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Inner Product Groups and Riesz Representation Theorem

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Abstract: In this paper, we introduce an inner product on abelian groups and, after investigating the basic properties of the inner product, we first show that each inner product group is a torsion-free abelian normed group. We give examples of such groups and describe the norms induced by such inner products. Among other results, Hilbert groups, midconvex and orthogonal subgroups are presented, and a Riesz representation theorem on divisible Hilbert groups is proved.

Keywords: inner product groups; Hilbert groups; Riesz representation theorem; midconvex subgroups; normed groups; torsion-free groups; invariant metrics

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1. Introduction and Preliminaries

In 1936, Birkhoff and Kakutani independently proved a significant theorem: *A Hausdorff group $\mathcal{K}$ is homeomorphic with a metric space, if and only if $\mathcal{K}$ satisfies the first countability axiom.* They also showed that this group has a right invariant metric. The theorem then became known as Birkhoff–Kakutani's metrization theorem for groups [1,2]. A metric d on a semigroup $\mathcal{K}$ is called left-invariant if $d(vx, vy) = d(x, y)$ and right-invariant if $d(xv, yv) = d(x, y)$ whenever $v, x, y \in \mathcal{K}$. The metric d is said to be invariant if it is both right and left-invariant. In 1950, V. L. Klee studied invariant metrics on groups to solve a problem of Banach [3]. In this article, we focus on normed groups closely related to the groups with invariant metrics and which recently played a role in the theory of topological groups [4–7]. Some may use the term "length function" instead of "norm" for groups [8,9]. Recently, new approaches have been presented to the theory of normed groups, such as probabilistic normed groups [10,11]; see [4] for a broader discussion about the history of normed groups. We start with some preliminaries as required in the paper.

Definition 1 ([4])**.** *Let $\mathcal{K}$ be a group with identity element e. A function $\|.\| : \mathcal{K} \to \mathbb{R}$ is called a group-norm if the following holds for all $v, s \in \mathcal{K}$:*

1. $\|v\| = \|v^{-1}\| \qquad (Symmetry)$;
2. $\|vs\| \leq \|v\| + \|s\| \quad (Triangle \quad inequality)$;
3. $\|v\| \geq 0 \quad and \quad \|v\| = 0 \quad iff \quad v = e \quad (Positivity)$.

Then, $\mathcal{K}$ equipped with a group-norm $\|.\|$ is said to be a normed group.

Definition 2 ([4])**.** *Let $(\mathcal{K}, \|.\|)$ be a normed group and $s \in \mathcal{K}$ and $\{s_n\}$ be a sequence in $\mathcal{K}$. Then,*

1. *The sequence $\{s_n\}$ converges to s if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists a positive integer n_0 depending on ϵ such that $\|s_n s^{-1}\| < \epsilon$ for every $n > n_0$. We denote this by $s = \lim_{n\to\infty} s_n$.*
2. *The sequence $\{s_n\}$ is called a Cauchy sequence if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$ there exists a positive integer n_0 depending on ϵ such that $\|s_n s_m^{-1}\| < \epsilon$ for every $n, m > n_0$.*
3. *The normed group $(\mathcal{K}, \|.\|)$ is called complete if any Cauchy sequence in $\mathcal{K}$ converges to an element of $\mathcal{K}$; i.e., it has a limit in group $\mathcal{K}$.*

4. *A Banach group is a normed group $(\mathcal{K}, \|.\|)$ that is complete with respect to the metric*

$$d(s,q) = \|sq^{-1}\|, \quad (s,q \in \mathcal{K}).$$

Definition 3 ([12])**.** *Let $\mathcal{K}$ be a group with identity element e. The order of an element $v \in \mathcal{K}$ is the smallest $n \in \mathbb{N}$ such that $v^n = e$. If no such n exists, v is said to have infinite order. An abelian group $\mathcal{K}$ is said to be torsion-free if no element other than the identity e is of finite order.*

Let $(\mathcal{K}, \|.\|)$ be a normed group. For $s \in \mathcal{K}$, the s-conjugate norm is defined by

$$\|v\|_s := \|svs^{-1}\|.$$

Note that the group-norm is abelian iff the norm is preserved under conjugacy [13]. It is obvious that each norm on an abelian group is an abelian norm. The following example shows how non-trivial cases can be considered.

Example 1. *Let $\mathcal{K}$ be the nonabelian dihedral group D_3. A matrix representation of this group is given by*

$$r_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$r_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad s_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$r_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad s_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}. \end{pmatrix}$$

The relations

$$\begin{aligned} r_i r_j &= r_{i+j} \\ r_i s_j &= s_{i+j} \\ s_i r_j &= s_{i-j} \\ s_i s_j &= r_{i-j}, \end{aligned}$$

holds for integers i and j such that $0 \leq i, j \leq 2$ and both $i+j$ and $i-j$ are computed modulo 3. Note that r_0 is the identity element, $r_i^{-1} = r_{3-i}$ and $s_i^{-1} = s_i$, for each $0 \leq i \leq 2$.
Now let $\|.\| : D_3 \to \mathbb{R}$ be defined by

$$\|V\| = 2 - tr(V) \qquad (\forall V \in D_3),$$

where $tr(V)$ denotes the trace of matrix V. Note that $\|.\|$ is abelian because

$$\|r_i r_j\| = \|r_{i+j}\| = \|r_{j+i}\| = \|r_j r_i\|,$$

$$\|r_i s_j\| = \|s_{i+j}\| = 2 = \|s_j r_i\|,$$

$$\|s_i s_j\| = \|r_{i-j}\| = \begin{cases} 0, & if \ \ i = j \\ 3, & if \ \ i \neq j \end{cases} = \|s_j s_i\|.$$

This shows that there is an abelian norm on a non-abelian group.

Definition 4 ([14])**.** *Let $\mathcal{K}$ be a group. An element $v \in \mathcal{K}$ is said to be divisible by $n \in \mathbb{Z}$ if $v = x^n$ has a solution x in $\mathcal{K}$. A group $\mathcal{K}$ is called infinitely divisible if each element in $\mathcal{K}$ is divisible by every positive integer.*

A group norm $\|.\|$ is $\mathbb{N}$-homogeneous if for each $n \in \mathbb{N}$

$$\|v^n\| = n\|v\| \ \ (\forall v \in \mathcal{K}).$$

Remark 1. *For a divisible element v in a $\mathbb{N}$-homogeneous normed group $\mathcal{K}$, let $s^n = v$; then, $\|s\| = \frac{1}{n}\|v\|$ and as $s^m = v^{\frac{m}{n}}$, we have $\frac{m}{n}\|v\| = \|v^{\frac{m}{n}}\|$.*

Definition 5 ([4])**.** *Let $(\mathcal{K}, \|.\|_{\mathcal{K}})$ and $(\mathcal{L}, \|.\|_{\mathcal{L}})$ be normed groups. A map $\alpha : \mathcal{K} \to \mathcal{L}$ is called continuous if for every $\varepsilon > 0$ there exists such a $\delta > 0$ that $\|vs^{-1}\|_{\mathcal{K}} < \delta$ implies $\|\alpha(v)\alpha(s)^{-1}\|_{\mathcal{L}} < \varepsilon$.*

2. Inner Product Groups

In this section, we introduce the notion of the inner product on abelian groups. Here, we derive and investigate in detail the induced norm properties such as *Cauchy–Schwartz inequality* for groups and the *Parallelogram law* for inner product groups. Besides, we show that every inner product group is a normed torsion-free group. Note that in this paper, the identity element of groups is denoted by e.

Definition 6. *A semi-inner product on a group $\mathcal{K}$ with identity element e is a function that associates a real number $\langle v, s\rangle$ with each pair of elements v and s in $\mathcal{K}$ in such a way that the following axioms are satisfied for all elements v, s and z in $\mathcal{K}$:*

1. $\langle v, s\rangle = \langle s^{-1}, v^{-1}\rangle$ *(Symmetry);*
2. $\langle vs, z\rangle = \langle v, z\rangle + \langle s, z\rangle$ *(Distributivity);*
3. $\langle v, v\rangle \geq 0$ *(Positivity).*

A group with a semi-inner product is called a semi-inner product group.

Note that combining (1) and (2) gives the equation

$$\langle v, sz\rangle = \langle v, s\rangle + \langle v, z\rangle \qquad (\forall v, s, z \in \mathcal{K}).$$

Example 2.

1. *Let $\mathcal{K} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ denote the discrete Heisenberg group and let $\langle ., .\rangle : \mathcal{K} \to \mathbb{R}$ be defined by*

$$\langle v, s\rangle = (a + c)(\acute{a} + \acute{c}) \qquad (\forall v, s \in \mathcal{K})$$

where $v = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & \acute{a} & \acute{b} \\ 0 & 1 & \acute{c} \\ 0 & 0 & 1 \end{pmatrix}$. Then $(\mathcal{K}, \langle ., .\rangle)$ is a semi-inner product group.

2. *Let $\mathcal{K} = SO(2)$. Define the group $SO(2)$ as 2×2 matrices by*

$$SO(2) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Note that the group operation is given by the matrix multiplication. Then, by

$$\langle A(\theta), A(\acute{\theta})\rangle = \theta\acute{\theta},$$

where $A(\theta)$ and $A(\acute{\theta})$ are elements in $SO(2)$, this group is a semi-inner product group.

3. *Let $\mathcal{K}$ be a finite abelian group with $\gcd(ord(v), ord(s)) = 1$ for all $v, s \in \mathcal{K}$, where $ord(v)$ denotes the order of an element $v \in \mathcal{K}$. For elements v and s in $\mathcal{K}$, with orders m and n, respectively, if m and n are co-prime, then vs has order mn [15].*
Then, for all $v, s \in \mathcal{K}$,

$$\langle v, s\rangle = (\log ord(v))(\log ord(s)),$$

is a semi-inner product on $\mathcal{K}$.

It is obvious that $\langle v, v \rangle \geq 0$ *and* $\langle v, s \rangle = \langle s^{-1}, v^{-1} \rangle$*. The principle of distributivity remains to be proven.*

$$\begin{aligned}\langle vs, z \rangle &= (\log ord(vs))(\log ord(z)) \\ &= (\log(ord(v)ord(s)))(\log ord(z)) \\ &= (\log ord(v) + \log ord(s))(\log ord(z)) \\ &= (\log ord(v))(\log ord(z)) + (\log ord(s))(\log ord(z)) \\ &= \langle v, z \rangle + \langle s, z \rangle.\end{aligned}$$

Proposition 1. *Let* $(\mathcal{K}, \langle ., . \rangle)$ *be a semi-inner product group. Here are some elementary properties of semi-inner product:*

1. $\langle v^{-1}, s \rangle = -\langle v, s \rangle \qquad (\forall v, s \in \mathcal{K})$;
2. $\langle v, s \rangle = \langle s, v \rangle \qquad (\forall v, s \in \mathcal{K})$;
3. $\langle v^n, s^m \rangle = nm\langle v, s \rangle \qquad (\forall v, s \in \mathcal{K} \quad and \quad n, m \in \mathbb{N})$;
4. $\langle v, e \rangle = 0 \qquad (\forall v \in \mathcal{K})$;

The definition of inner products usually comes immediately after defining semi-inner products; i.e., if after three axioms of Definition 6, $\langle v, v \rangle = 0$ implies $v = e$, we expect that $\langle ., . \rangle$ is an inner product for group $\mathcal{K}$. However, there is an additional condition for groups with an inner product. In other words, we show that the definition of our inner product imposes an abelian structure on groups.

Definition 7 ([12])**.** *Let* $\mathcal{K}$ *be a group. The subgroup of* $\mathcal{K}$ *generated by the set* $\{vsv^{-1}s^{-1} | v, s \in \mathcal{K}\}$ *is called a commutator subgroup of* $\mathcal{K}$ *and denoted by* $\acute{\mathcal{K}}$*. The elements* $vsv^{-1}s^{-1} (v, s \in \mathcal{K})$ *are called commutators.*

Theorem 1 ([12])**.** *Group* $\mathcal{K}$ *is abelian if and only if* $\acute{\mathcal{K}} = \{e\}$*.*

In fact, $\acute{\mathcal{K}}$ provides an indicator for measuring differences between group $\mathcal{K}$ and an abelian group. Now, let $(\mathcal{K}, \langle ., . \rangle)$ be an semi-inner product group and suppose that $\langle v, v \rangle = 0$ implies $v = e$. We show that $\acute{\mathcal{K}} = \{e\}$. Since for $v, s \in G$

$$\begin{aligned}\langle vsv^{-1}s^{-1}, vsv^{-1}s^{-1} \rangle &= \langle v, vsv^{-1}s^{-1} \rangle + \langle s, vsv^{-1}s^{-1} \rangle \\ &+ \langle v^{-1}, vsv^{-1}s^{-1} \rangle + \langle s^{-1}, vsv^{-1}s^{-1} \rangle \\ &= \langle v, vsv^{-1}s^{-1} \rangle + \langle s, vsv^{-1}s^{-1} \rangle \\ &- \langle v, vsv^{-1}s^{-1} \rangle - \langle s, vsv^{-1}s^{-1} \rangle \\ &= 0.\end{aligned}$$

Then, $vsv^{-1}s^{-1} = e$. This shows that all commutators of $\mathcal{K}$ are equal to e. So, we define inner products on abelian groups as follows:

Definition 8. *Let* $(\mathcal{K}, \langle ., . \rangle)$ *be a semi-inner product abelian group, in which for* $v \in \mathcal{K}$, $\langle v, v \rangle = 0$ *implies* $v = e$*. Then* $(\mathcal{K}, \langle ., . \rangle)$ *is called an abelian inner product group.*

In this paper, it is supposed that every inner product group is abelian. Therefore, when we talk of an inner product group or a Hilbert group, we mean that there is an inner product on an abelian group.

Example 3.

1. *Let* $\mathcal{K} = \mathbf{R}_+^*$ *denote the group of positive real numbers with multiplication as the group operation. By*

$$\langle v, s \rangle := (\log v)(\log s) \qquad (\forall v, s \in \mathcal{K}),$$

$\mathcal{K}$ will be an inner product group.

2. *Let $\mathcal{K} = 2^z := \{2^a | a \in \mathbb{Z}\}$, then $\mathcal{K}$ is an abelian group by real multiplication. By setting*

$$\langle 2^a, 2^{\acute{a}} \rangle := a\acute{a},$$

$\mathcal{K}$ is an inner product group.

3. *Let $Q(\sqrt{2}) = \{p + q\sqrt{2} | p, q \in Q\}$. Then, with a binary operation*

$$(p + q\sqrt{2}) + (\acute{p} + \acute{q}\sqrt{2}) = (p + \acute{p}) + (q + \acute{q})\sqrt{2},$$

and by setting an inner product

$$\langle v, s \rangle = vs \qquad (\forall v, s \in \mathcal{K}),$$

$\mathcal{K} = Q\sqrt{2}$ is an inner product group.

4. *Let $Z \times Z = \{(a, b) | a, b \in Z\}$. Then, with a binary operation*

$$(a, b) + (\acute{a}, \acute{b}) = (a + \acute{a}, b + \acute{b})$$

and by setting an inner product

$$\langle (a, b), (\acute{a}, \acute{b}) \rangle = (a + \sqrt{2}b)(\acute{a} + \sqrt{2}\acute{b})$$

$\mathcal{K} = Z \times Z$ is an inner product group.

Proposition 2. *Let $(\mathcal{K}, \langle ., . \rangle)$ be an inner product group. For $v, s \in \mathcal{K}$ such that $\langle v, z \rangle = \langle s, z \rangle$ for all $z \in \mathcal{K}$, then $v = s$.*

Proof. For $v, s \in \mathcal{K}$, let $\langle v, z \rangle = \langle s, z \rangle$ for all $z \in \mathcal{K}$. Therefore, $\langle vs^{-1}, z \rangle = 0$ for all $z \in \mathcal{K}$, and we have $\langle vs^{-1}, vs^{-1} \rangle = 0$. So, $vs^{-1} = e$, which means $v = s$. □

The proofs of the next two lemmas are straightforward, and thus we omit them.

Lemma 1. *Let $\mathcal{K}$ be an abelian group and $(\mathcal{L}, \langle ., . \rangle_{\mathcal{L}})$ be an inner product group. If $\varpi : \mathcal{K} \to \mathcal{L}$ is a group monomorphism, then*

$$\langle v, s \rangle_{\mathcal{K}} := \langle \varpi(v), \varpi(s) \rangle_{\mathcal{L}} \quad (\forall v, s \in \mathcal{K})$$

is also an inner product on $\mathcal{K}$.

Lemma 2. *Let $(\mathcal{K}, \circ, \langle ., . \rangle_{\mathcal{K}})$ and $(\mathcal{L}, \bullet, \langle ., . \rangle_{\mathcal{L}})$ be inner product groups.*

Then, the group $(\mathcal{K} \times \mathcal{L}, *, \langle ., . \rangle_{\mathcal{K} \times \mathcal{L}})$ with

$$(v, s) * (\acute{v}, \acute{s}) = (v \circ \acute{v}, s \bullet \acute{s})$$

is also an inner product group by

$$\langle (v, s), (\acute{v}, \acute{s}) \rangle_{\mathcal{K} \times \mathcal{L}} := \langle v, \acute{v} \rangle_{\mathcal{K}} + \langle s, \acute{s} \rangle_{\mathcal{L}}.$$

Definition 9. *Let $(\mathcal{K}, \langle ., . \rangle)$ be an inner product group. We define the map $\|.\| : \mathcal{K} \to \mathbb{R}$ for all $v \in \mathcal{K}$ by*

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Theorem 2. *Let $(\mathcal{K}, \langle ., . \rangle)$ be an inner product group. Then,*

$$|\langle v, s \rangle| \leq \|v\|.\|s\| \qquad (\forall v, s \in \mathcal{K}). \tag{1}$$

Proof. Let v and s be arbitrary in $\mathcal{K}$. If $s = e$, then the inequality is true. For $v, s \neq e$ and $n \in \mathbb{N}$, the positivity of the inner product shows that

$$\begin{aligned} 0 \leq \langle vs^{-n}, vs^{-n} \rangle &= \langle v, vs^{-n} \rangle + \langle s^{-n}, vs^{-n} \rangle \\ &= \langle s^n v^{-1}, v^{-1} \rangle + \langle s^n v^{-1}, s^n \rangle \\ &= -2n\langle v, s \rangle + \langle v, v \rangle + n^2 \langle s, s \rangle. \end{aligned}$$

Then,

$$\langle s, s \rangle n^2 - 2\langle v, s \rangle n + \langle v, v \rangle \geq 0.$$

Let $a = \langle s, s \rangle$, $b = -2\langle v, s \rangle$ and $c = \langle v, v \rangle$. Then, the equation becomes $an^2 + bn + c \geq 0$. This is a quadratic equation for $n \in \mathbb{N}$ with real coefficients. Since this polynomial takes only non-negative values, its discriminate $b^2 - 4ac$ must be non-positive

$$4\langle v, s \rangle^2 - 4\langle v, v \rangle \langle s, s \rangle \leq 0.$$

This implies that

$$|\langle v, s \rangle| \leq \sqrt{\langle v, v \rangle} \sqrt{\langle s, s \rangle} = \|v\|.\|s\| \qquad (\forall v, s \in \mathcal{K}).$$

□

Inequality (1) is called Cauchy–Schwartz inequality for groups.

Proposition 3. *Let $(\mathcal{K}, \langle ., . \rangle)$ be an inner product group. Then,*

$$\langle vs(ab)^{-1}, vs(ab)^{-1} \rangle^{\frac{1}{2}} \leq \langle va^{-1}, va^{-1} \rangle^{\frac{1}{2}} + \langle sb^{-1}, sb^{-1} \rangle^{\frac{1}{2}}.$$

Proof. Recall that every inner product group is abelian; then,

$$\begin{aligned} \langle vs(ab)^{-1}, vs(ab)^{-1} \rangle^{\frac{1}{2}} &= \langle vsa^{-1}b^{-1}, vsa^{-1}b^{-1} \rangle^{\frac{1}{2}} \\ &= \langle va^{-1}sb^{-1}, va^{-1}sb^{-1} \rangle^{\frac{1}{2}} \\ &= (\langle va^{-1}, va^{-1}sb^{-1} \rangle + \langle sb^{-1}, va^{-1}sb^{-1} \rangle)^{\frac{1}{2}} \\ &= (\langle va^{-1}, va^{-1} \rangle + \langle va^{-1}, sb^{-1} \rangle + \langle sb^{-1}, va^{-1} \rangle + \langle sb^{-1}, sb^{-1} \rangle)^{\frac{1}{2}} \\ &\leq (\langle va^{-1}, va^{-1} \rangle + 2\langle va^{-1}, va^{-1} \rangle^{\frac{1}{2}} \langle sb^{-1}, sb^{-1} \rangle^{\frac{1}{2}} + \langle sb^{-1}, sb^{-1} \rangle)^{\frac{1}{2}} \\ &= \langle va^{-1}, va^{-1} \rangle^{\frac{1}{2}} + \langle sb^{-1}, sb^{-1} \rangle^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 3. *Let $(\mathcal{K}, \langle ., . \rangle)$ be an inner product group. Then, it is a normed group with norm $\|v\| = \sqrt{\langle v, v \rangle}$ for all $v \in \mathcal{K}$.*

Proof. The positivity principle of the norm is clear. From Proposition 1, we have $\langle v, v \rangle = \langle v^{-1}, v^{-1} \rangle$ whenever $v \in \mathcal{K}$. Then,

$$\|v\|^2 = \langle v, v \rangle = \langle v^{-1}, v^{-1} \rangle = \|v^{-1}\|^2.$$

Now, we show that $\|.\|$ satisfies the triangle inequality. For all $v, s \in \mathcal{K}$, we have

$$\begin{aligned} \|vs\|^2 = \langle vs, vs \rangle &= \langle v, vs \rangle + \langle s, vs \rangle \\ &= \langle v, v \rangle + \langle v, s \rangle + \langle s, v \rangle + \langle s, s \rangle \\ &= \|v\|^2 + 2\langle v, s \rangle + \|s\|^2. \end{aligned}$$

So, the Cauchy–Schwartz inequality for groups implies that

$$\|vs\|^2 \leq \|v\|^2 + 2\|v\|\|s\| + \|s\|^2 = (\|v\| + \|s\|)^2.$$

Hence,

$$\|vs\| \leq \|v\| + \|s\|.$$

$\square$

Example 4. *Let $\mathcal{K}$ be the abelian group of matrices of the form $A(\theta)$, where $\theta \in \mathbb{R}$, and*

$$A(\theta) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}.$$

The group operation is given by the multiplication of matrices. It follows that $A(\theta).A(\acute{\theta}) = A(\theta + \acute{\theta})$. Now, we define

$$\langle A(\theta), A(\acute{\theta})\rangle := \theta\acute{\theta}.$$

Therefore, it is obvious that $\mathcal{K}$ is a normed group with $\|A(\theta)\| = |\theta|$.

Corollary 1. *Let $(\mathcal{K}, \langle ., .\rangle)$ be an inner product group. Then, the group norm induced by the inner product is $\mathbb{N}$-homogeneous; i.e., $\|v^n\| = n\|v\| \quad (\forall v \in \mathcal{K}, n \in \mathbb{N})$.*

Proof. It follows immediately from the fact that $\langle v^n, v^n\rangle = n^2\langle v, v\rangle$, since

$$\|v^n\|^2 = \langle v^n, v^n\rangle = n^2\langle v, v\rangle = n^2\|v\|^2.$$

$\square$

Theorem 4. *Let $(\mathcal{K}, \langle ., .\rangle)$ be an inner product group. Then, $\mathcal{K}$ is a torsion-free abelian group.*

Proof. Let $v \in \mathcal{K}$ such that $ord(v) = n$ and $n \neq 0$. Then, the $\mathbb{N}$-homogeneous property of the induced group norm implies that

$$\|v^n\| = n\|v\| = \|e\| = 0.$$

Since $n \neq 0$, then $\|v\| = 0$ and $v = e$. Hence, $\mathcal{K}$ is a torsion-free abelian group. $\square$

Lemma 3 (Parallelogram Law)**.** *Let $(\mathcal{K}, \langle ., .\rangle)$ be an inner product group. Then,*

$$\|vs\|^2 + \|vs^{-1}\|^2 = 2(\|v\|^2 + \|s\|^2) \qquad (\forall v, s \in \mathcal{K}).$$

Proof. We have

$$\begin{aligned} \|vs\|^2 + \|vs^{-1}\|^2 &= \langle vs, vs\rangle + \langle vs^{-1}, vs^{-1}\rangle \\ &= \|v\|^2 + \|s\|^2 + \langle v, s\rangle + \langle s, v\rangle \\ &+ \|v\|^2 + \|s\|^2 - \langle v, s\rangle - \langle s, v\rangle \\ &= 2(\|v\|^2 + \|s\|^2). \end{aligned}$$

$\square$

Theorem 5. *Let $(\mathcal{K}, \|.\|)$ be an abelian normed group. The norm $\|.\|$ is induced by an inner product iff the parallelogram law holds in $(\mathcal{K}, \|.\|)$.*

Proof. Let $(\mathcal{K}, \|.\|)$ be an abelian normed group whose norm satisfies the parallelogram law. Put

$$\langle v, s\rangle := \frac{1}{4}(\|vs\|^2 - \|vs^{-1}\|^2).$$

We wish to show that $\langle ., .\rangle$ is an inner product on $\mathcal{K}$. We begin by observing that $\langle v, v\rangle = \frac{1}{4}(\|v^2\|^2 - \|e\|^2)$ for all $v \in \mathcal{K}$, so $\langle ., .\rangle$ is non-negative.

Since

$$\begin{aligned}\langle v, s\rangle &= \frac{1}{4}(\|vs\|^2 - \|vs^{-1}\|^2)\\ &= \frac{1}{4}(\|s^{-1}v^{-1}\|^2 - \|s^{-1}v\|^2)\\ &= \langle s^{-1}, v^{-1}\rangle,\end{aligned}$$

the function $\langle ., .\rangle$ is also a symmetric function. The final step in this proof is showing that $\langle ., .\rangle$ satisfies the distributivity property. We have

$$2\|vz\|^2 + 2\|s\|^2 = \|vsz\|^2 + \|vs^{-1}z\|^2.$$

This gives

$$\|vsz\|^2 = 2\|vz\|^2 + 2\|s\|^2 - \|vs^{-1}z\|^2.$$

Exchanging v and s in the last equation gives

$$\|vsz\|^2 = 2\|sz\|^2 + 2\|v\|^2 - \|sv^{-1}z\|^2.$$

Then,

$$\|vsz\|^2 = \|v\|^2 + \|s\|^2 + \|vz\|^2 + \|sz\|^2 - \frac{1}{2}\|vs^{-1}z\|^2 - \frac{1}{2}\|sv^{-1}z\|^2.$$

Replacing z by z^{-1}, we have

$$\|vsz^{-1}\|^2 = \|v\|^2 + \|s\|^2 + \|vz^{-1}\|^2 + \|sz^{-1}\|^2 - \frac{1}{2}\|vs^{-1}z^{-1}\|^2 - \frac{1}{2}\|sv^{-1}z^{-1}\|^2.$$

Since $\|v\| = \|v^{-1}\|$ for all $v \in \mathcal{K}$, we get

$$\begin{aligned}\langle vs, z\rangle &= \frac{1}{4}(\|vsz\|^2 - \|vsz^{-1}\|^2)\\ &= \frac{1}{4}(\|vz\|^2 - \|vz^{-1}\|^2) + \frac{1}{4}(\|sz\|^2 - \|sz^{-1}\|^2)\\ &= \langle v, z\rangle + \langle s, z\rangle.\end{aligned}$$

This shows that the defined $\langle ., .\rangle$ satisfies all three criteria of an inner product. Therefore, $(\mathcal{K}, \langle ., .\rangle)$ is an inner product group. □

Lemma 4. *Let $(\mathcal{K}, \langle ., .\rangle)$ be an inner product group with induced norm $\|.\|$. Then, $\langle ., .\rangle : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is continuous.*

Proof. Let $v, s \in \mathcal{K}$ and for all $\acute{v}, \acute{s} \in \mathcal{K}$ put $\acute{v}v^{-1} = z$ and $\acute{s}s^{-1} = w$. Then, we have

$$\begin{aligned}
|\langle \acute{v}, \acute{s}\rangle - \langle v, s\rangle| &= |\langle zv, ws\rangle - \langle v, s\rangle| \\
&= |\langle z, ws\rangle + \langle v, ws\rangle - \langle v, s\rangle| \\
&= |\langle z, s\rangle + \langle z, w\rangle + \langle v, s\rangle + \langle v, w\rangle - \langle v, s\rangle| \\
&\leq \|z\|\|s\| + \|z\|\|w\| + \|v\|\|w\| \\
&= \|\acute{v}v^{-1}\|\|s\| + \|\acute{v}v^{-1}\|\|\acute{s}s^{-1}\| + \|v\|\|\acute{s}s^{-1}\|.
\end{aligned}$$

From which it follows that $\langle ., . \rangle$ is continuous. □

Definition 10. *Let $(\mathcal{K}, \|.\|_{\mathcal{K}})$ and $(\mathcal{L}, \|.\|_{\mathcal{L}})$ be normed groups. An isomorphism (bijective group homomorphism) $\varpi : \mathcal{K} \to \mathcal{L}$ is said to be isometric if $\|\varpi(v)\|_{\mathcal{L}} = \|v\|_{\mathcal{K}}$ for all $v \in \mathcal{K}$.*

Proposition 4. *Let $(\mathcal{K}, \langle ., . \rangle_{\mathcal{K}})$ and $(\mathcal{L}, \langle ., . \rangle_{\mathcal{L}})$ are inner product groups. Then, an isomorphism $\varpi : \mathcal{K} \to \mathcal{L}$ is isometric if and only if*

$$\langle \varpi(v), \varpi(s)\rangle_{\mathcal{L}} = \langle v, s\rangle_{\mathcal{K}} \qquad (\forall v, s \in \mathcal{K})$$

Proof. Suppose that the given identity holds. Then,

$$\|\varpi(v)\|_{\mathcal{L}} = \sqrt{\langle \varpi(v), \varpi(v)\rangle_{\mathcal{L}}} = \sqrt{\langle v, v\rangle_{\mathcal{K}}} = \|v\|_{\mathcal{K}} \quad (\forall v \in \mathcal{K}).$$

For the converse, suppose that ϖ is isometric, and $v, s \in \mathcal{K}$, then

$$\begin{aligned}
\langle \varpi(v), \varpi(s)\rangle_{\mathcal{L}} &= \frac{1}{4}(\|\varpi(v)\varpi(s)\|^2_{\mathcal{L}} - \|\varpi(v)\varpi(s)^{-1}\|^2_{\mathcal{L}}) \\
&= \frac{1}{4}(\varpi(vs)\|^2_{\mathcal{L}} - \|\varpi(vs^{-1})\|^2_{\mathcal{L}}) \\
&= \frac{1}{4}(\|vs\|^2_{\mathcal{K}} - \|vs^{-1}\|^2_{\mathcal{K}}) \\
&= \langle v, s\rangle_{\mathcal{K}}.
\end{aligned}$$

□

3. Hilbert Groups and a Riesz Representation Theorem

In this section, we intend to prove a *Riesz Representation Theorem* for groups. However, we first need to define the notion of orthogonality and midconvexity and prove related theorems and results on Hilbert groups.

Definition 11. *Let $(\mathcal{K}, \|.\|_{\mathcal{K}})$ and $(\mathcal{L}, \|.\|_{\mathcal{L}})$ be normed groups. Suppose that $\alpha : \mathcal{K} \to \mathcal{L}$ is an arbitrary function. Define*

$$\begin{aligned}
\|\alpha\| &:= \sup\{\|\alpha(v)\|_{\mathcal{L}} / \|v\|_{\mathcal{K}} \ : v \in \mathcal{K}\} \\
&= \inf\{M : \ \|\alpha(v)\|_{\mathcal{L}} \leq M\|v\|_{\mathcal{K}} \ (\forall v \in \mathcal{K})\}.
\end{aligned}$$

Then, α is a possibly infinite number that is called bounded if $\|\alpha\|$ is finite.

Remark 2. *Suppose that $(\mathcal{K}, \|.\|_{\mathcal{K}})$ and $(\mathcal{L}, \|.\|_{\mathcal{L}})$ are normed groups and $\alpha : \mathcal{K} \to \mathcal{L}$ is a bounded homomorphism. Then, α is continuous.*
We denote the set of all bounded homomorphisms from $\mathcal{K}$ into the $\mathcal{L}$ by $\mathcal{B}(\mathcal{K}, \mathcal{L})$. Clearly, $(\mathcal{B}(\mathcal{K}, \mathcal{L}), \|.\|)$, under the pointwise multiplication $(\alpha\gamma)(v) = \alpha(v)\gamma(v)$, is also a normed group.

Lemma 5. *Let $(\mathcal{K}, \|.\|_{\mathcal{K}})$ and $(\mathcal{L}, \|.\|_{\mathcal{L}})$ be normed groups. If $\mathcal{L}$ is Banach group, then so is $\mathcal{B}(\mathcal{K}, \mathcal{L})$.*

Proof. Suppose that $\{\alpha_n\}$ is a Cauchy sequence in the group $\mathcal{B}(\mathcal{K},\mathcal{L})$. Let $v \in V$. Since

$$\|\alpha_n(v)\alpha_m^{-1}(v)\|_{\mathcal{L}} \le \|\alpha_n\alpha_m^{-1}\|\|v\|_{\mathcal{K}},$$

it follows that $\{\alpha_n(v)\}$ is also a Cauchy sequence in the group $\mathcal{L}$. Now, define α_0 by

$$\alpha_0(v) := \lim_{n\to\infty}\alpha_n(v).$$

We have

$$\begin{aligned}\alpha_0(vs) &= \lim_{n\to\infty}\alpha_n(vs)\\ &= \lim_{n\to\infty}\alpha_n(v)\alpha_n(s)\\ &= \alpha_0(v)\alpha_0(s).\end{aligned}$$

To see that α_0 is bounded, note that $\{\alpha_n\}$ is bounded. Hence, there exists an $M > 0$ such that $\|\alpha_n\| \le M$ for each $n \in \mathbb{N}$.

Moreover, since

$$\|\alpha_0(v)\|_{\mathcal{L}} = \lim_{n\to\infty}\|\alpha_n(v)\| \le M\|v\|_{\mathcal{K}},$$

then, $\|\alpha_0\| \le M$.

Now, let $\varepsilon > 0$ and choose an $N \in \mathbb{N}$ so that if $n, m > N$, then $\|\alpha_n\alpha_m^{-1}\| < \varepsilon$.

Let $v \in \mathcal{K}$ with $\|v\|_{\mathcal{K}} \le 1$. Since $\|\alpha_n(v)\alpha_m^{-1}(v)\|_{\mathcal{L}} < \varepsilon$ for each $m \ge N$, we have

$$\|\alpha_0(v)\alpha_0^{-1}(v)\|_{\mathcal{L}} = \lim_{n\to\infty}\|\alpha_n(v)\alpha_m^{-1}(v)\|_{\mathcal{L}} \le \varepsilon.$$

In particular,

$$\alpha_0 = \lim_{n\to\infty}\alpha_n,$$

in $\mathcal{B}(\mathcal{K},\mathcal{L})$. □

Let $\mathcal{K}$ be a topological abelian group. We denote the set of all continuous homomorphisms $\alpha : \mathcal{K} \to \mathbb{R}$ by $\tilde{\mathcal{K}}$. Note that in this content, we consider $\mathbb{R}$ as an additive group.

Lemma 6. *Let $(\mathcal{K}, \langle .,. \rangle_{\mathcal{K}})$ be an inner product group. Fix $s \in \mathcal{K}$ and define $\phi_s : \mathcal{K} \to \mathbb{R}$ by $\phi_s(v) = \langle v, s\rangle_{\mathcal{K}}$. Then, $\phi_s \in \tilde{\mathcal{K}}$ and $\|\phi_s\| = \|s\|_{\mathcal{K}}$.*

Proof. $|\phi_s(v)| = |\langle v, s\rangle_{\mathcal{K}}| \le \|v\|_{\mathcal{K}}\|s\|_{\mathcal{K}}$, so $\phi_s \in \tilde{\mathcal{K}}$ and $\|\phi_s\| \le \|s\|_{\mathcal{K}}$. Since

$$|\phi_s(s)| = |\langle s, s\rangle_{\mathcal{K}}| = \|s\|^2_{\mathcal{K}},$$

then

$$\|\phi_s\| \ge \|s\|_{\mathcal{K}}.$$

So $\|\phi_s\| = \|s\|_{\mathcal{K}}$. □

Definition 12. *Let $(\mathcal{K}, \langle .,. \rangle)$ be an inner product group. Then, $\mathcal{K}$ is called a Hilbert group if $\mathcal{K}$ is complete with respect to the norm induced by the inner product.*

Example 5.

1. *Let $\mathcal{K} = (R^n, +)$. Then $\mathcal{K}$ with the inner product $\langle v, s\rangle = \sum_{j=1}^{n} v_j s_j$ is a Hilbert group.*
2. *Let $\mathcal{K} = (M_{m\times n}, +)$. Then $\mathcal{K}$ with the inner product $\langle K, Z\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij}b_{ij}$ is a Hilbert group.*

Moreover, the connected component of identity in an abelian Lie group is an infinitely divisible group [16]. So, considering Example 5, let $(A, \langle .,. \rangle)$ be the connected component of identity in R^n with the same inner product. Then, $(A, \langle ., . \rangle)$ is an infinitely divisible Hilbert group.

Definition 13. *Two elements v and s of an inner product group $\mathcal{K}$ are said to be orthogonal, written $v \perp s$, if $\langle v, s \rangle = 0$. We say that subsets A and B are orthogonal if $a \perp b$ for each $a \in A$ and $b \in B$. The orthogonal complement $A^\perp$ of a subset A is the set of elements $u \in \mathcal{K}$ such that u is orthogonal to all elements in A.*

Lemma 7. *Let A be a subset of Hilbert group $\mathcal{H}$. Then, the orthogonal complement of A is a closed subgroup of $\mathcal{H}$ and $A \cap A^\perp = \{e\}$.*

Proof. Let $\mathcal{H}$ be a Hilbert group and $A \subset \mathcal{H}$. If $a, b \in A^\perp$, then for arbitrary $c \in A$, we have

$$\begin{aligned} \langle c, ab^{-1} \rangle &= \langle c, a \rangle + \langle c, b^{-1} \rangle \\ &= \langle c, a \rangle - \langle c, b \rangle = 0. \end{aligned}$$

Thus, $A^\perp$ is a subgroup of $\mathcal{H}$.

To show that $A^\perp$ is closed, let $\{b_n\}$ be a sequence in $A^\perp$ such that converges to b. We show $b \in A^\perp$.

Let $a \in A$, then

$$\langle a, b \rangle = \langle a, \lim_{n \to \infty} b_n \rangle = \lim_{n \to \infty} \langle a, b_n \rangle = 0.$$

Hence, $b \in A^\perp$.

To prove $A \cap A^\perp = \{e\}$, let $a \in A \cap A^\perp$. Then, $\langle a, a \rangle = 0$ and $a = e$. □

Definition 14 ([4])**.** *Let $\mathcal{K}$ be a group. A subset C of $\mathcal{K}$ is called $\frac{1}{2}$-convex (or midconvex), if for every $v, s \in C$ there exists an element $z \in C$, denoted by $\sqrt{vs}$, such that $z^2 = vs$.*

Lemma 8. *Let $\mathcal{H}$ be a Hilbert group and A be a non-empty, closed and $\frac{1}{2}$-convex subset of $\mathcal{H}$. Then, A contains a unique element of the smallest norm; i.e.,*

$$\|v\| = \inf_{a \in A} \|a\|.$$

Proof. There exists a sequence v_n in A such that

$$\|v_n\| \to b = \inf_{a \in A} \|a\|.$$

By applying the Parallelogram law to v_n and v_m, we obtain

$$\frac{1}{4}\|v_n v_m^{-1}\|^2 = \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|v_m\|^2 - \|\sqrt{v_n v_m}\|^2.$$

Since $\|v_n\|^2 \to b^2$, given $\varepsilon > 0$ if N is large enough, then for $n > N$

$$2\|v_n\|^2 < 2b^2 + \frac{\varepsilon^2}{2}.$$

By $\frac{1}{2}$-convexity of A, we have $(v_n v_m)^{\frac{1}{2}} \in A$, so

$$\|\sqrt{v_n v_m}\|^2 \geq b^2.$$

Combining these estimates gives

$$n, m \geq N \Rightarrow \|v_n v_m^{-1}\|^2 \leq 4b^2 + \varepsilon^2 - 4b^2 = \varepsilon^2.$$

Since $\mathcal{H}$ is complete and A is closed, $v_n \to v \in A$. Moreover,

$$\|v\| = \lim_{n\to\infty} \|v_n\| = b.$$

Thus, v exists, and if v and $\acute{v}$ are two elements in A with $\|v\| = \|\acute{v}\|$, then

$$\|v\acute{v}^{-1}\|^2 = 2\|v\|^2 + 2\|\acute{v}\|^2 - 4\|\sqrt{v\acute{v}}\|^2 \leq 0.$$

Therefore, $v = \acute{v}$. □

Lemma 9. *Let A be a closed and $\frac{1}{2}$-convex subgroup of Hilbert group $\mathcal{H}$. Then, $\mathcal{H} = AA^{\perp}$. This mens that any $s \in \mathcal{H}$ has a unique decomposition $s = aa^{\perp}$ where $a \in A$ and $a^{\perp} \in A^{\perp}$.*

Proof. Let A be a closed and $\frac{1}{2}$-convex subgroup of $\mathcal{H}$. If $A = \mathcal{H}$, then $A^{\perp} = \{e\}$ and there is nothing to show. Consider $v \in \mathcal{H} \backslash A$ and set

$$B = vA = \{va : a \in A\} = \{\acute{v} \in \mathcal{H}; \acute{v} = va, a \in A\}.$$

where B is non-empty, because $e \in A$. So, for $\acute{b} = v\acute{a}$ and $\grave{b} = v\grave{a}$ in B, we have

$$(\acute{b}\grave{b})^{\frac{1}{2}} = v(\acute{a}\grave{a})^{\frac{1}{2}} \in B.$$

In addition, every sequence in B is of the form $\acute{v}_n = va_n$, where a_n is a sequence in A. Thus, $\acute{v}_n$ converges in B if and only if a_n converges in A. It follows that B is a closed subset of $\mathcal{H}$.

Now, we define $a^{\perp}$ to be the element of the smallest norm in vA; this exists as a result of Lemma 8. Put $a = v(a^{\perp})^{-1}$; it is clear that $a \in A$. Then, $v = aa^{\perp}$.

To prove the uniqueness of this, let $a\acute{a} = b\acute{b}$ for some elements $a, \acute{a} \in A$ and $b, \acute{b} \in A^{\perp}$. Then, $ba^{-1} = \acute{b}\acute{a}^{-1}$. Since $ba^{-1} \in A$, $\acute{b}\acute{a}^{-1} \in A^{\perp}$ and $A \bigcap A^{\perp} = \{e\}$, we have $a = b, \acute{a} = \acute{b}$. □

Definition 15. *Let $\mathcal{K}$ be a group and q_n be a sequence of rational numbers that converges to $r \in \mathbb{R}$. For each $v \in \mathcal{K}$ and $r \in \mathbb{R}$, define*

$$v^r := \lim_{n\to\infty} v^{q_n}.$$

Remark 3. *Let $(\mathcal{K}, \langle .,. \rangle)$ be an inner product group. We know that $\langle v^n, s\rangle = n\langle v, s\rangle$ for each $v, s \in \mathcal{K}$ and $n \in \mathbb{Z}$. Then $\langle v^q, s\rangle = q\langle v, s\rangle$ for each $v, s \in \mathcal{K}$ and $q \in \mathbb{Q}$, Since for each $q = \frac{m}{n}$*

$$\begin{aligned} n\frac{m}{n}\langle v, s\rangle &= \langle v^{n\frac{m}{n}}, s\rangle \\ &= n\langle v^{\frac{m}{n}}, s\rangle. \end{aligned}$$

Thus, $\frac{m}{n}\langle v, s\rangle = \langle v^{\frac{m}{n}}, s\rangle$. We wish to show that $\langle v^r, s\rangle = r\langle v, s\rangle$ for each $v, s \in \mathcal{K}$ and $r \in \mathbb{R}$.

Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, for each $r \in \mathbb{R}$, there is a sequence $\{q_n\}$ of $\mathbb{Q}$ that converges to r. Let $\varepsilon > 0$ and $N \in \mathbb{N}$; then, for each $n > N$,

$$|q_n - r| < \frac{\varepsilon}{|\langle v, s\rangle + 1|}.$$

Thus, for each $v, s \in \mathcal{K}$, we have

$$\begin{aligned} d(\langle v^{q_n}, s\rangle, r\langle v, s\rangle) &= |\langle v^{q_n}, s\rangle - r\langle v, s\rangle| \\ &= |q_n\langle v, s\rangle - r\langle v, s\rangle| \\ &= \langle v, s\rangle|q_n - r| < \varepsilon. \end{aligned}$$

Since $\lim_{n\to\infty}\langle v^{q_n}, s\rangle = \langle v^r, s\rangle$, *then*

$$\langle v^r, s\rangle = r\langle v, s\rangle.$$

Theorem 6. *Let $\mathcal{H}$ be an infinitely divisible Hilbert group and $\phi : \mathcal{H} \to \mathbb{R}$ be a continuous epimorphism. Then, there is a unique element s_0 in $\mathcal{H}$ such that $\phi(s) = \langle s, s_0\rangle$ for every $s \in \mathcal{H}$.*

Proof. Let $M = \ker\phi$. If $\phi = 0$ then $s_0 = e$, and thus assume that ϕ is a non-zero epimorphism. Because ϕ is continuous, then M is a closed $\frac{1}{2}$-convex subgroup of $\mathcal{H}$. As we suppose that $M \neq \mathcal{H}$, so $M^{\perp} \neq \{e\}$. Therefore, there is v in $M^{\perp}$ such that $\phi(v) = 1$.
Now, if $s \in \mathcal{H}$ and $a = \phi(s)$, then

$$\phi(sv^{-a}) = \phi(s) - \phi(v^a) = \phi(s) - a\phi(v) = 0.$$

Therefore, $sv^{-a} \in M$ and we have

$$\begin{aligned} 0 &= \langle sv^{-a}, v\rangle \\ &= \langle s, v\rangle + \langle v^{-a}, v\rangle \\ &= \langle s, v\rangle - a\langle v, v\rangle. \end{aligned}$$

Thus, if $s_0 = v^{\frac{1}{\langle v,v\rangle}}$, then $\phi(s) = \langle s, s_0\rangle$ for all $s \in \mathcal{H}$.
If $\acute{s_0} \in \mathcal{H}$ such that $\langle s, s_0\rangle = \langle s, \acute{s_0}\rangle$ for all $s \in \mathcal{H}$, then

$$\langle s_0, s\rangle - \langle \acute{s_0}, s\rangle = \langle s_0\acute{s_0}^{-1}, s\rangle = 0$$

In particular, $\langle s_0\acute{s_0}^{-1}, s_0\acute{s_0}^{-1}\rangle = 0$ and so $s_0 = \acute{s_0}$. $\square$

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