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Stability Analysis of an LTI System with Diagonal Norm Bounded Linear Differential Inclusions

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Abstract: In this article, we present a stability analysis of linear time-invariant systems in control theory. The linear time-invariant systems under consideration involve the diagonal norm bounded linear differential inclusions. We propose a methodology based on low-rank ordinary differential equations. We construct an equivalent time-invariant system (linear) and use it to acquire an optimization problem whose solutions are given in terms of a system of differential equations. An iterative method is then used to solve the system of differential equations. The stability of linear time-invariant systems with diagonal norm bounded differential inclusion is studied by analyzing the Spectrum of equivalent systems.

Keywords: linear differential inclusion; spectrum of an operator; differential equations

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1. Introduction

The dynamical system is time-varying if some time shift to input data on the time axis leads to equivalent shifting of output data on the same time axis while having no other changes. On the other hand, a time-varying system is a linear time-invariant if both linearity and time-invariant conditions are true in all situations. The input-output relation for linear time-invariant systems is straightforward. For linear time-invariant systems, the impulse response can characterize the systems under consideration in order to describe the overall behavior of systems for the input data.

The linear time-invariant analysis and synthesis have captured great attention in the recent past. If both state and output data to linear time-invariant systems are non-negative then such systems are internally positive [1,2]. The internally positive linear time-invariant systems vastly appear in various fields such as engineering, economics, pharmacy, and chemistry. An intensive amount of research work has been done for the analysis and synthesis of linear time-invariant systems by making use of convex optimization techniques [3–8]. If the output to the linear time-invariant system is non-negative for the non-negative input with zero initial rates, then such a system is externally positive [1,2].

Linear differential inclusions can be used to simplify stability analysis and to synthesis complex control systems. Linear differential inclusions help ensure that trajectories of such complex systems possess certain features, which can be used to analyze systems [9]. The trajectories of certain nonlinear systems with polytopic linear differential inclusions are extensively studies in [10,11].

The linear matrix inequalities technique [10,12,13] is used to solve a large class of central problems that occur in system theory. Linear matrix inequalities technique simplifies the control systems so that certain efficient convex optimization techniques can be applied to solve them [10]. In [9], a linear parameter varying system is used to represent a nonlinear control system. For this purpose they have used Mean Value Theorem [14–16]. Furthermore, linear matrix inequalities are used to study the quadratic stability of polytopic linear differential inclusions.

The norm bounded linear differential inclusions can be used to describe the non-linear control systems with very high dimensions [17,18]. J. Doyle has introduced the idea of structured feedback while considering the fact that admissible perturbation Δ possesses a block diagonal structure, we refer [10] and the references therein for more details. For a single block with equality constraints, S. Boyed [10] was able to use the norm bounded linear differential inclusion and diagonal norm bounded linear differential equations when there were non equality constraints. The diagonal norm bounded linear differential inclusion is well-posed if and only if $det(I - D_{qp}\Delta) > 0$ with $|\Delta_{ii}| = 1$ [19]. Another equivalent condition is to show that the pair $(I + D_{qp}, I - D_{qp})$ is a W_0 pair, for more details we refer interested reader to see [20]. The standard branch-and-bound methodologies [21–25] are proposed to determine the well-posedness of the diagonal norm bounded linear differential inclusion.

In this paper , we study stability analysis of linear time-invariant systems with diagonal norm bounded linear differential inclusions. We derive an equivalent dynamical system parallel to diagonal norm bounded linear differential inclusion. The computation of the spectrum of the equivalent dynamical system discusses the stability of a given linear time-invariant system. For this purpose, our aim is to introduce a technique based on differential equations that maximize the negative spectrum corresponding to an equivalent dynamical system. Our proposed technique helps us to construct an optimization problem that involves the computation of eigenvectors and a direction matrix. The solution of the optimization problem turns over a system of ordinary differential equations whose solutions describe the maximization of the negative spectrum.

Overview of the Article

The paper is organized as follows. In Section 2, we present the basic definitions and properties. We start by defining certain types of matrices such as positive definite matrices, positive semi-definite matrices, negative definite matrices, and negative semi-definite matrices. Furthermore, we also present the definitions of matrix inequalities and linear matrix inequalities.

The problem under consideration is discussed in Section 3. Also, we discuss how were are able to write down an equivalent dynamical system for a the linear time-invariant system having diagonal norm bounded linear differential inclusion. In Section 4 of our article, we construct and solve an optimization problem that turns over a system of ordinary differential equations to study the stability analysis of the dynamical system under consideration.

In Section 5 of our paper, we present an alternative approach to maximize the negative spectrum while constructing a correlation matrix against theoriginal matrix. Finally, a conclusion is presented in Sections 6.

2. Preliminaries

Definition 1. *$M \in \mathbb{R}^{n,n}$ (matrix) is called positive definite if $X^T MX > 0$, for all $X \neq 0 \in \mathbb{R}^{n,1}$.*

Definition 2. *$M \in \mathbb{R}^{n,n}$ is called positive semi-definite if $X^T MX \geq 0$, for all $X \in \mathbb{R}^{n,1}$.*

Definition 3. *$M \in \mathbb{R}^{n,n}$ is called negative definite if $X^T MX < 0$, for all $X \neq 0 \in \mathbb{R}^{n,1}$.*

Definition 4. *$M \in \mathbb{R}^{n,n}$ is called negative semi-definite if $X^T MX \leq 0$, for all $X \in \mathbb{R}^{n,1}$.*

Definition 5 ([10]). *The matrix inequality* $F : \mathbb{R}^m \to S^{n,n}$ *in* $X \in \mathbb{R}^{m,1}$ *is defined as*

$$F(X) := F_0 + \sum_{i=1}^{n} f_i(x)F_i \leq 0$$

with $X = (x_1, x_2, x_3, ..., x_n)^T,\ F_0 \in S^{n,n},\ F_i \in \mathbb{R},\ i = 1 : n.$

Definition 6 ([10]). *The linear matrix inequality* $F : \mathbb{R}^m \to S^{n,n}$ *in* $X \in \mathbb{R}^{m,1}$ *is defined as*

$$F(X) := F_0 + \sum_{i=1}^{n} x_i F_i \leq 0$$

and with $X = (x_1, x_2, x_3, ..., x_n)^T,\ F_i \in S^{n,n},\ i = 0 : n.$

3. Diagonal Norm Bounded Linear Differential Inclusion

We consider linear time-invariant system with non-linear diagonal perturbation as

$$\begin{cases} \dot{x}(t) = Ax(t) + B_p p, \quad x(0) = x_0 \\ q = C_q x(t) + D_{qp} p \\ p_i = \delta_i(t) q_i,\ \|\delta_i(t)\|_2 \leq 1. \end{cases} \tag{1}$$

The perturbations $\delta_i(t)\ \ \forall i$ in (1) is a time-varying scalar perturbation. The vectors $x(t), p, q$ are the state-vector, input-vector and output-vector respectively. The system (1) is quadratically stable if for some real symmetric matrix P we have $P = P^t > 0$ and it satisfies Lyapunov function $V(x(t))$, that is,

$$\frac{dV(x(t))}{dt} = \left(\frac{dx(t)}{dt}\right)^t Px(t) + x^t P \left(\frac{dx(t)}{dt}\right) < 0.$$

An Equivalent System

The system given in (1) is equivalent to a linear time-invariant system of the form

$$\frac{dx(t)}{dt} = \Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big)x(t). \tag{2}$$

The subscript i on matrices $D_{qp,i}$ and $C_{q,i}$ denotes the ith-row of the matrix. The system (2) is quadratically stable if for some real, symmetric matrix P we have $P = P^t > 0$ and it satisfies the matrix inequalities

$$\Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big)^t P + P\Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big) < 0,$$

or

$$x^t\Big[\Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big)^t P + P\Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big)\Big]x < 0.$$

Finally we may write this as

$$x^t\Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big)^t Px < -x^t\Big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\Big)^t Px. \tag{3}$$

For quadratic stability, we must have following facts to hold true:

1. $x^t\big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\big)x = 0,$
2. $x^t\big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\big)x > 0.$

But it's also highly possible that the quantity

- $x^t\big(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}\big)x < 0.$

  Now, the main task is to make above quadratic form to be strictly positive, that is,

$$x^t\Big(P(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i})\Big)x > 0,$$

or

$$P(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i}) > 0,$$

or

$$\lambda_i\Big(P(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i})\Big) > 0, \ \forall i.$$

4. Systems of ODE's to Shift Negative Spectrum

In this section, we aim to construct and solve a system of ordinary differential equations in order to shift the negative spectrum or the negative eigenvalues (say) $\lambda_1(t), \lambda_2(t)$ from the spectrum of the perturbed matrix $\Big(\widehat{A} + D + \epsilon E(t)\Big)$ where matrix $\widehat{A} = P(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i})$ and the perturbation level $\epsilon > 0$. The matrix D is a diagonal matrix such that $\widehat{A} + D$ have the unit diagonal. The matrix function $E(t)$ has zero diagonal while it's Frobenius norm is bounded above by 1. As a result, this will case to increase the eigenvalues $\lambda_1(t), \lambda_2(t)$ so that both becomes strictly positive.

4.1. Optimization Problem

Next, we aim to determine the direction $Z = \dot{E}$ such that the solution of system of ODE's obtained while solving optimization problem cause a maximal growth of $\lambda_1(t)$ and $\lambda_2(t)$. The local optimization problem is given as

$$\begin{aligned} & max(\eta_1{}^* Z \eta_1) \\ & Subject\ to \\ \eta_2{}^* Z \eta_2 &= \eta_1{}^* Z \eta_1 \\ \langle Z, E(t)\rangle &= 0 \\ diag(Z) &= 0. \end{aligned} \tag{4}$$

In (4), $\eta_1 = \eta_1(t), \eta_2 = \eta_2(t)$, the eigenvectors corresponding to eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ respectively for the perturbed matrix $P(A + B_p(I - D_{qp,i})^{-1}\delta_i(t)C_{q,i})$.

The solution Z_* of above optimization problem given in (4) is obtained as

$$Z_* = \dot{E} = (1-\mu)\eta_1^*\eta_1 + \mu\eta_2^*\eta_2 - \mu\{\langle \eta_1^*\eta_1 - \eta_2^*\eta_2, E(t)\rangle - \langle \eta_1^*\eta_1, E(t)\rangle\}. \tag{5}$$

4.2. Euler's Method

The solution to the system of ordinary differential equations in (5) is obtained by making use of Euler's method

$$E_{n+1}(t) = E_n(t) + h\dot{E}_n(t).$$

Thus, finally we compute all strictly positive eigenvalues, that is, $\lambda_i > 0 \ \forall i$ from the eigenvalue problem

$$\Big(\widehat{A} + D + \epsilon E(t)\Big)\eta(t) = \lambda(t)\eta(t).$$

5. Alternative Way to Compute Strictly Positive Spectrum

In this section, we aim to compute a matrix B for the perturbation matrix $\Big(\widehat{A} + D + \epsilon E(t)\Big)$ such that

1. $B^t = B$, Symmetric matrix
2. $b_{ij} = b_{ji} \in [-1, 1] \ \forall i, j,$

3. $Diag(B) = 1$,
4. $\lambda_i(B) > 0\ \forall i$.

For the computation of such a matrix B, we refer to see Section 5 of [26]. After the computation of B, the nearest correlation matrix, we aim to determine a matrix C such that

$$C = \epsilon\widehat{B} + (1-\epsilon)B$$

with the perturbation level $\epsilon \in [0,1]$ and the matrix B. The matrix C is also a nearest correlation matrix. To show this, we have

$$b_{ii} = 1\ \forall i, \quad b_{ij} = b_{ji}\ \forall i \neq j,$$

and

$$\widehat{b}_{ii} = 1\ \forall i, \quad \widehat{b}_{ij} = \widehat{b}_{ji}\ \forall i \neq j.$$

In view of above one may write

$$c_{ij} = \epsilon\widehat{b}_{ij} + (1-\epsilon)b_{ij}$$

or

$$c_{ij} = \epsilon\widehat{b}_{ji} + (1-\epsilon)b_{ji}.$$

To show that $Diag(C) = 1$, we have that

$$c_{ii} = \epsilon\widehat{b}_{ii} + (1-\epsilon)b_{ii}.$$

Since $\epsilon \in [0,1]$ and $b_{ii} = \widehat{b}_{ii} = 1\ \forall i$. Then clearly $c_{ii} = 1\ \forall i$.
Next, to show that $c_{ij} = c_{ji} \in [-1,1]\ \forall i, j$. For this, we have

$$c_{ij} = \epsilon\widehat{b}_{ij} + (1-\epsilon)b_{ij},$$

or

$$c_{ji} = \epsilon\widehat{b}_{ji} + (1-\epsilon)b_{ji}.$$

Since $b_{ij} = b_{ji}\ \forall i \neq j$ and $b_{ij}, \widehat{b}_{ji}\ \forall i \neq j$ lies in $[-1,1]$ and indicates that c_{ij} or c_{ji} also belongs to $[-1,1]$. Finally to make sure that the structure of C is positive semi-definite we take eigevector $\eta \in \mathbb{R}^{n,1}$ such that

$$\eta^t C\eta \geq 0.$$

Since $C = \epsilon\widehat{B} + (1-\epsilon)B$, this ensure that $\eta^t\widehat{B}\eta \geq 0$ and $\eta^t B\eta \geq 0$ because both $\widehat{B}$ and B are correlation matrices. Thus,

$$\eta^t C\eta = \epsilon(\eta^t\widehat{B}\eta) + (1-\epsilon)(\eta^t B\eta) \geq 0,$$

and this implies that $\eta^t C\eta \geq 0$.

Construction of $\widehat{B}$

The Cholesky decomposition of $B = LL^t$ with L^t being upper triangular Cholesky decomposition matrix. The column vectors of L are $L = (l_1, l_2, \cdots, l_{m-1}, l_n)$ with $l_{n-1} = (0, 0, \cdots, 0, a, b)^t$ and $l_n = (0, 0, \cdots, 0, 0)^t$. For instance if we take B to be a five dimensional matrix given by

$$B = \begin{pmatrix} 1 & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & 1 & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & 1 & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & 1 & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & 1 \end{pmatrix}.$$

Suppose we aim to adjust pair (b_{45}, b_{54}) of B. As $B = LL^t$ with

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 & 0 \\ l_{41} & l_{42} & l_{43} & a & 0 \\ l_{51} & l_{52} & l_{53} & b & c \end{pmatrix},$$

where a and b are computed using Cholesky decomposition of B. For the computation of upper and lower bounds of pair (a_{45}, a_{54}) one may use the formula $\widehat{\Omega} = \widehat{L}\widehat{L}^T = \widehat{l}_{n-1}\widehat{l}_{n-1}^T + \widehat{l}_n\widehat{l}_n^T + \sum_{i=1}^{n-2} \widehat{l}_i\widehat{l}_i^T$ given by [27]. The computation of upper and lower bounds of (a_{45}, a_{54}) gives a new matrix $\hat{B}$ whose all entries are entries of B except (a_{45}, a_{54}) and it is a correlation matrix if and only if bounds are inside or on the boundaries of closed interval given by [27]. Finally the updated correlation matrix is obtained by taking $\epsilon \in [0, 1]$.

6. Conclusions

In this article, we have studied the stability analysis of linear time-invariant systems in control having diagonal norm bounded linear differential inclusions. The main contribution is to introduce a low-rank ordinary differential equations based technique to construct and then solve an optimization problem. The optimization problem involves the computation of left and right eigenvectors corresponding to the Spectrum of the perturbed matrix. The solution to optimization problem allows us to check the behavior of the spectrum of perturbation matrix corresponding to an equivalent system. In near future, our aim is to discuss:

- Comparison of different numerical techniques to determine the suitable choice of perturbation level,
- Stability analysis of some practical examples from linear time-invariant systems with norm bounded differential inclusion and diagonal norm bounded linear differential inclusions,
- Stability analysis of linear time-variant systems in control.

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