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A Self-Adaptive Extra-Gradient Methods for a Family of Pseudomonotone Equilibrium Programming with Application in Different Classes of Variational Inequality Problems

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Abstract: The main objective of this article is to propose a new method that would extend Popov's extragradient method by changing two natural projections with two convex optimization problems. We also show the weak convergence of our designed method by taking mild assumptions on a cost bifunction. The method is evaluating only one value of the bifunction per iteration and it is uses an explicit formula for identifying the appropriate stepsize parameter for each iteration. The variable stepsize is going to be effective for enhancing iterative algorithm performance. The variable stepsize is updating for each iteration based on the previous iterations. After numerical examples, we conclude that the effect of the inertial term and variable stepsize has a significant improvement over the processing time and number of iterations.

Keywords: subgradient extragradient method; equilibrium problem; pseudomonotone equilibrium problems; lipschitz-type conditions; weak convergence; variational inequality problems

1. Introduction

Let C to be a nonempty convex, closed subset of a Hilbert space $\mathbb{E}$ and $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be a bifunction with $f(u,u) = 0$ for each $u \in C$. The *equilibrium problem* for f upon C is defined as follows:

$$\text{Find} \quad p^* \in C \quad \text{such that} \quad f(p^*, y) \geq 0, \quad \forall\, y \in C. \tag{1}$$

The equilibrium problem (EP) has many mathematical problems as a particular case, for example, the fixed point problems, complementarity problems, the variational inequality problems (VIP), the minimization problems, Nash equilibrium of noncooperative games, saddle point problems and problem of vector minimization (see [1–4]). The unique formulation of an equilibrium problem was specifically defined in 1992 by Muu and Oettli [5] and further developed by Blum and Oettli [1]. An equilibrium problem is also known as the Ky Fan inequality problem. Fan [6] presents a review and gives specific conditions on a bifunction for the existence of an equilibrium point. Many researchers have provided and generalized many results corresponding to the existence of a solution for the equilibrium problem (see [7–10]). A considerable number of methods are the earliest set up over the last few years concentrating on the different equilibrium problem classes and other particular forms of an equilibrium problem in abstract spaces (see [11–29]).

The Korpelevich and Antipin's extragradient method [30,31] are efficient two-step methods. Flam [12,20] employed the auxiliary problem principle to set up the extragradient method for the monotone equilibrium problems. The consideration on the extragradient method is to figure out two natural projections on C to achieve the next iteration. If the computing of a projection on a feasible set C is hard to compute, it is a challenge to solve two minimal distance problems for the next iteration, which may have an effect on method's performance and efficiency. In order to overcome it, Censor initiated a subgradient extragradient method [32] where the second projection is replaced by a half-plane projection that can be computed effectively. Iterative sequences set up with the above-mentioned extragradient-like methods need to make use of a certain stepsize constant based on the Lipschitz-type constants of a cost bifunction. The prior knowledge about these constant imposes some restrictions on developing an iterative sequence because these Lipschitz-type constants are normally not known or hard to compute.

In 2016, Lyashko et al. [33] developed an extragradient method for solving pseudomonotone equilibrium problems in a real Hilbert space. It is required to solve two optimization problems on a closed convex set for each next iteration, with a reasonable fixed stepsize depends upon on the Lipschitz-type constants. The superiority of the Lyashko et al. [33] method compared to the Tran et al. [20] extragradient method is that the value of the bifunction f is to determine only once for each iteration. Inertial-type methods are based on the discrete variant of a second-order dissipative dynamical system. In order to handle numerically smooth convex minimization problem, Polyak [34] proposed an iterative scheme that would require inertial extrapolation as a boost ingredient to improve the convergence rate of the iterative sequence. The inertial method is commonly a two-step iterative scheme and the next iteration is computed by use of previous two iterations and may be pointed out to as a method of pacing up the iterative sequence (see [34,35]). In the case of equilibrium problems, Moudafi established the second-order differential proximal method [36]. These inertial methods are employed to accelerate the iterative process for the desired solution. Numerical studies indicate that inertial effects generally enhance the performance of the method in terms of the number of iterations and execution time in this context. There are many methods established for the different classes of variational inequality problems (for more details see [37–41]).

In this study, we considered Lyashko et al. [33] and Liu et al. [42] extragradient methods and present its improvement by employing an inertial scheme. We also improved the stepsize to its second step. The stepsize was not fixed in our proposed method, but the stepsize was set up by an explicit formula based on some previous iterations. We formulated a weak convergence theorem for our proposed method for dealing with the problems of equilibriums involving pseudomonotone bifunction within specific conditions. We also examined how our results are linked to variational inequality problems. Apart from this, we considered the well-known Nash–Cournot equilibrium model as a test problem to support the validity of our results. Some applications for variational inequality problems were considered and other numerical examples were explained to back the appropriateness of our designed results.

The rest of the article is set up as follows: In Section 2 we give a few definitions and significant results to be utilized in this paper. Section 3 includes our first algorithm involving pseudomonotone bifunction, and gives the weak convergence result. Section 4 illustrates some application of our results in variational inequality problems. Section 5 sets out numerical examinations to describe numerical performance.

2. Preliminaries

In this part we cover some relevant lemmas, definitions and other notions that will be employed throughout the convergence analysis and numerical part. The notion $\langle ., . \rangle$ and $\|.\|$ presents for the inner product and norm on the Hilbert space $\mathbb{E}$. Let $G : \mathbb{E} \to \mathbb{E}$ be a well-defined operator and $VI(G, C)$ is the solution set of a variational inequality problem corresponding operator G over the set C. Moreover $EP(f, C)$ stands for the solution set of an equilibrium problem over the set C and p^* is any arbitrary element of $EP(f, C)$ or $VI(G, C)$.

Let $g : C \to \mathbb{R}$ be a convex function with *subdifferential of* g at $u \in C$ defined as:

$$\partial g(u) = \{z \in \mathbb{E} : g(v) - g(u) \geq \langle z, v - u \rangle, \ \forall \, v \in C\}.$$

A *normal cone of* C at $u \in C$ is given as

$$N_C(u) = \{z \in \mathbb{E} : \langle z, v - u \rangle \leq 0, \ \forall \, v \in C\}.$$

We consider various conceptions of a bifunction monotonicity (see [1,43] for details).

Definition 1. *The bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ on C for $\gamma > 0$ is*

(i) strongly monotone if $f(u, v) + f(v, u) \leq -\gamma \|u - v\|^2, \ \forall \, u, v \in C$;
(ii) monotone if $f(u, v) + f(v, u) \leq 0, \ \forall \, u, v \in C$;
(iii) strongly pseudomonotone if $f(u, v) \geq 0 \Longrightarrow f(v, u) \leq -\gamma \|u - v\|^2, \ \forall \, u, v \in C$;
(iv) pseudomonotone if $f(u, v) \geq 0 \Longrightarrow f(v, u) \leq 0, \ \forall \, u, v \in C$;
(v) satisfying the Lipschitz-type condition on C if there are two real numbers $c_1, c_2 > 0$ such that

$$f(u, w) \leq f(u, v) + f(v, w) + c_1 \|u - v\|^2 + c_2 \|v - w\|^2, \ \forall \, u, v, w \in C,$$

holds.

Definition 2. *[44] A metric projection $P_C(u)$ of u onto a closed, convex subset C of $\mathbb{E}$ is defined as follows:*

$$P_C(u) = \arg\min_{v \in C} \{\|v - u\|\}$$

Lemma 1. *[45] Let $P_C : \mathbb{E} \to C$ be metric projection from $\mathbb{E}$ upon C. Thus*

(i) For each $u \in C, v \in \mathbb{E}$,

$$\|u - P_C(v)\|^2 + \|P_C(v) - v\|^2 \leq \|u - v\|^2.$$

(ii) $w = P_C(u)$ if and only if

$$\langle u - w, v - w \rangle \leq 0, \ \forall \, v \in C.$$

This portion concludes with a few crucial lemmas which are advantageous in investigating the convergence of our proposed results.

Lemma 2. *[46] Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{E}$ and $h : C \to \mathbb{R}$ be a convex, subdifferentiable and lower semi-continuous function on C. Moreover, $x \in C$ is a minimizer of a* *function h if and only if* $0 \in \partial h(x) + N_C(x)$ *where* $\partial h(x)$ *and* $N_C(x)$ *stands for the subdifferential of h at x and the normal cone of C at x respectively.*

Lemma 3 ([47], Page 31)**.** *For every* $a, b \in \mathbb{E}$ *and* $\xi \in \mathbb{R}$ *the following relation is true:*

$$\|\xi a + (1-\xi)b\|^2 = \xi\|a\|^2 + (1-\xi)\|b\|^2 - \xi(1-\xi)\|a-b\|^2.$$

Lemma 4. *[48] If* α_n*,* β_n *and* γ_n *are sequences in* $[0, +\infty)$*,*

$$\alpha_{n+1} \leq \alpha_n + \beta_n(\alpha_n - \alpha_{n-1}) + \gamma_n,\ \forall\, n \geq 1, \quad \textit{with} \quad \sum_{n=1}^{+\infty} \gamma_n < +\infty$$

holds with $\beta > 0$ *such that* $0 \leq \beta_n \leq \beta < 1, \forall\, n \in \mathbb{N}$*. The following items are true.*

(i) $\sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty$*, with* $[p]_+ := \max\{p, 0\}$*;*

(ii) $\lim_{n\to+\infty} \alpha_n = \alpha^* \in [0, \infty)$.

Lemma 5. *[49] Let* $\{\eta_n\}$ *be a sequence in* $\mathbb{E}$ *and* $C \subset \mathbb{E}$ *such that*

(i) *For each* $\eta \in C$*,* $\lim_{n\to\infty} \|\eta_n - \eta\|$ *exists;*

(ii) *All sequentially weak cluster point of* $\{\eta_n\}$ *lies in C;*

Then $\{\eta_n\}$ *weakly converges to a element of C.*

Lemma 6. *[50] Assume* $\{a_n\}, \{b_n\}$ *are real sequences such that* $a_n \leq b_n\ \forall\, n \in \mathbb{N}$*. Take* $\varrho, \sigma \in (0, 1)$ *and* $\mu \in (0, \sigma)$*. Then there is a sequence* λ_n *in a manner that* $\lambda_n a_n \leq \mu b_n$ *and* $\lambda_n \in (\varrho\mu, \sigma)$*.*

Due to Lipschitz-like condition on a bifunction f through above lemma, we have the following inequality.

Corollary 1. *Assume that bifunction f satisfy the Lipschitz-type condition on C through positive constants* c_1 *and* c_2*. Let* $\varrho \in (0, 1)$*,* $\sigma < \min\left\{\frac{1-3\theta}{(1-\theta)^2+4c_1(\theta+\theta^2)}, \frac{1}{2c_2+4c_1(1+\theta)}\right\}$ *where* $\theta \in [0, \frac{1}{3})$ *and* $\mu \in (0, \sigma)$*. Then there exits a positive real number* λ *such that*

$$\lambda\big(f(u, w) - f(u, v) - c_1\|u - v\|^2 - c_2\|v - w\|^2\big) \leq \mu f(v, w)$$

and $\varrho\mu < \lambda < \sigma$ *where* $u, v, w \in C$*.*

Assumption 1. *Let a bifunction* $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ *satisfies*

f_1. $f(v, v) = 0$ *for all* $v \in C$ *and f is pseudomonotone on feasible set C.*

f_2. *f satisfy the Lipschitz-type condition on* $\mathbb{E}$ *with constants* c_1 *and* c_2*.*

f_3. $\limsup_{n\to\infty} f(x_n, v) \leq f(x^*, v)$ *for all* $v \in C$ *and* $\{x_n\} \subset C$ *satisfy* $x_n \rightharpoonup x^*$*.*

f_4. $f(u, .)$ *need to be convex and subdifferentiable over* $\mathbb{E}$ *for all fixed* $u \in \mathbb{E}$*.*

Since $f(u, .)$ is convex and subdifferentiable on $\mathbb{E}$ for each fixed $u \in \mathbb{E}$ and *subdifferential of* $f(u, .)$ at $x \in \mathbb{E}$ defined as:

$$\partial_2 f(u, .)(x) = \partial_2 f(u, x) = \{z \in \mathbb{E} : f(u, v) - f(u, x) \geq \langle z, v - x\rangle,\ \forall\, v \in \mathbb{E}\}.$$

3. An Algorithm and Its Convergence Analysis

We develop a method and provide a weak convergence result for it. We consider bifunction f that satisfies the conditions of Assumption 1 and $EP(f, C) \neq \emptyset$. The detailed method is written below.

Lemma 7. *If a sequence $\{u_n\}$ is set up by Algorithm 1. Then the following relationship holds.*

$$\mu\lambda_n f(v_n, y) - \mu\lambda_n f(v_n, u_{n+1}) \geq \langle w_n - u_{n+1}, y - u_{n+1}\rangle, \ \forall\, y \in E_n.$$

Proof. By definition of u_{n+1} we have

$$u_{n+1} = \arg\min_{y \in E_n}\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|w_n - y\|^2\}.$$

By using Lemma 2, we obtain

$$0 \in \partial_2\Big\{\mu\lambda_n f(v_n, y) + \frac{1}{2}\|w_n - y\|^2\Big\}(u_{n+1}) + N_{E_n}(u_{n+1}).$$

From the above expression there is a $\omega \in \partial_2 f(v_n, u_{n+1})$ and $\overline{\omega} \in N_{E_n(}u_{n+1})$ such that

$$\mu\lambda_n\omega + u_{n+1} - w_n + \overline{\omega} = 0.$$

Thus, we have

$$\langle w_n - u_{n+1}, y - u_{n+1}\rangle = \mu\lambda_n\langle\omega, y - u_{n+1}\rangle + \langle\overline{\omega}, y - u_{n+1}\rangle, \ \forall\, y \in E_n.$$

Since $\overline{\omega} \in N_{E_n}(u_{n+1})$ then $\langle\overline{\omega}, y - u_{n+1}\rangle \leq 0$, for all $y \in E_n$. Thus, we have

$$\mu\lambda_n\langle\omega, y - u_{n+1}\rangle \geq \langle w_n - u_{n+1}, y - u_{n+1}\rangle, \ \forall\, y \in E_n. \tag{2}$$

Since $\omega \in \partial_2 f(v_n, u_{n+1})$ we obtain

$$f(v_n, y) - f(v_n, u_{n+1}) \geq \langle\omega, y - u_{n+1}\rangle, \ \forall\, y \in \mathbb{E}. \tag{3}$$

Combining the expressions of Equations (2) and (3) we get

$$\mu\lambda_n f(v_n, y) - \mu\lambda_n f(v_n, u_{n+1}) \geq \langle w_n - u_{n+1}, y - u_{n+1}\rangle, \ \forall\, y \in E_n.$$

□

Lemma 8. *Let sequence $\{v_n\}$ be generated by Algorithm 1. Then the following inequality holds.*

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle w_{n+1} - v_{n+1}, y - v_{n+1}\rangle, \ \forall\, y \in C.$$

Proof. By definition of v_{n+1}, we have

$$0 \in \partial_2\Big\{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|w_{n+1} - y\|^2\Big\}(v_{n+1}) + N_C(v_{n+1}).$$

Thus, there is a $\omega \in \partial_2 f(v_n, v_{n+1})$ and $\overline{\omega} \in N_C(v_{n+1})$ such that

$$\lambda_{n+1}\omega + v_{n+1} - w_{n+1} + \overline{\omega} = 0.$$

The above expression implies that

$$\langle w_{n+1} - v_{n+1}, y - v_{n+1}\rangle = \lambda_{n+1}\langle\omega, y - v_{n+1}\rangle + \langle\overline{\omega}, y - v_{n+1}\rangle, \ \forall\, y \in C.$$

Since $\overline{\omega} \in N_C(v_{n+1})$ then $\langle\overline{\omega}, y - v_{n+1}\rangle \leq 0$, for all $y \in C$. This implies that

$$\lambda_{n+1}\langle\omega, y - v_{n+1}\rangle \geq \langle w_{n+1} - v_{n+1}, y - v_{n+1}\rangle, \ \forall\, y \in C. \tag{4}$$

By $\omega \in \partial_2 f(v_n, v_{n+1})$, we can obtain

$$f(v_n, y) - f(v_n, v_{n+1}) \geq \langle \omega, y - v_{n+1} \rangle, \ \forall\, y \in \mathbb{E}. \tag{5}$$

Combining the expressions in Equations (4) and (5) we get

$$\lambda_{n+1} f(v_n, y) - \lambda_{n+1} f(v_n, v_{n+1}) \geq \langle w_{n+1} - v_{n+1}, y - v_{n+1} \rangle, \ \forall\, y \in C.$$

□

Algorithm 1 (The Modified Popov's subgradient extragradient method for pseudomonotone *EP*)

Initialization: Choose $u_{-1}, v_{-1}, u_0, v_0 \in \mathbb{E}$, $\varrho \in (0,1)$, $\sigma < \min\left\{\frac{1-3\theta}{(1-\theta)^2+4c_1(\theta+\theta^2)}, \frac{1}{2c_2+4c_1(1+\theta)}\right\}$ for a nondecreasing sequence θ_n such that $0 \leq \theta_n \leq \theta < \frac{1}{3}$ and $\lambda_0 > 0$.

Iterative steps: Let u_{n-1}, v_{n-1}, u_n and v_n are known for $n \geq 0$. Construct a half-space

$$E_n = \{z \in \mathbb{E} : \langle w_n - \lambda_n \omega_{n-1} - v_n, z - v_n \rangle \leq 0\},$$

where $\omega_{n-1} \in \partial_2 f(v_{n-1}, v_n)$ and $w_n = u_n + \theta_n(u_n - u_{n-1})$.

Step 1: Compute

$$u_{n+1} = \arg\min_{y \in E_n} \{\mu \lambda_n f(v_n, y) + \frac{1}{2}\|w_n - y\|^2\}.$$

Step 2: Revised the stepsize as follows

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu f(v_n, u_{n+1})}{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - c_1\|v_{n-1} - v_n\|^2 - c_2\|u_{n+1} - v_n\|^2 + 1}\right\} \tag{6}$$

and compute

$$v_{n+1} = \arg\min_{y \in C} \{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|w_{n+1} - y\|^2\},$$

where $w_{n+1} = u_{n+1} + \theta_{n+1}(u_{n+1} - u_n)$.

Step 3: If $v_n = v_{n-1}$ and $u_{n+1} = w_n$, then stop. Else, take $n := n+1$ and return back to **Iterative steps**.

Lemma 9. *Let $\{u_n\}$ and $\{v_n\}$ are sequences generated by Algorithm 1. Then the following inequality is true.*

$$\lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle w_n - v_n, u_{n+1} - v_n \rangle.$$

Proof. Since $u_{n+1} \in E_n$ then by the definition of E_n gives that

$$\langle w_n - \lambda_n \omega_{n-1} - v_n, u_{n+1} - v_n \rangle \leq 0.$$

The above implies that

$$\lambda_n \langle \omega_{n-1}, u_{n+1} - v_n \rangle \geq \langle w_n - v_n, u_{n+1} - v_n \rangle. \tag{7}$$

By $\omega_{n-1} \in \partial f(v_{n-1}, v_n)$ with $y = u_{n+1}$, we reach the following

$$f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) \geq \langle \omega_{n-1}, u_{n+1} - v_n \rangle, \ \forall y \in \mathbb{E}. \tag{8}$$

By combining Equations (7) and (8), we obtain

$$\lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle w_n - v_n, u_{n+1} - v_n \rangle.$$

□

Lemma 10. *If $u_{n+1} = w_n$ and $v_n = v_{n-1}$ in Algorithm 1. Then, v_n is the solution of Equation (1).*

Proof. Setting $u_{n+1} = w_n$ and $v_n = v_{n-1}$ in Lemma 9, we get

$$\lambda_n f(v_n, u_{n+1}) \geq 0. \tag{9}$$

By the means of $u_{n+1} = w_n$ in Lemma 7, we get

$$\mu\lambda_n f(v_n, y) \geq \mu\lambda_n f(v_n, u_{n+1}) \geq 0, \ \forall\, y \in E_n. \tag{10}$$

Since $\mu \in (0,1)$ and $\lambda_n \in (0, \infty)$ then $f(v_n, y) > 0$, for all $y \in C \subset E_n$. □

Remark 1. *(i). If $u_{n+1} = v_n = w_n$ in Algorithm 1, then $v_n \in EP(f, C)$. It is obvious from Lemma 7.*
(ii). If $w_{n+1} = v_{n+1} = v_n$ in Algorithm 1, then $v_n \in EP(f, C)$. It is obvious from Lemma 8.

Lemma 11. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is satisfying the assumptions (f_1–f_4). Thus, for each $p^* \in EP(f, C) \neq \emptyset$, we have*

$$\begin{aligned} \|u_{n+1} - p^*\|^2 \leq \|w_n - p^*\|^2 - (1 - \lambda_{n+1})\|u_{n+1} - w_n\|^2 + 4c_1\lambda_{n+1}\lambda_n\|w_n - v_{n-1}\|^2 \\ - \lambda_{n+1}(1 - 4c_1\lambda_n)\|w_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2. \end{aligned}$$

Proof. By substituting $y = p^*$ into Lemma 7, we get

$$\mu\lambda_n f(v_n, p^*) - \mu\lambda_n f(v_n, u_{n+1}) \geq \langle w_n - u_{n+1}, p^* - u_{n+1}\rangle, \ \forall\, y \in E_n. \tag{11}$$

By make use of $p^* \in EP(f, C)$ implies that $f(p^*, v_n) \geq 0$. Due to the pseudomonotonicity of a bifunction f we get $f(v_n, p^*) \leq 0$. Therefore, from Equation (11) we get

$$\langle w_n - u_{n+1}, u_{n+1} - p^*\rangle \geq \mu\lambda_n f(v_n, u_{n+1}). \tag{12}$$

Corollary 1 implies that λ_{n+1} in Equation (6) is well-defined and

$$\begin{aligned} &\mu f(v_n, u_{n+1}) \\ &\geq \lambda_{n+1}\big(f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - c_1\|v_{n-1} - v_n\|^2 - c_2\|v_n - u_{n+1}\|^2\big). \end{aligned} \tag{13}$$

The expressions in Equations (12) and (13) imply that

$$\begin{aligned} \langle w_n - u_{n+1}, u_{n+1} - p^*\rangle \geq \lambda_{n+1}\Big[&\lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \\ &- c_1\lambda_n\|v_{n-1} - v_n\|^2 - c_2\lambda_n\|u_{n+1} - v_n\|^2\Big]. \end{aligned} \tag{14}$$

Since $u_{n+1} \in E_n$ and using Lemma 9, we have

$$\lambda_n\{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n)\} \geq \langle w_n - v_n, u_{n+1} - v_n\rangle. \tag{15}$$

Combining the expressions in Equations (14) and (15) we get

$$\begin{aligned} \langle w_n - u_{n+1}, u_{n+1} - p^*\rangle \geq \lambda_{n+1}\Big[&\langle w_n - v_n, u_{n+1} - v_n\rangle \\ &- c_1\lambda_n\|v_{n-1} - v_n\|^2 - c_2\lambda_n\|u_{n+1} - v_n\|^2\Big]. \end{aligned} \tag{16}$$

By vector algebra we have the following facts:

$$2\langle w_n - u_{n+1}, u_{n+1} - p^* \rangle = \|w_n - p^*\|^2 - \|u_{n+1} - w_n\|^2 - \|u_{n+1} - p^*\|^2.$$

$$2\langle w_n - v_n, u_{n+1} - v_n \rangle = \|w_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|w_n - u_{n+1}\|^2.$$

From the above last two inequalities and Equation (16) we obtain

$$\begin{aligned}\|u_{n+1} - p^*\|^2 \le \|w_n - p^*\|^2 - (1 - \lambda_{n+1})\|u_{n+1} - w_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2 \\ - \lambda_{n+1}\|w_n - v_n\|^2 + \lambda_{n+1}(2c_1\lambda_n)\|v_{n-1} - v_n\|^2\end{aligned}$$

By triangle inequality and elementary algebra gives the following inequality

$$\|v_{n-1} - v_n\|^2 \le \left(\|v_{n-1} - w_n\| + \|w_n - v_n\|\right)^2 \le 2\|v_{n-1} - w_n\|^2 + 2\|w_n - v_n\|^2.$$

From the above two inequalities we have the desired result

$$\begin{aligned}\|u_{n+1} - p^*\|^2 \le \|w_n - p^*\|^2 - (1 - \lambda_{n+1})\|u_{n+1} - w_n\|^2 + 4c_1\lambda_n\lambda_{n+1}\|w_n - v_{n-1}\|^2 \\ - \lambda_{n+1}(1 - 4c_1\lambda_n)\|w_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2.\end{aligned}$$

□

Theorem 1. *Suppose a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is satisfying the Assumption 1. Then for all $p^* \in EP(f, C) \neq \emptyset$, the sequences $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ are generated by Algorithm 1 weakly converge to $p^* \in EP(f, C)$.*

Proof. From Lemma 11 we have

$$\begin{aligned}\|u_{n+1} - p^*\|^2 \le \|w_n - p^*\|^2 - (1 - \lambda_{n+1})\|u_{n+1} - w_n\|^2 + 4c_1\lambda_n\lambda_{n+1}\|w_n - v_{n-1}\|^2 \\ - \lambda_{n+1}(1 - 4c_1\lambda_n)\|w_n - v_n\|^2 - \lambda_{n+1}(1 - 2c_2\lambda_n)\|u_{n+1} - v_n\|^2.\end{aligned} \tag{17}$$

By definition of w_n in the Algorithm 1 we may write

$$\begin{aligned}\|w_n - v_{n-1}\|^2 &= \|u_n + \theta_n(u_n - u_{n-1}) - v_{n-1}\|^2 \\ &= \|(1 + \theta_n)(u_n - v_{n-1}) - \theta_n(u_{n-1} - v_{n-1})\|^2 \\ &= (1 + \theta_n)\|u_n - v_{n-1}\|^2 - \theta_n\|u_{n-1} - v_{n-1}\|^2 + \theta_n(1 + \theta_n)\|u_n - u_{n-1}\|^2 \\ &\le (1 + \theta)\|u_n - v_{n-1}\|^2 + \theta(1 + \theta)\|u_n - u_{n-1}\|^2.\end{aligned} \tag{18}$$

Adding the value $4c_1\sigma\lambda_{n+1}(1 + \theta)\|u_{n+1} - v_n\|^2$ on both sides of expression in Equation (17) and for each $n \ge 1$, we obtain

$$
\begin{aligned}
&\|u_{n+1}-p^*\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2\\
&\leq \|w_n-p^*\|^2-(1-\sigma)\|u_{n+1}-w_n\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2\\
&\quad+4c_1\sigma\lambda_n\Big[(1+\theta)\|u_n-v_{n-1}\|^2+\theta(1+\theta)\|u_n-u_{n-1}\|^2\Big]\\
&\quad-\lambda_{n+1}(1-4c_1\sigma)\|w_n-v_n\|^2-\lambda_{n+1}(1-2c_2\sigma)\|u_{n+1}-v_n\|^2 \qquad (19)\\
&\leq \|w_n-p^*\|^2-(1-\sigma)\|u_{n+1}-w_n\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2\\
&\quad+4c_1\sigma(\theta+\theta^2)\|u_n-u_{n-1}\|^2-\lambda_{n+1}(1-4c_1\sigma)\|w_n-v_n\|^2\\
&\quad-\lambda_{n+1}(1-2c_2\sigma-4c_1\sigma(1+\theta))\|u_{n+1}-v_n\|^2 \qquad (20)\\
&\leq \|w_n-p^*\|^2-(1-\sigma)\|u_{n+1}-w_n\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2\\
&\quad+4c_1\sigma(\theta+\theta^2)\|u_n-u_{n-1}\|^2\\
&\quad-\frac{\lambda_{n+1}}{2}(1-2c_2\sigma-4c_1\sigma(1+\theta))\Big[2\|u_{n+1}-v_n\|^2+2\|w_n-v_n\|^2\Big] \qquad (21)\\
&\leq \|w_n-p^*\|^2-(1-\sigma)\|u_{n+1}-w_n\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2\\
&\quad+4c_1\sigma(\theta+\theta^2)\|u_n-u_{n-1}\|^2\\
&\quad-\frac{\lambda_{n+1}}{2}(1-2c_2\sigma-4c_1\sigma(1+\theta))\|u_{n+1}-w_n\|^2. \qquad (22)
\end{aligned}
$$

By Algorithm 1, $0<\lambda_n\leq\sigma<\frac{1}{2c_2+4c_1(1+\theta)}$ and the above inequality ensures

$$
\begin{aligned}
&\|u_{n+1}-p^*\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2\\
&\leq \|w_n-p^*\|^2-(1-\sigma)\|u_{n+1}-w_n\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2+4c_1\sigma(\theta+\theta^2)\|u_n-u_{n-1}\|^2. \qquad (23)
\end{aligned}
$$

From definition of w_n in Algorithm 1 we obtain

$$
\begin{aligned}
\|w_n-p^*\|^2&=\|u_n+\theta_n(u_n-u_{n-1})-p^*\|^2\\
&=\|(1+\theta_n)(u_n-p^*)-\theta_n(u_{n-1}-p^*)\|^2\\
&=(1+\theta_n)\|u_n-p^*\|^2-\theta_n\|u_{n-1}-p^*\|^2+\theta_n(1+\theta_n)\|u_n-u_{n-1}\|^2. \qquad (24)
\end{aligned}
$$

By definition of w_{n+1} and through Cauchy inequality, we achieve

$$
\begin{aligned}
\|u_{n+1}-w_n\|^2&=\|u_{n+1}-u_n-\theta_n(u_n-u_{n-1})\|^2\\
&=\|u_{n+1}-u_n\|^2+\theta_n^2\|u_n-u_{n-1}\|^2-2\theta_n\langle u_{n+1}-u_n,u_n-u_{n-1}\rangle \qquad (25)\\
&\geq\|u_{n+1}-u_n\|^2+\theta_n^2\|u_n-u_{n-1}\|^2-2\theta_n\|u_{n+1}-u_n\|\|u_n-u_{n-1}\|\\
&\geq\|u_{n+1}-u_n\|^2+\theta_n^2\|u_n-u_{n-1}\|^2-\theta_n\|u_{n+1}-u_n\|^2-\theta_n\|u_n-u_{n-1}\|^2\\
&=(1-\theta_n)\|u_{n+1}-u_n\|^2+(\theta_n^2-\theta_n)\|u_n-u_{n-1}\|^2. \qquad (26)
\end{aligned}
$$

By combining the expressions of Equations (23), (24) and (26) we have

$$
\begin{aligned}
&\|u_{n+1}-p^*\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2\\
&\le(1+\theta_n)\|u_n-p^*\|^2-\theta_n\|u_{n-1}-p^*\|^2+\theta_n(1+\theta_n)\|u_n-u_{n-1}\|^2\\
&\quad-(1-\sigma)\Big[(1-\theta_n)\|u_{n+1}-u_n\|^2+(\theta_n^2-\theta_n)\|u_n-u_{n-1}\|^2\Big]\\
&\quad+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2+4c_1\sigma(\theta+\theta^2)\|u_n-u_{n-1}\|^2 \qquad (27)\\
&\le(1+\theta_n)\|u_n-p^*\|^2-\theta_n\|u_{n-1}-p^*\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2\\
&\quad+\Big[\theta(1+\theta)-(1-\sigma)(\theta_n^2-\theta_n)+4c_1\sigma(\theta+\theta^2)\Big]\|u_n-u_{n-1}\|^2\\
&\quad-(1-\sigma)(1-\theta_n)\|u_{n+1}-u_n\|^2 \qquad (28)\\
&\le(1+\theta_n)\|u_n-p^*\|^2-\theta_n\|u_{n-1}-p^*\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2\\
&\quad+\phi_n\|u_n-u_{n-1}\|^2-\psi_n\|u_{n+1}-u_n\|^2, \qquad (29)
\end{aligned}
$$

where

$$\phi_n=\Big[\theta(1+\theta)-(1-\sigma)(\theta_n^2-\theta_n)+4c_1\sigma(\theta+\theta^2)\Big];$$

$$\psi_n=(1-\sigma)(1-\theta_n).$$

Suppose that

$$\Psi_n=\Phi_n+\phi_n\|u_n-u_{n-1}\|^2$$

where $\Phi_n=\|u_n-p^*\|^2-\theta_n\|u_{n-1}-p^*\|^2+4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2$. We compute the following by expression in Equation (29) we obtain

$$
\begin{aligned}
&\Psi_{n+1}-\Psi_n\\
&=\|u_{n+1}-p^*\|^2-\theta_{n+1}\|u_n-p^*\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2+\phi_{n+1}\|u_{n+1}-u_n\|^2\\
&\quad-\|u_n-p^*\|^2+\theta_n\|u_{n-1}-p^*\|^2-4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2-\phi_n\|u_n-u_{n-1}\|^2\\
&\le\|u_{n+1}-p^*\|^2-(1+\theta_n)\|u_n-p^*\|^2+\theta_n\|u_{n-1}-p^*\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2\\
&\quad+\phi_{n+1}\|u_{n+1}-u_n\|^2-4c_1\sigma\lambda_n(1+\theta)\|u_n-v_{n-1}\|^2-\phi_n\|u_n-u_{n-1}\|^2\\
&\le-(\psi_n-\phi_{n+1})\|u_{n+1}-u_n\|^2. \qquad (30)
\end{aligned}
$$

Next, we are going to compute

$$
\begin{aligned}
(\psi_n-\phi_{n+1})&=(1-\sigma)(1-\theta_n)-\theta(1+\theta)+(1-\sigma)(\theta_{n+1}^2-\theta_{n+1})-4c_1\sigma(\theta+\theta^2)\\
&\ge(1-\sigma)(1-\theta)^2-\theta(1+\theta)-4c_1\sigma(\theta+\theta^2)\\
&=(1-\theta)^2-\theta(1+\theta)-\sigma(1-\theta)^2-4c_1\sigma(\theta+\theta^2)\\
&=1-3\theta-\sigma\big((1-\theta)^2+4c_1(\theta+\theta^2)\big)\\
&\ge 0. \qquad (31)
\end{aligned}
$$

Equations (30) and (31) with some $\delta\ge 0$, imply that

$$\Psi_{n+1}-\Psi_n\le-(\psi_n-\phi_{n+1})\|u_{n+1}-u_n\|^2\le-\delta\|u_{n+1}-u_n\|^2\le 0. \qquad (32)$$

The relationship in Equation (32) implies that the sequence $\{\Psi_n\}$ is nonincreasing. Furthermore, by definition of Ψ_{n+1} we have

$$
\begin{aligned}
\Psi_{n+1}&=\|u_{n+1}-p^*\|^2-\theta_{n+1}\|u_n-p^*\|^2+\phi_{n+1}\|u_{n+1}-u_n\|^2+4c_1\sigma\lambda_{n+1}(1+\theta)\|u_{n+1}-v_n\|^2\\
&\ge-\theta_{n+1}\|u_n-p^*\|^2.
\end{aligned} \qquad (33)
$$

Additionally, by definition of Ψ_n we have

$$\begin{aligned}\|u_n - p^*\|^2 &\le \Psi_n + \theta_n\|u_{n-1} - p^*\|^2 \\ &\le \Psi_1 + \theta\|u_{n-1} - p^*\|^2 \\ &\le \cdots \le \Psi_1(\theta^{n-1} + \cdots + 1) + \theta^n\|u_0 - p^*\|^2 \\ &\le \frac{\Psi_1}{1-\theta} + \theta^n\|u_0 - p^*\|^2. \end{aligned} \tag{34}$$

On the basis of Equations (33) and (34) we have

$$\begin{aligned}-\Psi_{n+1} &\le \theta_{n+1}\|u_n - p^*\|^2 \\ &\le \theta\|u_n - p^*\|^2 \\ &\le \theta\frac{\Psi_1}{1-\theta} + \theta^{n+1}\|u_0 - p^*\|^2. \end{aligned} \tag{35}$$

It follows from expressions in Equations (32) and (35) we have

$$\begin{aligned}\delta\sum_{n=1}^{k}\|u_{n+1} - u_n\|^2 &\le \Psi_1 - \Psi_{k+1} \\ &\le \Psi_1 + \theta\frac{\Psi_1}{1-\theta} + \theta^{k+1}\|u_0 - p^*\|^2 \\ &\le \frac{\Psi_1}{1-\theta} + \|u_0 - p^*\|^2, \end{aligned} \tag{36}$$

letting $k \to \infty$ in Equation (36) we have

$$\sum_{n=1}^{\infty}\|u_{n+1} - u_n\|^2 < +\infty \quad \text{implies} \quad \lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{37}$$

By the relationship in Equations (25) with (37) we have

$$\|u_{n+1} - w_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{38}$$

The expression in Equation (35) implies that

$$-\Phi_{n+1} \le \theta\frac{\Psi_1}{1-\theta} + \theta^{n+1}\|u_0 - p^*\|^2 + \phi_{n+1}\|u_{n+1} - u_n\|^2. \tag{39}$$

By Equation (21) we have

$$\begin{aligned}&\lambda_{n+1}(1 - 2c_2\sigma - 4c_1\sigma(1+\theta))\left[\|u_{n+1} - v_n\|^2 + \|w_n - v_n\|^2\right] \\ &\le \Phi_n - \Phi_{n+1} + \theta(1+\theta)\|u_n - u_{n-1}\|^2 + 4c_1\sigma\theta(1+\theta)\|u_{n+1} - u_n\|^2. \end{aligned} \tag{40}$$

Fix $k \in \mathbb{N}$ and use above equation for $n = 1, 2, \cdots, k$. Summing up, we get

$$\begin{aligned}
&\lambda_{n+1}(1-2c_2\sigma-4c_1\sigma(1+\theta))\sum_{n=1}^{k}\Big[\|u_{n+1}-v_n\|^2+\|w_n-v_n\|^2\Big] \\
&\leq \Phi_0-\Phi_{k+1}+\theta(1+\theta)\sum_{n=1}^{k}\|u_n-u_{n-1}\|^2+4c_1\sigma\theta(1+\theta)\sum_{n=1}^{k}\|u_n-u_{n-1}\|^2 \\
&\leq \Phi_0+\theta\frac{\Psi_1}{1-\theta}+\theta^{k+1}\|u_0-p^*\|^2+\phi_{k+1}\|u_{k+1}-u_k\|^2 \\
&\quad+\theta(1+\theta)\sum_{n=1}^{k}\|u_n-u_{n-1}\|^2+4c_1\sigma\theta(1+\theta)\sum_{n=1}^{k}\|u_{n+1}-u_n\|^2, \qquad (41)
\end{aligned}$$

letting $k \to \infty$ in above expression we have

$$\sum_n \|u_{n+1}-v_n\|^2 < +\infty \quad \text{and} \quad \sum_n \|w_n-v_n\|^2 < +\infty \qquad (42)$$

and

$$\lim_{n\to\infty}\|u_{n+1}-v_n\| = \lim_{n\to\infty}\|w_n-v_n\| = 0. \qquad (43)$$

By using the triangular inequality we can easily derive the following from the above-mentioned expressions

$$\lim_{n\to\infty}\|u_n-v_n\| = \lim_{n\to\infty}\|u_n-w_n\| = \lim_{n\to\infty}\|v_{n-1}-v_n\| = 0. \qquad (44)$$

Moreover, we follow the relationship in Equation (27) such that

$$\begin{aligned}
\|u_{n+1}-p^*\|^2 &\leq (1+\theta_n)\|u_n-p^*\|^2-\theta_n\|u_{n-1}-p^*\|^2+\theta(1+\theta)\|u_n-u_{n-1}\|^2 \\
&\quad+4c_1\sigma(1+\theta)\|u_n-v_{n-1}\|^2+4c_1\sigma(\theta+\theta^2)\|u_n-u_{n-1}\|^2. \qquad (45)
\end{aligned}$$

The above expression with Equations (37) and (42) and Lemma 4 suggest that limits of $\|u_n-p^*\|$, $\|w_n-p^*\|$ and $\|v_n-p^*\|$ exist for each $p^* \in EP(f,C)$ and imply that the sequences $\{u_n\}$, $\{w_n\}$ and $\{v_n\}$ are bounded. We require to establish that every weak sequential limit point of the sequence $\{u_n\}$ lies in $EP(f,C)$. Take z to be any sequential weak cluster point of the sequence $\{u_n\}$, i.e., if there exists a weak convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges to z, it implies that $\{v_{n_k}\}$ also weakly converge to z. Our purpose is to prove $z \in EP(f,C)$. Using Lemma 7 with Equations (13) and (15) we obtain

$$\begin{aligned}
\mu\lambda_n f(v_{n_k},y) &\geq \mu\lambda_n f(v_{n_k},u_{n_k+1})+\langle w_{n_k}-u_{n_k+1},y-u_{n_k+1}\rangle \\
&\geq \lambda_n\lambda_{n+1}f(v_{n_k-1},u_{n_k+1})-\lambda_n\lambda_{n+1}f(v_{n_k-1},v_{n_k})-c_1\lambda_n\lambda_{n+1}\|v_{n_k-1}-v_{n_k}\|^2 \\
&\quad-c_2\lambda_n\lambda_{n+1}\|v_{n_k}-u_{n_k+1}\|^2+\langle w_{n_k}-u_{n_k+1},y-u_{n_k+1}\rangle \\
&\geq \lambda_{n+1}\langle w_{n_k}-v_{n_k},u_{n_k+1}-v_{n_k}\rangle-c_1\lambda_n\lambda_{n+1}\|v_{n_k-1}-v_{n_k}\|^2 \\
&\quad-c_2\lambda_n\lambda_{n+1}\|v_{n_k}-u_{n_k+1}\|^2+\langle w_{n_k}-u_{n_k+1},y-u_{n_k+1}\rangle \qquad (46)
\end{aligned}$$

for any member y in $C \subset E_n$. The expressions in Equations (38), (43) and (44) as well as the boundedness of the sequence $\{u_n\}$ mean the right side of the above-mentioned inequality is zero. Taking $\mu, \lambda_n > 0$, condition (f_3) in (Assumption 1) and $v_{n_k} \rightharpoonup z$, we obtain

$$0 \leq \limsup_{k\to\infty} f(v_{n_k},y) \leq f(z,y),\ \forall\, y \in E_n. \qquad (47)$$

Then $z \in C$ implies that $f(z,y) \geq 0$ for all $y \in C \subset E_n$. This determines that $z \in EP(f,C)$. By Lemma 5, the sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ weakly converges to $p^* \in EP(f,C)$. $\square$

We make $\theta_n = 0$ in the Algorithm 1 and by following Theorem 1 we have an improved variant of Liu et al. [42] extragradient method in terms of stepsize.

Corollary 2. *Let a bifunction $f : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ satisfies Assumption 1. For every $p^* \in EP(f, C) \neq \emptyset$, the sequence $\{u_n\}$ and $\{v_n\}$ are set up in the subsequent manner:*

Initialization: *Given $u_0, v_{-1}, v_0 \in \mathbb{E}$, $\varrho \in (0, 1)$, $\sigma < \min\{1, \frac{1}{2c_2+4c_1}\}$, $\mu \in (0, \sigma)$ and $\lambda_0 > 0$.*
Iterative steps: *For given u_n, v_{n-1} and v_n, construct a half-space*

$$E_n = \{z \in \mathbb{E} : \langle u_n - \lambda_n \omega_{n-1} - v_n, z - v_n \rangle \leq 0\},$$

where $\omega_{n-1} \in \partial_2 f(v_{n-1}, v_n)$.
Step 1: *Compute*

$$u_{n+1} = \arg\min_{y \in E_n} \{\mu \lambda_n f(v_n, y) + \frac{1}{2}\|u_n - y\|^2\}.$$

Step 2: *Update the stepsize as follows*

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu f(v_n, u_{n+1})}{f(v_{n-1}, u_{n+1}) - f(v_{n-1}, v_n) - c_1\|v_{n-1} - v_n\|^2 - c_2\|u_{n+1} - v_n\|^2 + 1}\right\}$$

and compute

$$v_{n+1} = \arg\min_{y \in C} \{\lambda_{n+1} f(v_n, y) + \frac{1}{2}\|u_{n+1} - y\|^2\}.$$

Then $\{u_n\}$ and $\{v_n\}$ weakly converge to the solution $p^* \in EP(f, C)$.

4. Solving Variational Inequality Problems with New Self-Adaptive Methods

We consider the application of our above-mentioned results to solve variational inequality problems involving pseudomonotone and Lipschitz-type continuous operator. The variational inequality problem is written in the following way:

$$\text{find} \quad p^* \in C \quad \text{so that} \quad \langle G(p^*), v - p^* \rangle \geq 0, \ \forall\, v \in C.$$

An operator $G : \mathbb{E} \to \mathbb{E}$ is

(i). *monotone* on C if $\langle G(u) - G(v), u - v \rangle \geq 0, \ \forall\, u, v \in C$;
(ii). *L-Lipschitz continuous* on C if $\|G(u) - G(v)\| \leq L\|u - v\|, \ \forall\, u, v \in C$;
(iii). *pseudomonotone* on C if $\langle G(u), v - u \rangle \geq 0 \Rightarrow \langle G(v), u - v \rangle \leq 0, \ \forall\, u, v \in C$.

Note: If we choose the bifunction $f(u, v) := \langle G(u), v - u \rangle$ for all $u, v \in C$ then the equilibrium problem transforms into the above variational inequality problem with $L = 2c_1 = 2c_2$. This means that from the definitions of v_{n+1} in the Algorithm 1 and according to the above definition of bifunction f we have

$$\begin{aligned}
v_{n+1} &= \arg\min_{y\in C}\Big\{\lambda_{n+1}f(v_n,y)+\frac{1}{2}\|w_{n+1}-y\|^2\Big\}\\
&= \arg\min_{y\in C}\Big\{\lambda_{n+1}\langle G(v_n),y-v_n\rangle+\frac{1}{2}\|w_{n+1}-y\|^2\Big\}\\
&= \arg\min_{y\in C}\Big\{\lambda_{n+1}\langle G(v_n),y-w_{n+1}\rangle+\frac{1}{2}\|w_{n+1}-y\|^2+\lambda_{n+1}\langle G(v_n),w_{n+1}-v_n\rangle\Big\}\\
&= \arg\min_{y\in C}\Big\{\frac{1}{2}\|y-(w_{n+1}-\lambda_{n+1}G(v_n)\|^2\Big\}-\frac{\lambda_{n+1}^2}{2}\|G(v_n)\|^2\\
&= P_C(w_{n+1}-\lambda_{n+1}G(v_n)).
\end{aligned}\tag{48}$$

Similarly to the expression in Equation (48) the value u_{n+1} from Algorithm 1 converts into

$$u_{n+1}=P_{E_n}(w_n-\mu\lambda_nG(v_n)).$$

Due to $\omega_{n-1}\in\partial_2f(v_{n-1},v_n)$ and by subdifferential definition we obtain

$$\begin{aligned}
\langle\omega_{n-1},z-v_n\rangle &\le \langle G(v_{n-1}),z-v_{n-1}\rangle-\langle G(v_{n-1}),v_n-v_{n-1}\rangle,\ \forall z\ \in\mathbb{E}\\
&= \langle G(v_{n-1}),z-v_n\rangle,\ \forall\, z\in\mathbb{E}
\end{aligned}\tag{49}$$

and consequently $0\le\langle G(v_{n-1})-\omega_{n-1},z-v_n\rangle$ for all $z\in\mathbb{E}$. This implies that

$$\begin{aligned}
&\langle w_n-\lambda_nG(v_{n-1})-v_n,z-v_n\rangle\\
&\le\langle w_n-\lambda_nG(v_{n-1})-v_n,z-v_n\rangle+\lambda_n\langle G(v_{n-1})-\omega_{n-1},z-v_n\rangle\\
&=\langle w_n-\lambda_n\omega_{n-1}-v_n,z-v_n\rangle.
\end{aligned}\tag{50}$$

Assumption 2. *We assume that G is satisfying the following assumptions:*

G_1^*. *G is monotone on C and $VI(G,C)$ is nonempty;*
G_1. *G is pseudomonotone on C and $VI(G,C)$ is nonempty;*
G_2. *G is L-Lipschitz continuous on C through positive parameter $L>0$;*
G_3. $\limsup_{n\to\infty}\langle G(u_n),v-u_n\rangle\le\langle G(x^*),v-x^*\rangle$ *for every $v\in C$ and $\{u_n\}\subset C$ satisfying $u_n\rightharpoonup x^*$.*

We have reduced the following results from our main results applicable to solve variational inequality problems.

Corollary 3. *Assume that $G:C\to\mathbb{E}$ is satisfying (G_1,G_2,G_3) in Assumption 2. Let $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ be the sequences obtained as follows:*

> ***Initialization:*** *Choose $u_{-1},v_{-1},u_0,v_0\in\mathbb{E}$, $\varrho\in(0,1)$, $\sigma<\min\left\{\frac{1-3\theta}{(1-\theta)^2+2L(\theta+\theta^2)},\frac{1}{3L+2\theta L)}\right\}$ for a nondecreasing sequence θ_n such that $0\le\theta_n\le\theta<\frac{1}{3}$ and $\lambda_0>0$.*
> ***Iterative steps:*** *For given u_{n-1},v_{n-1},u_n and v_n, construct a half space*
>
> $$E_n=\{z\in\mathbb{E}:\langle w_n-\lambda_nGv_{n-1}-v_n,z-v_n\rangle\le0\}$$
>
> *where $w_n=u_n+\theta_n(u_n-u_{n-1})$.*
> ***Step 1:*** *Compute*
>
> $$u_{n+1}=P_{E_n}(w_n-\mu\lambda_nG(v_n)).$$

Step 2: *The stepsize* λ_{n+1} *is updated as follows*

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu\langle Gv_n, u_{n+1} - v_n\rangle}{\langle Gv_{n-1}, u_{n+1} - v_n\rangle - \frac{L}{2}\|v_{n-1} - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}$$

and compute

$$v_{n+1} = P_C(w_{n+1} - \lambda_{n+1}G(v_n)) \quad \textit{where} \quad w_{n+1} = u_{n+1} + \theta_{n+1}(u_{n+1} - u_n).$$

Then the sequence $\{w_n\}$*,* $\{u_n\}$ *and* $\{v_n\}$ *weakly converge to* p^* *of* $VI(G, C)$*.*

Corollary 4. *Assume that* $G : C \to \mathbb{E}$ *is satisfying* (G_1, G_2, G_3) *in Assumption 2. Let* $\{u_n\}$ *and* $\{v_n\}$ *be the sequences obtained as follows:*

Initialization: *Choose* $v_{-1}, u_0, v_0 \in \mathbb{E}$*,* $\varrho \in (0, 1)$*,* $\sigma < \min\{1, \frac{1}{3L}\}$ *and* $\lambda_0 > 0$*.*
Iterative steps: *For given* v_{n-1}, u_n *and* v_n*, construct a half space*

$$E_n = \{z \in \mathbb{E} : \langle u_n - \lambda_n Gv_{n-1} - v_n, z - v_n\rangle \leq 0\}.$$

Step 1: *Compute*

$$u_{n+1} = P_{E_n}(u_n - \mu\lambda_n G(v_n)).$$

Step 2: *The stepsize* λ_{n+1} *is updated as follows*

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu\langle Gv_n, u_{n+1} - v_n\rangle}{\langle Gv_{n-1}, u_{n+1} - v_n\rangle - \frac{L}{2}\|v_{n-1} - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}$$

and compute

$$v_{n+1} = P_C(u_{n+1} - \lambda_{n+1}G(v_n)).$$

Thus $\{u_n\}$ *and* $\{v_n\}$ *converge weakly to the solution* p^* *of* $VI(G, C)$*.*

We examine that if G is monotone then condition (G_3) can be removed. The assumption (G_3) is required to specify $f(u, v) = \langle G(u), v - u\rangle$ complies with the condition (f_3). In addition, condition (f_3) is required to show $z \in EP(f, C)$ after the inequality in Equation (47). This implies that the condition (G_3) is used to prove $z \in VI(G, C)$. Now we will prove that $z \in VI(G, C)$ by using the monotonicity of operator G. Since G is monotone, we have

$$\langle G(y), y - v_n\rangle \geq \langle G(v_n), y - v_n\rangle, \ \forall\, y \in \mathbb{E}. \tag{51}$$

By $f(u, v) = \langle G(u), v - u\rangle$ and Equation (46) we have

$$\limsup_{k\to\infty}\langle G(v_{n_k}), y - v_{n_k}\rangle \geq 0, \ y \in E_n. \tag{52}$$

Combining Equations (51) with (52), we deduce that

$$\limsup_{k\to\infty}\langle G(y), y - v_{n_k}\rangle \geq 0, \ y \in E_n. \tag{53}$$

Since $v_{n_k} \rightharpoonup z \in C$ and $\langle G(y), y - z\rangle \geq 0$ for all $y \in C$. Let $v_t = (1 - t)z + ty$ for all $t \in [0, 1]$. Due to convexity of C the value $v_t \in C$ for each $t \in (0, 1)$. We obtain

$$0 \leq \langle G(v_t), v_t - z\rangle = t\langle G(v_t), y - z\rangle \tag{54}$$

That is $\langle G(v_t), y - z \rangle \geq 0$ for all $t \in (0,1)$. By $v_t \to z$ as $t \to 0$ and the continuity of G gives $\langle G(z), y - z \rangle \geq 0$ for each $y \in C$, which implies that $z \in VI(G, C)$.

Corollary 5. *Assume that $G : C \to \mathbb{E}$ is satisfying (G_1^*, G_2) in Assumption 2. Let $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ be the sequences obtained as follows:*

Initialization: *Choose $u_{-1}, v_{-1}, u_0, v_0 \in \mathbb{E}$, $\varrho \in (0,1)$, $\sigma < \min\left\{\frac{1-3\theta}{(1-\theta)^2+2L(\theta+\theta^2)}, \frac{1}{3L+2\theta L)}\right\}$ for a nondecreasing sequence θ_n such that $0 \leq \theta_n \leq \theta < \frac{1}{3}$ and $\lambda_0 > 0$.*
Iterative steps: *For given u_{n-1}, v_{n-1}, u_n and v_n, construct a half space*

$$E_n = \{z \in \mathbb{E} : \langle w_n - \lambda_n G v_{n-1} - v_n, z - v_n \rangle \leq 0\}$$

where $w_n = u_n + \theta_n(u_n - u_{n-1})$.
Step 1:

$$u_{n+1} = P_{E_n}(w_n - \mu \lambda_n G(v_n)).$$

Step 2: *The stepsize λ_{n+1} is updated as follows*

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu \langle G v_n, u_{n+1} - v_n \rangle}{\langle G v_{n-1}, u_{n+1} - v_n \rangle - \frac{L}{2}\|v_{n-1} - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}$$

and compute

$$v_{n+1} = P_C(w_{n+1} - \lambda_{n+1} G(v_n)) \quad \textit{where} \quad w_{n+1} = u_{n+1} + \theta_{n+1}(u_{n+1} - u_n).$$

Then the sequences $\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ converges weakly to p^* of $VI(G, C)$.

Corollary 6. *Assume that $G : C \to \mathbb{E}$ is satisfying (G_1^*, G_2) in Assumption 2. Let $\{u_n\}$ and $\{v_n\}$ be the sequences obtained as follows:*

Initialization: *Choose $v_{-1}, u_0, v_0 \in \mathbb{E}$, $\varrho \in (0,1)$, $\sigma < \min\left\{1, \frac{1}{3L}\right\}$ and $\lambda_0 > 0$.*
Iterative steps: *For given v_{n-1}, u_n and v_n, construct a half space*

$$E_n = \{z \in \mathbb{E} : \langle u_n - \lambda_n G v_{n-1} - v_n, z - v_n \rangle \leq 0\}.$$

Step 1:

$$u_{n+1} = P_{E_n}(u_n - \mu \lambda_n G(v_n)).$$

Step 2: *The stepsize λ_{n+1} is updated as follows*

$$\lambda_{n+1} = \min\left\{\sigma, \frac{\mu \langle G v_n, u_{n+1} - v_n \rangle}{\langle G v_{n-1}, u_{n+1} - v_n \rangle - \frac{L}{2}\|v_{n-1} - v_n\|^2 - \frac{L}{2}\|u_{n+1} - v_n\|^2 + 1}\right\}$$

and compute

$$v_{n+1} = P_C(u_{n+1} - \lambda_{n+1} G(v_n)).$$

Then $\{u_n\}$ and $\{v_n\}$ converge weakly to p^* of $VI(G, C)$.

5. Computational Experiment

Numerical results produced in this section show the performance of our proposed methods. The MATLAB codes were running in MATLAB version 9.5 (R2018b) on a PC Intel(R) Core(TM)i5-6200 CPU @ 2.30GHz 2.40GHz, RAM 8.00 GB. In these examples, the x-axis indicates the number of iterations or the execution time (in seconds) and y-axes represents the values $D_n = \|u_{n+1} - u_n\|$. We present the comparison of Algorithm 1 (Algo3) with the Lyashko et al. [33] (Algo1) and Liu et al. [42] (Algo2).

Example 1. *Suppose that* $f : C \times C \to \mathbb{R}$ *is defined by*

$$f(u, v) = \langle Au + Bv + d, v - u \rangle,$$

where $d \in \mathbb{R}^n$ *and* A, B *are matrices of order* n *where* B *an symmetric positive semidefinite and* $B - A$ *is symmetric negative definite with Lipschitz constants are* $c_1 = c_2 = \frac{1}{2}\|A - B\|$ *(for more details see [20]). During Example 1, matrices* A, B *are randomly produced (Two matrices are randomly generated* E *and* F *with entries from* $[-1, 1]$*. The matrix* $B = E^T E$, $S = F^T F$ *and* $A = S + B$*.) and entries of* d *randomly belongs to* $[-1, 1]$*. The constraint set* $C \subset \mathbb{R}^n$ *as*

$$C := \{u \in \mathbb{R}^n : -10 \le u_i \le 10\}.$$

The numerical findings are shown in Figures 1–6 and Table 1 with $v_{-1} = (4, \cdots, 4)$, $u_{-1} = (3, \cdots, 3)$, $u_0 = (1, \cdots, 1)$, $v_0 = (2, \cdots, 2)$, $\lambda = \frac{1}{12c_1}$, $\sigma = \frac{5}{42c_1}$, $\mu = \frac{5}{44c_1}$, $\theta_n = 0.12$ *and* $\lambda_0 = \frac{1}{4c_1}$.

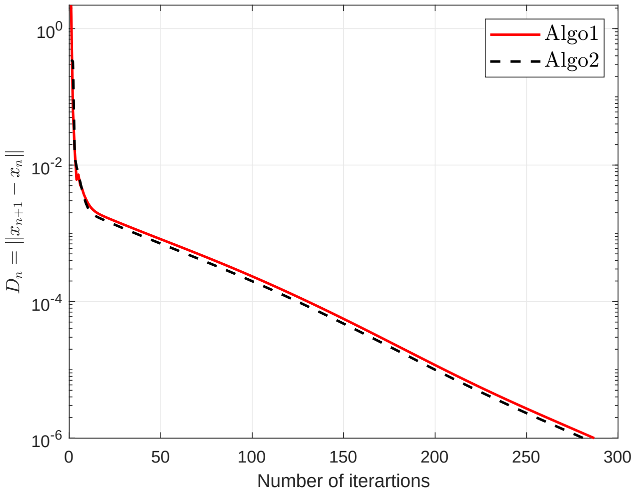

Figure 1. Example 1 when $n = 5$.

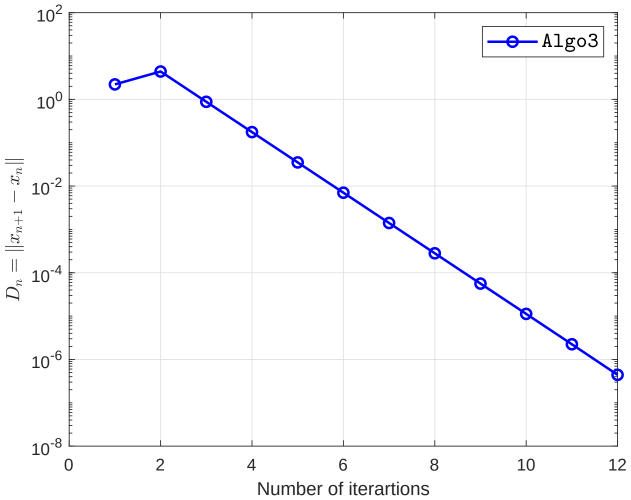

Figure 2. Example 1 when $n = 5$.

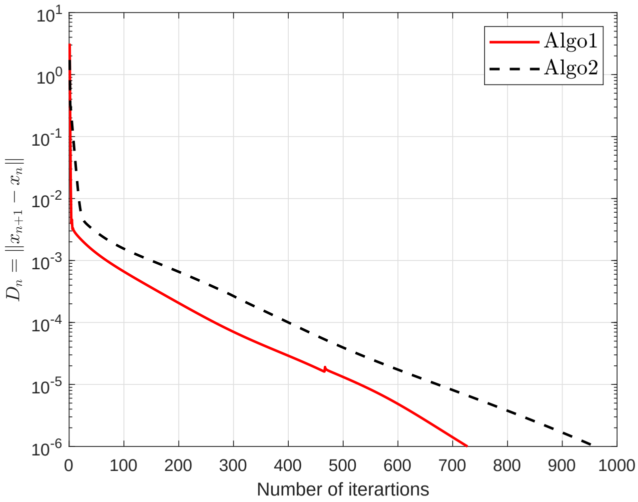

Figure 3. Example 1 when $n = 10$.

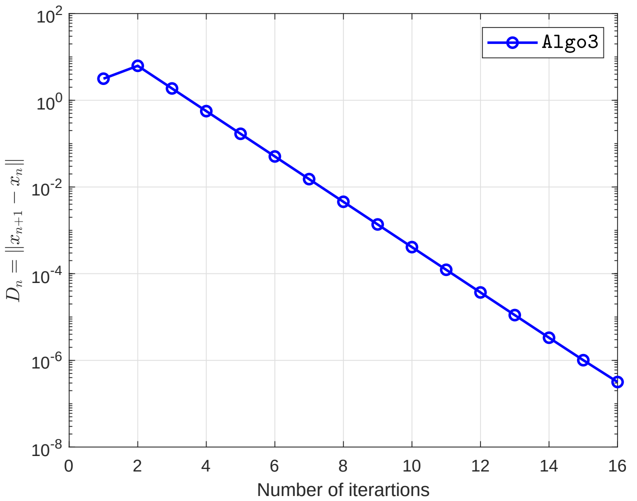

Figure 4. Example 1 when $n = 10$.

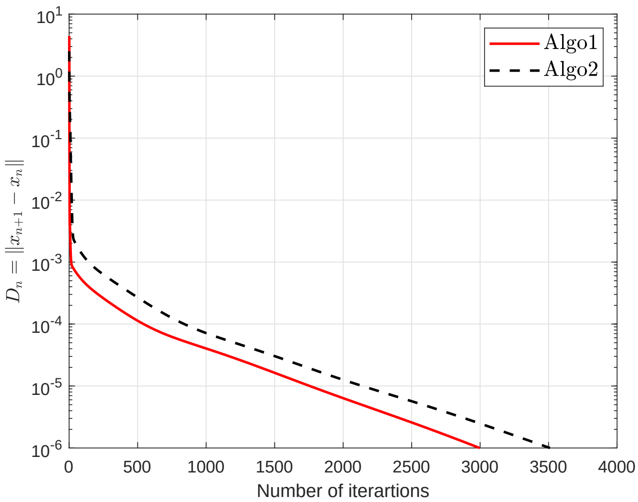

Figure 5. Example 1 when $n = 20$.

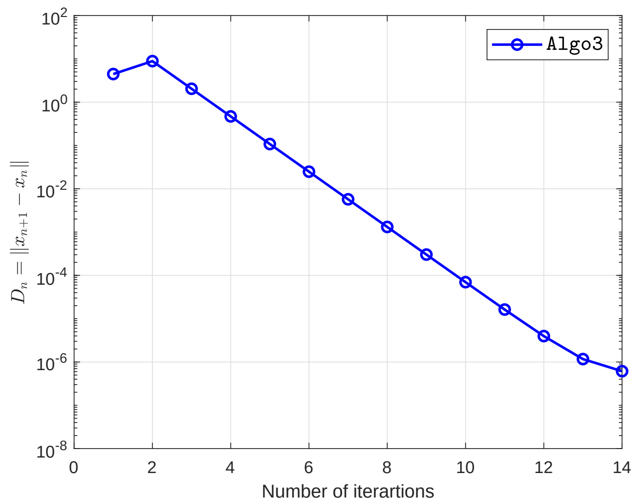

Figure 6. Example 1 when $n = 20$.

Table 1. Example 1: The numerical results for Figures 1–6.

	Algo1		Algo2		Algo3	
n	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**
5	287	5.9342	281	3.5302	12	0.1204
10	727	19.8789	960	12.8186	16	0.1584
20	2997	72.7622	3510	3510	14	0.1624

Example 2. *Let $f : C \times C \to \mathbb{R}$ be defined as*

$$f(u, v) = (u_1 + u_2 - 1)(v_1 - u_1) + (u_1 + u_2 - 1)(v_2 - u_2)$$

where $C = [-2, 5] \times [-2, 5]$. We see that

$$f(u, v) + f(v, u) = -(u_1 - v_1 + u_2 - v_2)^2 \leq 0$$

which gives that bifunction f is monotone. The numerical findings are shown in Figures 7–14 and Table 2 with $v_{-1} = u_{-1} = u_0 = (-1, 1)$, $\lambda = 0.03$, $\sigma = 0.476$, $\mu = 0.455$, $\theta_n = 0.15$ and $\lambda_0 = 0.1$.

Table 2. Example 2: The numerical results for Figures 7–14.

	Algo1		Algo2		Algo3	
v_0	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**
$(-1.0, 2.0)$	180	1.7844	172	0.7740	20	0.1025
$(1.5, 1.7)$	187	2.1016	181	0.8069	23	0.1125
$(2.7, 4.6)$	190	1.9044	184	0.7979	17	0.0881
$(2.0, 3.0)$	188	1.8635	182	0.7792	20	0.1063

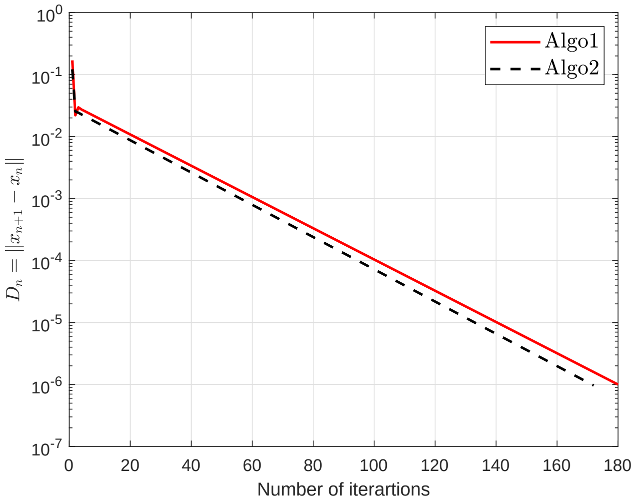

Figure 7. Example 2 when $v_0 = (-1.0, 2.0)$.

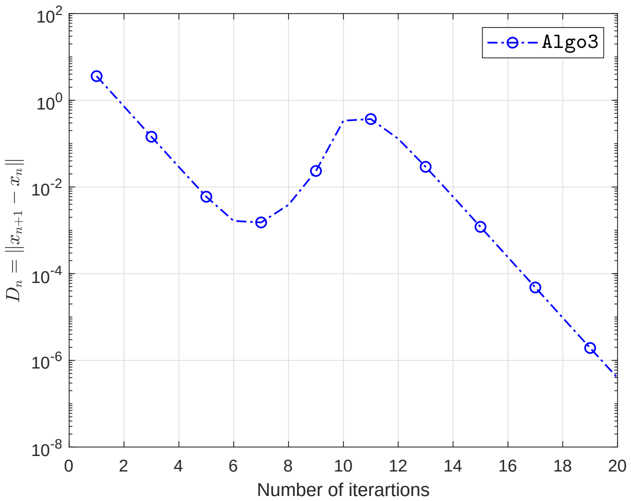

Figure 8. Example 2 when $v_0 = (-1.0, 2.0)$.

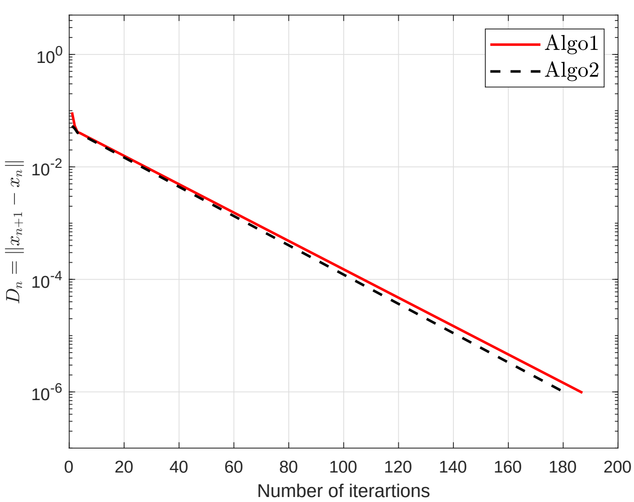

Figure 9. Example 2 when $v_0 = (1.5, 1.7)$.

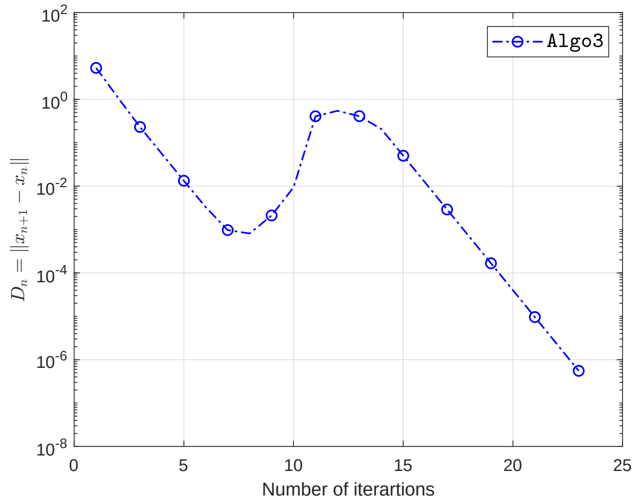

Figure 10. Example 2 when $v_0 = (1.5, 1.7)$.

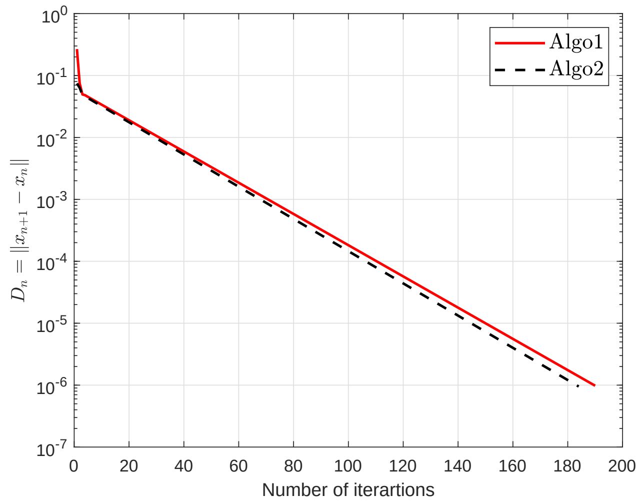

Figure 11. Example 2 when $v_0 = (2.7, 4.6)$.

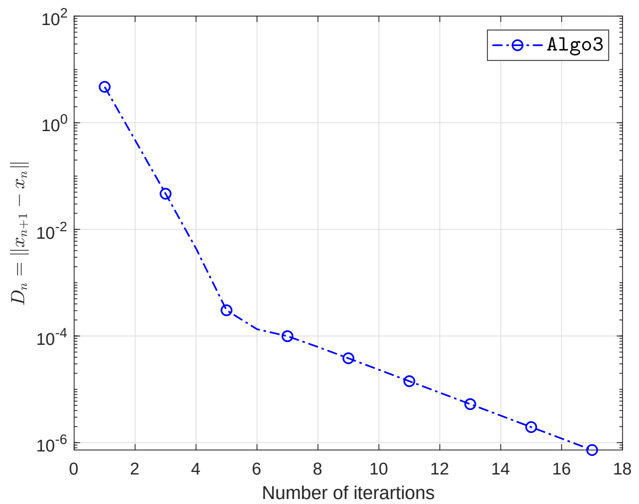

Figure 12. Example 2 when $v_0 = (2.7, 4.6)$.

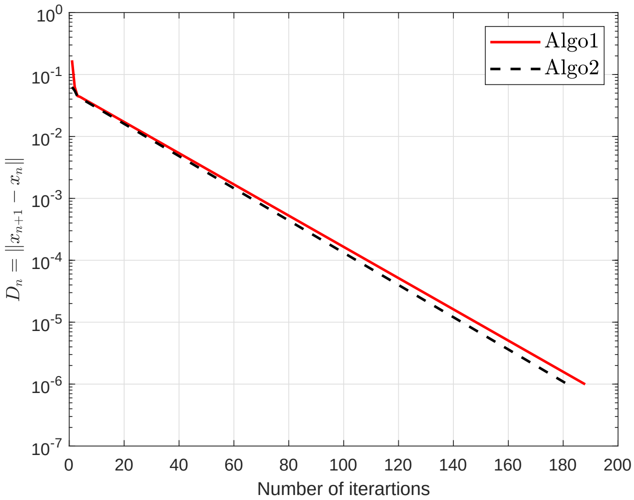

Figure 13. Example 2 when $v_0 = (2.0, 3.0)$.

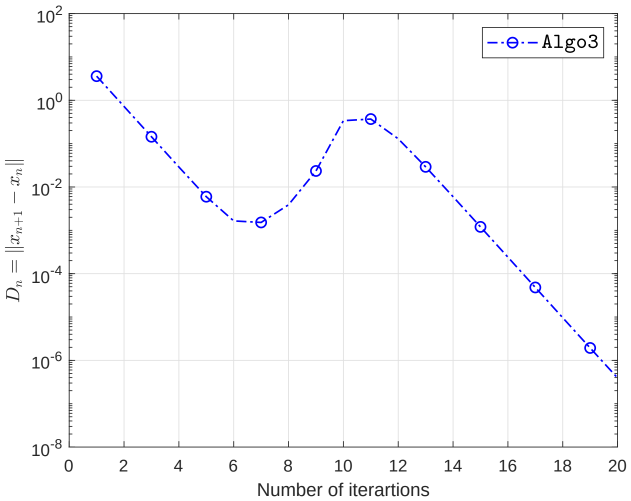

Figure 14. Example 2 when $v_0 = (2.0, 3.0)$.

Example 3. *Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$G(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}$$

and let $C = \{u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \leq 1\}$. The operator G is Lipschitz continuous with $L = 5$ and pseudomonotone. During this experiment we use $u_{-1} = (1, 1)$, $v_{-1} = (2, 2)$, $u_0 = (3, 4)^T$ with stepsize $\lambda = 10^{-8}$ according Lyashko et al. [33] and Liu et al. [42]. We take $\lambda_0 = 0.1$, $\sigma = 0.0392$ and $\mu = 0.0377$. The experimental results are shown in Table 3 and Figures 15–18.

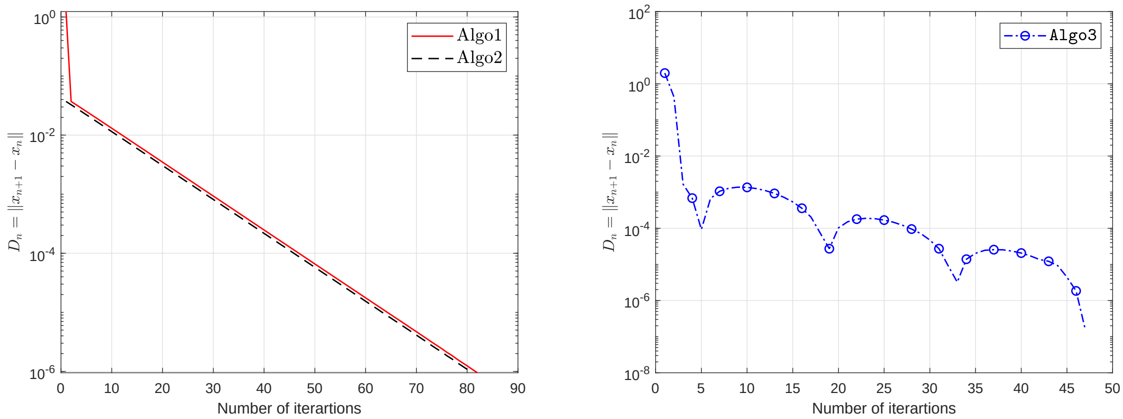

Figure 15. Example 3 when $u_0 = (1.5, 1.7)$.

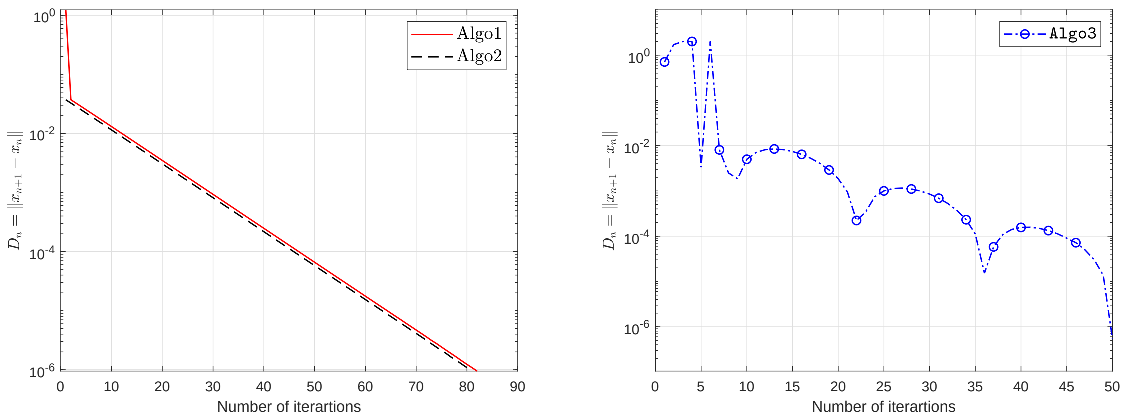

Figure 16. Example 3 when $u_0 = (2.0, 3.0)$.

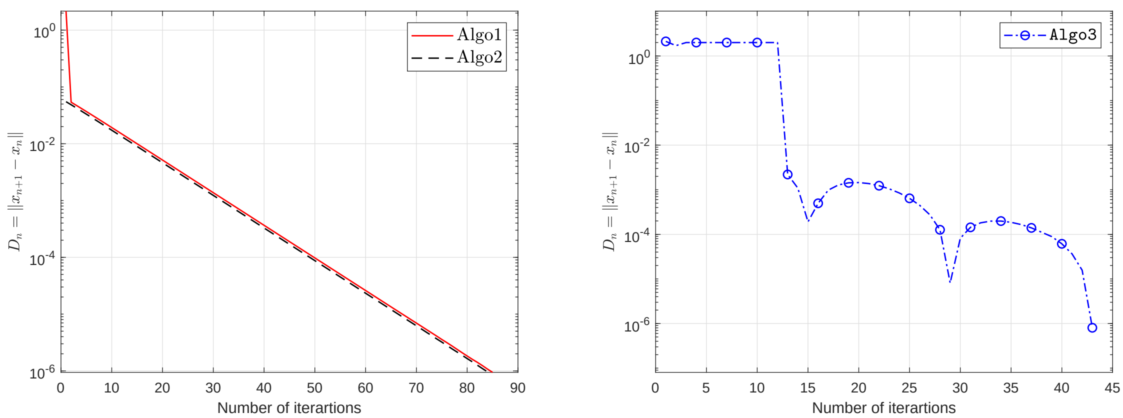

Figure 17. Example 3 when $u_0 = (1.0, 2.0)$.

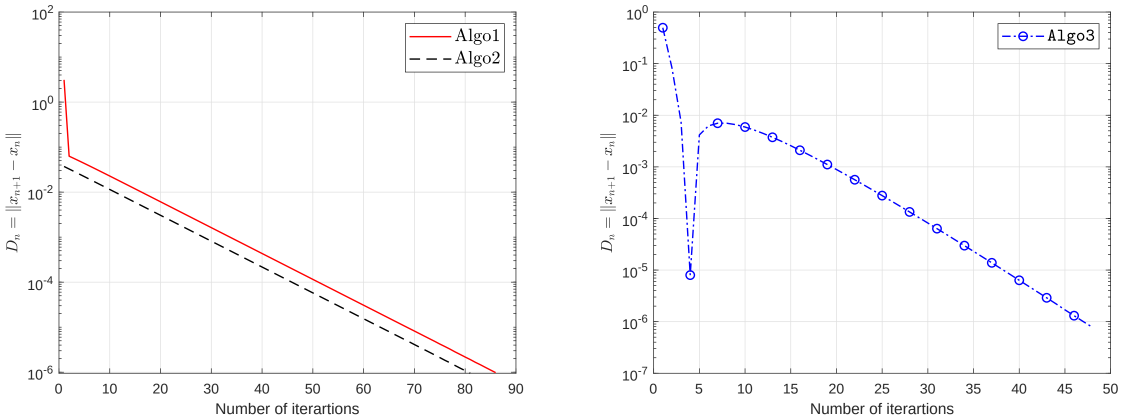

Figure 18. Example 3 when $u_0 = (2.7, 2.6)$.

Table 3. Example 3: The numerical results for Figures 15–18.

	Algo1		Algo2		Algo3	
v_0	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**
$(1.5, 1.7)$	82	2.6525	81	1.3557	47	0.9015
$(2.0, 3.0)$	82	2.7698	81	1.3698	50	1.4948
$(1.0, 2.0)$	85	2.9042	84	1.4026	43	1.2657
$(2.7, 2.6)$	86	2.8937	81	1.3990	48	1.4540

Example 4. *Take $G : \mathbb{R}^n \to \mathbb{R}^n$ to be defined through*

$$G(u) = Au + B(u)$$

where A is a $n \times n$ symmetric semidefinite matrix and $B(u)$ is the proximal mapping through the function $h(u) = \frac{1}{4}\|u\|^4$ such that

$$B(u) = \arg\min_{v \in \mathbb{R}^n} \left\{ \frac{\|v\|^4}{4} + \frac{1}{2}\|v - u\|^2 \right\}$$

The property of A and the proximal mapping B implies that G is monotone upon C [45]. The following is a feasible set

$$C := \{u \in \mathbb{R}^5 : -2 \le u_i \le 5\}.$$

The numerical results are shown in Table 4 and Figures 19.

Table 4. Example 4: The numerical results for Figure 19.

	Algo1		Algo3	
n	**Iter.**	**Exeu.time**	**Iter.**	**Exeu.time**
5	338	12.6364	112	8.8393

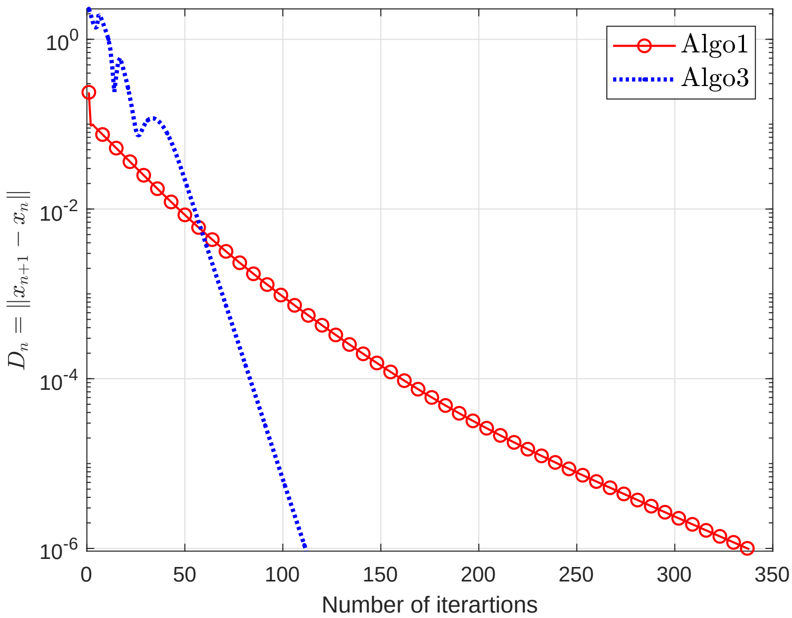

Figure 19. Example 4 when $n = 5$.

6. Conclusions

We have developed extragradient-like methods to solve pseudomonotone equilibrium problems and different classes of variational inequality problems in real Hilbert space. The advantage of our method is in designing an explicit formula for step size evaluation. For each iteration the stepsize formula is updated based on the previous iterations. Numerical results were reported to demonstrate numerical effectiveness of our results relative to other methods. These numerical studies suggest that inertial effects in this sense also generally improve the effectiveness of the iterative sequence.

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Conflicts of Interest: The authors declare that they have conflict of interest.

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