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On the Recurrence Properties of Narayana's Cows Sequence

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Abstract: In this paper, we consider the recurrence properties of two generalized forms of Narayana's cows sequence. On the one hand, we study Narayana's cows sequence at negative indices and express it as the linear combination of the sequence at positive indices. On the other hand, we study the convolved Narayana number and obtain a computation formula for it.

Keywords: Narayana's cows sequence; negative indices; linear recurrence sequence; convolution formula

MSC: 11B37; 11B83

1. Introduction

Narayana's cows sequence originated from a problem with cows proposed by the Indian mathematician Narayana in the 14th century. In this problem, we suppose that there is one cow in the beginning and that every cow produces one calf each year from the age of 4 years old. Narayana's cow problem counts the number of calves produced every year [1,2]. This problem appears to be similar to Fibonacci's rabbit problem. So too do the answers, known as the Narayana sequence and the Fibonacci sequence.

Narayana's cows sequence (A000930 in [3]) satisfies a third-order recurrence relation:

$$G_n = G_{n-1} + G_{n-3}, \quad \text{for } n \geq 3.$$

This has the initial values $G_0 = 0$ and $G_1 = G_2 = G_3 = 1$ [1]. Explicitly, the characteristic equation of G_n is:

$$x^3 - x^2 - 1 = 0,$$

and the characteristic roots are:

$$\begin{aligned}
\alpha &= \frac{1}{3}\left(\sqrt[3]{\frac{1}{2}\left(29 - 3\sqrt{93}\right)} + \sqrt[3]{\frac{1}{2}\left(3\sqrt{93} + 29\right)} + 1\right), \\
\beta &= \frac{1}{3} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29 - 3\sqrt{93}\right)} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(3\sqrt{93} + 29\right)}, \\
\gamma &= \frac{1}{3} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(29 - 3\sqrt{93}\right)} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(3\sqrt{93} + 29\right)}.
\end{aligned} \tag{1}$$

Then, the Narayana sequence can be obtained by Binet's formula:

$$G_n = A\alpha^n + B\beta^n + C\gamma^n. \tag{2}$$

For $n \in \mathbb{Z}_{\geq 0}$, the generating function of the Narayana sequence is:

$$g(x) = \frac{1}{1 - x - x^3} = \sum_{n=0}^{\infty} G_{n+1}x^n. \tag{3}$$

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With the Vieta theorem, we have:

$$\begin{cases} \alpha+\beta+\gamma=1, \\ \alpha\beta+\beta\gamma+\alpha\gamma=0, \\ \alpha\beta\gamma=1. \end{cases} \tag{4}$$

From Formula (2), we obtain:

$$\begin{aligned} G_0 &= A+B+C=0, \\ G_1 &= A\alpha+B\beta+C\gamma=1, \\ G_2 &= A\alpha^2+B\beta^2+C\gamma^2=1, \end{aligned}$$

which implies:

$$A=\frac{1-\beta-\gamma}{(\alpha-\beta)(\alpha-\gamma)}, \qquad B=\frac{1-\alpha-\gamma}{(\beta-\alpha)(\beta-\gamma)}, \qquad C=\frac{1-\alpha-\beta}{(\gamma-\beta)(\gamma-\alpha)}.$$

With Formula (4), we can simplify A, B, and C and obtain:

$$A=\frac{\alpha}{\alpha^2-\alpha\beta-\alpha\gamma+\beta\gamma}=\frac{\alpha}{\alpha^2+2\beta\gamma}=\frac{\alpha^2}{\alpha^3+2}$$

and

$$B=\frac{\beta^2}{\beta^3+2}, \quad C=\frac{\gamma^2}{\gamma^3+2}. \tag{5}$$

The Narayana sequence was originally defined at positive indices. Actually, it can be extended to negative indices by defining:

$$G_{-n}=\frac{A}{\alpha^n}+\frac{B}{\beta^n}+\frac{C}{\gamma^n}, \tag{6}$$

The following recurrence relation holds for all integral indices.

$$G_n=G_{n-1}+G_{n-3}, \quad n\in\mathbb{Z}. \tag{7}$$

Through a simple computation, the first few terms of G_n at negative indices can be obtained from Formulas (5) and (6), so that $G_{-1}=0, G_{-2}=1, G_{-3}=0, G_{-4}=-1$, which also satisfies Relation (7).

The Narayana sequence has a close connection to some famous numbers or sequences and plays an important role in cryptography and combinatorics. For instance, it can be seen as the number of compositions of n into parts 1 and 3. For $n\geq 3$, the Narayana sequence can be expressed as the row sums of Pascal's triangle with triplicated diagonals, while the Fibonacci number F_n is the row sums of Pascal's triangle with slope diagonals of 45 degrees [3]. Narayana's sequence has a beautiful distribution pattern, the ratio of whose consecutive terms approximates the supergolden ratio, which is closely related to the golden ratio [2]. Moreover, the Narayana sequence satisfies good cross-correlation and autocorrelation properties, which provide wide applications in data coding, information theory, and cryptography, and especially in multiparty computation [2,4,5]. More research about the Narayana sequence and its applications can be found in [6–8].

In this paper, we study two natural generalized forms of the Narayana sequence G_n (i.e., $\{G_n \mid n\in\mathbb{Z}_{<0}\}$ and $\{g^h(x) \mid h\in\mathbb{R}_{>0}\}$) and construct the recurrence relation between these generalized forms and G_n. Many mathematical efforts have been made concerning these two generalizations of the linear recurrence sequence, such as the Fibonacci sequence

F_n (A000035 in [3]), the Tribonacci sequence T_n (A000073 in [3]), the Lucas sequence L_n (A000032 in [3]), and so on.

For the first aspect, Falcon [9,10] obtained the connection between F_n and L_n and their values at negative indices. He proved that for $n \in \mathbb{Z}_{\geq 0}$, the following identities hold.

$$F_{-n} = (-1)^{n+1} F_n \quad \text{and} \quad L_{-n} = (-1)^{n+1} L_n.$$

Halici and Akyuz [11] proposed a series of formulae expressing F_n and L_n at negative indices as the linear combination of themselves. For example, they obtained that for any integer $n \geq 0$ and $k \geq 1$, one has:

$$\begin{aligned} F_{-nk} &= F_{-n} L_n^{k-1} \sum_{i=0}^{[\frac{k-1}{2}]} \binom{k-1-i}{i} L_n^{-2i} (-1)^i, \quad n \text{ and } k \text{ are even.} \\ L_{-nk} &= 5^{\frac{k-1}{2}} L_{-n} \sum_{i=0}^{[\frac{k-1}{2}]} \binom{k-1-i}{i} 5^{-i} F_n^{k-1-2i} (-1)^i, \quad n \text{ and } k \text{ are odd,} \end{aligned}$$

where $\binom{n}{i}$ denotes the binomial coefficients. More works about the linear recurrence sequence at negative indices can be seen in [12–15].

For the second aspect, Kim et al. [16] and Chen and Qi [17] considered the convolved Fibonacci number $F_n(x)$, which is defined by the generating function:

$$\left(\frac{1}{1-t-t^2} \right)^x = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \quad x \in \mathbb{R}_{>0}.$$

With the combinatorial methods, they expressed the $F_n(x)$ by the linear combination of $F_n(x)$ itself [16] or L_n [17].

$$\begin{aligned} F_n(x) &= \sum_{l=0}^{n} \binom{n}{l} F_l(r) F_{n-l}(x-r). \\ F_n(x) &= \frac{1}{2} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \langle x \rangle_i \langle x \rangle_{n-i} L_{n-2i}, \end{aligned}$$

where $\langle x \rangle_0 = 1$ and $\langle x \rangle_i = x(x+1)(x+2)\cdots(x+n-1)$. As a corollary of [17], one has the following identity for $k \in \mathbb{Z}_{\geq 0}$:

$$\sum_{a_1+a_2+\cdots+a_k=n} F_{a_1} F_{a_2} \cdots F_{a_k} = \frac{1}{2(k-1)!^2} \sum_{i=0}^{n} (-1)^i \frac{(k+i-1)!(k+n-i-1)!}{i!(n-i)!} L_{n-2i}.$$

Zhou and Chen [18] studied the convolved Tribonacci number $T_n(x)$, defined by:

$$\left(\frac{1}{1-t-t^2-t^3} \right)^x = \sum_{n=0}^{\infty} T_n(x) t^n, \quad x \in \mathbb{R}_{>0}.$$

They obtained:

$$\begin{aligned} T_n(x) &= \frac{1}{6} \sum_{u+v+w=n} \frac{\langle x \rangle_u}{u!} \frac{\langle x \rangle_v}{v!} \frac{\langle x \rangle_w}{w!} (3T_{w+1-u} - 2T_{w-u} - T_{w-u-1}) \\ &\quad \times (3T_{w+1-v} - 2T_{w-v} - T_{w-v-1}) \\ &\quad - \frac{1}{6} \sum_{u+v+w=n} \frac{\langle x \rangle_u}{u!} \frac{\langle x \rangle_v}{v!} \frac{\langle x \rangle_w}{w!} (3T_{3w+1-n} - 2T_{3w+n} - T_{3w-1-n}). \end{aligned}$$

In particular, for $k \in \mathbb{Z}_{\geq 0}$, [18] obtained the computational formula for:

$$\sum_{a_1+a_2+\cdots+a_k=n} T_{a_1+1}T_{a_2+1}\cdots T_{a_k+1}.$$

Other works related to convolved numbers can be found in [19–24].

Recently, Professor Tianxin Cai visited Northwest University and gave a talk about a series of linear recurrence sequences and their properties, which incited our interest in this field. There are many recursive identities concerning the Fibonacci, Tribonacci, and Lucas sequences, etc. However, few studies have been conducted regarding the Narayana sequence. Professor Cai proposed an open problem:

Whether and how can the Narayana sequence at negative indices be expressed by the sequence itself at positive indices?

Inspired by the question above, our work focuses on the new recurrence relations around the Narayana sequence. On the one hand, we study the recursive properties of the Narayana sequence at negative indices. For $n \in \mathbb{Z}$, we express G_{-n} as the linear combination of G_n. On the other hand, we consider the convolved Narayana number $G_{n+1}(h)$ defined as:

$$g^h(x) = \left(\frac{1}{1-x-x^3}\right)^h = \sum_{n=0}^{\infty} G_{n+1}(h)x^n, \quad h \in \mathbb{R}_{>0}. \tag{8}$$

When $h = 1$, $G_{n+1}(h)$ becomes the Narayana number G_{n+1}. We propose a computation formula for $G_{n+1}(h)$ involved with G_{n+1}. As a corollary, we obtain an identity between $G_{n+1}(h)$ and G_{n+1}.

Our main results are presented as follows.

Theorem 1. *For $n \in \mathbb{Z}$, we have:*

$$G_{-n} = G_{2n} - 3G_nG_{n+1} + 2G_n^2.$$

Theorem 1 solves Professor Cai's problem completely. It illustrates the connection between the Narayana sequence at the positive index and the negative index. By Theorem 1, we obtain the recurrence property of the sequence at the negative index, which deepens our knowledge of the nature of the sequence.

Theorem 2. *Let $h \in \mathbb{R}_{>0}$. For $n \in \mathbb{Z}_{\geq 0}$. We have:*

$$\begin{aligned} G_{n+1}(h) \quad = \quad & \frac{1}{6}\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} \\ & \times(3G_{i-k+1}-2G_{i-k})(3G_{j-k+1}-2G_{j-k}) \\ & -\frac{1}{6}\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!}(3G_{i+j-2k+1}-2G_{i+j-2k}). \end{aligned}$$

Theorem 2 gives a computation formula for $G_{n+1}(h)$ involving G_n. If h is a positive integer, we can deduce the corollary as follows.

Corollary 1. *Let $h \in \mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{\geq 0}$. We have:*

$$\begin{aligned}
&\sum_{a_1+a_2+\cdots+a_h=n} G_{a_1+1}G_{a_2+1}\cdots G_{a_h+1} \\
= \ & \frac{1}{6}\sum_{i+j+k=n}\binom{h+i-1}{i}\binom{h+j-1}{j}\binom{h+k-1}{k} \\
& \times(3G_{i-k+1}-2G_{i-k})(3G_{j-k+1}-2G_{j-k}) \\
& -\frac{1}{6}\sum_{i+j+k=n}\binom{h+i-1}{i}\binom{h+j-1}{j}\binom{h+k-1}{k}(3G_{i+j-2k+1}-2G_{i+j-2k}).
\end{aligned}$$

The formulae in the theorems and corollary above are recursive and computable. To better reflect our results, we calculated some values for the formula in Theorem 1 and Corollary 1; they are listed in Appendix A. The numerical results for G_{-n} and $G_{n+1}(h)$ ($h \in \mathbb{Z}_{>0}$) meet their definitions, which prove our results numerically. The numerical experiments were performed in Mathematica 12.0.

2. Preliminary

To prove our theorems, we propose some lemmas.

Lemma 1. *For $n \in \mathbb{Z}_{\geq 0}$, denote:*

$$\begin{aligned}
T_n &= \alpha^n+\beta^n+\gamma^n, \\
S_n &= \alpha^n\beta^n+\alpha^n\gamma^n+\beta^n\gamma^n,
\end{aligned}$$

where α, β, and γ are as defined in Formula (1). We have $T_0=3$, $T_1=1$, $T_2=1$; $S_0=3$, $S_1=0$, $S_2=-2$; and the following recurrence relation for $n \geq 3$.

$$T_n = T_{n-1}+T_{n-3}, \quad \text{and} \quad S_n = -S_{n-2}+S_{n-3}. \tag{9}$$

T_n and S_n satisfy the identity so that:

$$2S_n = T_n^2 - T_{2n}. \tag{10}$$

Proof. With Formula (4), we obtain the initial value of T_n and S_n so that $T_0=3$, $T_1=1$, $S_0=3$, $S_1=0$, and:

$$\begin{aligned}
T_2 &= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=1, \\
S_2 &= (\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)=-2.
\end{aligned}$$

For $n \geq 3$, we have:

$$\begin{aligned}
T_1T_{n-1} &= (\alpha+\beta+\gamma)(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}) \\
&= T_n+\alpha\beta\left(\alpha^{n-2}+\beta^{n-2}\right)+\alpha\gamma\left(\alpha^{n-2}+\gamma^{n-2}\right)+\beta\gamma\left(\beta^{n-2}+\gamma^{n-2}\right) \\
&= T_n+\alpha\beta\left(T_{n-2}-\gamma^{n-2}\right)+\alpha\gamma\left(T_{n-2}-\beta^{n-2}\right)+\beta\gamma\left(T_{n-2}-\alpha^{n-2}\right) \\
&= T_n+T_{n-2}(\alpha\beta+\alpha\gamma+\beta\gamma)-\alpha\beta\gamma\left(\alpha^{n-3}+\beta^{n-3}+\gamma^{n-3}\right) \\
&= T_n-T_{n-3} \ = \ T_{n-1}
\end{aligned}$$

and

$$\begin{aligned} S_1 S_{n-1} &= (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha^{n-1}\beta^{n-1} + \alpha^{n-1}\gamma^{n-1} + \beta^{n-1}\gamma^{n-1}) \\ &= S_n + \alpha\beta\gamma\Big(\alpha^{n-2}\beta^{n-2}(\alpha+\beta) + \alpha^{n-2}\gamma^{n-2}(\alpha+\gamma) + \beta^{n-2}\gamma^{n-2}(\beta+\gamma)\Big) \\ &= S_n + S_{n-2} - \alpha\beta\gamma(\alpha^{n-3}\beta^{n-3} + \alpha^{n-3}\gamma^{n-3} + \beta^{n-3}\gamma^{n-3}) \\ &= S_n + S_{n-2} - S_{n-3} \;=\; 0. \end{aligned}$$

From the definition of T_n and S_n, we obtain:

$$\begin{aligned} 2S_n &= \alpha^n(\beta^n + \gamma^n) + \beta^n(\alpha^n + \gamma^n) + \gamma^n(\alpha^n + \beta^n) \\ &= T_n(\alpha^n + \beta^n + \gamma^n) - (\alpha^{2n} + \beta^{2n} + \gamma^{2n}) \\ &= T_n^2 - T_{2n}. \end{aligned}$$

The proof of Lemma is 1 completed. □

Note: In fact, the index n of sequence T_n and S_n in Lemma 1 can be defined with $\mathbb{Z}$. Since $\alpha\beta\gamma = 1$, we have $T_n = S_{-n}$ for $n \in \mathbb{Z}$. With symmetry, it suffices to consider that $n \in \mathbb{Z}_{\geq 0}$.

Lemma 2. *For $n \in \mathbb{Z}_{\geq 0}$, we have:*

$$T_n = 3G_{n+1} - 2G_n \quad \textit{and} \quad G_n = \frac{9T_{n+2} - 3T_{n+1} - 2T_n}{31}. \tag{11}$$

Proof. With Formula (5), we have:

$$A(\alpha^3 + 2) = \alpha^2, \quad B(\beta^3 + 2) = \beta^2, \quad C(\gamma^3 + 2) = \gamma^2.$$

Then, we obtain the recurrence relation for α^n, β^n, and γ^n:

$$\begin{cases} \alpha^n = A(\alpha^3 + 2)\alpha^{n-2} = A\alpha^{n+1} + 2A\alpha^{n-2}, \\ \beta^n = B(\beta^3 + 2)\beta^{n-2} = B\beta^{n+1} + 2B\beta^{n-2}, \\ \gamma^n = C(\gamma^3 + 2)\gamma^{n-2} = C\gamma^{n+1} + 2C\gamma^{n-2}. \end{cases} \tag{12}$$

Summing the formulae above, we obtain:

$$\begin{aligned} T_n &= (A\alpha^{n+1} + B\beta^{n+1} + C\gamma^{n+1}) + 2(A\alpha^{n-2} + B\beta^{n-2} + C\gamma^{n-2}) \\ &= G_{n+1} + 2G_{n-2} \;=\; 3G_{n+1} - 2G_n. \end{aligned}$$

Now, we prove the second formula. Denote:

$$G_n = A\alpha^n + B\beta^n + C\gamma^n = \frac{\alpha^{n+2}}{\alpha^3 + 2} + \frac{\beta^{n+2}}{\beta^3 + 2} + \frac{\gamma^{n+2}}{\gamma^3 + 2} = \frac{N_n}{D}.$$

The nominator N_n and denominator D of G_n are:

$$\begin{aligned} N_n &= 4\alpha^{n+2} + 4\beta^{n+2} + 4\gamma^{n+2} + \beta^3\gamma^3\alpha^{n+2} + \alpha^3\gamma^3\beta^{n+2} + \alpha^3\beta^3\gamma^{n+2} \\ &\quad + 2\beta^3\alpha^{n+2} + 2\gamma^3\alpha^{n+2} + 2\alpha^3\beta^{n+2} + 2\alpha^3\gamma^{n+2} + 2\gamma^3\beta^{n+2} + 2\beta^3\gamma^{n+2} \\ &= 4\Big(\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2}\Big) + \alpha^3\beta^3\gamma^3\Big(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1}\Big) \\ &\quad + 2\alpha^3\beta^3\Big(\alpha^{n-1} + \beta^{n-1}\Big) + 2\alpha^3\gamma^3\Big(\alpha^{n-1} + \gamma^{n-1}\Big) + 2\beta^3\gamma^3\Big(\beta^{n-1} + \gamma^{n-1}\Big) \end{aligned}$$

and

$$\begin{aligned} D &= \left(\alpha^3+2\right)\left(\beta^3+2\right)\left(\gamma^3+2\right) \\ &= \alpha^3\beta^3\gamma^3+4\alpha^3+4\beta^3+4\gamma^3+2\alpha^3\beta^3+2\alpha^3\gamma^3+2\beta^3\gamma^3+8. \end{aligned}$$

With Formula (4), we compute N_n and D, so that:

$$\begin{aligned} D &= 1+16+6+8 = 31, \\ N_n &= 4T_{n+2}+7T_{n-1}-2T_{n-4}, \end{aligned}$$

This implies that:

$$G_n = \frac{4T_{n+2}+7T_{n-1}-2T_{n-4}}{31} = \frac{9T_{n+2}-3T_{n+1}-2T_n}{31}.$$

The last equality holds from Formula (9). □

Lemma 3. *For $n \in \mathbb{Z}_{\geq 0}$, we have:*

$$S_n = G_{-n}+3G_{-(n+2)} \quad \textit{and} \quad G_{-n} = \frac{6S_{n+2}+9S_{n+1}+4S_n}{31}.$$

Proof. With Formulas (4) and (5), we have:

$$A = \frac{\alpha^2}{\alpha^3+2} = \frac{\beta\gamma}{1+2\beta^3\gamma^3} \quad \text{and} \quad A(1+2\beta^3\gamma^3) = \beta\gamma.$$

Similar to Formula (12), we obtain:

$$\begin{cases} \beta^n\gamma^n = A\beta^{n-1}\gamma^{n-1}+2A\beta^{n+2}\gamma^{n+2}, \\ \alpha^n\gamma^n = B\alpha^{n-1}\gamma^{n-1}+2B\alpha^{n+2}\gamma^{n+2}, \\ \alpha^n\beta^n = C\alpha^{n-1}\beta^{n-1}+2C\alpha^{n+2}\beta^{n+2}. \end{cases}$$

Combining the formulae above, we express S_n by G_n.

$$\begin{aligned} S_n &= (A\beta^{n-1}\gamma^{n-1}+B\alpha^{n-1}\gamma^{n-1}+C\alpha^{n-1}\beta^{n-1}) \\ &\quad +2(A\beta^{n+2}\gamma^{n+2}+B\alpha^{n+2}\gamma^{n+2}+C\alpha^{n+2}\beta^{n+2}) \\ &= G_{-(n-1)}+2G_{-(n+2)} = G_{-n}+3G_{-(n+2)}. \end{aligned}$$

Now, we prove the second formula. With Formula (6), we have:

$$G_{-n} = \frac{1}{\alpha^{n+1}+2\alpha^{n-2}} + \frac{1}{\beta^{n+1}+2\beta^{n-2}} + \frac{1}{\gamma^{n+1}+2\gamma^{n-2}} = \frac{N_{-n}}{D_{-n}},$$

whose denominator and nominator are:

$$D_{-n} = \alpha^n\beta^n\gamma^n\left(\alpha^3+2\right)\left(\beta^3+2\right)\left(\gamma^3+2\right) = D = 31$$

and

$$\begin{aligned} N_{-n} &= 4\gamma^2\alpha^n\beta^n + 4\beta^2\alpha^n\gamma^n + 4\alpha^2\beta^n\gamma^n + \gamma^2\alpha^{n+3}\beta^{n+3} + \beta^2\alpha^{n+3}\gamma^{n+3} \\ &\quad +\alpha^2\beta^{n+3}\gamma^{n+3} + \Big(2\gamma^2\alpha^{n+3}\beta^n + 2\beta^2\alpha^{n+3}\gamma^n + 2\gamma^2\alpha^n\beta^{n+3} + 2\beta^2\alpha^n\gamma^{n+3} \\ &\quad +2\alpha^2\beta^{n+3}\gamma^n + 2\alpha^2\beta^n\gamma^{n+3}\Big) \\ &= 4\alpha^2\beta^2\gamma^2 S_{n-2} + \alpha^2\beta^2\gamma^2 S_{n+1} + 2\alpha^2\beta^2\gamma^2\Big(\alpha^{n+1}\beta^{n-2} + \alpha^{n+1}\gamma^{n-2} \\ &\quad +\alpha^{n-2}\beta^{n+1} + \alpha^{n-2}\gamma^{n+1} + \beta^{n+1}\gamma^{n-2} + \beta^{n-2}\gamma^{n+1}\Big). \end{aligned}$$

After a simple computation, the last term of the formula above can also be expressed by S_n.

$$\begin{aligned} &\alpha^{n+1}\beta^{n-2} + \alpha^{n+1}\gamma^{n-2} + \alpha^{n-2}\beta^{n+1} + \alpha^{n-2}\gamma^{n+1} + \beta^{n+1}\gamma^{n-2} + \beta^{n-2}\gamma^{n+1} \\ =\ &\alpha^{n-2}\beta^{n-2}(\alpha^3+\beta^3) + \alpha^{n-2}\gamma^{n-2}(\alpha^3+\gamma^3) + \beta^{n-2}\gamma^{n-2}(\beta^3+\gamma^3) \\ =\ &4S_{n-2} - \alpha^3\beta^3\gamma^3(\alpha^{n-5}\beta^{n-5} + \alpha^{n-5}\gamma^{n-5} + \beta^{n-5}\gamma^{n-5}) \\ =\ &4S_{n-2} - S_{n-5}. \end{aligned}$$

Then, we have:

$$N_{-n} = S_{n+1} + 12S_{n-2} - 2S_{n-5}.$$

Thus:

$$G_{-n} = \frac{S_{n+1} + 12S_{n-2} - 2S_{n-5}}{31} = \frac{6S_{n+2} + 9S_{n+1} + 4S_n}{31}.$$

□

Lemma 4. *Let $h \in \mathbb{R}_{>0}$. For $n \in \mathbb{Z}_{\geq 0}$, we have:*

$$\begin{aligned} &\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!}\alpha^i\beta^j\gamma^k \\ =\ &\frac{1}{6}\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!}(3G_{i-k+1} - 2G_{i-k})(3G_{j-k+1} - 2G_{j-k}) \\ &-\frac{1}{6}\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!}(3G_{i+j-2k+1} - 2G_{i+j-2k}). \end{aligned}$$

Proof. For any positive integer i, j, and k, we have:

$$\begin{aligned} T_{i-k}T_{j-k} &= (\alpha^{i-k} + \beta^{i-k} + \gamma^{i-k})(\alpha^{j-k} + \beta^{j-k} + \gamma^{j-k}) \\ &= \alpha^{i+j-2k} + \beta^{i+j-2k} + \gamma^{i+j-2k} + \alpha^{i-k}\beta^{j-k} + \alpha^{i-k}\gamma^{j-k} \\ &\quad +\beta^{i-k}\alpha^{j-k} + \beta^{i-k}\gamma^{j-k} + \gamma^{i-k}\alpha^{j-k} + \gamma^{i-k}\beta^{j-k}. \end{aligned}$$

Note that: $\alpha^{i-k}\beta^{j-k} = (\alpha^{i-k}\beta^{j-k})(\alpha^k\beta^k\gamma^k) = \alpha^i\beta^j\gamma^k$. Then, the formula above can be reduced to:

$$\begin{aligned} T_{i-k}T_{j-k} &= \alpha^{i+j-2k} + \beta^{i+j-2k} + \gamma^{i+j-2k} + \alpha^i\beta^j\gamma^k + \alpha^i\beta^k\gamma^j + \alpha^j\beta^i\gamma^k \\ &\quad +\alpha^k\beta^i\gamma^j + \alpha^j\beta^k\gamma^i + \alpha^k\beta^j\gamma^i. \end{aligned} \tag{13}$$

From Formula (11), we have:

$$\begin{aligned}
&\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} T_{i-k}T_{j-k} \\
= &\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} (3G_{i-k+1}-2G_{i-k})(3G_{j-k+1}-2G_{j-k}).
\end{aligned} \tag{14}$$

With Formulas (11) and (13), we obtain:

$$\begin{aligned}
&\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} T_{i-k}T_{j-k} \\
= &\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} (3G_{i+j-2k+1}-2G_{i+j-2k}) \\
&+6\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} \alpha^i\beta^j\gamma^k.
\end{aligned} \tag{15}$$

Comparing Formulas (14) and (15), we obtain:

$$\begin{aligned}
&6\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} \alpha^i\beta^j\gamma^k \\
= &\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} (3G_{i-k+1}-2G_{i-k})(3G_{j-k+1}-2G_{j-k}) \\
&-\sum_{i+j+k=n} \frac{\langle h\rangle_i}{i!}\frac{\langle h\rangle_j}{j!}\frac{\langle h\rangle_k}{k!} (3G_{i+j-2k+1}-2G_{i+j-2k}).
\end{aligned}$$

This proves Lemma 4. □

3. Proof of Theorems

Now, we complete the proof of Theorems.

Proof of Theorem 1. For $n \geq 0$, we have:

$$\begin{aligned}
G_nT_n &= (A\alpha^n + B\beta^n + C\gamma^n)(\alpha^n+\beta^n+\gamma^n) \\
&= A\alpha^{2n} + B\beta^{2n} + C\gamma^{2n} + (A+B)\alpha^n\beta^n + (A+C)\alpha^n\gamma^n + (B+C)\beta^n\gamma^n \\
&= G_{2n} + (A+B+C)G_{-n} - G_{-n} \;=\; G_{2n} - G_{-n}.
\end{aligned}$$

With Lemma 2, we obtain:

$$G_{-n} = G_{2n} - 3G_nG_{n+1} + 2G_n^2.$$

Similarly, for $n < 0$ we have:

$$\begin{aligned}
G_{-n}T_{-n} &= \frac{A}{\alpha^{2n}} + \frac{B}{\beta^{2n}} + \frac{C}{\gamma^{2n}} + (A+B)\gamma^n + (A+C)\beta^n + (B+C)\alpha^n \\
&= G_{-2n} - G_n.
\end{aligned}$$

With Lemma 3 and Formula (7), we obtain:

$$G_n = G_{-2n} - G_n(3G_{-n+1} - 2G_{-n}) = G_{-2n} - 3G_{-n}G_{-n+1} + 2G_{-n}^2.$$

This proves Theorem 1. □

Proof of Theorem 2. Note that the expansion of power series:

$$\frac{1}{(1-x)^h} = \sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} x^n, \quad |x| < 1.$$

Decomposing $g^h(x)$, we obtain:

$$\begin{aligned} g^h(x) &= \left(\frac{1}{1-x-x^3}\right)^h = \frac{1}{(1-\alpha x)^h (1-\beta x)^h (1-\gamma x)^h} \\ &= \left(\sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \alpha^n x^n\right)\left(\sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \beta^n x^n\right)\left(\sum_{n=0}^{\infty} \frac{\langle h \rangle_n}{n!} \gamma^n x^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i+j+k=n} \frac{\langle h \rangle_i}{i!}\frac{\langle h \rangle_j}{j!}\frac{\langle h \rangle_k}{k!} \alpha^i \beta^j \gamma^k\right) x^n, \end{aligned}$$

where α, β, and γ are the roots of the characteristic equation $x^3 - x^2 - 1 = 0$. With Formula (8), we obtain:

$$G_{n+1}(h) = \sum_{i+j+k=n} \frac{\langle h \rangle_i}{i!}\frac{\langle h \rangle_j}{j!}\frac{\langle h \rangle_k}{k!} \alpha^i \beta^j \gamma^k.$$

Theorem 2 follows from Lemma 4. □

Proof of Corollary 1. Let $h \in \mathbb{Z}_{>0}$. With Formulas (3) and (8), we have:

$$\begin{aligned} g^h(x) &= \left(\frac{1}{1-x-x^3}\right)^h = \left(\sum_{n=0}^{+\infty} G_{n+1} x^n\right)^h \\ &= \sum_{n=0}^{+\infty}\left(\sum_{a_1+a_2+\cdots+a_h=n} G_{a_1+1}G_{a_2+1}\cdots G_{a_h+1}\right) x^n. \end{aligned}$$

Since $\frac{\langle h \rangle_n}{n!} = \binom{h+n-1}{n}$ for $h \in \mathbb{Z}_{>0}$ Corollary 1 can be obtained by Theorem 2. □

4. Discussion

The main results of this paper propose new recurrence properties of Narayana's cow sequence in two generalized forms. First, we consider Narayana's cows sequence G_n at negative indices and construct the relationship between the sequence and itself at positive indices. This illustrates the recurrence property of the sequence at the negative index. Meanwhile, we prove that this connection holds for all integers, and not only for positive integers. Our results solve an open problem proposed by Professor Tianxin Cai completely. In addition, we obtain a computable recurrence formula for the convolved Narayana number. As a corollary, we obtain an identity related to G_n.

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Appendix A

To better illustrate the main results, we present numerical experiments for Narayana's cows sequence at negative indices and the convolved Narayana number for our main results. Some values of Narayana's cows sequence at both negative and positive indices are listed in Table A1, where $G_0 = 0$.

Table A1. Some values of G_n.

n	G_{-n}	G_n	n	G_{-n}	G_n	n	G_{-n}	G_n
1	0	1	21	−26	1278	41	−1407	2,670,964
2	1	1	22	28	1873	42	472	3,914,488
3	0	1	23	19	2745	43	1740	5,736,961
4	−1	2	24	−54	4023	44	−1879	8,407,925
5	1	3	25	9	5896	45	−1268	12,322,413
6	1	4	26	73	8641	46	3619	18,059,374
7	−2	6	27	−63	12,664	47	−611	26,467,299
8	0	9	28	−64	18,560	48	−4887	38,789,712
9	3	13	29	136	27,201	49	4230	56,849,086
10	−2	19	30	1	39,865	50	4276	83,316,385
11	−3	28	31	−200	58,425	51	−9117	122,106,097
12	5	41	32	135	85,626	52	−46	178,955,183
13	1	60	33	201	125,491	53	13,393	262,271,568
14	−8	88	34	−335	183,916	54	−9071	384,377,665
15	4	129	35	−66	269,542	55	−13,439	563,332,848
16	9	189	36	536	395,033	56	22,464	825,604,416
17	−12	277	37	−269	578,949	57	4368	1,209,982,081
18	−5	406	38	−602	848,491	58	−35,903	1,773,314,929
19	21	595	39	805	1,243,524	59	18,096	2,598,919,345
20	−7	872	40	333	1,822,473	60	40,271	3,808,901,426

Let $h \in \mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{\geq 0}$, we have:

$$G_h(n+1) = \sum_{a_1+a_2+\cdots+a_h=n} G_{a_1+1}G_{a_2+1}\cdots G_{a_h+1}.$$

We list some values of $G_h(n+1)$ in Table A2.

Table A2. Some values of $G_h(n+1)$.

$G_h(n+1)$ \ h / n	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	2	3	4	6	9	13	19	28
2	1	2	3	6	11	18	30	50	81	130	208
3	1	3	6	13	27	51	94	171	303	527	906
4	1	4	10	24	55	116	234	460	879	1640	3006
5	1	5	15	40	100	231	505	1065	2175	4320	8391
6	1	6	21	62	168	420	987	2220	4815	10,122	20,733
7	1	7	28	91	266	714	1792	4278	9807	21,721	46,732

The values in Tables A1 and A2 were computed from Theorem 1 and Corollary 1. It is easy to verify that the values match their definitions completely, which proves our main results numerically.

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