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On (2-d)-Kernels in Two Generalizations of the Petersen Graph

Paweł Bednarz *,† and Natalia Paja †

The Faculty of Mathematics and Applied Physics, Rzeszów University of Technology, al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland; nbednarz@prz.edu.pl
* Correspondence: pbednarz@prz.edu.pl
† These authors contributed equally to this work.

Abstract: A subset J is a (2-d)-kernel of a graph if J is independent and 2-dominating simultaneously. In this paper, we consider two different generalizations of the Petersen graph and we give complete characterizations of these graphs which have (2-d)-kernel. Moreover, we determine the number of (2-d)-kernels of these graphs as well as their lower and upper kernel number. The property that each of the considered generalizations of the Petersen graph has a symmetric structure is useful in finding (2-d)-kernels in these graphs.

Keywords: domination; independence; (2-d)-kernel; generalized Petersen graphs

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1. Introduction

In general, we use the standard terminology and notation of graph theory (see [1]). Let G be an undirected, connected, and simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of the graph G is the number of vertices in G. The size of the graph G is its number of edges. By P_n, $n \geq 1$ and C_n, $n \geq 3$, we mean a path and a cycle of order n, respectively.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a subgraph of G, written as $G' \subseteq G$. If $G' \subseteq G$ and G' contain all the edges $xy \in E$ with $x, y \in V'$, then G' is an induced subgraph of G and we write $G' := \langle V' \rangle_G$. Graphs G and G' are called isomorphic, and denoted by $G \cong G'$, if there exists a bijection $\phi : V \to V'$ with $xy \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. The complement of the graph G is a graph $\overline{G}$ such that $V(G) = V(\overline{G})$ and two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A graph G is called bipartite if $V(G)$ admits a partition into two classes such that every edge has its ends in different classes.

A subset $D \subseteq V(G)$ is a dominating set of G if each vertex of G not belonging to D is adjacent to at least one vertex of D. A subset $S \subseteq V(G)$ is called an independent set of G if no two vertices of S are adjacent in G. A subset J being independent and dominating is a kernel of G.

The concept of kernels was initiated in 1953 by von Neumann and Morgenstern in digraphs with regard to game theory (see [2]). One of the pioneers studying the kernels in digraphs was C. Berge (see [3–5]). In literature, we can find many types and generalizations of kernels in digraphs (for results and applications, see, for example, [6–11]). The problem of the existence of kernels in undirected graphs is trivial because every maximal independent set is a kernel. Currently, distinct kind of kernels in undirected graphs are being studied quite intensively and many papers are available. For results and application, see, for example, [12–18]. Among many types of kernels in undirected graphs, there are kernels related to multiple domination, introduced by Fink and Jacobson in [19]. Let $p \geq 1$ be an integer. A subset S is said to be p-dominating if every vertex outside S has at least p neighbors in S. If $p = 1$, then we obtain a dominating set in the classical sense. If $p = 2$, we get a 2-dominating set. A set which is 2-dominating and independent is named a 2-dominating kernel ((2-d)-kernel in short). The concept of (2-d)-kernels was introduced by

A. Włoch in [20]. Some properties of (2-d)-kernels were studied in [21–24]. In particular, in [23], it was proved that the problem of the existence of (2-d)-kernels is $\mathcal{NP}$-complete for general graphs. In [25], Nagy extended the concept of (2-d)-kernels to k-dominating kernels. He considered a k-dominating set instead of the 2-dominating set, which he called k-dominating independent sets. Some properties of these sets were studied in [26,27].

The number of (2-d)-kernels in the graph G is denoted by $\sigma(G)$. Let G be a graph with the (2-d)-kernel. The minimum cardinality of the (2-d)-kernel of G is called a lower (2-d)-kernel number and denoted by $\gamma_{(2-d)}(G)$. The maximum cardinality of the (2-d)-kernel of G is called an upper (2-d)-kernel number and is denoted by $\Gamma_{(2-d)}(G)$.

In this paper, we consider two different generalizations of the Petersen graph. Various types of domination in the class of generalized Petersen graphs have been extensively studied in the literature (see [28–32]). Referring to this research, we will consider (2-d)-kernels for two different generalizations of the Petersen graph. We solve the problem of the existence of (2-d)-kernels, their number, and their cardinality in these graphs. Moreover, we determine a lower and an upper kernel number in these graphs. It is worth noting that each of presented generalizations of the Petersen graph has a symmetric structure. This property is useful in finding (2-d)-kernels in these graphs.

2. Main Results

In this section, we consider the problem of the existence of (2-d)-kernels in two different generalizations of the Petersen graph. In particular, we give complete characterizations of these generalizations, which have the (2-d)-kernel. We determine the number of (2-d)-kernels in these graphs as well as the lower and the upper (2-d)-kernel number.

In the further part of the paper, we will use green color to mark vertices belonging to the (2-d)-kernel, and red color to indicate vertices that cannot belong to it.

2.1. Generalized Petersen Graph

Let $n \geq 3$, $k < \frac{n}{2}$ be integers. The graph $P(n,k)$ is called the generalized Petersen graph, if $V(P(n,k)) = \bigcup_{i=0}^{n-1} \{u_i, v_i\}$ and $E(P(n,k)) = \bigcup_{i=0}^{n-1} \{u_i u_{i+1}, u_i v_i, v_i v_{i+k}\}$, where subscripts are reduced modulo n. These graphs were first defined by Watkins in [33]. Figure 1 shows generalized Petersen graphs $P(10,3)$, $P(5,2)$ and examples of (2-d)-kernels in these graphs.

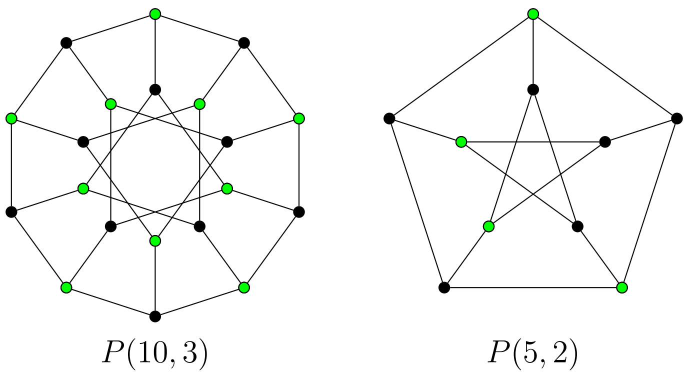

Figure 1. Examples of (2-d)-kernels in $P(10,3)$ and $P(5,2)$.

We start with the problem of existence of (2-d)-kernels. At the beginning, we give a sufficient condition, emerging from the property of bipartite graphs. We have the following complete characterization of bipartite generalized Petersen graphs.

Proposition 1 ([34])**.** *Let $n \geq 3$, $k < \frac{n}{2}$ be integers. The graph $P(n,k)$ is bipartite if and only if n is even and k is odd.*

From this characterization we directly obtain the sufficient condition for the existence of $(2\text{-}d)$-kernels.

Proposition 2. *Let $n \geq 3, k < \frac{n}{2}$ be integers. If n is even and k is odd, then the graph $P(n,k)$ has at least two $(2\text{-}d)$-kernels which are a partition of the vertex set.*

Proof. Let n, k be as in the statement of the proposition. From Proposition 1, it follows that the graph $P(n,k)$ is a bipartite graph. Thus, there exist two independent sets of vertices V_1, V_2 that are a partition of the set $V(P(n,k))$. Moreover, the graph $P(n,k)$ is a 3-regular graph. Therefore, sets V_1, V_2 are $(2\text{-}d)$-kernels of the graph $P(n,k)$. □

Now, we improve the above proposition to obtain the complete characterization of the generalized Petersen graph having $(2\text{-}d)$-kernel.

Theorem 1. *Let $n \geq 3, k < \frac{n}{2}$ be integers. The graph $P(n,k)$ has a $(2\text{-}d)$-kernel if and only if*

(i) *n is even and k is odd or*
(ii) $n \equiv 0(\mathrm{mod}\ 5)$ *and* $k \equiv 2(\mathrm{mod}\ 5)$ *or*
(iii) $n \equiv 0(\mathrm{mod}\ 5)$ *and* $k \equiv 3(\mathrm{mod}\ 5)$.

Proof. If $n = 3,4$, then the result is obvious. Let $n \geq 5, k < \frac{n}{2}$ be integers. If n is even and k is odd, then by Proposition 2, (i) follows. Let $n \equiv 0(\mathrm{mod}\ 5), k \equiv j(\mathrm{mod}\ 5), j = 2,3$. We will show that the set $J = \{u_i; i \in \{0,5,\ldots,n\}\} \cup \{u_{i+2}; i \in \{0,5,\ldots,n\}\} \cup \{v_{i+3}; i \in \{0,5,\ldots,n\}\} \cup \{v_{i+4}; i \in \{0,5,\ldots,n\}\}$ is a $(2\text{-}d)$-kernel of $P(n,k)$. The independence of J follows from the definition of $P(n,k)$. Let us assume that $x \in V(P(n,k)) \setminus J$. Then, either $x = u_s$, $s \in \{0,1,\ldots,n-1\}$, $s \equiv a(\mathrm{mod}\ 5)$, $a = 1,3,4$ or $x = v_t$, $t \in \{0,1,\ldots,n-1\}$, $t \equiv b(\mathrm{mod}\ 5)$, $b = 0,1,2$. We consider two cases.

1. $x = u_s$.

If $s \equiv 1(\mathrm{mod}\ 5)$, then $\{u_{s-1}, u_{s+1}\} \subseteq N(u_s)$ and $u_{s-1}, u_{s+1} \in J$. If $s \equiv 3(\mathrm{mod}\ 5)$, then $\{u_{s-1}, v_s\} \subseteq N(u_s)$ and $u_{s-1}, v_s \in J$. If $s \equiv 4(\mathrm{mod}\ 5)$, then $\{u_{s+1}, v_s\} \subseteq N(u_s)$ and $u_{s+1}, v_s \in J$.

2. $x = v_t$.

Let $t \equiv 0(\mathrm{mod}\ 5)$. If $k \equiv 2(\mathrm{mod}\ 5)$, then $\{u_t, v_{t-2}\} \subseteq N(v_t)$ and $u_t, v_{t-2} \in J$. If $k \equiv 3(\mathrm{mod}\ 5)$, then $\{u_t, v_{t+3}\} \subseteq N(v_t)$ and $u_t, v_{t+3} \in J$. If $t \equiv 1(\mathrm{mod}\ 5)$, then $\{v_{t-k}, v_{t+k}\} \subseteq N(v_t)$ and $v_{t-k}, v_{t+k} \in J$, $k \equiv j(\mathrm{mod}\ 5)$, $j = 2,3$. Let $t \equiv 2(\mathrm{mod}\ 5)$. If $k \equiv 2(\mathrm{mod}\ 5)$, then $\{u_t, v_{t+2}\} \subseteq N(v_t)$ and $u_t, v_{t+2} \in J$. If $k \equiv 3(\mathrm{mod}\ 5)$, then $\{u_t, v_{t-3}\} \subseteq N(v_t)$ and $u_t, v_{t-3} \in J$.

Summing up all the above cases we obtain that every vertex $x \in V(P(n,k)) \setminus J$ is 2-dominated by J. Hence, J is a $(2\text{-}d)$-kernel of $P(n,k)$.

Conversely, let $n \geq 5, k < \frac{n}{2}, i \in \{0,1,\ldots,n-1\}$ be integers and let J be a $(2\text{-}d)$-kernel of $P(n,k)$. If $u_i, u_{i+1}, u_{i+2} \notin J$, then the vertex u_{i+1} is not 2-dominated by J. Thus, each connected component of the graph $\left\langle \bigcup_{i=0}^{n-1} u_i \right\rangle_{P(n,k)} \setminus J$ is isomorphic to either P_1 or P_2. We will show that in the graph $P(n,k)$ having a $(2\text{-}d)$-kernel, the configurations of these paths P_1, P_2 on the outer cycle, which are shown in the Figure 2 are forbidden.

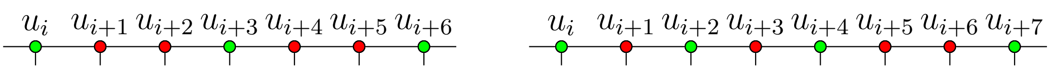

Figure 2. Forbidden configurations of the paths P_1, P_2 for the graph $P(n,k)$ with the $(2\text{-}d)$-kernel.

Let us consider the following cases.

1. First, we will prove that the configuration of the paths P_1, P_2 shown on the left side of the Figure 2 is forbidden. Suppose that $u_i, u_{i+3}, u_{i+6} \in J$ for some i, as in Figure 3. Then,

$v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5} \in J$; otherwise, vertices $u_{i+1}, u_{i+2}, u_{i+4}, u_{i+5}$ are not 2-dominated by J. Therefore, for every k vertices $v_{i+1+k}, v_{i+2+k}, v_{i+4+k}, v_{i+5+k} \notin J$.

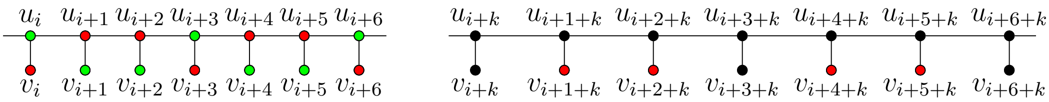

Figure 3. The case when $u_i, u_{i+3}, u_{i+6} \in J$.

We have the next two possibilities.

1.1. $v_{i+3+k} \notin J$ for some i (see Figure 4).

Since $v_{i+3+k} \notin J$, the vertex $u_{i+3+k} \in J$ and $u_{i+2+k}, u_{i+4+k} \notin J$. Then $v_{i+2+2k}, v_{i+3+2k}, v_{i+4+2k} \in J$. This means that $u_{i+2+2k}, u_{i+3+2k}, u_{i+4+2k} \notin J$. Hence, the vertex u_{i+3+2k} is not 2-dominated by J, a contradiction.

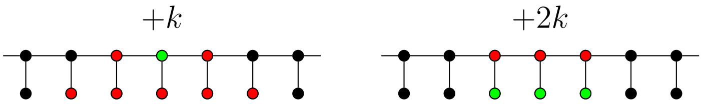

Figure 4. The case when $u_i, u_{i+3}, u_{i+6} \in J$ (the first subcase).

1.2. $v_{i+3+k} \in J$ for some i (see Figure 5).

Then, $u_{i+3+k} \notin J$ and $u_{i+2+k}, u_{i+4+k} \in J$; otherwise, they are not 2-dominated by J. Because J is an independent set, $u_{i+1+k}, u_{i+5+k} \notin J$. Moreover, $u_{i+k}, u_{i+6+k} \in J$ to 2-dominate u_{i+1+k}, u_{i+5+k}. Hence, $v_{i+k}, v_{i+6+k} \notin J$. To 2-dominate v_{i+1+k}, v_{i+5+k}, we must have $v_{i+1+2k}, v_{i+5+2k} \in J$. Moreover, $u_{i+1+2k}, u_{i+5+2k}, v_{i+3+2k} \notin J$. Since v_{i+k}, v_{i+6+k} have exactly one neighbour in J, vertices $v_{i+2k}, v_{i+6+2k} \in J$ and $u_{i+2k}, u_{i+6+2k} \notin J$. Next, $u_{i+2+2k}, u_{i+4+2k} \in J$ to 2-dominate u_{i+1+2k}, u_{i+5+2k} and $v_{i+2+2k}, v_{i+4+2k}, u_{i+3+2k} \notin J$. Thus, $v_{i+2+3k}, v_{i+3+3k}, v_{i+4+3k} \in J$ to 2-dominate $v_{i+2+2k}, v_{i+3+2k}, v_{i+4+2k}$. Therefore, $u_{i+2+3k}, u_{i+3+3k}, u_{i+4+3k} \notin J$. This means that u_{i+3+3k} is not 2-dominated, a contradiction.

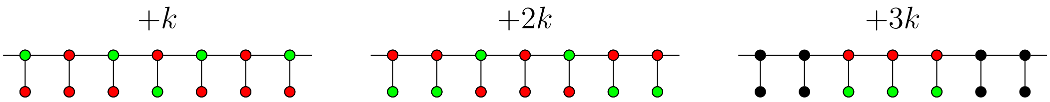

Figure 5. The case when $u_i, u_{i+3}, u_{i+6} \in J$ (the second subcase).

Hence, for each n and k, it is not possible that the vertices u_i, u_{i+3}, u_{i+6} belong to a (2-d)-kernel of $P(n,k)$.

2. Now, we will prove that the configuration of the paths P_1, P_2 shown on the right side of the Figure 2 is forbidden. Suppose that $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$ for some i, as in Figure 6. Then, v_{i+5}, v_{i+6}, which causes $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7} \notin J$.

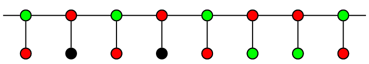

Figure 6. The case when $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$.

We consider four subcases.

2.1. $v_{i+1}, v_{i+3} \notin J$ for some i (see Figure 7).

Then, $v_{i+1-k}, v_{i+3-k}, v_{i+1+k}, v_{i+3+k} \in J$. Since v_{i+2} must be 2-dominated, so $v_{i+2-k} \in J$ or $v_{i+2+k} \in J$. Without loss of generality, assume that $v_{i+2+k} \in J$. Thus, $u_{i+1+k}, u_{i+2+k}, u_{i+3+k} \notin J$. Hence, the vertex u_{i+2+k} is not 2-dominated, a contradiction.

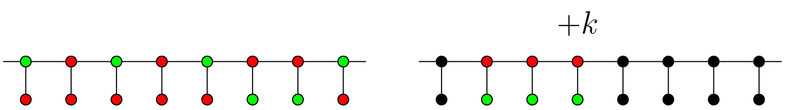

Figure 7. The case when $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$ (the first subcase).

2.2. $v_{i+1} \notin J$ and $v_{i+3} \in J$ for some i (see Figure 8).
Then, $v_{i+3+k}, v_{i+5+k}, v_{i+6+k} \notin J$. Since v_{i+7} must be 2-dominated, we obtain that $v_{i+7-k} \in J$ or $v_{i+7+k} \in J$. Without loss of generality, assume that $v_{i+7+k} \in J$. Thus, $u_{i+7+k} \notin J$ and $u_{i+6+k} \in J$. Because J is an independent set and $u_{i+6+k} \in J$, $u_{i+5+k} \notin J$. Therefore, $u_{i+4+k} \in J$, which causes $u_{i+3+k}, v_{i+4+k} \notin J$. Moreover, $v_{i+3+2k}, v_{i+4+2k}, v_{i+5+2k} \in J$, and finally $u_{i+3+2k}, u_{i+4+2k}, u_{i+5+2k} \notin J$. Hence, the vertex u_{i+4+2k} is not 2-dominated, a contradiction.

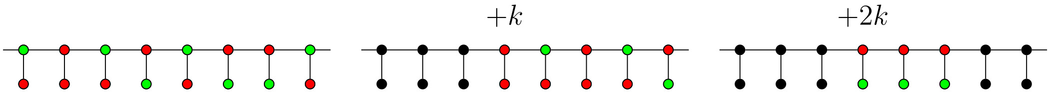

Figure 8. The case when $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$ (the second subcase).

2.3. $v_{i+1} \in J$ and $v_{i+3} \notin J$ for some i (see Figure 9).
Then, $v_{i+3+k} \in J$ and $v_{i+1+k}, v_{i+5+k}, v_{i+6+k}, u_{i+3+k} \notin J$. Since v_{i+4} must be 2-dominated, $v_{i+4-k} \in J$ or $v_{i+4+k} \in J$. Without loss of generality, assume that $v_{i+4+k} \in J$. Thus, $u_{i+4+k} \notin J$. Moreover, $u_{i+2+k}, u_{i+5+k} \in J$, which causes $u_{i+1+k}, u_{i+6+k}, v_{i+2+k} \notin J$. To 2-dominate u_{i+1+k}, we must have $u_{i+k} \in J$. Then, $v_{i+k} \notin J$ and $v_{i+2k}, v_{i+1+2k}, v_{i+2+2k} \in J$. From the independence of the set J, we get that $u_{i+2k}, u_{i+1+2k}, u_{i+2+2k} \notin J$. Hence, the vertex u_{i+1+2k} is not 2-dominated, a contradiction.

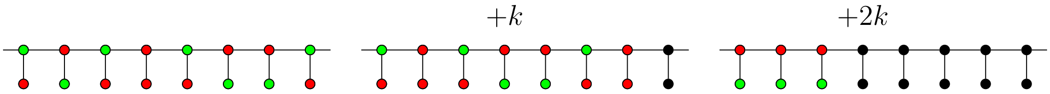

Figure 9. The case when $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$ (the third subcase).

2.4. $v_{i+1}, v_{i+3} \in J$ for some i.
Proving analogously as in subcase 2.3., we obtain a contradiction with the assumption that J is a $(2\text{-}d)$-kernel.
Therefore, for each n and k, it is not possible that the vertices $u_i, u_{i+2}, u_{i+4}, u_{i+7}$ belong to a $(2\text{-}d)$-kernel of $P(n,k)$.

Hence, for the graph with the $(2\text{-}d)$-kernel, the configurations of P_1, P_2 shown in the Figure 10 are the only ones that may be possible. Now, we will show that they are indeed possible.

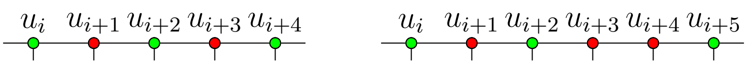

Figure 10. Possible configurations of the paths P_1, P_2 for the graph $P(n,k)$ with the $(2\text{-}d)$-kernel.

3. Suppose that $u_i, u_{i+2}, u_{i+4} \in J$ for some i, as in Figure 11. Then, $u_{i+1}, u_{i+3}, v_i, v_{i+2}, v_{i+4} \notin J$.

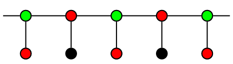
Figure 11. The case when $u_i, u_{i+2}, u_{i+4} \in J$.

We consider four subcases.
3.1. $v_{i+1}, v_{i+3} \notin J$ for some i (see Figure 12).
Since v_{i+2} must be 2-dominated, we obtain that $v_{i+2+k} \in J$ or $v_{i+2-k} \in J$. Without loss of generality, assume that $v_{i+2+k} \in J$. Moreover, $v_{i+1+k}, v_{i+3+k} \in J$ and $u_{i+1+k}, u_{i+2+k}, u_{i+3+k} \notin J$. Hence, the vertex u_{i+2+k} is not 2-dominated, a contradiction.

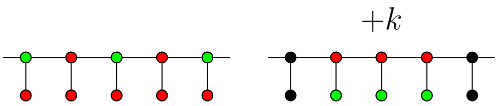

Figure 12. The case when $u_i, u_{i+2}, u_{i+4} \in J$ (the first subcase).

3.2. $v_{i+1} \notin J$ and $v_{i+3} \in J$ for some i (see Figure 13).
Then, $v_{i+1+k} \in J$ and $v_{i+3+k} \notin J$. Since v_{i+2} must be 2-dominated, $v_{i+2+k} \in J$ or $v_{i+2-k} \in J$. Without loss of generality, assume that $v_{i+2+k} \in J$. Thus, $u_{i+1+k}, u_{i+2+k} \notin J$ and $u_{i+k}, u_{i+3+k} \in J$, which causes $v_{i+k}, u_{i+4+k} \notin J$ and $v_{i+4+k} \in J$. Moreover, v_{i+1+2k}, $v_{i+2+2k}, v_{i+4+2k} \notin J$, $v_{i+2k} \in J$, $u_{i+2k} \notin J$, $u_{i+1+2k} \in J$, $u_{i+2+2k} \notin J$, $u_{i+3+2k} \in J$ and $u_{i+4+2k}, v_{i+3+2k} \notin J$. Finally, $v_{i+2+3k}, v_{i+3+3k}, v_{i+4+3k} \in J$ and $u_{i+2+3k}, u_{i+3+3k}, u_{i+4+3k} \notin J$. Hence, the vertex u_{i+3+3k} is not 2-dominated, a contradiction.

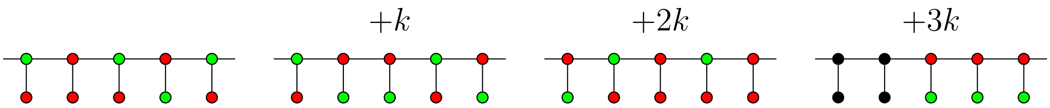

Figure 13. The case when $u_i, u_{i+2}, u_{i+4} \in J$ (the second subcase).

3.3. $v_{i+1} \in J$ and $v_{i+3} \notin J$ for some i.
Proving analogously as in subcase 3.2., we obtain a contradiction with the assumption that J is a $(2\text{-}d)$-kernel.
3.4. $v_{i+1}, v_{i+3} \in J$ for some i (see Figure 14).
Then, $v_{i+1+k}, v_{i+3+k} \notin J$. First, we will show that v_{i+k} and v_{i-k} must belong to a $(2\text{-}d)$-kernel J. Suppose on contrary that $v_{i+k} \notin J$. Since v_{i+k} must be 2-dominated, $u_{i+k} \in J$. Thus, $u_{i+1+k} \notin J$ and $u_{i+2+k} \in J$. Moreover, $v_{i+2k}, v_{i+1+2k}, v_{i+2+2k} \in J$ and u_{i+2k}, u_{i+1+2k}, $u_{i+2+2k} \in J$. Hence, the vertex u_{i+1+2k} is not 2-dominated, a contradiction.

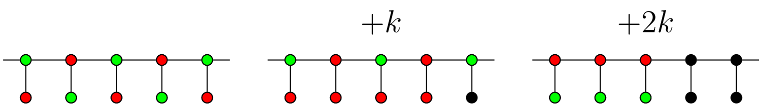

Figure 14. The case when $u_i, u_{i+2}, u_{i+4} \in J$ (the fourth subcase).

This means that $v_{i+k}, v_{i-k} \in J$ and also $v_{i+2+k}, v_{i+4+k}, u_{i+1+k}, u_{i+3+k}$ belong to a $(2\text{-}d)$-kernel (see Figure 15).

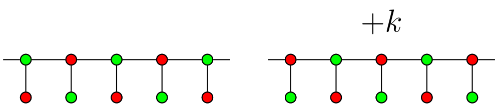

Figure 15. The case when $u_i, u_{i+2}, u_{i+4} \in J$ implies that $v_{i+k}, v_{i+2+k}, v_{i+4+k}, u_{i+1+k}, u_{i+3+k} \in J$.

Hence, n must be even, and from the definition of $P(n,k)$, we conclude that k must be odd, which proves (i).
4. Suppose that $u_i, u_{i+2}, u_{i+5} \in J$ for some i. Then, $u_{i+1}, u_{i+3}, u_{i+4}, v_i, v_{i+2}, v_{i+5} \notin J$. Since u_{i+3}, u_{i+4} must be 2-dominated, $v_{i+3}, v_{i+4} \in J$. First, we prove that $v_{i+1} \notin J$. Suppose on contrary that $v_{i+1} \in J$, as in Figure 16. Then, $v_{i+1+k}, v_{i+3+k}, v_{i+4+k} \notin J$. Since v_i must be 2-dominated, $v_{i-k} \in J$ or $v_{i+k} \in J$. Without loss of generality, assume that $v_{i+k} \in J$. Thus, $u_{i+k} \notin J$, $u_{i+1+k} \in J$ and $u_{i+2+k} \notin J$. Moreover, $u_{i+3+k} \in J$, $u_{i+4+k} \notin J$, $u_{i+5+k} \in J$, $v_{i+5+k} \notin J$, and $v_{i+2+k} \in J$. Proving analogously as in subcase 3.3., we obtain a contradiction with the assumption that J is a $(2\text{-}d)$-kernel.

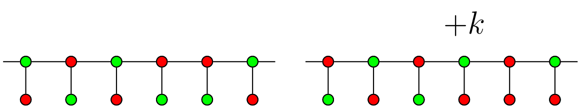

Figure 16. The case when $u_i, u_{i+2}, u_{i+5}, v_{i+1} \in J$.

Hence, $v_{i+1} \notin J$ (see Figure 17). Moreover, $v_{i+1+k} \in J$ and $u_{i+1+k}, v_{i+3+k}, v_{i+4+k} \notin J$. We consider two subcases.

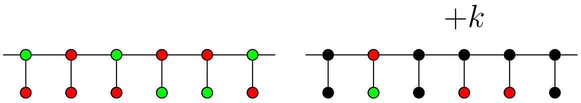

Figure 17. The case when $u_i, u_{i+2}, u_{i+5} \in J$, and $v_{i+1} \notin J$.

4.1. $v_{i+2+k} \notin J$ for some i (see Figure 18).
Then, $u_{i+2+k} \in J$, $u_{i+3+k} \notin J$, $u_{i+4+k} \in J$, $u_{i+5+k} \notin J$, and $v_{5+i+k} \in J$. Moreover, $v_{i+k} \in J$ and $u_{i+k} \notin J$; otherwise, we obtain the same configuration as in subcase 3.3.

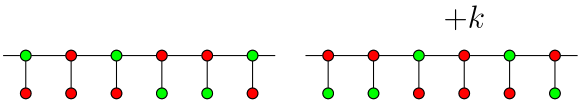

Figure 18. The case when $u_i, u_{i+2}, u_{i+5} \in J$ (the first subcase).

Hence, n must be divisible by 5, and from the definition of $P(n,k)$, we conclude that $k \equiv 2(\text{mod } 5)$, which proves (ii).
4.2. $v_{i+2+k} \in J$ for some i (see Figure 19).
Then, $u_{i+2+k} \notin J$, $u_{i+k}, u_{i+3+k} \in J$ and $v_{i+k}, u_{i+4+k} \notin J$. Moreover, $u_{i+5+k} \in J$ and $v_{i+5+k} \notin J$

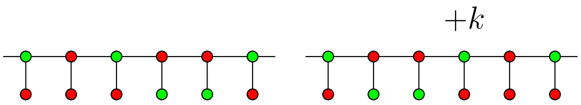

Figure 19. The case when $u_i, u_{i+2}, u_{i+5} \in J$ (the second subcase).

Hence, n must be divisible by 5, and from the definition of $P(n,k)$, we conclude that $k \equiv 3(\text{mod } 5)$, which proves (iii), which ends the proof. □

Basing on the proof of Theorem 1, the following corollaries are obtained. They concern the number of $(2\text{-}d)$-kernels in the generalized Petersen graph as well as the lower and upper $(2\text{-}d)$-kernel numbers. By a rotation of configurations shown on Figure 10, condition (i) of Theorem 1 gives two $(2\text{-}d)$-kernels in generalized Petersen graph and conditions (ii) and (iii) give five $(2\text{-}d)$-kernels. Therefore, if n and k satisfy more than one of these conditions, we obtain more $(2\text{-}d)$-kernels. Moreover, the proof of the Theorem 1 presents the constructions of the $(2\text{-}d)$-kernels in the generalized Petersen graph $P(n,k)$. Figure 20 shows the smallest and the largest $(2\text{-}d)$-kernel in the graph $P(20,7)$.

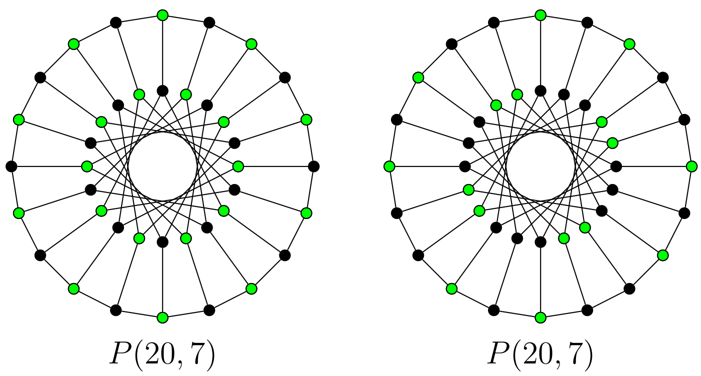

Figure 20. The largest (left side) and the smallest (right side) $(2\text{-}d)$-kernel in the graph $P(20,7)$.

Corollary 1. *Let $n \geq 3, k < \frac{n}{2}$ be integers. Then,*

$$\sigma(P(n,k)) = \begin{cases} 7 & \text{for } n \equiv 0 (\text{mod } 10) \text{ and } k \equiv a (\text{mod } 10), a = 3,7 \\ 5 & \begin{array}{l} \text{for } n \equiv 5 (\text{mod } 5) \text{ and } k \equiv a (\text{mod } 5), a = 2,3 \text{ or} \\ \text{for } n \equiv 0 (\text{mod } 10) \text{ and } k \equiv a (\text{mod } 10), a = 2,8 \end{array} \\ 2 & \begin{array}{l} \text{for } n \equiv 0 (\text{mod } 10) \text{ and } k \equiv a (\text{mod } 10), a = 1,5,9 \text{ or} \\ \text{for even } n, n \not\equiv 0 (\text{mod } 10) \text{ and odd } k. \end{array} \end{cases}$$

Corollary 2. *Let $n \geq 3$, $k < \frac{n}{2}$ be integers. If $n \equiv 0(\text{mod } 10)$ and $k \equiv a(\text{mod } 10)$, $a = 3,7$, then*

$$\gamma_{(2-d)}(P(n,k)) = \frac{4}{5}n \quad \text{and} \quad \Gamma_{(2-d)}(P(n,k)) = n.$$

Corollary 3. *Let $n \geq 3$, $k < \frac{n}{2}$ be integers. If $n \equiv 5(\text{mod } 10)$ and $k \equiv a(\text{mod } 5)$, $a = 2,3$ or $n \equiv 0(\text{mod } 10)$ and $k \equiv a(\text{mod } 10)$, $a = 2,8$, then*

$$\gamma_{(2-d)}(P(n,k)) = \Gamma_{(2-d)}(P(n,k)) = \frac{4}{5}n.$$

Corollary 4. *Let $n \geq 3$, $k < \frac{n}{2}$ be integers. If $n \equiv 0(\text{mod } 10)$ and $k \equiv a(\text{mod } 10)$, $a = 1,5,9$ or n is even, $n \not\equiv 0(\text{mod } 10)$ and k is odd, then*

$$\gamma_{(2-d)}(P(n,k)) = \Gamma_{(2-d)}(P(n,k)) = n.$$

The above corollaries characterize all possible graphs $P(n,k)$, which have the $(2\text{-}d)$-kernel.

2.2. The Second Generalization of the Petersen Graph

Now, we consider another generalization of the Petersen graph. Let $n \geq 5$ be an integer. Let C_n be a cycle and $\overline{C_n}$ its complement such that $V(C_n) = \{x_1, x_2, \ldots, x_n\}$, $V(\overline{C_n}) = \{x_1^c, x_2^c, \ldots, x_n^c\}$ with the numbering of vertices in the natural order. Let $G(n)$ be the graph such that $V(G(n)) = V(C_n) \cup V(\overline{C_n})$ and $E(G(n)) = E(C_n) \cup E(\overline{C_n}) \cup \{x_i x_i^c; i \in \{1, 2, \ldots, n\}\}$. Figure 21 shows an example of a $(2\text{-}d)$-kernel in $G(13)$. It is easy to check that if $n = 5$, then $G(5)$ is isomorphic to the Petersen graph.

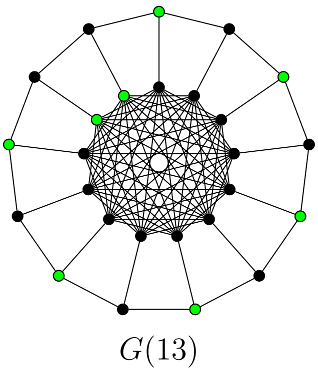

Figure 21. An example of a $(2\text{-}d)$-kernel in $G(13)$.

The next Theorem shows a complete characterization of graphs $G(n)$ with the $(2\text{-}d)$-kernel.

Theorem 2. *Let $n \geq 5$ be integer. The graph $G(n)$ has a $(2\text{-}d)$-kernel if and only if n is odd.*

Proof. Let $n \geq 5$ be odd. We will show that $J = \{x_2^c, x_3^c, x_1, x_4, x_6, \ldots, x_{n-1}\}$ is the $(2\text{-}d)$-kernel of the graph $G(n)$. The independence of J is obvious. It is sufficient to show that J

is a 2-dominating set. By the definition of the graph $G(n)$, we can assume that $x_{n+1} = x_1$. Suppose that $y \in V(G(n)) \setminus J$. Hence, $y \in V(C_n)$ or $y \in V(\overline{C_n})$. Let $y \in V(C_n)$. Thus $y = x_k, k \in \{2, 3, 5, \ldots, n\}$. If $x_k^c \notin J$, then there exist vertices $x_{k-1}, x_{k+1} \in J$ adjacent to x_k. If $x_k^c \in J$, then $k = 2$ or $k = 3$. For $k = 2$, the vertex x_2 is adjacent to $x_1, x_2^c \in J$. Moreover, if $k = 3$, then the vertex x_3 is adjacent to $x_4, x_3^c \in J$. Hence, every vertex from the set $V(C_n)$ is 2-dominated by the set J. Let now $y \in V(\overline{C_n})$. Thus $y = x_k^c, k \in \{1, 4, 5, \ldots, n\}$. Then, the vertex $x_k^c, k \in \{5, 6, \ldots, n\}$ is adjacent to $x_2^c, x_3^c \in J$. If $k = 1$, then $x_1^c x_1, x_1^c x_3^c \in E(G(n))$. Moreover, for $k = 4$ the vertex x_4^c is adjacent to x_4, x_2^c. Therefore, vertices from the set $V(\overline{C_n})$ are 2-dominated by J and hence J is a $(2\text{-}d)$-kernel of $G(n)$.

Conversely, suppose that a graph $G(n)$ has a $(2\text{-}d)$-kernel J. We will show that n is odd. By the definition of the graph $G(n)$, we obtain that $J \cap V(\overline{C_n}) \neq \emptyset$. Otherwise, vertices from the set $V(\overline{C_n})$ are not 2-dominated by the set J. Let $x_1^c \in J$. Then either $x_2^c \in J$ or $x_n^c \in J$. Otherwise, x_2^c or x_n^c is not 2-dominated. Hence, $|J \cap V(\overline{C_n})| = 2$. Without loss of generality assume that $x_1^c, x_2^c \in J$. This means that $x_i^c, i \in \{4, 5, \ldots, n-1\}$ is 2-dominated by J and x_3^c, x_n^c are dominated by J. Let $J^* = J \setminus \{x_1^c, x_2^c\}$. Then, $J^* \subset V(C)$. Since J is the $(2\text{-}d)$-kernel, $x_3, x_n \in J^*$; otherwise, x_3^c, x_n^c are not 2-dominated by J. Therefore, the graph $\langle \{x_3, x_4, \ldots, x_n\} \rangle_{G(n)} \cong P_{n-2}$ must have a $(2\text{-}d)$-kernel to 2-dominate vertices from $V(C_n) \setminus J^*$. This means that n must be odd. Thus, $J^* = \{x_3, x_5, \ldots, x_n\}$, which ends the proof. □

Finally, it turns out that if a graph $G(n)$ has $(2\text{-}d)$-kernel, then the number of $(2\text{-}d)$-kernels depends linearly on the number of vertices. Moreover, each $(2\text{-}d)$-kernel of $G(n)$ has the same cardinality.

Corollary 5. *If $n \geq 5$ is odd, then $\sigma(G(n)) = n$ and*

$$\gamma_{(2-d)}(G(n)) = \Gamma_{(2-d)}(G(n)) = \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

Proof. Let $n \geq 5$ be odd. From the construction of a $(2\text{-}d)$-kernel described in the proof of Theorem 2, we conclude that exactly two not adjacent vertices from the set $V(\overline{C_n}) \subset V(G(n))$ belong to a $(2\text{-}d)$-kernel. The selection of these two vertices will determine the $(2\text{-}d)$-kernel in $G(n)$. Since two not adjacent vertices can be chosen on n ways, $\sigma(G(n)) = n$. Moreover, from the construction of $(2\text{-}d)$-kernels in $G(n)$, it follows that all of them have the same cardinality. Hence, $\gamma_{(2-d)}(G(n)) = \Gamma_{(2-d)}(G(n)) = \lfloor \frac{n}{2} \rfloor + 2$, which ends the proof. □

3. Concluding Remarks

In this paper, we considered two different generalizations of the Petersen graph, and we discussed the problem of the existence of $(2\text{-}d)$-kernels in these graphs. In particular, we determined the number of $(2\text{-}d)$-kernels in these graphs and their lower and upper $(2\text{-}d)$-kernel number. The generalized Petersen graphs considered in this paper are special cases of I-graphs (see, for example, [35]). The I-graph $I(n, j, k)$ is a graph with a vertex set $V(I(n, j, k)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and an edge set $E(I(n, j, k)) = \{u_i u_{i+j}, u_i v_i, v_i v_{i+k}; i \in \{1, 2, \ldots, n\}\}$, where subscripts are reduced modulo n. Because $P(n, k) = I(n, 1, k)$, the results obtained could be a starting point to studying and counting $(2\text{-}d)$-kernels in I-graphs. It could also be interesting to investigate the number of $(2\text{-}d)$-kernels in other generalizations of generalized Petersen graphs. For more generalizations, see, for example, [36].

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