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Initial Coefficient Estimates and Fekete–Szegö Inequalities for New Families of Bi-Univalent Functions Governed by $(p-q)$-Wanas Operator

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Abstract: The motivation of the present article is to define the $(p-q)$-Wanas operator in geometric function theory by the symmetric nature of quantum calculus. We also initiate and explore certain new families of holormorphic and bi-univalent functions $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ and $S_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ which are defined in the unit disk U associated with the $(p-q)$-Wanas operator. The upper bounds for the initial Taylor–Maclaurin coefficients and Fekete–Szegö-type inequalities for the functions in these families are obtained. Furthermore, several consequences of our results are pointed out based on the various special choices of the involved parameters.

Keywords: holormorphic function; bi-univalent function; upper bounds; Fekete–Szegö functional; $(p-q)$-Wanas operator

MSC: 30C45; 30C20

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1. Introduction

Indicate by $\mathcal{A}$ the family of all holomorphicfunctions of the form

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{1}$$

in the open unit disk $U = \{z \in C : |z| < 1\}$.

We also denote by T the subfamily of $\mathcal{A}$ consisting of functions which are also univalent in U.

The famous Koebe one-quarter theorem [1] ensures that the image of U under each univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Furthermore, each function $f \in T$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$ and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2 b_3 + b_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is named bi-univalent in U if both f and f^{-1} are univalent in U. The family of all bi-univalent functions in U is denoted by E.

From the work of Srivastava et al. [2], we choose to recall the following examples of functions in the family E:

$$\frac{z}{1-z}, \quad -\log(1-z) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

In fact, Srivastava et al. [2] have actually revived the study of analytic and bi-univalent functions in the recent years. This was followed by works such as those by Frasin and Aouf [3], Ali et al. [4], Bulut et al. [5], Srivastava and et al. [6] and others (see, for example, [7–15]).

We notice that the family E is not empty. However, the Koebe function is not a member of E.

The problem to obtain the general coefficient bounds on the Taylor–Maclaurin coefficients

$$|b_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

for functions $f \in E$ is still not completely addressed for many of the subfamilies of E. The origin of the Fekete-Szegö functional $\left|b_3 - \eta b_2^2\right|$ for $f \in T$ was in the disproof [16] by Fekete and Szegö of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. Researchers in the Theory of Geometric Function have recently obtained remarkable results on this topic (see, for example, [17–21]).

With a view to recalling the principle of subordination between holomorphic functions, let the functions f and g be holomorphic in U. The function f is subordinate to g, if there exists a Schwarz function ω, which is analytic in U with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in U),$$

such that

$$f(z) = g\big(\omega(z)\big).$$

The subordination is denoted by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in U).$$

It is well known that (see [22]), if the function g is univalent in U, then

$$f \prec g \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subseteq g(U).$$

For $0 < q < p \leq 1$, the (p,q)-derivative operator or (p,q)-difference operator for a function f is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z} \qquad (z \in U^* = U \setminus \{0\}), \tag{2}$$

and

$$D_{p,q}f(0) = f'(0).$$

For more details on the concepts of (p,q)-calculus see [20,23–27].

For function $f \in \mathcal{A}$, we deduce that

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} b_k z^{k-1}, \tag{3}$$

where the (p,q)-bracket number or twin-basic $[k]_{p,q}$ is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p-q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + \cdots + pq^{k-2} + q^{k-1} \quad (p \neq q),$$

which is a natural generalization of the q-number, that is, we have (see [28–30])

$$\lim_{p\to 1^-}[k]_{p,q}=[k]_q=\frac{1-q^k}{1-q}.$$

It is clear that the notation $[k]_{p,q}$ is symmetric, that is,

$$[k]_{p,q}=[k]_{q,p}.$$

In 2019, Wanas [31] introduced the following operator, which can also be called Wanas operator $W_{s,t}^{\sigma,\delta}:\mathcal{A}\longrightarrow\mathcal{A}$ defined by

$$W_{s,t}^{\sigma,\delta}f=z+\sum_{k=2}^{\infty}\left[\sum_{\tau=1}^{\sigma}\binom{\sigma}{\tau}(-1)^{\tau+1}\left(\frac{s^\tau+kt^\tau}{s^\tau+t^\tau}\right)\right]^{\delta}b_kz^k, \tag{4}$$

where $s\in\mathbb{R}$, $t\in\mathbb{R}_0^+$ with $s+t>0$, $k-1,\sigma\in\mathbb{N}$ and $\delta\in\mathbb{N}_0$.

Now, for $f\in\mathcal{A}$, we define the $(p-q)$-difference Wanas operator as given below

$$\begin{aligned}
W_{0,1,p,q}^{1,0}f(z)&=f(z)\\
W_{0,1,p,q}^{1,1}f(z)&=zW_{p,q}f(z)\\
W_{0,1,p,q}^{1,\delta}f(z)&=zW_{p,q}(W_{p,q}^{\delta-1}f(z))\\
W_{s,t,p,q}^{\sigma,\delta}f(z)&=z+\sum_{k=2}^{\infty}\left(\frac{[\Psi_k(\sigma,s,t)]_{p,q}}{[\Psi_1(\sigma,s,t)]_{p,q}}\right)^{\delta}b_kz^k
\end{aligned}$$

where

$$W_{p,q}f(z)=z+\sum_{k=2}^{\infty}\frac{p^k-q^k}{p-q}b_kz^k,\ W_{p,q}^{\delta-1}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{p^k-q^k}{p-q}\right)^{\delta-1}b_kz^k,$$

$$\Psi_k(\sigma,s,t)=\sum_{\tau=1}^{\sigma}\binom{\sigma}{\tau}(-1)^{\tau+1}(s^\tau+kt^\tau),\ \Psi_1(\sigma,s,t)=\sum_{\tau=1}^{\sigma}\binom{\sigma}{\tau}(-1)^{\tau+1}(s^\tau+t^\tau),$$

$$s\in\mathbb{R},t\in\mathbb{R}_0^+\text{ with }s+t>0,k-1,\sigma\in\mathbb{N},\delta\in\mathbb{N}_0,0<q<p\leq 1\text{ and }z\in U.$$

Remark 1. *The operator $W_{s,t,p,q}^{\sigma,\delta}$ is a generalization of several known operators studied in earlier investigations which are being recalled below.*

1. *For $p=\sigma=t=1$, $\delta=-\nu$, $\Re(\nu)>1$ and $s\in\mathbb{C}\setminus\mathbb{Z}_0^-$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the q-Srivastava–Attiya operator $J_{q,s}^{\nu}$ [32];*
2. *For $p=\sigma=t=1$, $\delta=-1$ and $s>-1$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the q-Bernardi operator [33];*
3. *For $p=\sigma=s=t=1$ and $\delta=-1$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the q-Libera operator [33];*
4. *For $s=0$ and $p=\sigma=t=1$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the q-Sălăgean operator [34];*
5. *For $q\longrightarrow 1^-$ and $p=\sigma=1$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the operator $I_{s,t}^{\delta}$ that was introduced and studied by Swamy [35];*
6. *For $q\longrightarrow 1^-$, $p=\sigma=t=1$, $\delta=-\nu$, $\Re(\nu)>1$ and $s\in\mathbb{C}\setminus\mathbb{Z}_0^-$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the operator J_s^{ν} that was investigated by Srivastava and Attiya [36]. The operator J_s^{ν} is now popularly known in the literature as the Srivastava–Attiya operator;*
7. *For $q\longrightarrow 1^-$, $p=\sigma=t=1$ and $s>-1$, the operator $W_{s,t,p,q}^{\sigma,\delta}$ reduces to the operator I_s^{δ} that was investigated by Cho and Srivastava [37];*

8. *For* $q \longrightarrow 1^-$, $p = \sigma = s = t = 1$, *the operator* $W^{\sigma,\delta}_{s,t,p,q}$ *reduces to the operator* I^{δ} *that was considered by Uralegaddi and Somanatha [38];*
9. *For* $q \longrightarrow 1^-$, $p = \sigma = s = t = 1$, $\delta = -\xi$ *and* $\xi > 0$, *the operator* $W^{\sigma,\delta}_{s,t,p,q}$ *reduces to the operator* I^{ξ} *that was introduced by Jung et al. [39]. The operator* I^{ξ} *is the Jung–Kim–Srivastava integral operator;*
10. *For* $q \longrightarrow 1^-$, $p = \sigma = t = 1$, $\delta = -1$ *and* $s > -1$, *the operator* $W^{\sigma,\delta}_{s,t,p,q}$ *reduces to the Bernardi operator [40];*
11. *For* $q \longrightarrow 1^-$, $s = 0$, $p = \sigma = t = 1$ *and* $\delta = -1$, *the operator* $W^{\sigma,\delta}_{s,t,p,q}$ *reduces to the Alexander operator [41];*
12. *For* $q \longrightarrow 1^-$, $p = \sigma = 1$, $s = 1 - t$ *and* $t \geq 0$, *the operator* $W^{\sigma,\delta}_{s,t,p,q}$ *reduces to the operator* D^{δ}_{t} *that was given by Al-Oboudi [42];*
13. *For* $q \longrightarrow 1^-$, $p = \sigma = 1$, $s = 0$ *and* $t = 1$, *the operator* $W^{\sigma,\delta}_{s,t,p,q}$ *reduces to the operator* S^{δ} *that was considered by Sălăgean [43].*

We shall need the following Lemma in our investigation.

Lemma 1 ([44], p. 41 and [45], p. 41)**.** *Let the function* $x \in P$ *be given by the following series:*

$$x(z) = 1 + x_1 z + x_2 z^2 + \cdots \quad (z \in U).$$

The sharp estimate given by

$$|x_n| \leqq 2 \quad (n \in \mathbb{N})$$

holds true.

2. A Set of Main Results

Indicate by $\vartheta(z)$ the holomorphic function with positive real part in U such that

$$\vartheta(0) = 1, \quad \vartheta'(0) > 0$$

and $\vartheta(U)$ is symmetric with respect to the real axis, which is of the type:

$$\vartheta(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \tag{5}$$

where $B_1 > 0$.

Using the $(p - q)$-Wanas operator, we now provide the following subfamilies of holomorphic and bi-univalent functions.

Definition 1. *For* $0 \leqq \lambda \leqq 1$, *a function* $f \in E$ *is said to be in the family* $\mathcal{A}_E(\lambda, \sigma, \delta, s, t, p, q; \vartheta)$ *if it fulfills the subordinations:*

$$(1-\lambda)\frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}{W^{\sigma,\delta}_{s,t,p,q}f(z)} + \lambda\left(1 + \frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}\right) \prec \vartheta(z)$$

and

$$(1-\lambda)\frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}{W^{\sigma,\delta}_{s,t,p,q}g(w)} + \lambda\left(1 + \frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}\right) \prec \vartheta(w),$$

where $g(w) = f^{-1}(w)$.

Definition 2. *For $\mu \geqq 1$ and $\gamma \geqq 0$, a function $f \in E$ is said to be in the family $\mathcal{S}_E(\mu, \gamma, \sigma, \delta, s, t, p, q; \vartheta)$ if it fulfills the subordinations:*

$$(1-\mu)\frac{W^{\sigma,\delta}_{s,t,p,q}f(z)}{z} + \mu\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)' + \gamma z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'' \prec \vartheta(z)$$

and

$$(1-\mu)\frac{W^{\sigma,\delta}_{s,t,p,q}g(w)}{w} + \mu\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)' + \gamma w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'' \prec \vartheta(w),$$

where $g(w) = f^{-1}(w)$.

In particular, if we choose

$$\vartheta(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 < \alpha \leqq 1) \quad \text{and} \quad \vartheta(z) = \frac{1+(1-2\beta)}{1-z} \quad (0 \leqq \beta < 1),$$

the family $\mathcal{A}_E(\lambda, \sigma, \delta, s, t, p, q; \vartheta)$ reduces to the families $H_E(\lambda, \sigma, \delta, s, t, p, q; \alpha)$ and $H_E(\lambda, \sigma, \delta, s, t, p, q; \beta)$ which are families of the functions $f \in E$ satisfying

$$\left|\arg\left((1-\lambda)\frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}{W^{\sigma,\delta}_{s,t,p,q}f(z)} + \lambda\left(1 + \frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}\right)\right)\right| < \frac{\alpha\pi}{2},$$

$$\left|\arg\left((1-\lambda)\frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}{W^{\sigma,\delta}_{s,t,p,q}g(w)} + \lambda\left(1 + \frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}\right)\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\Re\left((1-\lambda)\frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}{W^{\sigma,\delta}_{s,t,p,q}f(z)} + \lambda\left(1 + \frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}\right)\right) > \beta,$$

$$\Re\left((1-\lambda)\frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}{W^{\sigma,\delta}_{s,t,p,q}g(w)} + \lambda\left(1 + \frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}\right)\right) > \beta,$$

respectively.

In addition, the family $\mathcal{S}_E(\mu, \gamma, \sigma, \delta, s, t, p, q; \vartheta)$ reduces to the families $T_E(\mu, \gamma, \sigma, \delta, s, t, p, q; \alpha)$ and $T_E(\mu, \gamma, \sigma, \delta, s, t, p, q; \beta)$ which are families of the functions $f \in E$ satisfying

$$\left|\arg\left((1-\mu)\frac{W^{\sigma,\delta}_{s,t,p,q}f(z)}{z} + \mu\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)' + \gamma z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''\right)\right| < \frac{\alpha\pi}{2},$$

$$\left|\arg\left((1-\mu)\frac{W^{\sigma,\delta}_{s,t,p,q}g(w)}{w} + \mu\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)' + \gamma w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\Re\left((1-\mu)\frac{W^{\sigma,\delta}_{s,t,p,q}f(z)}{z} + \mu\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)' + \gamma z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''\right) > \beta,$$

$$\Re\left((1-\mu)\frac{W^{\sigma,\delta}_{s,t,p,q}g(w)}{w} + \mu\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)' + \gamma w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''\right) > \beta,$$

respectively.

Remark 2. *The families $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ and $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ are a generalization of several known families studied in earlier investigations which are being recalled below.*

1. *For $\delta=0$ and $\vartheta(z)=\frac{a+(b-ap_1)rz}{1-p_1rz-q_1z^2}+1-a$, $r\in\mathbb{R}$, a,b,p_1 and q_1 are real constants, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $M_E(\lambda,r)$ which was studied by Abirami et al. [1];*
2. *For $\delta=0$, $\lambda=1$ and $\vartheta(z)=\frac{a+(b-ap_1)rz}{1-p_1rz-q_1z^2}+1-a$, $r\in\mathbb{R}$, a,b,p_1 and q_1 are real constants, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{K}_E(r)$ which was introduced by Abirami et al. [1];*
3. *For $\delta=\gamma=0$, $\mu=1$ and $\vartheta(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$, $0<\alpha\leqq 1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{H}_E(\alpha)$ which was investigated by Srivastava et al. [2];*
4. *For $\delta=\gamma=0$, $\mu=1$ and $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}$, $0\leqq\beta<1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{H}_E(\beta)$ which was defined by Srivastava et al. [2].*
5. *For $\delta=\gamma=0$ and $\vartheta(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$, $0<\alpha\leqq 1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,q;\vartheta)$ reduces to the family $\mathcal{B}_E(\alpha,\mu)$ which was considered by Frasin and Aouf [3];*
6. *For $\delta=\gamma=0$ and $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}$, $0\leqq\beta<1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{B}_E(\beta,\mu)$ which was studied by Frasin and Aouf [3];*
7. *For $\delta=0$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{M}_E(\lambda;\vartheta)$ which was introduced by Ali et al. [4];*
8. *For $\delta=\lambda=0$ and $\vartheta(z)=\frac{1}{1-2tz+z^2}$, $t\in(\frac{1}{2},1]$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $S_E^*(t)$ which was introduced by Bulut et al. [5];*
9. *For $\delta=\lambda=0$ and $\vartheta(z)=\frac{a+(b-ap_1)rz}{1-p_1rz-q_1z^2}+1-a$, $r\in\mathbb{R}$, a,b,p_1 and q_1 are real constants, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{W}_E(r)$ which was defined by Srivastava et al. [10];*
10. *For $\delta=0$ and $\vartheta(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$, $0<\alpha\leqq 1$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $M_E(\alpha,\lambda)$ which was considered by Liu and Wang [46];*
11. *For $\delta=0$ and $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}$, $0\leqq\beta<1$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $M_E(\lambda)$ which was studied by Liu and Wang [46];*
12. *For $\delta=\lambda=0$ and $\vartheta(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$, $0<\alpha\leqq 1$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $S_E^*(\alpha)$ which was considered by Brannan and Taha [47];*
13. *For $\delta=\lambda=0$ and $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}$, $0\leqq\beta<1$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $S_E^*(\beta)$ which was investigated by Brannan and Taha [47];*
14. *For $\delta=0$ and $\vartheta(z)=\frac{1}{1-2tz+z^2}$, $t\in(\frac{\sqrt{2}}{2},1]$, the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $H_E(\lambda,t)$ which was studied by Altınkaya and Yalçin [48];*
15. *For $\delta=0$, $\mu=1$ and $\vartheta(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$, $0<\alpha\leqq 1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{H}_E(\alpha,\gamma)$ which was considered by Frasin [49];*
16. *For $\delta=0$, $\mu=1$ and $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}$, $0\leqq\beta<1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{H}_E(\gamma,\beta)$ which was studied by Frasin [49];*
17. *For $\delta=0$ and $\vartheta(z)=\frac{1+(1-2\beta)z}{1-z}$, $0\leqq\beta<1$, the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ reduces to the family $\mathcal{N}_E(\beta,\mu,\gamma)$ which was defined by Bulut [50];*

Theorem 1. *Let f given by (1) be in the family $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$. Then,*

$$|b_2|\leqq\min\left\{\frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}},\right.$$
$$\left.\frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}B_1^{\frac{3}{2}}}{\sqrt{\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\Big((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\Big)\right|}}\right\}$$

and

$$|b_3| \leqq \min\left\{ \frac{B_1[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}} + \frac{B_2[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}}{2(2\lambda+1)[1+t]^{s}_{q}[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}[\Psi_3(\sigma,s,t)]^{\delta}_{p,q} - (3\lambda+1)[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}}, \right.$$
$$\left. \frac{B_1[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}} + \frac{B_1^2[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}}{(\lambda+1)^2[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}} \right\},$$

where the coefficients B_1 and B_2 are defined as in (5).

Proof. Let $f \in \mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ and $g = f^{-1}$. Then, there are holomorphic functions $S, T : U \longrightarrow U$ with $S(0) = T(0) = 0$, which fulfill the following conditions:

$$(1-\lambda)\frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}{W^{\sigma,\delta}_{s,t,p,q}f(z)} + \lambda\left(1 + \frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}\right) = \vartheta(S(z)), \quad z \in U \tag{6}$$

and

$$(1-\lambda)\frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}{W^{\sigma,\delta}_{s,t,p,q}g(w)} + \lambda\left(1 + \frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}\right) = \vartheta(T(w)), \quad w \in U. \tag{7}$$

Define the functions x and y by

$$x(z) = \frac{1+S(z)}{1-S(z)} = 1 + x_1 z + x_2 z^2 + \cdots$$

and

$$y(z) = \frac{1+T(z)}{1-T(z)} = 1 + y_1 z + y_2 z^2 + \cdots.$$

Then, x and y are analytic in U with $x(0) = y(0) = 1$. Since we have $S, T : U \longrightarrow U$, each of the functions x and y has a positive real part in U.

Solving for $S(z)$ and $T(z)$, we have

$$S(z) = \frac{x(z)-1}{x(z)+1} = \frac{1}{2}\left[x_1 z + \left(x_2 - \frac{x_1^2}{2}\right)z^2\right] + \cdots \quad (z \in U) \tag{8}$$

and

$$T(z) = \frac{y(z)-1}{y(z)+1} = \frac{1}{2}\left[y_1 z + \left(y_2 - \frac{y_1^2}{2}\right)z^2\right] + \cdots \quad (z \in U). \tag{9}$$

By substituting (8) and (9) into (6) and (7) and applying (5), we obtain

$$(1-\lambda)\frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}{W^{\sigma,\delta}_{s,t,p,q}f(z)} + \lambda\left(1 + \frac{z\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}f(z)\right)'}\right)$$
$$= \vartheta\left(\frac{x(z)-1}{x(z)+1}\right) = 1 + \frac{1}{2}B_1 x_1 z + \left[\frac{1}{2}B_1\left(x_2 - \frac{x_1^2}{2}\right) + \frac{1}{4}B_2 x_1^2\right]z^2 + \cdots \tag{10}$$

and

$$(1-\lambda)\frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}{W^{\sigma,\delta}_{s,t,p,q}g(w)}+\lambda\left(1+\frac{w\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)''}{\left(W^{\sigma,\delta}_{s,t,p,q}g(w)\right)'}\right)$$
$$=\vartheta\left(\frac{y(w)-1}{y(w)+1}\right)=1+\frac{1}{2}B_1y_1w+\left[\frac{1}{2}B_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}B_2y_1^2\right]w^2+\cdots \quad (11)$$

Equating the coefficients in (10) and (11), yields

$$\frac{(\lambda+1)[\Psi_2(\sigma,s,t)]^{\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}b_2=\frac{1}{2}B_1x_1, \quad (12)$$

$$\frac{2(2\lambda+1)[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}b_3-\frac{(3\lambda+1)[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}}b_2^2=\frac{1}{2}B_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}B_2x_1^2, \quad (13)$$

$$-\frac{(\lambda+1)[\Psi_2(\sigma,s,t)]^{\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}b_2=\frac{1}{2}B_1y_1 \quad (14)$$

and

$$\frac{2(2\lambda+1)[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}(2b_2^2-b_3)-\frac{(3\lambda+1)[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}}b_2^2=\frac{1}{2}B_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}B_2y_1^2. \quad (15)$$

From (12) and (14), we have

$$x_1=-y_1 \quad (16)$$

and

$$\frac{2(\lambda+1)^2[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}}b_2^2=\frac{1}{4}B_1^2(x_1^2+y_1^2). \quad (17)$$

If we add (13) to (15), we obtain

$$\frac{4(2\lambda+1)[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}b_2^2-\frac{2(3\lambda+1)[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}}{[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}}b_2^2$$
$$=\frac{1}{2}B_1\left[x_2+y_2-\left(\frac{x_1^2+y_1^2}{2}\right)\right]+\frac{1}{4}B_2[x_1^2+y_1^2] \quad (18)$$

Substituting the value of $x_1^2+y_1^2$ from (17) in the right-hand side of (18), we deduce that

$$b_2^2=\frac{B_1^3[\Psi_1(\sigma,s,t)]^{2\delta}_{p,q}(x_2+y_2)}{4\left[2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}+[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right]}. \quad (19)$$

Applying Lemma 1 for the coefficients x_1, x_2, y_1, y_2 in (17) and (19), we obtain

$$|b_2|\leqq\frac{[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}B_1^{\frac{3}{2}}}{\sqrt{\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}[\Psi_3(\sigma,s,t)]^{\delta}_{p,q}+[\Psi_2(\sigma,s,t)]^{2\delta}_{p,q}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|}},$$

$$|b_2|\leqq\frac{B_1[\Psi_1(\sigma,s,t)]^{\delta}_{p,q}}{(\lambda+1)[\Psi_2(\sigma,s,t)]^{\delta}_{p,q}},$$

which gives the estimates of $|b_2|$.

Furthermore, in order to find the bound on $|b_3|$, we subtract (15) from (13) and also apply (16). We obtain $x_1^2 = y_1^2$, hence,

$$\frac{4(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}(b_3 - b_2^2) = \frac{1}{2}B_1(x_2 - y_2), \tag{20}$$

then, by substituting the value of b_2^2 from (17) into (20), gives

$$b_3 = \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2 - y_2)}{8(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(x_1^2 + y_1^2)}{8(\lambda+1)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}.$$

So, we have

$$|b_3| \leqq \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(\lambda+1)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}.$$

In addition, substituting the value of b_2^2 from (18) into (20), we obtain

$$b_3 = \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2 - y_2)}{8(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{\left(B_1(x_2+y_2) + \frac{1}{2}(x_1^2+y_1^2)(B_2 - B_1)\right)[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{8(2\lambda+1)[1+t]_q^s[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} - 4(3\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}$$

and we have

$$|b_3| \leqq \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{B_2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{2(2\lambda+1)[1+t]_q^s[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} - (3\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}},$$

which gives us the desired estimates of the coefficient $|b_3|$. □

Taking $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leqq 1)$ in Theorem 1, we obtain the next corollary.

Corollary 1. *Let f given by (1) be in the family $H_E(\lambda,\sigma,\delta,s,t,p,q;\alpha)$, where $(0 < \alpha \leqq 1)$. Then,*

$$|b_2| \leqq \min\left\{\frac{2\alpha[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}, \frac{2\alpha\sqrt{2\alpha}[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{\sqrt{\left|4\alpha^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} + [\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left(2\alpha(1-\alpha)(\lambda+1)^2 - 4\alpha^2(3\lambda+1)\right)\right|}}\right\}$$

and

$$|b_3| \leqq \min\left\{\frac{\alpha[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{2\alpha^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{2(2\lambda+1)[1+t]_q^s[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} - (3\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}, \frac{\alpha[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{4\alpha^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(\lambda+1)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}\right\}.$$

Taking $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leqq \beta < 1)$ in Theorem 1, we obtain the next corollary.

Corollary 2. *Let the function f given by (1) be in the function family $H_E(\lambda,\sigma,\delta,s,t,p,q;\beta)$, where $(0 \leqq \beta < 1)$. Then,*

$$|b_2| \leqq \min\left\{\frac{2(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}, \frac{\sqrt{2(1-\beta)}[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{\sqrt{\left|2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} - [\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(3\lambda+1)\right|}}\right\}$$

and

$$|b_3| \leqq \min\left\{ \frac{(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{2(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{2(2\lambda+1)[1+t]_q^s[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} - (3\lambda+1)[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}, \right.$$
$$\left. \frac{(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{4(1-\beta)^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(\lambda+1)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}} \right\}.$$

Theorem 2. *Let f given by (1) be in the family $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$. Then,*

$$|b_2| \leqq \min\left\{ \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+\mu+2\gamma)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}, \right.$$
$$\left. \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta} B_1^{\frac{3}{2}}}{\sqrt{\left| B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} + [\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2) \right|}} \right\}$$

and

$$|b_3| \leqq \min\left\{ \frac{B_2[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}, \right.$$
$$\left. \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}} \right\},$$

where the coefficients B_1 and B_2 are defined as in (5).

Proof. Let $f \in \mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ and $g = f^{-1}$. Then, there are holomorphic functions $S, T : U \longrightarrow U$ such that

$$(1-\mu)\frac{W_{s,t,p,q}^{\sigma,\delta} f(z)}{z} + \mu\left(W_{s,t,p,q}^{\sigma,\delta} f(z)\right)' + \gamma z\left(W_{s,t,p,q}^{\sigma,\delta} f(z)\right)'' = \vartheta(S(z)), \quad z \in U \tag{21}$$

and

$$(1-\mu)\frac{W_{s,t,p,q}^{\sigma,\delta} g(w)}{w} + \mu\left(W_{s,t,p,q}^{\sigma,\delta} g(w)\right)' + \gamma w\left(W_{s,t,p,q}^{\sigma,\delta} g(w)\right)'' = \vartheta(T(w)), \quad w \in U, \tag{22}$$

where $S(z)$ and $T(z)$ have the forms (8) and (9). From (21), (22) and (5), we deduce that

$$(1-\mu)\frac{W_{s,t,p,q}^{\sigma,\delta} f(z)}{z} + \mu\left(W_{s,t,p,q}^{\sigma,\delta} f(z)\right)' + \gamma z\left(W_{s,t,p,q}^{\sigma,\delta} f(z)\right)''$$
$$= \vartheta\left(\frac{x(z)-1}{x(z)+1}\right) = 1 + \frac{1}{2}B_1 x_1 z + \left[\frac{1}{2}B_1\left(x_2 - \frac{x_1^2}{2}\right) + \frac{1}{4}B_2 x_1^2\right] z^2 + \cdots \tag{23}$$

and

$$(1-\mu)\frac{W_{s,t,p,q}^{\sigma,\delta} g(w)}{w} + \mu\left(W_{s,t,p,q}^{\sigma,\delta} g(w)\right)' + \gamma w\left(W_{s,t,p,q}^{\sigma,\delta\delta} g(w)\right)''$$
$$= \vartheta\left(\frac{y(w)-1}{y(w)+1}\right) = 1 + \frac{1}{2}B_1 y_1 w + \left[\frac{1}{2}B_1\left(y_2 - \frac{y_1^2}{2}\right) + \frac{1}{4}B_2 y_1^2\right] w^2 + \cdots \tag{24}$$

Equating the coefficients in (23) and (24), yields

$$\frac{(1+\mu+2\gamma)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}} b_2 = \frac{1}{2}B_1 x_1, \tag{25}$$

$$\frac{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}b_3=\frac{1}{2}B_1\left(x_2-\frac{x_1^2}{2}\right)+\frac{1}{4}B_2x_1^2, \tag{26}$$

$$-\frac{(1+\mu+2\gamma)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}b_2=\frac{1}{2}B_1y_1 \tag{27}$$

and

$$\frac{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}(2b_2^2-b_3)=\frac{1}{2}B_1\left(y_2-\frac{y_1^2}{2}\right)+\frac{1}{4}B_2y_1^2. \tag{28}$$

From (25) and (27), we have

$$x_1=-y_1 \tag{29}$$

and

$$\frac{2(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}b_2^2=\frac{1}{4}B_1^2(x_1^2+y_1^2). \tag{30}$$

If we add (26) to (28), we obtain

$$\frac{2(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}b_2^2=\frac{1}{2}B_1\left[x_2+y_2-\left(\frac{x_1^2+y_1^2}{2}\right)\right]+\frac{1}{4}B_2[x_1^2+y_1^2] \tag{31}$$

Substituting the value of $x_1^2+y_1^2$ from (30) in the right-hand side of (31), we deduce that

$$b_2^2=\frac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(x_2+y_2)}{4\left[B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right]}. \tag{32}$$

Applying Lemma 1 for the coefficients x_1, x_2, y_1, y_2 in (30) and (32), we obtain

$$|b_2|\leqq\frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}B_1^{\frac{3}{2}}}{\sqrt{\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}},$$

$$|b_2|\leqq\frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+\mu+2\gamma)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}},$$

which gives the estimates of $|b_2|$.

Furthermore, in order to find the bound of $|b_3|$, we subtract (28) from (26) and also apply (29). Then, we obtain $x_1^2=y_1^2$, and hence,

$$\frac{2(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}(b_3-b_2^2)=\frac{1}{2}B_1(x_2-y_2), \tag{33}$$

then, by substituting the value of b_2^2 from (30) into (33), gives

$$b_3=\frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2-y_2)}{4(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}+\frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(x_1^2+y_1^2)}{8(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}.$$

So, we have

$$|b_3|\leqq\frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}+\frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}$$

In addition, substituting the value of b_2^2 from (31) into (33), we obtain

$$b_3 = \frac{2B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}x_2 + \frac{1}{2}(B_2 - B_1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_1^2 + y_1^2)}{4(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}$$

and we have

$$|b_3| \leqq \frac{B_2[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}},$$

which gives us the desired estimates of the coefficient $|b_3|$. □

Taking $\vartheta(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leqq 1)$ in Theorem 2, we obtain the next corollary.

Corollary 3. *Let f given by (1) be in the family $T_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\alpha)$, where $(0 < \alpha \leqq 1)$. Then,*

$$|b_2| \leqq \min\left\{\frac{2\alpha[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+\mu+2\gamma)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}, \frac{2\alpha\sqrt{2\alpha}[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{\sqrt{\left|4\alpha^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta} + 2\alpha(1-\alpha)(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\right|}}\right\}$$

and

$$|b_3| \leqq \min\left\{\frac{2\alpha^2[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}, \frac{2\alpha[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{4\alpha^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}\right\}.$$

Taking $\vartheta(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leqq \beta < 1)$ in Theorem 2, we obtain the next corollary.

Corollary 4. *Let the function f given by (1) be in the function family $T_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\beta)$, where $(0 \leqq \beta < 1)$. Then,*

$$|b_2| \leqq \min\left\{\frac{2(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+\mu+2\gamma)[\Psi_2(\sigma,s,t)]_{p,q}^{\delta}}, \frac{\sqrt{2(1-\beta)}[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{\sqrt{\left|(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}\right|}}\right\}$$

and

$$|b_3| \leqq \min\left\{\frac{2(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}, \frac{2(1-\beta)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} + \frac{4(1-\beta)^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}}{(1+\mu+2\gamma)^2[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}}\right\}.$$

Remark 3. *The problem of maximizing the absolute value of the functional $|b_3 - \eta b_2^2|$ is called the Fekete–Szegö problem. Many authors obtained Fekete–Szegö inequalities for different classes of functions. Obtaining Fekete–Szegö inequalities for different classes of functions defined by operators, the study of bi-univalent functions using operators and the study on coefficients of the functions is a topic of interest at this time (see [1–5,7,10,11,14,46–51]).*

1. *In [1], the authors obtained Fekete–Szegö inequalities and coefficient inequalities for certain classes of bi-univalent functions defined by Horamad Polynomials;*
2. *In [8], the authors obtained Fekete–Szegö inequalities for classes of analytic and bi-univalent functions defined by (p, q)-derivative operator;*
3. *In [23], the authors obtained Fekete–Szegö inequalities for subclasses of analytic and bi-univalent functions defined by subordinations using the Sălăgean operator;*
4. *In [52], the author obtained Fekete–Szegö inequalities for analytic and bi-univalent functions subordinate to (p,q)-Lucas Polynomials;*
5. *In [53], the authors obtained Fekete–Szegö inequalities for analytic and bi-univalent functions subordinate to Gegenbauer polynomials;*
6. *In [54], the authors obtained Fekete–Szegö inequalities for analytic and bi-univalent functions subordinate to Cebyshev polynomials;*
7. *In [55], the authors obtained Fekete–Szegö inequalities and coefficients bounds for new classes of bi-univalent functions defined by the Sălăgean integro-differential operator;*
8. *In [56], the author obtained Fekete–Szegö inequalities for classes of bi-univalent functions defined in terms of subordinations.*

In the next theorems, we provide the Fekete–Szegö type inequalities for the functions of the families $\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ and $\mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$.

Theorem 3. *For $\eta\in\mathbb{R}$, let $f\in\mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ be of the form (1). Then,*

$$\left|b_3-\eta b_2^2\right|\leqq\begin{cases}\dfrac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}};\\ |\eta-1|\leqq\dfrac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|}{2B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(2\lambda+1)},\\ \dfrac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}|\eta-1|}{\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|};\\ |\eta-1|\geqq\dfrac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|}{2B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(2\lambda+1)}.\end{cases}$$

Proof. It follows from (19) and (20) that

$$\begin{aligned}b_3-\eta b_2^2&=\frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2-y_2)}{8(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}+(1-\eta)b_2^2\\&=\frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2-y_2)}{8(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}\\&+\frac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(x_2+y_2)(1-\eta)}{4\left[2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right]}\\&=\frac{B_1}{4}\left[\left(\Upsilon(\eta)+\frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}\right)x_2+\left(\Upsilon(\eta)-\frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}\right)y_2\right],\end{aligned}$$

where

$$\Upsilon(\eta)=\frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(1-\eta)}{2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)}.$$

According to Lemma 1 and (5), we find that

$$\left|b_3-\eta b_2^2\right|\leqq\begin{cases}\dfrac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}, & 0\leqq|\Upsilon(\eta)|\leqq\dfrac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}},\\ B_1|\Upsilon(\eta)|, & |\Upsilon(\eta)|\geqq\dfrac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}.\end{cases}$$

After some computations, we obtain

$$\left|b_3-\eta b_2^2\right| \leqq \begin{cases} \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}; \\ |\eta-1| \leqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|}{2B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(2\lambda+1)}, \\ \frac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}|\eta-1|}{\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|}; \\ |\eta-1| \geqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|2B_1^2(2\lambda+1)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}\left((\lambda+1)^2(B_1-B_2)-(3\lambda+1)B_1^2\right)\right|}{2B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(2\lambda+1)}. \end{cases}$$

□

Putting $\eta = 1$ in Theorem 3, we obtain the following result.

Corollary 5. *If $f \in \mathcal{A}_E(\lambda,\sigma,\delta,s,t,p,q;\vartheta)$ is of the form (1), then*

$$\left|b_3-b_2^2\right| \leqq \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{2(2\lambda+1)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}.$$

Theorem 4. *For $\eta \in \mathbb{R}$, let $f \in \mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ be of the form (1). Then,*

$$\left|b_3-\eta b_2^2\right| \leqq \begin{cases} \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}; \\ |\eta-1| \leqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(1+2\mu+6\gamma)}, \\ \frac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}|\eta-1|}{\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}; \\ |\eta-1| \geqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(1+2\mu+6\gamma)}. \end{cases}$$

Proof. It follows from (32) and (33) that

$$\begin{aligned} b_3-\eta b_2^2 &= \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2-y_2)}{4(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}+(1-\eta)b_2^2 \\ &= \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}(x_2-y_2)}{4(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}} \\ &+ \frac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(x_2+y_2)(1-\eta)}{4\left[B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right]} \\ &= \frac{B_1}{4}\left[\left(\Omega(\eta)+\frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}\right)x_2+\left(\Omega(\eta)-\frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}\right)y_2\right], \end{aligned}$$

where

$$\Omega(\eta) = \frac{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}(1-\eta)}{B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)}.$$

According to Lemma 1 and (5), we find that

$$\left|b_3-\eta b_2^2\right| \leqq \begin{cases} \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}, & 0 \leqq |\Omega(\eta)| \leqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}, \\ B_1|\Omega(\eta)|, & |\Omega(\eta)| \geqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}. \end{cases}$$

After some computations, we obtain

$$\left|b_3-\eta b_2^2\right| \leqq \begin{cases} \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}; \\ |\eta-1| \leqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(1+2\mu+6\gamma)}, \\ \frac{B_1^3[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}|\eta-1|}{\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}; \\ |\eta-1| \geqq \frac{[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}\left|B_1^2(1+2\mu+6\gamma)[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}+[\Psi_2(\sigma,s,t)]_{p,q}^{2\delta}(1+\mu+2\gamma)^2(B_1-B_2)\right|}{B_1^2[\Psi_1(\sigma,s,t)]_{p,q}^{2\delta}[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}(1+2\mu+6\gamma)}. \end{cases}$$

□

Putting $\eta = 1$ in Theorem 4, we obtain the following result.

Corollary 6. *If* $f \in \mathcal{S}_E(\mu,\gamma,\sigma,\delta,s,t,p,q;\vartheta)$ *is of the form* (1)*, then*

$$\left|b_3-b_2^2\right| \leqq \frac{B_1[\Psi_1(\sigma,s,t)]_{p,q}^{\delta}}{(1+2\mu+6\gamma)[\Psi_3(\sigma,s,t)]_{p,q}^{\delta}}.$$

3. Conclusions

As future research directions, the symmetry properties of this newly introduced operator can be studied.

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