2)(y_1, y_2)) + (\nu_{\mathcal{P}_1}[\nu_{\mathcal{P}_2}])(x_1, x_2) \\
&= \sum_{x_1=y_1, x_2y_2\in E_2} \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2, x_1y_1\in E_1} \nu_{\mathcal{P}_2}(y_2) \vee \nu_{\mathcal{P}_2}(x_2) \vee \nu_{\mathcal{Q}_1}(x_1y_1) + \nu_{\mathcal{P}_1}(x_1) \vee \nu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

**Theorem 10.** *Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{Q}_1)$ and $\mathcal{G}_2 = (\mathcal{P}_2, \mathcal{Q}_2)$ be two q-ROFGs. For any $(x_1, x_2) \in V_1 \times V_2$,*
*(1) If $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$ and $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$, then*

$$(td_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) = (td_\mu)_{\mathcal{G}_2}(x_2) + |V_2|\,(td_\mu)_{\mathcal{G}_1}(x_1) - (|V_2| - 1)\mu_{\mathcal{P}_1}(x_1) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2);$$

*(2) If $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$ and $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$, then*

$$(td_\nu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) = (td_\nu)_{\mathcal{G}_2}(x_2) + |V_2|\,(td_\nu)_{\mathcal{G}_1}(x_1) - (|V_2| - 1)\nu_{\mathcal{P}_1}(x_1) - \nu_{\mathcal{P}_1}(x_1) \wedge \nu_{\mathcal{P}_2}(x_2).$$

*In the above equalities, $|V_2|$ represents the number of vertices in $\mathcal{G}_2$.*

**Proof.** For any vertex $(x_1, x_2) \in V_1 \times V_2$,

$$\begin{aligned}
(1)(td_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(x_1, x_2) &= \sum_{x_1=y_1, x_2y_2\in E_2} \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2, x_1y_1\in E_1} \mu_{\mathcal{P}_2}(y_2) \wedge \mu_{\mathcal{P}_2}(x_2) \wedge \mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) \wedge \mu_{\mathcal{P}_2}(x_2) \\
&= \sum_{x_1=y_1} 1 \times \sum_{x_2y_2\in E_2} \mu_{\mathcal{Q}_2}(x_2y_2) + \sum_{x_2=y_2} 1 \times \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) \\
&\quad + \sum_{x_2\neq y_2} 1 \times \sum_{x_1y_1\in E_1} \mu_{\mathcal{Q}_1}(x_1y_1) + \mu_{\mathcal{P}_1}(x_1) + \mu_{\mathcal{P}_2}(x_2) - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2) \\
&\quad (\text{Since } \mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2},\ \mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}) \\
&= (td_\mu)_{\mathcal{G}_2}(x_2) + |V_2|\,(td_\mu)_{\mathcal{G}_1}(x_1) - (|V_2| - 1)\mu_{\mathcal{P}_1}(x_1) \\
&\quad - \mu_{\mathcal{P}_1}(x_1) \vee \mu_{\mathcal{P}_2}(x_2).
\end{aligned}$$

Analogously, we can prove (2). □

**Example 6.** *Consider two q-ROFGs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 2, where $\mu_{\mathcal{P}_1} \geq \mu_{\mathcal{Q}_2}$, $\mu_{\mathcal{P}_2} \geq \mu_{\mathcal{Q}_1}$ and $\nu_{\mathcal{P}_1} \leq \nu_{\mathcal{Q}_2}$, $\nu_{\mathcal{P}_2} \leq \nu_{\mathcal{Q}_1}$ and their lexicographic product $\mathcal{G}_1[\mathcal{G}_2]$ is shown in Figure 7.*

*By Theorem 9, we have*

$$\begin{aligned}
(d_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(l, p) &= |V_2|\,(d_\mu)_{\mathcal{G}_1}(l) + (d_\mu)_{\mathcal{G}_2}(p) \\
&= 3 \times 0.1 + 0.7 + 0.2 = 1.2, \\
(d_\nu)_{\mathcal{G}_1[\mathcal{G}_2]}(l, p) &= |V_2|\,(d_\nu)_{\mathcal{G}_1}(l) + (d_\nu)_{\mathcal{G}_2}(p) \\
&= 3 \times 0.8 + 0.6 + 0.7 = 3.7.
\end{aligned}$$

*Therefore,* $d_{\mathcal{G}_1[\mathcal{G}_2]}(l,p) = (1.2, 3.7)$. *In addition, by Theorem 10, we must have*

$$\begin{aligned}(td_\mu)_{\mathcal{G}_1[\mathcal{G}_2]}(l,p) &= |V_2|\,(td_\mu)_{\mathcal{G}_1}(l) + (td_\mu)_{\mathcal{G}_2}(p) - (|V_2|-1)\mu_{\mathcal{P}_1}(l) - \mu_{\mathcal{P}_1}(l) \vee \mu_{\mathcal{P}_2}(p)\\ &= 3 \times (0.1+0.9) + 0.7 + 0.2 + 0.9 - (3-1) \times 0.9 - 0.9 \vee 0.9 = 2.1,\\ (td_\nu)_{\mathcal{G}_1[\mathcal{G}_2]}(l,p) &= |V_2|\,(td_\nu)_{\mathcal{G}_1}(l) + (td_\nu)_{\mathcal{G}_2}(p) - (|V_2|-1)\nu_{\mathcal{P}_1}(l) - \nu_{\mathcal{P}_1}(l) \wedge \nu_{\mathcal{P}_2}(p)\\ &= 3 \times (0.8+0.6) + 0.6 + 0.7 + 0.5 - (3-1) \times 0.6 - 0.6 \wedge 0.5 = 4.3.\end{aligned}$$

*Therefore,* $td_{\mathcal{G}_1[\mathcal{G}_2]}(l,p) = (2.1, 4.3)$. *Likewise, we can get the degree and total degree of each vertex in* $\mathcal{G}_1[\mathcal{G}_2]$.

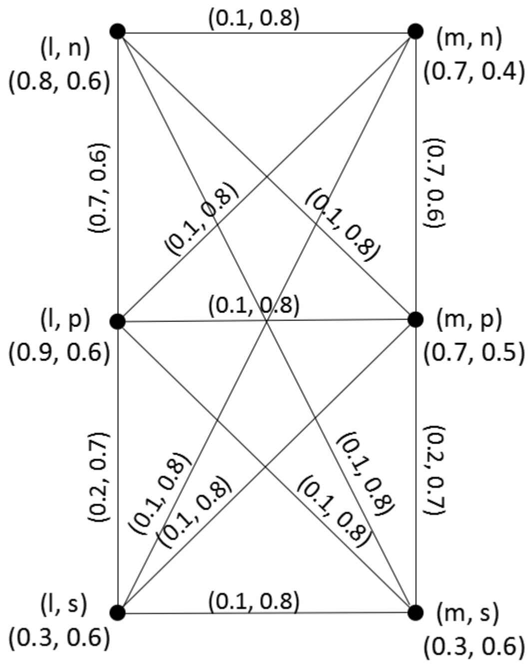


**Figure 7.** Lexicographic product of two $q$-ROFGs.

## 5. Conclusions

Our paper contributes to the literature on fuzzy graphs in several ways. First, the degree and total degree of a vertex in $q$-ROFGs are defined. The implications of the degree and total degree of a vertex in $q$-ROFGs are illustrated by the example of road network. The degree and total degree of a vertex help one understand the properties of the product operations on $q$-ROFGs. Second, product operations on $q$-ROFGs, including direct product, Cartesian product, semi-strong product, strong product and lexicographic product, are defined. The product operations on $q$-ROFGs simplify the number of $q$-ROFGs and will be helpful when the $q$-ROFGs are very large. Third, some general theorems about the degree and total degree under the defined product operations on $q$-ROFGs are obtained. We illustrate these theorems through some examples. These theorems improve the similar results in SVNGs and PFGs. More specifically, these theorems show that the degree (or total degree) of a vertex in product operations on $q$-ROFGs are not only related to the degree (or total degree) of vertices but also the number of adjacent points, which is omitted in the SVNGs and PFGs.

In the future, we are working to extend our study to: (1) $q$-rung orthopair fuzzy soft graphs; (2) Rough $q$-rung orthopair fuzzy graphs; (3) Simplified interval-valued $q$-rung orthopair fuzzy graphs and; (4) Hesitant $q$-rung orthopair fuzzy graphs.

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