symmetry

MDPI

Article

Further Results on the IDCPE Class of Life Distributions

Haiyan Wang [1], Diantong Kang [2] and Lei Yan [1,*]

[1] Business School, Zhejiang Wanli University, Ningbo 315100, China; wanghaiyan@zwu.edu.cn
[2] School of Mathematics and Statistics, Hexi University, Zhangye 734000, China; dtkang@hxu.edu.cn
* Correspondence: yanlei@zwu.edu.cn

Abstract: Navarro et al. (2010) proposed the increasing dynamic cumulative past entropy (IDCPE) class of life distributions. In this paper, we investigate some characterizations of this class. Closure and reversed closure properties of the IDCPE class are obtained. As applications of a main result, we explore the preservation and reversed preservation properties of this class in several stochastic models. We also investigate preservation and reversed preservation of the IDCPE class for coherent systems with dependent and identically distributed components.

Keywords: dynamic cumulative past entropy; IDCPE class; closure property; proportional reversed hazard rate model; record values model

check for updates

Citation: Wang, H.; Kang, D.; Yan, L. Further Results on the IDCPE Class of Life Distributions. *Symmetry* **2021**, *13*, 1964. https://doi.org/10.3390/sym13101964

Academic Editor: Paolo Emilio Ricci

Received: 19 September 2021
Accepted: 12 October 2021
Published: 18 October 2021

1. Introduction

The subject of partial order relation is crucial to compare the variability of two random variables. This research field faces an interesting problem, namely, the transmission of individual (or a unit) properties to population (or a system) properties, or population properties to individual properties. This relationship was called closure or reversed closure. According to the definition of symmetry, a mathematical object is symmetric with respect to a given mathematical operation, if, when applied to the object, this operation preserves some property of the object (Morris [1]). In fact, under the partial order relation, the transmission of individual (or a unit) properties to population (or a system) properties is usually of the anti-symmetry. In this paper, we investigate closure and reversed closure properties of the IDCPE (increasing dynamic cumulative past entropy) class, and this symmetry or anti-symmetry is conducive to the concrete realization of risk management.

Let X be an absolutely continuous non-negative random variable representing the random lifetime of a device or a living thing. Assume that X has probability density function $f_X(x)$. The Shannon differential entropy is a classical measure of uncertainty for X defined by

$$H_X = -\mathbb{E}[\ln f_X(X)] = -\int_0^{+\infty} f_X(x) \ln f_X(x) \mathrm{d}x.$$

It was introduced by Shannon [2] and Wiener [3], and developed subsequently by Ebrahimi and Pellerey [4], Ebrahimi [5], Ebrahimi and Kirmani [6], Crescenzo and Longobardi [7], Navarro et al. [8], etc. Furthermore, some generalizations of H_X have been proposed, see, for example, Di Crescenzo and Longobardi [9,10], Nanda and Paul [11–13], Abbasnejad et al. [14], Kundu et al. [15], Kumar and Taneja [16], Khorashadizadeh et al. [17], Nanda et al. [18], Kayal [19], Vineshkumar [20], Kang [21], Kang and Yan [22], Yan and Kang [23], and others.

Rao et al. [24] defined a new uncertainty measure, the cumulative residual entropy (CRE), by

$$\mathcal{E}_X = -\int_0^{+\infty} \overline{F}_X(x) \ln \overline{F}_X(x) \mathrm{d}x, \tag{1}$$

as an alternative measure of uncertainty.

The residual life of X at time t is defined by $X_t = [X - t \mid X > t]$, and the inactivity time of X at time t is defined as $X_{(t)} = [t - X \mid X \leq t]$, for all $t \geq 0$. Then, the survival functions of X_t and $X_{(t)}$ are given by, respectively,

$$\overline{F}_{X_t}(x) = \frac{\overline{F}_X(x+t)}{\overline{F}_X(t)}, \quad \text{for all } x \geq 0; \quad \overline{F}_{X_{(t)}}(x) = \frac{F_X(t-x)}{F_X(t)}, \quad \text{for all } 0 \leq x \leq t.$$

Then, the mean inactivity time (MIT) of X is given by

$$m_X(t) = \mathbb{E}[X_{(t)}] = \frac{1}{F_X(t)} \int_0^t F_X(x) \mathrm{d}x, \quad \text{for all } t \geq 0.$$

Asadi and Zohrevand [25] introduced a new measure of uncertainty, the CRE of residual life X_t. This function is called the dynamic cumulative residual entropy (DCRE) and given by

$$\mathcal{E}_X(t) = \mathcal{E}_{X_t} = -\int_0^{+\infty} \overline{F}_{X_t}(x) \ln \overline{F}_{X_t}(x) \mathrm{d}x, \quad \text{for all } t \geq 0. \tag{2}$$

Namely,

$$\mathcal{E}_X(t) = -\int_t^{+\infty} \frac{\overline{F}_X(x)}{\overline{F}_X(t)} \ln\left[\frac{\overline{F}_X(x)}{\overline{F}_X(t)}\right] \mathrm{d}x, \quad \text{for all } t \geq 0. \tag{3}$$

Obviously, $\mathcal{E}_X(0) = \mathcal{E}_X$.

Navarro et al. [8] introduced the dynamic cumulative past entropy (DCPE) of X, defined as the CRE of inactivity time $X_{(t)}$, and denoted by $\widetilde{\mathcal{E}}_X(t)$. Then, $\widetilde{\mathcal{E}}_X(t)$ is given by

$$\widetilde{\mathcal{E}}_X(t) = \mathcal{E}_{X_{(t)}} = -\int_0^t \frac{F_X(x)}{F_X(t)} \ln\left[\frac{F_X(x)}{F_X(t)}\right] \mathrm{d}x, \quad \text{for all } t \geq 0. \tag{4}$$

It is worth mentioning that a generalization of the DCPE is the dynamic fractional generalized cumulative entropy studied in Section 4 of Di Crescenzo et al. [26].

To prove our main results, we first introduce the following lemma taken from Barlow and Proschan [27], which plays a key role in the proofs of this paper and are repeatedly used in the sequel.

Lemma 1. *Let W be a measure on the interval (a, b), not necessarily non-negative, where $-\infty \leq a < b \leq +\infty$. Let h be a non-negative and decreasing function defined on (a, b). If $\int_a^t dW(x) \geq 0$, for all $t \in (a, b)$, then $\int_a^t h(x) dW(x) \geq 0$, for all $t \in (a, b)$.*

Recall that a non-negative function h defined on $[0, \infty)$ is said to be convex (concave), if for all $x, y \in [0, \infty)$ and all $\theta \in (0, 1)$, h satisfies

$$h(\theta x + (1-\theta) y) \leq [\geq] \theta h(x) + (1-\theta) h(y).$$

Throughout this paper, the term *increasing* stands for monotone non-decreasing and *decreasing* stands for monotone non-increasing. Assume that the random variables under consideration are continuous and non-negative, the integrals involved are always finite. All ratios are always supposed to exist whenever they are written.

In this article, we mainly study characterizations, closure and reverse closure properties of IDCPE class. In Section 2, we investigate characterizations of the IDCPE class. In Section 3, we consider closure and reversed closure properties of this class. As applications of a main result, in Section 4, we study the closure and reversed closure properties of the IDCPE class in several stochastic models, including the proportional reversed hazard rate and hazard rate models, the proportional odds model, and the record values model. In Section 5, we also investigate preservation and reversed preservation of the IDCPE class for coherent systems with dependent and identically distributed components. Finally, we give the conclusions of this research in Section 6.

2. Characterization Results of the IDCPE Class

In this section, we explore characterizations of the IDCPE class.

Navarro et al. [8] proposed the following two classes of life distributions based on the DCPE functions.

Definition 1. *A non-negative random variable X is said to be increasing (decreasing) DCPE, denoted by $X \in$ IDCPE (DDCPE), if $\widetilde{\mathcal{E}}_X(t)$ is an increasing (decreasing) function of $t \geq 0$.*

First, we need a lemma from Navarro et al. [8].

Lemma 2. *A non-negative random variable $X \in$ IDCPE $\Leftrightarrow \widetilde{\mathcal{E}}_X(t) \leq m_X(t)$ for all $t \geq 0$.*

$$\Leftrightarrow -\int_0^t \frac{F_X(x)}{F_X(t)} \ln\left[\frac{F_X(x)}{F_X(t)}\right] dx \leq \int_0^t \frac{F_X(x)}{F_X(t)} dx, \quad \text{for all } t \geq 0.$$

The following Theorem 1 will be useful in the proofs of results throughout the paper.

Theorem 1. *A non-negative random variable $X \in$ IDCPE if, and only if,*

$$\int_0^t F_X(x)\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] dx \geq 0, \quad \text{for all } t \geq 0. \tag{5}$$

Proof. From Lemma 2 we have X is IDCPE if, and only if,

$$\widetilde{\mathcal{E}}_X(t) = -\int_0^t \frac{F_X(x)}{F_X(t)} \ln\left(\frac{F_X(x)}{F_X(t)}\right) \mathrm{d}x \leq m_X(t) = \int_0^t \frac{F_X(x)}{F_X(t)} \mathrm{d}x, \quad \text{for all } t \geq 0. \tag{6}$$

Rewriting (6) as (5), the proof is complete. □

The quantile function of F_X is defined by

$$Q_X(u) = F_X^{-1}(u) = \inf\{x |\, F(x) \geq u\}, \quad \text{for all } u \in [0,1], \tag{7}$$

and the quantile density function is defined by $q_X(u) = \frac{\mathrm{d}}{\mathrm{d}u} Q_X(u) = [f_X(Q_X(u))]^{-1}$, for all $u \in [0,1]$.

A continuous lifetime distribution can be specified either in terms of the distribution function or by the quantile function. Recently, the quantile-based methods have been employed effectively for the analysis of lifetime data by many authors in the literature, see, for example, Li and Shaked [28], Nair et al. [29], Bartoszewicz and Benduch [30], Nair and Sankaran [31], Nair and Vineshkumar [32,33], Nair et al. [34], Midhu et al. [35], Nair et al. [36], Nair and Sankaran [37], Franco-Pereira and Shaked [38], Nanda et al. [18], Vineshkumar et al. [20], etc.

By letting $F_X(x) = p$ and denoting $F_X(t) = u$, the following corollary is a direct consequence of Theorem 1.

Corollary 1. *A non-negative random variable X is IDCPE if, and only if,*

$$\int_0^u p\left[\ln\left(\frac{p}{u}\right) + 1\right] q_X(p) dp \geq 0, \quad \text{for all } u \in [0,1]. \tag{8}$$

The following example illustrates the usefulness of Corollary 1.

Example 1. *Now consider a random variable studied in Franco-Pereira and Shaked [38]. Let X be a non-negative continuous random variable with distribution function*

$$F_X(x) = \begin{cases} 1 - e^{-x}, & 0 < x \le 1; \\ 1 - e^{-\frac{x+1}{2}}, & 1 < x \le 2; \\ 1 - e^{-(x-\frac{1}{2})}, & x > 2. \end{cases}$$

Inverting F_X in the three different regions we have

$$Q_X(u) = \begin{cases} -\ln(1-u), & 0 < u \le 1 - e^{-1}; \\ -1 - 2\ln(1-u), & 1 - e^{-1} < u \le 1 - e^{-\frac{3}{2}}; \\ \frac{1}{2} - \ln(1-u), & 1 - e^{-\frac{3}{2}} < u < 1. \end{cases}$$

Differentiating Q_X we get

$$q_X(u) = \begin{cases} \frac{1}{1-u}, & 0 < u \le 1 - e^{-1}; \\ \frac{2}{1-u}, & 1 - e^{-1} < u \le 1 - e^{-\frac{3}{2}}; \\ \frac{1}{1-u}, & 1 - e^{-\frac{3}{2}} < u < 1. \end{cases}$$

In view of (8), denote the function

$$I(u) := \int_0^u p\Big[\ln\Big(\frac{p}{u}\Big) + 1\Big] q_X(p)dp, \quad \textit{for all } u \in [0,1].$$

When $0 < u \le 1 - e^{-1}$, we have

$$I(u) = \int_0^u p\Big[\ln\Big(\frac{p}{u}\Big) + 1\Big] q_X(p)dp = \int_0^u p\Big[\ln\Big(\frac{p}{u}\Big) + 1\Big] \frac{1}{1-p}dp$$

$$= \frac{1}{8}[1 - (1-t)^4] \ge 0.$$

$$I(u) = \int_0^u p\Big[\ln\Big(\frac{p}{u}\Big) + 1\Big] q_X(p)dp = \frac{1}{8}[1 - (1-t)^4] \ge 0.$$

When $t \in (\frac{1}{2}, 1]$,

$$I(t) = I(\frac{1}{2}) - \frac{3}{4}\int_{\frac{1}{2}}^t (1-x)^3 dx = I(\frac{1}{2}) + \frac{3}{16}\Big[(1-t)^4 - (\frac{1}{2})^4\Big] \ge I(1) = \frac{27}{256},$$

where the inequality is due to the decreasing property of $I(t)$. When $t > 1$, $I(t) \ge 0$ trivially holds. Hence, we get that $I(t) \ge 0$ for all $t \ge 0$. By using Corollary 1 we see that $X \in$ IDCPE.

Theorem 2. *Let $a > 0$ be a real constant. If $X \in$ IDCPE, then $aX \in$ IDCPE.*

Proof. Suppose that $X \in$ IDCPE. Then, from (5) we have

$$\int_0^t F_X(x)\Big[\ln\Big(\frac{F_X(x)}{F_X(t)}\Big) + 1\Big]\mathrm{d}x \ge 0, \quad \text{for all } t \ge 0. \tag{9}$$

On the other hand, aX is IDCPE if, and only if, for all $t \ge 0$,

$$\int_0^t F_{aX}(x) \cdot \Big[\ln\Big(\frac{F_{aX}(x)}{F_{aX}(t)}\Big) + 1\Big]\mathrm{d}x = a\int_0^{t/a} F_X(x) \cdot \Big[\ln\Big(\frac{F_X(x)}{F_X(t/a)}\Big) + 1\Big]\mathrm{d}x \ge 0, \tag{10}$$

letting $t/a = u$ in the second integral of the above in (10) yields that if (9) holds, then (10) holds, as claimed. □

Remark 1. *Theorem 2 indicates that the IDCPE class has closure property under a positive scale transform.*

Theorem 3. *Let X be a uniform random variable on interval $(0,1)$, then $X \in$ IDCPE.*

Proof. Suppose that $X \sim U(0,1)$. Then, $F_X(x) = x$, $x \in (0,1)$. It can be verified that

$$\int_0^t F_X(x)\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0.$$

By Theorem 1 we see that $X \in$ IDCPE. As claimed. □

Let $X \sim U(0,1)$, then, for any $a > 0$, $X/a \sim U(0,a)$, where $U(0,a)$ denotes a uniform distribution on interval $(0,a)$. By means of Theorems 2 and 3 we have reached the following result.

Theorem 4. *Let $X \sim U(0,a)$, then X is IDCPE for any $a > 0$.*

Theorem 5. *Let $X \sim U(0,a)$, let X_t be the residual life of X at time t $(0 < t < a)$, then X_t is also IDCPE for any $a > 0$, and all $t \in (0,a)$.*

Proof. Let $X \sim U(0,a)$. It can be verified that $X_t \sim U(0, a-t)$ for any t $(0 < t < a)$. From Theorem 4 we see that X_t is also IDCPE, as claimed. □

Theorem 6. *Let $X \sim U(0,a)$, $a > 0$. Let $X_{(t)}$ be the inactivity time of X at time t $(0 < t < a)$, then $X_{(t)}$ is also IDCPE.*

Proof. Let $X \sim U(0,a)$. It can be checked that $X_{(t)} \sim U(0,t)$ for any t $(0 < t < a)$. From Theorem 4 we see that $X_{(t)}$ is also IDCPE, as desired. □

3. Closure and Reversed Closure Properties of the IDCPE Class

In this section, we study the closure and reverse closure properties of the IDCPE class. First, we consider the closure or reversed closure properties for a series and a parallel system.

Let X be a non-negative and continuous random variable with distribution function F_X and survival function $\overline{F}_X$, respectively. Denote

$$X_{1:n} = \min\{X_1, \ldots, X_n\}, \quad X_{n:n} = \max\{X_1, \ldots, X_n\}.$$

where $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) copies of X, representing the lifetimes of components composed of the system. Then $X_{1:n}$ and $X_{n:n}$ represent the lifetimes of a series system and of a parallel system, respectively. Denote by $F_{X_{1:n}}$ and $F_{X_{n:n}}$ the distribution functions of $X_{1:n}$ and $X_{n:n}$, respectively.

Theorem 7. *If X is IDCPE, then $\min\{X_1, \ldots, X_n\}$ is IDCPE.*

Proof. Suppose that X is IDCPE. Then, from (5) we have

$$\int_0^t F_X(x) \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{11}$$

Since the function $h(x) = \sum_{i=1}^{n} \left[\overline{F}_X(x)\right]^{i-1}$ is non-negative and decreasing, making using of (11) and Lemma 1 we get that

$$\int_0^t F_X(x) \cdot \sum_{i=1}^{n} \left[\overline{F}_X(x)\right]^{i-1} \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{12}$$

It is easy to see that, for all $x \geq 0$,

$$F_{X_{1:n}}(x) = 1 - \overline{F}_{X_{1:n}}(x) = 1 - \left[\overline{F}_X(x)\right]^n = F_X(x) \cdot \sum_{i=1}^{n} \left[\overline{F}_X(x)\right]^{i-1}, \tag{13}$$

and then

$$\begin{aligned} &\int_0^t F_{X_{1:n}}(x) \cdot \left[\ln\left(\frac{F_{X_{1:n}}(x)}{F_{X_{1:n}}(t)}\right) + 1\right] \mathrm{d}x \\ &= \int_0^t F_X(x) \sum_{i=1}^{n} \left[\overline{F}_X(x)\right]^{i-1} \left[\ln\left(\frac{F_X(x)}{F_X(t)} \cdot \frac{\sum_{i=1}^{n} \left[\overline{F}_X(x)\right]^{i-1}}{\sum_{i=1}^{n} \left[\overline{F}_X(t)\right]^{i-1}}\right) + 1\right] \mathrm{d}x \\ &\geq \int_0^t F_X(x) \cdot \sum_{i=1}^{n} \left[\overline{F}_X(x)\right]^{i-1} \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x. \end{aligned} \tag{14}$$

On using (12) and (14) we obtain that

$$\int_0^t F_{X_{1:n}}(x) \cdot \left[\ln\left(\frac{F_{X_{1:n}}(x)}{F_{X_{1:n}}(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{15}$$

That is, $\min\{X_1, \ldots, X_n\}$ is IDCPE. Therefore, the proof is complete. □

Remark 2. *Theorem 7 indicates that the IDCPE class has closure property under series operation. Theorem 7 also says that the IDCPE class has closure property with respect to a series system.*

Example 2. *Let X be a uniform random variable on interval $(0,1)$. By Theorem 3 we see that X is IDCPE. On the other hand, $X_{1:n}$ has its own distribution function $F_{X_{1:n}}(x) = 1 - (1-x)^n$, $x \in (0,1)$. That is, $X_{1:n}$ has a Beta distribution $Beta(1,n)$. On using Theorem 7 we get that $X_{1:n}$ is IDCPE.*

Theorem 8. *If $\max\{X_1, \ldots, X_n\}$ is IDCPE, then X is also IDCPE.*

Proof. Suppose that $\max\{X_1, \ldots, X_n\}$ is IDCPE. Then, from (5) we have for all $t \geq 0$,

$$\begin{aligned} 0 &\leq \int_0^t F_{X_{n:n}}(x) \cdot \left[\ln\left(\frac{F_{X_{n:n}}(x)}{F_{X_{n:n}}(t)}\right) + 1\right] \mathrm{d}x \\ &= \int_0^t F_X(x) \cdot [F_X(x)]^{n-1} \cdot \left[n \ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \\ &\leq \int_0^t F_X(x) \cdot [F_X(x)]^{n-1} \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x. \end{aligned} \tag{16}$$

Since the function $h(x) = 1 \big/ [F_X(x)]^{n-1}$ is non-negative and decreasing in x, on using (16) and Lemma 1 we get that

$$\int_0^t F_X(x) \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0.$$

Again, by Theorem 1 X is IDCPE. This completes the proof. □

Remark 3. *Theorem 8 says that the IDCPE class has reversed closure property with respect to a parallel system. Theorem 8 also indicates that the parallel operations reversely preserve the IDCPE class.*

Let X be an absolutely continuous non-negative random variables with distribution function F_X. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. copies of X. Assume that N is a positive integer-valued random variable independent of X_i's, and N has probability mass function $p_N(n) = P\{N = n\}$, $n = 1, 2, \ldots$. Next, we consider to extend the results in Theorem 7 and Theorem 8 from a finite number n to a random number N. Denote by

$$X_{1:N} = \min\{X_1, \ldots, X_N\}, \quad X_{N:N} = \max\{X_1, \ldots, X_N\}.$$

Then $X_{1:N}$ and $X_{N:N}$ have distribution functions, respectively,

$$F_{X_{1:N}}(x) = F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n) \right] \tag{17}$$

and

$$F_{X_{N:N}}(x) = F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left[(F_X(x))^{n-1} \right] p_N(n) \right]. \tag{18}$$

The following Theorem 9 can be viewed as an extension of Theorem 7.

Theorem 9. *If X is IDCPE, then* $\min\{X_1, \ldots, X_N\}$ *is also IDCPE.*

Proof. Suppose that $\min\{X_1, \ldots, X_N\}$ is IDCPE. Then, from (5) we have

$$\int_0^t F_X(x) \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \right) + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{19}$$

In view of (17),

$$F_{X_{1:N}}(x) = F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n) \right].$$

Since the function $h_X(x) = \sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n)$ is non-negative and decreasing, from (19) and Lemma 1 we get that for all $t \geq 0$,

$$\int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n) \right] \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \right) + 1 \right] \mathrm{d}x \geq 0. \tag{20}$$

On the other hand, $\min\{X_1, \ldots, X_N\}$ is IDCPE if, and only if, for all $t \geq 0$,

$$\begin{aligned} \int_0^t F_{X_{1:N}}(x) \cdot \left[\ln\left(\frac{F_{X_{1:N}}(x)}{F_{X_{1:N}}(t)} \right) + 1 \right] \mathrm{d}x &= \int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n) \right] \\ &\cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \cdot \frac{\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n)}{\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(t) \right) p_N(n)} \right) + 1 \right] \mathrm{d}x \geq 0. \end{aligned} \tag{21}$$

Moreover,

$$\begin{aligned} &\int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n) \right] \\ &\qquad \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \cdot \frac{\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n)}{\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(t) \right) p_N(n)} \right) + 1 \right] \mathrm{d}x \\ &\geq \int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left(\sum_{i=1}^{n} \overline{F}_X^{i-1}(x) \right) p_N(n) \right] \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \right) + 1 \right] \mathrm{d}x. \end{aligned} \tag{22}$$

Hence, by means of (20) and (22) we see that (21) holds. That is, $\min\{X_1, \dots, X_N\}$ is IDCPE. Therefore, the proof follows. □

Remark 4. *Theorem 9 indicates that the IDCPE class has the closure property under random series operations. Theorem 9 also says that the IDCPE class has the closure property with respect to a random series system.*

Example 3. *Let X be a uniform random variable on interval $(0,1)$. By Theorem 3 we see that X is IDCPE. Let N, again, be a positive integer-valued random variable with probability mass function $\mathbb{P}(N=1) = 1/2$ and $\mathbb{P}(N=2) = 1/2$. Then, $X_{1:N}$ has its own distribution function*

$$F_{X_{1:N}}(x) = 1 - \frac{1}{2}[(1-x) + (1-x)^2] \quad \text{for all } x \in (0,1).$$

According to Theorem 9 we know that $X_{1:N}$ is IDCPE.

Remark 5. *In Theorem 9, if N takes a positive integer n almost surely, then $X_{1:N} =_d X_{1:n}$, Theorem 9 becomes as Theorem 7. Hence, Theorem 9 can be viewed as an extension of Theorem 7.*

The following Theorem 10 can be viewed as an extension of Theorem 8.

Theorem 10. *If $\max\{X_1, \dots, X_N\}$ is IDCPE, then X is also IDCPE.*

Proof. Suppose that $\max\{X_1, \dots, X_N\}$ is IDCPE. Then, from (5) and (18) we get

$$\int_0^t F_{X_{N:N}}(x) \cdot \left[\ln\left(\frac{F_{X_{N:N}}(x)}{F_{X_{N:N}}(t)}\right) + 1\right] \mathrm{d}x = \int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} \left[(F_X(x))^{n-1}\right] p_N(n)\right] \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \cdot \frac{\sum_{n=1}^{+\infty}\left[(F_X(x))^{n-1}\right] p_N(n)}{\sum_{n=1}^{+\infty}\left[(F_X(t))^{n-1}\right] p_N(n)}\right) + 1\right] \mathrm{d}x \geq 0 \tag{23}$$

for all $t \geq 0$. Whereas

$$\int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} (F_X(x))^{n-1} p_N(n)\right] \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)} \cdot \frac{\sum_{n=1}^{+\infty}\left[(F_X(x))^{n-1}\right] p_N(n)}{\sum_{n=1}^{+\infty}\left[(F_X(t))^{n-1}\right] p_N(n)}\right) + 1\right] \mathrm{d}x$$
$$\leq \int_0^t F_X(x) \cdot \left[\sum_{n=1}^{+\infty} (F_X(x))^{n-1} p_N(n)\right] \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x, \quad \text{for all } t \geq 0. \tag{24}$$

So, from (23) and (24) we have, for all $t \geq 0$,

$$\int_0^t F_{X_{N:N}}(x) \cdot \left[\sum_{n=1}^{+\infty} \left[(F_X(x))^{n-1}\right] p_N(n)\right] \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0. \tag{25}$$

Since the function $h(x) = 1 \Big/ \left[\sum_{n=1}^{+\infty} (F_X(x))^{n-1} p_N(n)\right]$ is non-negative and decreasing in x, from inequality (25) and Lemma 1 we obtain that

$$\int_0^t F_X(x) \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0, \tag{26}$$

which is equivalent to that X is IDCPE. Therefore, the proof is complete. □

Remark 6. *Theorem 10 indicates that the IDCPE class has reversed closure property under random parallel operations. Theorem 10 also says that the IDCPE class has reversed closure property with respect to a random parallel system.*

Theorem 11. *Let $\phi(\cdot)$ be a non-negative increasing and concave function defined on an interval $\Im = [0, a) \subseteq \mathbb{R}_+ \equiv [0, +\infty)$ such that $\phi(0) = 0$. If X is IDCPE, then $\phi(X)$ is also IDCPE.*

Proof. Suppose that X is IDCPE. Then, from (5) we have

$$\int_0^t F_X(x)\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{27}$$

Since $\phi(x)$ is increasing concave implies that $\phi'(x)$ is non-negative and decreasing, by using (27) and Lemma 1 we get that

$$\int_0^t F_X(x)\phi'(x) \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{28}$$

Additionally, from (5), $\phi(X)$ is IDCPE if, and only if,

$$\int_0^t F_{\phi(X)}(x) \cdot \left[\ln\left(\frac{F_{\phi(X)}(x)}{F_{\phi(X)}(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{29}$$

Moreover, since

$$F_{\phi(X)}(x) = F_X(\phi^{-1}(x)), \tag{30}$$

making use of (28) we obtain

$$\begin{aligned}\int_0^t F_{\phi(X)}(x) \cdot \left[\ln\left(\frac{F_{\phi(X)}(x)}{F_{\phi(X)}(t)}\right) + 1\right] \mathrm{d}x &= \int_0^t F_X(\phi^{-1}(x)) \cdot \left[\ln\left(\frac{F_X(\phi^{-1}(x))}{F_X(\phi^{-1}(t))}\right) + 1\right] \mathrm{d}x \\ &= \int_0^t \phi'(x)) F_X(x) \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0. \end{aligned} \tag{31}$$

That is, the inequality (29) holds, which asserts that $\phi(X)$ is IDCPE. This completes the proof. □

Let X be a non-negative continuous random variable, and $\phi(\cdot)$ be a non-negative increasing function defined on an interval $\Im = [0, a) \subseteq \mathbb{R}_+ \equiv [0, +\infty)$ with $\phi(0) = 0$. We call $\phi(X)$ the generalized scale transform of X.

Refer to the research of Kang and Yan [22], Yan and Kang [23], for a real-valued function defined on an interval $\Im = [0, a) \subseteq \mathbb{R}_+ \equiv [0, +\infty)$ with $\phi(0) = 0$, If $\phi(\cdot)$ is increasing convex (concave), then $\phi(\cdot)$ is called a risk preference (aversion) function, and $\phi(X)$ is called the risk preference (aversion) transform of X.

Remark 7. *Theorem 11 says that the IDCPE class has closure property under a concave generalized scale transform. Theorem 11 also indicates that the IDCPE class has closure property under a risk aversion transform.*

Example 4. *Let X be a uniform random variable on interval $(0, 1)$. By Theorem 3 we see that X is IDCPE. Let Y be a Beta random variable with the distribution function $F_Y(x) = x^\alpha$, $x \in (0, 1)$, where constant $\alpha > 1$ is a parameter. That is, Y has a Beta distribution $Beta(\alpha, 1)$. Then, $F_Y(x)$ has its inverse function $F_Y^{-1}(u) = u^{1/\alpha}$, for all $u \in [0, 1)$. Take $\phi(u) = F_Y^{-1}(u) = u^{1/\alpha}$, $u \in [0, 1)$. Then $\phi(u)$ is a non-negative increasing and concave function with $\phi(0) = 0$. Due to the fact that $\phi(X) = F_Y^{-1}(X) = Y$ and the fact that X is IDCPE, by means of Theorem 11 we get that Y is IDCPE. That is, this power random variable is IDCPE.*

Remark 8. *In Theorem 11, the condition "$\phi(\cdot)$ is a non-negative increasing and concave function" is only a sufficient condition, but not necessary.*

Counterexample 1. Let X be a power random variable with distribution function

$$F_X(x) = x^\alpha, \quad \text{for all } x \in (0,1),$$

where constant $\alpha > 1$ is a parameter. From Example 4 we know that X is IDCPE. Take $\phi(x) = F_X(x) = x^\alpha$, $x \in (0,1)$. One has

$$\phi(X) = F_X(X) = X^\alpha.$$

It is easy to see that $\phi(X)$ is a uniform random variable on interval $(0,1)$. Hence $\phi(X)$ is IDCPE. Clearly, $\phi(x) = x^\alpha$ is not increasing and concave. Hence, the condition in Theorem 11 "$\phi(\cdot)$ is a non-negative increasing and concave function" is only a sufficient condition, but not necessary.

On using a method similar to above Theorem 11 we easily have the following theorem.

Theorem 12. *Let $\phi(\cdot)$ be an increasing convex function defined on an interval $\Im = [0,a) \subseteq \mathbb{R}_+ \equiv [0,+\infty)$, such that $\phi(0) = 0$. If $\phi(X)$ is IDCPE, then X is also IDCPE.*

Remark 9. *Theorem 12 says that the IDCPE class has reversed closure property under a convex generalized scale transform. Theorem 12 also indicates that the IDCPE class has reversed closure property under a risk preference transform.*

Example 5. *Let X be a uniform random variable on interval $(0,1)$. Let Y be a Beta random variable with the distribution function $F_Y(x) = 1-(1-x)^\alpha$, $x \in [0,1)$, where constant α $(0 < \alpha < 1)$ is a parameter. That is, Y has a Beta distribution Beta$(1,\alpha)$. Take $\phi(x) = F_Y(x) = 1-(1-x)^\alpha$, $x \in [0,1)$. It is easy to verify that $\phi(x)$ is a non-negative increasing and convex function with $\phi(0) = 0$. Due to the fact that $\phi(Y) = F_Y(Y) = X$ and the fact that X is IDCPE, by Theorem 11 we obtain that Y is IDCPE. That is, this Beta random variable is also IDCPE.*

Remark 10. *In Theorem 12, the condition "$\phi(\cdot)$ is an increasing convex function" is only a sufficient condition, but not necessary.*

4. Preservation of the IDCPE Class in Several Stochastic Models

In this section, we investigate the preservation of the IDCPE class in the proportional reversed failure rate model, the proportional hazard rate model, the proportional odds model, and the k-record values model.

First, we deal with the following proportional reversed hazard rate model. For more details on the proportional reversed hazard rate model, we refer to Di Crescenzo [39], Gupta and Gupta [40], Di Crescenzo and Longobardi [10], and Shaked and Shanthikumar [41].

Let X be a non-negative random variable with the distribution functions F_X. For any real $\theta > 0$, let $X(\theta)$ denote another random variable with the distribution function $(F_X)^\theta$. Suppose that X has 0 as the left endpoint of its support. Then, we have the following results.

Theorem 13. *Let X, Y, $X(\theta)$, and $Y(\theta)$ be non-negative random variables as described above.*
(a) If $0 < \theta \leq 1$, then X is IDCPE $\Longrightarrow X(\theta)$ is IDCPE;
(b) If $\theta \geq 1$, then $X(\theta)$ is IDCPE $\Longrightarrow X$ is IDCPE.

Proof. Since $F_{X(\theta)}(x) = [F_X(x)]^\theta$. From (5) we have that X is IDCPE if, and only if, the inequality

$$\int_0^t F_X(x) \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0 \tag{32}$$

holds, and that $X(\theta)$ is IDCPE if, and only if, the inequality

$$\int_0^t F_{X(\theta)}(x)\left[\ln\left(\frac{F_{X(\theta)}(x)}{F_{X(\theta)}(t)}\right)+1\right]\mathrm{d}x = \int_t^{+\infty}[F_X(x)]^\theta\cdot\left[\theta\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x \geq 0 \quad (33)$$

holds for all $t \geq 0$.

(a) Assume that X is IDCPE. Since the function $h(x) = [F_X(x)]^{\theta-1}$ is non-negative and decreasing in $x \geq 0$ whenever $0 < \theta \leq 1$, by using Lemma 1 and inequality (32) we obtain

$$\int_0^t [F_X(x)]^\theta\cdot\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \quad (34)$$

Moreover, for $0 < \theta \leq 1$,

$$\int_0^t [F_X(x)]^\theta\cdot\left[\theta\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x \geq \int_0^t [F_X(x)]^\theta\cdot\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x. \quad (35)$$

Making use of (34) and (35), we see that (33) holds. That is, $X(\theta)$ is IDCPE.

(b) Assume that $X(\theta)$ is IDCPE. If $\theta \geq 1$, since $\ln\left(\frac{F_X(x)}{F_X(t)}\right) \leq 0$, then

$$\int_0^t [F_X(x)]^\theta\cdot\left[\theta\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x \leq \int_0^t [F_X(x)]^\theta\cdot\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x. \quad (36)$$

On using inequalities (33) and (36) we get that

$$\int_0^t [F_X(x)]^\theta\cdot\left[\ln\left(\frac{F_X(x)}{F_X(t)}\right)+1\right]\mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \quad (37)$$

Since the function $h(x) = [F_X(x)]^{1-\theta}$ is non-negative decreasing in $x \geq 0$ whenever $\theta \geq 1$. In view of (37) and Lemma 1, we know that (32) holds. That is, X is IDCPE. This completes the proof. □

Below we investigate the preservation in a proportional hazard rate model. For more details about the model, one can refer to Nanda and Paul [11], Abbasnejad et al. [14], Shaked and Shanthikumar [41], Kang and Yan [22], and Yan and Kang[23].

Let X be a non-negative random variable with survival function $\overline{F}_X$. For $\theta > 0$, let $X(\theta)$ denote a random variable with survival function $(\overline{F}_X)^\theta$. We have the following results, the proofs are similar to that of Theorem 13, and hence, are omitted here.

Theorem 14. *Let X, Y, $X(\theta)$ and $Y(\theta)$ be non-negative random variables as described above.*
(a) If $\theta \geq 1$, then X is IDCPE $\Longrightarrow$ $X(\theta)$ is IDCPE;
(b) If $0 < \theta \leq 1$, then $X(\theta)$ is IDCPE $\Longrightarrow$ X is IDCPE.

Marshall and Olkin [42], Sankaran and Jayakumar [43] and Navarro et al. [8] studied the following proportional odds models. Let X be a non-negative continuous random variable with the distribution function F_X and density function f_X. The proportional odds random variable, denoted by X_p, is defined by the distribution function

$$F_{X_\mathrm{p}}(x) = \frac{\theta F_X(x)}{1-(1-\theta)F_X(x)}$$

for $\theta > 0$, where θ is a proportional constant. It is easy to see that the reversed hazard rate function of X_p is

$$a_{X_\mathrm{p}}(x) = \frac{f_{X_\mathrm{p}}(x)}{F_{X_\mathrm{p}}(x)} = \frac{a_X(x)}{1-(1-\theta)F_X(x)}.$$

Clearly,

$$\frac{a_{X_p}(x)}{a_X(x)} = \frac{1}{1-(1-\theta)F_X(x)}.$$

Thus, we have reached the following results.

Lemma 3. *Let X and X_p be as described above.*
(a) If $\theta \geq 1$, then $a_{X_p}(x)/a_X(x)$ is decreasing in $x \geq 0$;
(b) If $0 < \theta \leq 1$, then $a_{X_p}(x)/a_X(x)$ is increasing in $x \geq 0$.

A real-valued function on $D \subseteq \mathbb{R}^r$ is called increasing (decreasing) if it is increasing (decreasing) in each variable when the other variables are held fixed. For the convenient of citation, we introduce the following lemma which will be useful in the proofs of next theorems. This result is motivated by Lemma 2.2 of Khaledi et al. [44], and the proof utilizes a similar manner there. The monotonicity assumption of the lemma is related to the conception of relative RHR order proposed in Definition 2.1 of Rezaei et al. [45].

Lemma 4. *Let X and Y be two non-negative random variables with corresponding reversed hazard rate functions a_X and a_Y. If $a_Y(u)/a_X(u)$ is increasing in $u \geq 0$, then, the function*

$$\varphi(x,y) = \frac{\ln G_Y(x) - \ln G_Y(y)}{\ln F_X(x) - \ln F_X(y)}$$

is increasing in $(x,y) \in \{(u,v) : 0 \leq u \leq v\}$.

Proof. Denote $h_1(x) = a_X(x)$, $h_2(x) = a_Y(x)$, and define

$$\psi_i(x,y) = \int_0^{\infty} 1_{\{x \leq u \leq y\}} h_i(u) \mathrm{d}u, \quad i \in \{1,2\},$$

where 1_A is the indicator function of set A. Since

$$\ln F_X(y) - \ln F_X(x) = \int_0^{\infty} 1_{\{x \leq u \leq y\}} a_X(u) \mathrm{d}u$$

and

$$\ln G_Y(y) - \ln G_Y(x) = \int_0^{\infty} 1_{\{x \leq u \leq y\}} a_Y(u) \mathrm{d}u,$$

we get that

$$\varphi(x,y) = \frac{\psi_2(x,y)}{\psi_1(x,y)}.$$

Note that $a_Y(u)/a_X(u)$ is increasing in $u \geq 0$ means that $h_i(u)$ is TP_2 in $(i,u) \in \{1,2\} \times \mathbb{R}_+$. It is easy to verify that $1_{\{x \leq u \leq y\}}$ is TP_2 in $(u,x) \in \mathbb{R}_+ \times [0,y]$ for any $y \in \mathbb{R}_+$, and is TP_2 in $(u,y) \in \mathbb{R}_+ \times [x,+\infty)$ for any $x \in \mathbb{R}_+$. Utilizing these facts, by using the basic composition formula, we conclude that $\psi_i(x,y)$ is TP_2 in $(i,x) \in \{1,2\} \times [0,y]$ for each $y \in \mathbb{R}_+$, and is TP_2 in $(i,y) \in \{1,2\} \times [x,+\infty)$ for each $x \in \mathbb{R}_+$. This proves the desired result. □

For the proportional odds models we obtain the following results.

Theorem 15. *Let X and X_p be as described above.*
(a) If $\theta \geq 1$, then $X \in IDCPE \Longrightarrow X_p \in IDCPE$;
(b) If $0 < \theta \leq 1$, then $X_p \in IDCPE \Longrightarrow X \in IDCPE$.

Proof. Denote the function

$$h(u) = \frac{\theta u}{1-(1-\theta)u}, \quad u \in [0,1] \tag{38}$$

for any $\theta > 0$. It is easy to see that

(i) If $\theta \geq 1$, then $h(u)$ is non-negative, increasing and concave on $[0,1]$;

(ii) If $0 < \theta \leq 1$, then $h(u)$ is non-negative, increasing and convex on $[0,1]$.

From the definition of X_{p} we have

$$F_{X_{\mathrm{p}}}(x) = h[F_X(x)], \quad \text{for all } x \geq 0. \tag{39}$$

By Theorem 1 we have $X \in$ IDCPE if, and only if,

$$\int_0^t F_X(x) \cdot \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0; \tag{40}$$

and that $X_{\mathrm{p}} \in$ IDCPE if, and only if,

$$\int_0^t F_{X_{\mathrm{p}}}(x) \left[\ln\left(\frac{F_{X_{\mathrm{p}}}(x)}{F_{X_{\mathrm{p}}}(t)}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0,$$

or, from (39), equivalently,

$$\int_0^t h[F_X(x)] \left[\ln\left(\frac{h(F_X(x))}{h(F_X(t))}\right) + 1\right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{41}$$

(a) If $\theta \geq 1$, since $h(u)$ is non-negative, increasing, and concave non-negative and increasing concave in $u \in [0,1]$, then, the function

$$g(x) := \frac{F_{X_{\mathrm{p}}}(x)}{F_X(x)} = \frac{h[F_X(x)]}{F_X(x)} \quad \text{is non-negative and decreasing in } x \geq 0. \tag{42}$$

Hence,

$$h[F_X(x)]/h[F_X(t)] \geq F_X(x)/F_X(t), \quad \text{for all } t \geq x \geq 0.$$

We obtain that, for all $t \geq 0$,

$$\int_0^t h[F_X(x)] \left[\ln\left(\frac{h(F_X(x))}{h(F_X(t))}\right) + 1\right] \mathrm{d}x \geq \int_0^t h[F_X(x)] \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x. \tag{43}$$

Assume that $X \in$ IDCPE. By using (40), (43) and Lemma 1, we get that

$$\int_0^t h[F_X(x)] \left[\ln\left(\frac{F_X(x)}{F_X(t)}\right) + 1\right] \mathrm{d}x \geq 0. \tag{44}$$

From (43) and (44) we see that inequality (41) holds, which asserts that $X_{\mathrm{p}} \in$ IDCPE.

(b) The proof is similar to that of above (a). Therefore, the proof is complete. □

Next, we study the closure property of the IDCPE class for the record values model. Chandler [46] introduced and studied some properties of record values. For more details about record values and their applications, one can refer to Khaledi et al. [44], Kundu et al. [47], Zhao and Balakrishnan [48], Zarezadeh and Asadi [49], Li and Zhang [50], Kang and Yan [22], and the references therein.

According to Kang and Yan [22], Yan and Kang[23], let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables (rv's) from an absolutely continuous non-negative random variable X with the survival function $\overline{F}_X(\cdot)$ and the density function $f_X(\cdot)$. The rv's T_n^X, defined recursively by $T_1^X = 1$ and

$$T_{n+1}^X = \min\{j > T_n^X : X_j > X_{T_n^X}\}, \quad n \geq 1,$$

are called the n-th record times. The quantities $X_{T_n^X}$, denoted by R_n^X, are called the n-th record values.

It can be proven that the probability density, distribution and reversed hazard rate functions of R_n^X are given, respectively, by

$$f_{R_n^X}(x) = \frac{1}{\Gamma(n)}\Lambda_X^{n-1}(x)f_X(x), \tag{45}$$

$$F_{R_n^X}(x) = \overline{F}_X(x)\sum_{j=n}^{+\infty}\frac{(\Lambda_X(x))^j}{j!} = \Gamma_n(\Lambda_X(x)), \tag{46}$$

for all $x \geq 0$, where $\Gamma_n(\cdot)$ is the distribution function of a Gamma random variable with a shape parameter n and a scale parameter 1, and $\Lambda_X(x) = -\ln\overline{F}_X(x)$ is the cumulative failure rate function of X.

We now recall two stochastic orders which will be used in the next. one can refer to Shaked and Shanthikumar [41] for more details.

Let X and Y be two non-negative continuous random variables with the density functions f_X and g_Y and the distribution functions F_X and G_Y, respectively.

(a) X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g_Y(x)/f_X(x)$ is increasing in $x \geq 0$;

(b) X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leq_{\mathrm{rh}} Y$) if $G_Y(x)/F_X(x)$ is increasing in $x \geq 0$.

It is well-known that

$$X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{rh}} Y.$$

For the preservation property of the IDCPE class in the record values model, we obtain the following result.

Theorem 16. *Let X and R_n^X be as described above, m and n be positive integers. Then*

$$R_n^X \in IDCPE \Longrightarrow R_m^X \in IDCPE, \quad \textit{for all } n > m \geq 1.$$

Proof. Suppose that $R_n^X \in$ IDCPE. Then, from Theorem 1 we have, for all $t \geq 0$,

$$\int_0^t F_{R_n^X}(x)\left[\ln\left(\frac{F_{R_n^X}(x)}{F_{R_n^X}(t)}\right) + 1\right]\mathrm{d}x \geq 0. \tag{47}$$

From (45) we see that the function

$$\frac{f_{R_m^X}(x)}{f_{R_n^X}(x)} = \frac{\Gamma(n)}{\Gamma(m)}(\Lambda_X(x))^{m-n} \quad \text{is increasing in } x \geq 0 \text{ for } n > m.$$

Hence, we have $R_m^X \leq_{\mathrm{lr}} R_n^Y$. So, $R_m^X \leq_{\mathrm{rh}} R_n^Y$. Thus, we obtain that

$$\frac{F_{R_m^X}(x)}{F_{R_n^X}(x)} \quad \text{is non-negative and decreasing in } x \geq 0. \tag{48}$$

This leads to

$$\frac{F_{R_m^X}(x)}{F_{R_m^X}(t)} \geq \frac{F_{R_n^X}(x)}{F_{R_n^X}(t)} \quad \text{for all } t \geq x \geq 0.$$

We, hence, find that

$$\int_0^t F_{R_m^X}(x)\left[\ln\left(\frac{F_{R_m^X}(x)}{F_{R_m^X}(t)}\right) + 1\right]\mathrm{d}x \geq \int_0^t F_{R_n^X}(x)\left[\ln\left(\frac{F_{R_n^X}(x)}{F_{R_n^X}(t)}\right) + 1\right]\mathrm{d}x. \tag{49}$$

In view of (47) and Theorem 1 the desired result follows. □

5. Preservation and Reversed Preservation of the IDCPE Class for Coherent Systems with Dependent and Identically Distributed Components

In this section, we explore the preservation and reversed preservation of the IDCPE class for a coherent system with dependent and identically distributed components.

A distortion distribution associated to a distribution function F and to an increasing continuous distortion function $q : [0,1] \to [0,1]$ with $q(0) = 0$ and $q(1) = 1$ is defined by

$$F_q(t) = q(F(t)). \tag{50}$$

By means of the distortion function, Navarro et al. [51] gave a convenient representation of a coherent system reliability $\overline{F}_T$. They proved the following result which plays a key role to obtain the results included in this section. For the ease of citation, we give this result as a lemma.

Lemma 5 (Navarro et al. [51])**.** *Let $T = \phi(X_1, \ldots, X_n)$ be the lifetime of a coherent system based on possibly dependent components with lifetimes $X_1, \ldots, X_n$, having a common reliability function $\overline{F}_X(t) = \Pr(X_i > t)$. Assume that h is a distortion function. Then, the system reliability function can be written as*

$$\overline{F}_T(t) = h(\overline{F}_X(t)), \tag{51}$$

where h only depends on ϕ and on the survival copula of $(X_1, \ldots, X_n)$.

Making use of (51), the distribution function of the coherent system lifetime T is given by

$$F_T(t) = 1 - \overline{F}_T(t) = 1 - h(1 - F_X(t)) = g(F_X(t)), \tag{52}$$

where $g(u) = 1 - h(1 - u)$, $u \in (0, 1)$. Notice that h and g depend on both ϕ and K, but they do not depend on $\overline{F}_X$(Navarro et al.) [51]. Moreover, h (or g) is an increasing function in $(0, 1)$ from $h(0) = 0$ to $h(1) = 1$. In the general case, the function h in Equation (51) is called structure and dependence function (see, for example, Navarro et al. [51] and Navarro and Gomis [52].

We now study the preservation of the IDCPE class for a coherent system with dependent and identically distributed components. We get the following result:

Theorem 17. *Let X be a non-negative continuous random variables with survival function $\overline{F}_X(t)$. Let $T = \phi(X_1, \ldots, X_n)$ be the lifetime of a coherent system with structure function ϕ and with identically distributed component lifetimes $X_1, \ldots, X_n$ having common continuous survival functions $\overline{F}_X(t) = \Pr(X_i > t)$. Let h be the domination function of the coherent system.*

(a) Assume $g(u)/u$ is decreasing in $u \in (0, 1)$. If $X \in$ IDCPE, then $T \in$ IDCPE;

(b) Assume $g(u)/u$ is increasing in $u \in (0, 1)$. If $T \in$ IDCPE, then $X \in$ IDCPE.

Proof. In view of Theorem 1, we have that $X \in$ IDCPE if, and only if,

$$\int_0^t F_X(x) \left[\ln\left(\frac{F_X(x)}{F_X(t)} \right) + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0; \tag{53}$$

Further, that $T \in$ IDCPE if, and only if,

$$\int_0^t F_T(x) \left[\ln\left(\frac{F_T(x)}{F_T(t)} \right) + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0,$$

by using (52), equivalently,

$$\int_0^t g[F_X(x)] \left[\ln\left(\frac{g(F_X(x))}{g(F_X(t))} \right) + 1 \right] \mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{54}$$

(a) Assume that $X \in$ IDCPE. If $g(u)/u$ is decreasing in $u \in (0,1)$, then, the function

$$g[F_X(x)]/F_X(x) \quad \text{is non-negative and decreasing in } x \geq 0. \tag{55}$$

Thus,

$$\frac{g[F_X(x)]}{g[F_X(t)]} \geq \frac{F_X(x)}{F_X(t)}, \quad \text{for all } t \geq x \geq 0.$$

Hence, for all $t \geq 0$,

$$\int_0^t g[F_X(x)]\left[\ln\left(\frac{g(F_X(x))}{g(F_X(t))}\right)+1\right]\mathrm{d}x \geq \int_0^t g[F_X(x)]\left[\ln\left(\frac{F_T(x)}{F_T(t)}\right)+1\right]\mathrm{d}x. \tag{56}$$

Moreover, By (53), (55) and Lemma 1, we get that

$$\int_0^t g[F_X(x)]\left[\ln\left(\frac{F_T(x)}{F_T(t)}\right)+1\right]\mathrm{d}x \geq 0, \quad \text{for all } t \geq 0. \tag{57}$$

On using (57), (56) and Lemma 1, we see that the inequality (54) holds, which asserts by Theorem 1 that $T \in$ IDCPE.

(b) The proof is similar with that of (a). Therefore, the proof is complete. □

6. Conclusions

In this paper, we investigate some characterizations of the IDCPE class, and we mainly obtain the closure and reversed closure properties of this class. Meanwhile, we examine the preservation and reversed preservation properties of this class in several stochastic models.

We get that the IDCPE class is:

(1) Closed respect to a series system (see Theorem 7); but
(i) The inverse proposition of Theorem 7 does not hold;
(ii) Not reversely closed respect to a series system.
These two cases can all be viewed as a kind of anti-symmetry.
(2) Reversely closed respect to a parallel system (see Theorem 8); but
(i) The inverse proposition of Theorem 8 does not hold;
(ii) Not closed respect to a parallel system.
These two cases can all be viewed as a kind of anti-symmetry.
(3) Closed respect to a random series system (see Theorem 9); but
(i) The inverse proposition of Theorem 9 does not hold;
(ii) Not reversely closed respect to a random series system.
These two cases can all be viewed as a kind of anti-symmetry.
(4) Reversely closed respect to a random parallel system (see Theorem 10); but
(i) The inverse proposition of Theorem 10 does not hold;
(ii) Not closed respect to a random parallel system.
These two cases can all be viewed as a kind of anti-symmetry.
(5) Closed under a non-negative, increasing and concave transform (see Theorem 11); but
(i) The inverse proposition of Theorem 11 does not hold;
(ii) Not reversely closed under a non-negative, increasing and concave transform.
These two cases can all be viewed as a kind of anti-symmetry.
(6) Reversely closed under a non-negative, increasing and convex transform (see Theorem 12); but
(i) The inverse proposition of Theorem 12 does not hold;
(ii) Not closed under a non-negative increasing convex transform.
These two cases can all be viewed as a kind of anti-symmetry.
(7) Closed but not reversely closed under some appropriate condition in the proportional reversed hazard rate models (see Theorem 13 (a)). This case can be viewed as a kind of anti-symmetry;

(8) Reversely closed but not closed under the other condition in the proportional reversed hazard rate models (see Theorem 13 (b)). This case can be viewed as a kind of anti-symmetry;

(9) Closed but not reversely closed under some appropriate condition in the proportional hazard rate models (see Theorem 14 (a)). This case can be viewed as a kind of anti-symmetry;

(10) Reversely closed but not closed under the other appropriate condition in the proportional hazard rate models (see Theorem 14 (b)). This case can be viewed as a kind of anti-symmetry;

(11) Closed but not reversely closed under some appropriate condition in the proportional odds model (see Theorem 15 (a)). This case can be viewed as a kind of anti-symmetry;

(12) Reversely closed but not closed under the other appropriate condition in the proportional odds model (see Theorem 15 (b)). This case can be viewed as a kind of anti-symmetry;

(13) Reversely closed in the record-value models (see Theorem 16). This case can be viewed as a kind of anti-symmetry;

(14) Closed under some appropriate condition for a coherent system (see Theorem 17 (a)); but the inverse proposition does not hold. This case can be viewed as a kind of anti-symmetry;

(15) Reversely closed under the other condition for a coherent system (see Theorem 17 (b)); but the inverse proposition does not hold. This case can be viewed as a kind of anti-symmetry.

Author Contributions: Conceptualization, H.W., D.K. and L.Y.; methodology, H.W., D.K. and L.Y.; software, H.W., D.K. and L.Y.; validation, H.W., D.K. and L.Y.; formal analysis, H.W., D.K. and L.Y.; investigation, H.W., D.K. and L.Y.; resources, H.W., D.K. and L.Y.; data curation, H.W., D.K. and L.Y.; writing—original draft preparation, H.W., D.K. and L.Y.; writing—review and editing, H.W., D.K. and L.Y.; visualization, H.W., D.K. and L.Y.; supervision, H.W., D.K. and L.Y.; project administration, H.W., D.K. and L.Y.; funding acquisition, H.W., D.K. and L.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This paper was supported by the scientific research and innovation team of “digital economy serving port economy research” of Zhejiang Wanli University (Grant No. 202036).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Morris, C.W. *Academic Press Dictionary of Science and Technology*; Gulf Professional Publishing: Oxford, UK, 1992; Volume 10.
2. Shannon, C.E. A mathematical theory of communication. *ACM SIGMOBILE Mob. Comput. Commun. Rev.* **2001**, *5*, 3–55. [CrossRef]
3. Wiener, N. *Cybernetics*; MIT Press, Wiley: New York, NY, USA, 1948.
4. Ebrahimi, N.; Pellerey, F. New partial ordering of survival functions based on the notion of uncertainty. *J. Appl. Probab.* **1995**, *32*, 202–211. [CrossRef]
5. Ebrahimi, N. How to measure uncertainty in the residual life time distribution. *Sankhyā Indian J. Stat. Ser. A* **1996**, *58*, 48–56.
6. Ebrahimi, N.; Kirmani, S. Some results on ordering of survival functions through uncertainty. *Stat. Probab. Lett.* **1996**, *29*, 167–176. [CrossRef]
7. Di Crescenzo, A.; Longobardi, M. Entropy-based measure of uncertainty in past lifetime distributions. *J. Appl. Probab.* **2002**, *39*, 434–440. [CrossRef]
8. Navarro, J.; del Aguila, Y.; Asadi, M. Some new results on the cumulative residual entropy. *J. Stat. Plan. Inference* **2010**, *140*, 310–322. [CrossRef]
9. Di Crescenzo, A.; Longobardi, M. A measure of discrimination between past lifetime distributions. *Stat. Probab. Lett.* **2004**, *67*, 173–182. [CrossRef]
10. Di Crescenzo, A.; Longobardi, M. On cumulative entropies. *J. Stat. Plan. Inference* **2009**, *139*, 4072–4087. [CrossRef]

11. Nanda, A.K.; Paul, P. Some properties of past entropy and their applications. *Metrika* **2006**, *64*, 47–61. [CrossRef]
12. Nanda, A.K.; Paul, P. Some results on generalized past entropy. *J. Stat. Plan. Inference* **2006**, *136*, 3659–3674. [CrossRef]
13. A.K. Nanda, P.P. Some results on generalized residual entropy. *Inf. Sci.* **2006**, *176*, 27–47. [CrossRef]
14. Abbasnejad, M.; Arghami, N.R.; Morgenthaler, S.; Borzadaran, G. On the dynamic survival entropy. *Stat. Probab. Lett.* **2010**, *79*, 1962–1971. [CrossRef]
15. Kundu, C.; Nanda, A.K.; Maiti, S.S. Some distributional results through past entropy. *J. Stat. Plan. Inference* **2010**, *140*, 1280–1291. [CrossRef]
16. Kumar, V.; Taneja, H. Some characterization results on generalized cumulative residual entropy measure. *Stat. Probab. Lett.* **2011**, *81*, 1072–1077. [CrossRef]
17. Khorashadizadeh, M.; Roknabadi, A.R.; Borzadaran, G.M. Doubly truncated (interval) cumulative residual and past entropy. *Stat. Probab. Lett.* **2013**, *83*, 1464–1471. [CrossRef]
18. Nanda, A.K.; Sankaran, P.; Sunoj, S. Renyi's residual entropy: A quantile approach. *Stat. Probab. Lett.* **2014**, *85*, 114–121. [CrossRef]
19. Kayal, S. On generalized dynamic survival and failure entropies of order (α, β). *Stat. Probab. Lett.* **2015**, *96*, 123–132. [CrossRef]
20. Vineshkumar, B.; Nair, N.U.; Sankaran, P. Stochastic orders using quantile-based reliability functions. *J. Korean Stat. Soc.* **2015**, *44*, 221–231. [CrossRef]
21. Kang, D.T. Further results on closure properties of LPQE order. *Stat. Methodol.* **2016**, *25*, 23–35. [CrossRef]
22. Kang, D.T.; Yan, L. On the dynamic cumulative residual quantile entropy ordering. *Stat. Methodol.* **2016**, *32*, 14–35. [CrossRef]
23. Yan, L.; Kang, D.T. Some new results on the Rényi quantile entropy Ordering. *Stat. Methodol.* **2016**, *33*, 55–70. [CrossRef]
24. Rao, M.; Chen, Y.; Vemuri, B.C.; Wang, F. Cumulative residual entropy: A new measure of information. *IEEE Trans. Inf. Theory* **2004**, *50*, 1220–1228. [CrossRef]
25. Asadi, M.; Zohrevand, Y. On the dynamic cumulative residual entropy. *J. Stat. Plan. Inference* **2007**, *137*, 1931–1941. [CrossRef]
26. Di Crescenzo, A.; Kayal, S.; Meoli, A. Fractional generalized cumulative entropy and its dynamic version. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *102*, 105899. [CrossRef]
27. Barlow, R.E.; Proschan, F. *Statistical Theory of Reliability and Life Testing: Probability Models*; Technical Report; Florida State University: Tallahassee, FL, USA, 1975.
28. Li, X.; Shaked, M. The observed total time on test and the observed excess wealth. *Stat. Probab. Lett.* **2004**, *68*, 247–258. [CrossRef]
29. Nair, N.U.; Sankaran, P.; Kumar, B.V. Total time on test transforms of order n and their implications in reliability analysis. *J. Appl. Probab.* **2008**, *45*, 1126–1139. [CrossRef]
30. Bartoszewicz, J.; Benduch, M. Some properties of the generalized TTT transform. *J. Stat. Plan. Inference* **2009**, *139*, 2208–2217. [CrossRef]
31. Nair, N.U.; Sankaran, P. Quantile-based reliability analysis. *Commun. Stat. Theory Methods* **2009**, *38*, 222–232. [CrossRef]
32. Nair, N.U.; Vineshkumar, B. L-moments of residual life. *J. Stat. Plan. Inference* **2010**, *140*, 2618–2631. [CrossRef]
33. Nair, N.U.; Vineshkumar, B. Ageing concepts: An approach based on quantile function. *Stat. Probab. Lett.* **2011**, *81*, 2016–2025. [CrossRef]
34. Nair, N.U.; Sankaran, P.; Kumar, B.V. Modelling lifetimes by quantile functions using Parzen's score function. *Statistics* **2012**, *46*, 799–811. [CrossRef]
35. Midhu, N.; Sankaran, P.; Nair, N.U. A class of distributions with the linear mean residual quantile function and it's generalizations. *Stat. Methodol.* **2013**, *15*, 1–24. [CrossRef]
36. Nair, N.U.; Sankaran, P.; Balakrishnan, N. *Quantile-Based Reliability Analysis*; Springer: New York, NY, USA, 2013.
37. Nair, N.U.; Sankaran, P. Some new applications of the total time on test transforms. *Stat. Methodol.* **2013**, *10*, 93–102. [CrossRef]
38. Franco-Pereira, A.M.; Shaked, M. The total time on test transform and the decreasing percentile residual life aging notion. *Stat. Methodol.* **2014**, *18*, 32–40. [CrossRef]
39. Di Crescenzo, A. Some results on the proportional reversed hazards model. *Stat. Probab. Lett.* **2000**, *50*, 313–321. [CrossRef]
40. Gupta, R.C.; Gupta, R.D. Proportional reversed hazard rate model and its applications. *J. Stat. Plan. Inference* **2007**, *137*, 3525–3536. [CrossRef]
41. Shaked, M.; Shanthikumar, J.G. *Stochastic Orders*; Springer: New York, NY, USA, 2007.
42. Marshall, A.W.; Olkin, I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika* **1997**, *84*, 641–652. [CrossRef]
43. Sankaran, P.; Jayakumar, K. On proportional odds models. *Stat. Pap.* **2008**, *49*, 779–789. [CrossRef]
44. Khaledi, B.E.; Amiripour, F.; Hu, T.; Shojaei, S.R. Some new results on stochastic comparisons of record values. *Commun. Stat. Theory Methods* **2009**, *38*, 2056–2066. [CrossRef]
45. Rezaei, M.; Gholizadeh, B.; Izadkhah, S. On relative reversed hazard rate order. *Commun. Stat. Theory Methods* **2015**, *44*, 300–308. [CrossRef]
46. Chandler, K. The distribution and frequency of record values. *J. R. Stat. Soc. Ser. B* **1952**, *14*, 220–228. [CrossRef]
47. Kundu, C.; Nanda, A.K.; Hu, T. A note on reversed hazard rate of order statistics and record values. *J. Stat. Plan. Inference* **2009**, *139*, 1257–1265. [CrossRef]
48. Zhao, P.; Balakrishnan, N. Stochastic comparison and monotonicity of inactive record values. *Stat. Probab. Lett.* **2009**, *79*, 566–572. [CrossRef]

49. Zarezadeh, S.; Asadi, M. Results on residual Rényi entropy of order statistics and record values. *Inf. Sci.* **2010**, *180*, 4195–4206. [CrossRef]
50. Li, X.; Zhang, S. Some new results on Rényi entropy of residual life and inactivity time. *Probab. Eng. Inf. Sci.* **2011**, *25*, 237–250. [CrossRef]
51. Navarro, J.; del Águila, Y.; Sordo, M.A.; Suárez-Llorens, A. Stochastic ordering properties for systems with dependent identically distributed components. *Appl. Stoch. Model. Bus. Ind.* **2013**, *29*, 264–278. [CrossRef]
52. Navarro, J.; Gomis, M.C. Comparisons in the mean residual life order of coherent systems with identically distributed components. *Appl. Stoch. Model. Bus. Ind.* **2016**, *32*, 33–47. [CrossRef]