symmetry

MDPI

Article

Separate Fractional (p,q)-Integrodifference Equations via Nonlocal Fractional (p,q)-Integral Boundary Conditions

Thongchai Dumrongpokaphan [1], Sotiris K. Ntouyas [2,3] and Thanin Sitthiwirattham [4,*]

[1] Research Group in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; thongchai.d@cmu.ac.th
[2] Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece; sntouyas@uoi.gr
[3] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[4] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
* Correspondence: thanin_sit@dusit.ac.th

Abstract: In this paper, we study a boundary value problem involving (p,q)-integrodifference equations, supplemented with nonlocal fractional (p,q)-integral boundary conditions with respect to asymmetric operators. First, we convert the given nonlinear problem into a fixed-point problem, by considering a linear variant of the problem at hand. Once the fixed-point operator is available, existence and uniqueness results are established using the classical Banach's and Schaefer's fixed-point theorems. The application of the main results is demonstrated by presenting numerical examples. Moreover, we study some properties of (p,q)-integral that are used in our study.

Keywords: fractional (p,q)-integral; fractional (p,q)-difference; nonlocal boundary value problems; existence

Citation: Dumrongpokaphan, T.; Ntouyas, S.K.; Sitthiwirattham, T. Separate Fractional (p,q)-Integrodifference Equations via Nonlocal Fractional (p,q)-Integral Boundary Conditions. *Symmetry* **2021**, *13*, 2212. https://doi.org/10.3390/sym13112212

Academic Editor: José Carlos R. Alcantud

Received: 26 October 2021
Accepted: 17 November 2021
Published: 19 November 2021

1. Introduction

Quantum calculus or q-calculus is the modern name for the study of calculus without limits. q-calculus was first introduced by Jackson [1,2] in 1910. Quantum calculus has many applications in mathematics and physics, for example, in orthogonal polynomials, combinatorics, number theory, simple hypergeometric functions, dynamics and theory of relativity, to name a few.

An extension of quantum calculus, the (p,q)-calculus or post-quantum calculus, was introduced by Chakrabarti and Jagannathan in [3]. (p,q)-calculus is a generalization of q calculus including two independent quantum parameters p and q, reduced to q-calculus for the case $p=1$ and to the classical q calculus when $q\to 1$. Furthermore, (p,q)-calculus has many applications, such as physical sciences, combinatorics, hypergeometric functions, number theory, mechanics, Bézier curves and surfaces, etc. (for instances, see [4–8]). Many researchers have recently begun working on (p,q)-calculus and some results can be found in [9–14] and references cited therein. Recently, in [15], the authors introduced the fractional (p,q)-difference operators and studied their properties.

Recently, (p,q)-calculus was applied to establish several new types of inequalities (see [16,17] and references cited therein). In the literature, there exist few papers studying boundary value problems for (p,q)-difference equations, because the (p,q)-fractional operator has been introduced recently. In [18], the following (p,q) boundary value problem for second order (p,q)-difference equations with separated boundary conditions was studied:

$$\begin{cases} D^2_{p,q}x(t) = f(t, x(p^2t)), \quad t \in [0, T/p^2], \\ \alpha_1 x(0) + \alpha_2 D_{p,q}x(0) = \alpha_3, \\ \beta_1 x(T) + \beta_2 D_{p,q}x(T/p) = \beta_3, \end{cases} \tag{1}$$

where $0 < q < p \leq 1$ are two quantum numbers, $D^2_{p,q}$ is the second order (p,q)-difference operator and $f \in C([0, T/p^2] \times \mathbb{R}, \mathbb{R})$, $T > 0$, α_i, β_i, $i = 1,2,3$ are given real constants. A variety of new existence and uniqueness results were established using Banach's, Schaefer's and Krasnoselskii's fixed-point theorems, as well as Leray–Schauder's nonlinear alternative.

Some more results on (p,q) boundary value problems can be found in [19–21]. For existence results for boundary value problems fractional (p,q)-difference Schrödinger equations, we refer to [22].

Recently, in [15], the authors introduced the fractional (p,q)-integrodifference operators and studied their properties. Boundary value problems for fractional (p,q)-integrodifference equations with Robin boundary conditions were studied in [23], where the authors established existence and uniqueness results for the following problem:

$$\begin{aligned} &D^\alpha_{p,q}u(t) = F\Big[t, u(t), \Psi^\gamma_{p,q}u(t), D^\nu_{p,q}u(t)\Big], \quad t \in I^T_{p,q}, \\ &\lambda_1 u(\eta) + \lambda_2 D^\beta_{p,q}u(\eta) = \phi_1(u), \quad \eta \in I^T_{p,q} - \left\{0, \frac{T}{p}\right\}, \\ &\mu_1 u\left(\frac{T}{p}\right) + \mu_2 D^\beta_{p,q}u\left(\frac{T}{p}\right) = \phi_2(u), \end{aligned} \tag{2}$$

where $I^T_{p,q} := \left\{\frac{q^k}{p^{k+1}}T : k \in \mathbb{N}_0\right\} \cup \{0\}$, $0 < q < p \leq 1$, $\alpha \in (1,2]$; $\beta, \gamma, \nu \in (0,1]$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$, $F \in C\big(I^T_{p,q} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ is given function, $\phi_1, \phi_2 : C\big(I^T_{p,q}, \mathbb{R}\big) \to \mathbb{R}$ are given functionals and

$$\Psi^\gamma_{p,q}u(t) := \Big(\mathcal{I}^\gamma_{p,q}\varphi\, u\Big)(t) = \frac{1}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)} \int_0^t (t - qs)^{\underline{\gamma-1}}_{p,q} \varphi(t,s)\, u\left(\frac{s}{p^{\gamma-1}}\right) d_{p,q}s, \tag{3}$$

are (p,q)-integral operators defined for $\varphi \in C\big(I^T_{p,q} \times I^T_{p,q}, [0,\infty)\big)$.

Motivated by the the aforementioned papers, our goal in this paper is to enrich the literature on boundary value problems for fractional (p,q)-integrodifference equations. More precisely, we introduce and study a nonlocal boundary value problem for (p,q)-integrodifference equations subject to fractional (p,q)-integral boundary conditions of the form

$$\begin{aligned} &D^\alpha_{p,q}u(t) = \lambda F\big[t, u(t), (\Psi^\gamma_{p,q}u)(t)\big] + \mu H\Big[t, u(t), \Big(\Upsilon^\nu_{p,q}u\Big)(t)\Big], \quad t \in I^T_{p,q}, \\ &\mathcal{I}^\beta_{p,q}g_1(\eta)u(\eta) = \phi_1(u), \quad \eta \in I^T_{p,q} - \left\{0, \frac{T}{p}\right\}, \\ &\mathcal{I}^\beta_{p,q}g_2\left(\frac{T}{p}\right)u\left(\frac{T}{p}\right) = \phi_2(u), \end{aligned} \tag{4}$$

where $I^T_{p,q} := \left\{\left(\frac{q}{p}\right)^k \frac{T}{p} : k \in \mathbb{N}_0\right\} \cup \{0\}$; $0 < q < p \leq 1$; $\alpha \in (1,2]$, $\beta, \gamma, \nu \in (0,1]$, $\lambda, \mu \in \mathbb{R}^+$, $F, H \in C\big(I^T_{p,q} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ and $g_1, g_2 \in C\big(I^T_{p,q}, \mathbb{R}^+\big)$ are given functions, $\phi_1, \phi_2 : C\big(I^T_{p,q}, \mathbb{R}\big) \to \mathbb{R}$ are given functionals and

$$(\Psi^\gamma_{p,q}u)(t) := \Big(\mathcal{I}^\gamma_{p,q}\varphi\, u\Big)(t) = \int_0^t \frac{(t - qs)^{\underline{\gamma-1}}_{p,q}}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)} \varphi\left(t, \frac{s}{p^{\gamma-1}}\right) u\left(\frac{s}{p^{\gamma-1}}\right) d_{p,q}s, \tag{5}$$

$$\left(\Upsilon^{\nu}_{p,q}u\right)(t) := \left(D^{\nu}_{p,q}\psi\, u\right)(t) = \int_0^t \frac{(t-qs)^{\underline{-\nu-1}}_{p,q}}{p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)}\psi\left(t, \frac{s}{p^{-\nu-1}}\right) u\left(\frac{s}{p^{-\nu-1}}\right) d_{p,q}s. \quad (6)$$

are operators defined for $\varphi, \psi \in C\big(I^T_{p,q} \times I^T_{p,q}, [0,\infty)\big)$.

We notice that the the boundary value problem (4) is of general type, concerning both (p,q)-fractional integral and (p,q)-fractional derivative operators. In addition, it contains nonlocal boundary conditions; it is well known that the study of nonlocal boundary value problems is of significance, since they have applications in physics and other areas of applied mathematics. We emphasize that the novelty of our paper lies in both the equation and the boundary conditions, contributing significantly to the existing literature on the topic. Our existence and uniqueness results rely on the standard tools of functional analysis. The methods used in our analysis are standard; however, their exposition in the framework of the boundary value problem (4) is new.

The remaining part of this manuscript is organized as follows. Section 2 contains some basic notions and known results of (p,q)-calculus. Furthermore, an auxiliary result is proved which plays a key role in transforming the given problem into a fixed-point problem. In Section 3, we prove the existence of a unique solution for the boundary value problem (4) via the Banach contraction mapping principle. An existence result is proved in Section 4, by using Schaefer's fixed-point theorem. Finally, examples illustrating the applicability of the main results are presented in Section 5. The papers ends with a section that illustrates the conclusions.

2. Preliminaries

In this section, some concepts regarding our study are recalled. Let $0 < q < p \le 1$. The following notations are used:

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q} := \begin{cases} \dfrac{p^k-q^k}{p-q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \displaystyle\prod_{i=1}^{k}\frac{p^i-q^i}{p-q}, & k \in \mathbb{N} \\ 1, & k = 0. \end{cases}$$

By

$$\sigma^k_{p,q}(t) := \left(\frac{q}{p}\right)^k t \quad \text{and} \quad \rho^k_{p,q}(t) := \left(\frac{p}{q}\right)^k t, \quad \text{for } k \in \mathbb{N},$$

we denote, respectively, the (p,q)-forward jump and the (p,q)-backward jump operators. For the power function $(a-b)^{\underline{n}}_q$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$, the q-analogue is given by

$$(a-b)^{\underline{0}}_q := 1, \qquad (a-b)^{\underline{n}}_q := \prod_{i=0}^{n-1}(a-bq^i), \qquad a,b \in \mathbb{R}.$$

and

$$(a-b)^{\underline{0}}_{p,q} := 1, \qquad (a-b)^{\underline{n}}_{p,q} := \prod_{k=0}^{n-1}(ap^k-bq^k), \qquad a,b \in \mathbb{R},$$

is the (p,q)-analogue of the power function $(a-b)^{\underline{n}}_{p,q}$, with $n \in \mathbb{N}_0$.

For $\alpha \in \mathbb{R}$, we define a general form

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^i}{1-\left(\frac{b}{a}\right)q^{\alpha+i}},\ a \neq 0.$$

$$(a-b)_{p,q}^{\underline{\alpha}} = p^{\binom{\alpha}{2}}(a-b)_{\frac{q}{p}}^{\underline{\alpha}} \ = a^\alpha \prod_{i=0}^{\infty} \frac{1}{p^\alpha}\left[\frac{1-\frac{b}{a}\left(\frac{q}{p}\right)^i}{1-\frac{b}{a}\left(\frac{q}{p}\right)^{i+\alpha}}\right],\ a \neq 0.$$

Note that $a_q^{\underline{\alpha}} = a^\alpha,\ a_{p,q}^{\underline{\alpha}} = \left(\dfrac{a}{p}\right)^\alpha$ and $(0)_q^{\underline{\alpha}} = (0)_{p,q}^{\underline{\alpha}} = 0$ for $\alpha > 0$.

The (p,q)-gamma and (p,q)-beta functions are defined by

$$\begin{aligned}
\Gamma_{p,q}(x) \quad &:= \quad \begin{cases} \dfrac{(p-q)_{p,q}^{\underline{x-1}}}{(p-q)^{x-1}} = \dfrac{\left(1-\frac{q}{p}\right)_{p,q}^{\underline{x-1}}}{\left(1-\frac{q}{p}\right)^{x-1}}, & x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\} \\ [x-1]_{p,q}!, & x \in \mathbb{N} \end{cases} \\
B_{p,q}(x,y) \quad &:= \quad \int_0^1 t^{x-1}(1-qt)_{p,q}^{\underline{y-1}} d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_q p,q(y)}{\Gamma_{p,q}(x+y)},
\end{aligned}$$

respectively.

Definition 1. *For $0 < q < p \leq 1$ and $f : [0,T] \to \mathbb{R}$, we define the (p,q)-difference of f as*

$$D_{p,q}f(t) \quad := \quad \begin{cases} \dfrac{f(pt)-f(qt)}{(p-q)(t)}, & \textit{for} \ \ t \neq 0 \\ f'(0), & \textit{for} \ \ t = 0 \end{cases}$$

if f is differentiable at 0. If $D_{p,q}f(t)$ exists for all $t \in I_{p,q}^T$, then f is called (p,q)-differentiable on $I_{p,q}^T$.

Definition 2. *Let us assume that I is a closed interval of $\mathbb{R}$ containing a, b and 0 and $f : I \to \mathbb{R}$ is a given function. The (p,q)-integral of the function f from a to b is defined by*

$$\int_a^b f(s)d_{p,q}s := \int_0^b f(s)d_{p,q}s - \int_0^a f(s)d_{p,q}s,$$

where

$$\mathcal{I}_{p,q}f(t) = \int_0^t f(s)d_{p,q}s = (p-q)t\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}t\right), \quad t \in I,$$

provided that the series converges at $t = a$ and $t = b$. If f is (p,q)-integrable on $[a,b]$ for all $a, b \in I$, then f is called (p,q)-integrable on $[a,b]$.

Next, we define an operator $\mathcal{I}_{p,q}^N$ as

$$\mathcal{I}_{p,q}^0 f(x) = f(x) \ \text{ and } \ \mathcal{I}_{p,q}^N f(x) = \mathcal{I}_{p,q}\mathcal{I}_{p,q}^{N-1}f(x), N \in \mathbb{N}.$$

The relations between (p,q)-difference and (p,q)-integral operators are given by

$$D_{p,q}\mathcal{I}_{p,q}f(x) = f(x) \ \text{ and } \ \mathcal{I}_{p,q}D_{p,q}f(x) = f(x) - f(0).$$

Next, we introduce the Riemann–Liouville type of fractional (p,q)-integral and fractional (p,q)-difference.

Definition 3. *For $\alpha > 0$, $0 < q < p \le 1$ and f defined on $I_{p,q}^T$, the fractional (p,q)-integral is defined by*

$$\begin{aligned} \mathcal{I}_{p,q}^{\alpha} f(t) &:= \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &= \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(t - \left(\frac{q}{p}\right)^{k+1} t\right)_{p,q}^{\underline{\alpha-1}} f\left(\frac{q^k}{p^{k+\alpha}} t\right), \end{aligned}$$

and $(\mathcal{I}_{p,q}^0 f)(t) = f(t)$, where the notation $\binom{\alpha}{2}$ is a combination.

Definition 4. *For $\alpha > 0$, $0 < q < p \le 1$ and f defined on $I_{p,q}^T$. The Riemann–Liouville type fractional (p,q)-difference operator of of order α is defined by*

$$\begin{aligned} D_{p,q}^{\alpha} f(t) &:= D_{p,q}^{N} \mathcal{I}_{p,q}^{N-\alpha} f(t) \\ &= \frac{1}{p^{\binom{-\alpha}{2}}\Gamma_{p,q}(-\alpha)} \int_0^t (t-qs)_{p,q}^{\underline{-\alpha-1}} f\left(\frac{s}{p^{-\alpha-1}}\right) d_{p,q}s, \end{aligned}$$

and $D_{p,q}^0 f(t) = f(t)$, where $N-1 < \alpha < N$, $N \in \mathbb{N}$.

Next, we introduce lemmas that are used in the main results.

Lemma 1 ([15])**.** *Let $\alpha \in (N-1, N)$, $N \in \mathbb{N}$, $0 < q < p \le 1$ and $f : I_{p,q}^T \to \mathbb{R}$. Then,*

$$\mathcal{I}_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) = f(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, N$.

Lemma 2 ([15])**.** *Let $0 < q < p \le 1$ and $f : I_{p,q}^T \to \mathbb{R}$ a continuous at 0 function. Then, we have*

$$\int_0^x \int_0^s f(\tau)\, d_{p,q}\tau\, d_{p,q}s = \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\, d_{p,q}s\, d_{p,q}\tau.$$

Lemma 3 ([15])**.** *Let $\alpha, \beta > 0$, $0 < q < p \le 1$. Then,*

$$\begin{aligned} (a) \quad & \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} s^{\beta}\, d_{p,q}s = t^{\alpha+\beta} B_{p,q}(\beta+1, \alpha), \\ (b) \quad & \int_0^t \int_0^x (t-qx)_{p,q}^{\underline{\alpha-1}} (x-qs)_{p,q}^{\underline{\beta-1}}\, d_{p,q}s\, d_{p,q}x = \frac{B_{p,q}(\beta+1,\alpha)}{[\beta]_{p,q}} t^{\alpha+\beta}. \end{aligned}$$

Lemma 4 ([23])**.** *Let $\alpha, \beta > 0$, $0 < q < p \le 1$ and $n \in \mathbb{Z}$. Then,*

$$\begin{aligned} (a) \quad & \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}}\, d_{p,q}s = p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)} t^{\alpha}, \\ (b) \quad & \int_0^t (t-qs)_{p,q}^{\underline{-\beta-1}} \left(\frac{s}{p^{-\beta-1}}\right)^{\alpha-n} d_{p,q}s = p^{\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha-n+1)\Gamma_{p,q}(-\beta)}{\Gamma_{p,q}(\alpha-\beta-n+1)} t^{\alpha-\beta-n}. \end{aligned}$$

We employ the above lemmas to obtain the new results as follows.

Lemma 5. *Let* $\alpha, \beta > 0$, $0 < q < p \leq 1$ *and* $n \in \mathbb{Z}$. *Then,*

$$\int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t-qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x \;=\; p^{\binom{\alpha}{2}+\binom{\beta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}{\Gamma_{p,q}(\alpha+\beta+1)}\, t^{\alpha+\beta}.$$

Proof. From Lemma 3 (a) and the definition of the (p,q)-beta function, we have

$$\begin{aligned}
\int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t-qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x &= \frac{B_{p,q}(1,\alpha)}{p^{\alpha(\beta-1)}} \int_0^t (t-qx)_{p,q}^{\underline{\beta-1}} x^{\alpha}\, d_{p,q}x \\
&= \frac{B_{p,q}(1,\alpha)}{p^{\alpha(\beta-1)}} B_{p,q}(\alpha+1,\beta)\, t^{\alpha+\beta} \\
&= p^{\binom{\alpha}{2}+\binom{\beta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}{\Gamma_{p,q}(\alpha+\beta+1)}\, t^{\alpha+\beta}.
\end{aligned}$$

The proof is complete. □

The following lemma, concerning a linear variant of problem (4), plays a significant role in the forthcoming analysis.

Lemma 6. *Let* $\Lambda \neq 0$, $\alpha \in (1,2]$, $\beta \in (0,1]$, $0 < q < p \leq 1$, $h \in C(I_{p,q}^T, \mathbb{R})$ *and* $g_1, g_2 \in C(I_{p,q}^T, \mathbb{R}^+)$ *be given functions and* $\phi_1, \phi_2 : C(I_{p,q}^T, \mathbb{R}) \to \mathbb{R}$ *be given functionals. Then, the boundary value problem*

$$D_{p,q}^{\alpha} u(t) = h(t), \quad t \in I_{p,q}^T, \tag{7}$$

$$\mathcal{I}_{p,q}^{\beta} g_1(\eta) u(\eta) = \phi_1(u), \quad \eta \in I_{p,q}^T - \left\{0, \frac{T}{p}\right\}, \tag{8}$$

$$\mathcal{I}_{p,q}^{\beta} g_2\left(\frac{T}{p}\right) u\left(\frac{T}{p}\right) = \phi_2(u), \tag{9}$$

has the unique solution

$$\begin{aligned}
u(t) \;=\; & \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\
& -\frac{t^{\alpha-1}}{\Lambda}\left\{ \boldsymbol{B}_T \mathcal{O}_\eta[\phi_1,h] - \boldsymbol{B}_\eta \mathcal{O}_T[\phi_2,h] \right\} \\
& +\frac{t^{\alpha-2}}{\Lambda}\left\{ \boldsymbol{A}_T \mathcal{O}_\eta[\phi_1,h] - \boldsymbol{A}_\eta \mathcal{O}_T[\phi_2,h] \right\}
\end{aligned} \tag{10}$$

where the functionals $\mathcal{O}_\eta[\phi_1,h], \mathcal{O}_T[\phi_2,h]$ *are defined by*

$$\begin{aligned}
\mathcal{O}_\eta[\phi_1,h] :=\, & \phi_1(u) - \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \times \\
& \int_0^\eta \int_0^{\frac{x}{p^{\beta-1}}} (\eta-qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} g_1\left(\frac{x}{p^{\beta-1}}\right) h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathcal{O}_T[\phi_2,h] :=\, & \phi_2(u) - \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \times \\
& \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p} - qx\right)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} g_2\left(\frac{x}{p^{\beta-1}}\right) h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x,
\end{aligned} \tag{12}$$

and the constants $\boldsymbol{A}_\eta, \boldsymbol{A}_T, \boldsymbol{B}_\eta, \boldsymbol{B}_T$ *and* Λ *are defined by*

$$\boldsymbol{A}_\eta := \frac{1}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} \int_0^\eta (\eta-qs)_{p,q}^{\underline{\beta-1}} g_1\left(\frac{s}{p^{\beta-1}}\right)\left(\frac{s}{p^{\beta-1}}\right)^{\alpha-1} d_{p,q}s \tag{13}$$

$$\boldsymbol{A}_T := \frac{1}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\underline{\beta-1}} g_2\left(\frac{s}{p^{\beta-1}}\right)\left(\frac{s}{p^{\beta-1}}\right)^{\alpha-1} d_{p,q}s \tag{14}$$

$$\boldsymbol{B}_\eta := \frac{1}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} \int_0^{\eta} (\eta - qs)_{p,q}^{\underline{\beta-1}} g_1\left(\frac{s}{p^{\beta-1}}\right)\left(\frac{s}{p^{\beta-1}}\right)^{\alpha-2} d_{p,q}s \tag{15}$$

$$\boldsymbol{B}_T := \frac{1}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\underline{\beta-1}} g_2\left(\frac{s}{p^{\beta-1}}\right)\left(\frac{s}{p^{\beta-1}}\right)^{\alpha-2} d_{p,q}s \tag{16}$$

$$\Lambda := \boldsymbol{A}_T\boldsymbol{B}_\eta - \boldsymbol{A}_\eta\boldsymbol{B}_T. \tag{17}$$

Proof. Taking fractional (p,q)-integral of order α for (7) and using Lemma 1, we have

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s. \tag{18}$$

Then, we take fractional (p,q)-integral of order β for (18); we have

$$\begin{aligned}\mathcal{I}_{p,q}^{\beta}u(t) = {} & \frac{C_1 t^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)} + \frac{C_2 t^{\alpha+\beta-2}}{\Gamma_{p,q}(\alpha+\beta-1)} + \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \times \\ & \int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t-qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x.\end{aligned} \tag{19}$$

Substituting $t = \eta$ into (19) and employing the condition (8), we have

$$C_1\mathbf{A}_\eta + C_2\mathbf{B}_\eta = \mathcal{O}_\eta[\phi_1, h]. \tag{20}$$

Taking $t = \dfrac{T}{p}$ into (19) and employing the condition (9), we have

$$C_1\mathbf{A}_T + C_2\mathbf{B}_T = \mathcal{O}_T[\phi_2, h]. \tag{21}$$

Solving (20) and (21), we find that

$$C_1 = \frac{\mathbf{B}_\eta\mathcal{O}_T - \mathbf{B}_T\mathcal{O}_\eta}{\Lambda} \quad \text{and} \quad C_2 = \frac{\mathbf{A}_T\mathcal{O}_\eta - \mathbf{A}_\eta\mathcal{O}_T}{\Lambda},$$

where $\mathcal{O}_\eta[\phi_1,h], \mathcal{O}_T[\phi_2,h], \mathbf{A}_\eta, \mathbf{A}_T, \mathbf{B}_\eta, \mathbf{B}_T$ and Λ are defined by (11)–(17), respectively. Substituting the constants C_1, C_2 into (18), we obtain (10). We can prove the converse by direct computation. This completes the proof. □

3. Existence and Uniqueness Result

In this section, an existence and uniqueness result for the problem (4) is proved, via the Banach contraction mapping principle. By $\mathcal{C} = C\left(I_{p,q}^T, \mathbb{R}\right)$, we denote the Banach space furnished with the norm

$$\|u\|_{\mathcal{C}} = \|u\| + \left\|D_{p,q}^{\nu}u\right\|,$$

where $\|u\| = \max\limits_{t\in I_{p,q}^T}\{|u(t)|\}$ and $\|D_{p,q}^{\nu}u\| = \max\limits_{t\in I_{p,q}^T}\left\{\left|D_{p,q}^{\nu}u(t)\right|\right\}$.

By Lemma 6, replacing $h(t)$ by $\lambda F[t, u(t), (\Psi_{p,q}^{\gamma}u)(t)] + \mu H\left[t, u(t), \left(\Upsilon_{p,q}^{\nu}u\right)(t)\right]$, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{A}u)(t) := \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{\underline{\alpha-1}} \left(\lambda F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), (\Psi_{p,q}^{\gamma}u)\left(\frac{s}{p^{\alpha-1}}\right)\right]\right.$$

$$
\begin{aligned}
&+\mu H\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Upsilon^{\nu}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]\Bigg)\,d_{p,q}s\\
&-\frac{t^{\alpha-1}}{\Lambda}\Big\{\mathbf{B}_T\mathcal{O}^*_\eta[\phi_1,F_u+H_u]-\mathbf{B}_\eta\mathcal{O}^*_T[\phi_2,F_u+H_u]\Big\}\\
&+\frac{t^{\alpha-2}}{\Lambda}\Big\{\mathbf{A}_T\mathcal{O}^*_\eta[\phi_1,F_u+H_u]-\mathbf{A}_\eta\mathcal{O}^*_T[\phi_2,F_u+H_u]\Big\}
\end{aligned}\tag{22}
$$

where the functionals $\mathcal{O}^*_\eta[\phi_1,F_u+H_u]$, $\mathcal{O}^*_T[\phi_2,F_u+H_u]$ are defined by

$$
\begin{aligned}
&\mathcal{O}^*_\eta[\phi_1,F_u+H_u]\\
&:=\phi_1(u)-\frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^\eta\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qx)^{\underline{\beta-1}}_{p,q}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q}\times\\
&g_1\left(\frac{x}{p^{\beta-1}}\right)\Bigg(\lambda F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Psi^{\gamma}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]\\
&+\mu H\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Upsilon^{\nu}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]\Bigg)\,d_{p,q}s\,d_{p,q}x,
\end{aligned}\tag{23}
$$

$$
\begin{aligned}
&\mathcal{O}^*_T[\phi_2,F_u+H_u]\\
&:=\phi_2(u)-\frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)^{\underline{\beta-1}}_{p,q}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q}\times\\
&g_2\left(\frac{x}{p^{\beta-1}}\right)\Bigg(\lambda F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Psi^{\gamma}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]\\
&+\mu H\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Upsilon^{\nu}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]\Bigg)\,d_{p,q}s\,d_{p,q}x,
\end{aligned}\tag{24}
$$

and the constants $\mathbf{A}_\eta,\mathbf{A}_T,\mathbf{B}_\eta,\mathbf{B}_T,\Lambda$ are defined by (13)–(17), respectively.

Notice that a fixed point of the operator $\mathcal{A}$ is a solution of the problem (4).

Theorem 1. *Let us assume that $F,H:I^T_{p,q}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous, $\varphi,\psi:I^T_{p,q}\times I^T_{p,q}\to[0,\infty)$ are continuous with $\varphi_0=\max\{\varphi(t,s):(t,s)\in I^T_{p,q}\times I^T_{p,q}\}$ and $\psi_0=\max\{\psi(t,s):(t,s)\in I^T_{p,q}\times I^T_{p,q}\}$. In addition, we suppose that:*

(H_1) *There exist constants $M_i>0$ such that, for each $t\in I^T_{p,q}$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$*

$$\big|F[t,u_1,u_2]-F[t,v_1,v_2]\big|\ \le\ M_1|u_1-v_1|+M_2|u_2-v_2|.$$

(H_2) *There exist constants $N_i>0$ such that, for each $t\in I^T_{p,q}$ and $u_i,v_i\in\mathbb{R}$, $i=1,2$*

$$\big|H[t,u_1,u_2]-H[t,v_1,v_2]\big|\ \le\ N_1|u_1-v_1|+N_2|u_2-v_2|.$$

(H_3) *There exist constants $\omega_1,\omega_2>0$ such that, for each $u,v\in\mathcal{C}$*

$$|\phi_1(u)-\phi_1(v)|\le\omega_1\|u-v\|_{\mathcal{C}}\quad\text{and}\quad|\phi_2(u)-\phi_2(v)|\le\omega_2\|u-v\|_{\mathcal{C}}.$$

(H_4) *For each $t\in I^T_{p,q}$, $g_1\le g_1(t)\le G_1$ and $g_2\le g_2(t)\le G_2$.*

(H_5) $\mathcal{X}:=\mathcal{L}\left[\Phi+\dfrac{G_1\eta^{\alpha+\beta}\Theta^*_T+G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\Theta^*_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right]+\omega_1\Theta^*_T+\omega_2\Theta^*_\eta\ <\ 1,$

where

$$\mathcal{L} := \lambda\left(M_1 + M_2\varphi_0 \frac{\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}\right) + \mu\left(N_1 + N_2\psi_0 \frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right), \tag{25}$$

$$\Phi := \frac{\left(\frac{T}{p}\right)^{\alpha}}{\Gamma_{p,q}(\alpha+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\gamma}}{\Gamma_{p,q}(\alpha+\gamma+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha-\nu}}{\Gamma_{p,q}(\alpha-\nu+1)}, \tag{26}$$

$$\Theta_T^* := \Theta_T + \bar{\Theta}_T, \tag{27}$$

$$\Theta_\eta^* := \Theta_\eta + \bar{\Theta}_\eta, \tag{28}$$

$$\Theta_T := \frac{1}{\min|\Lambda|}\left[\max|\boldsymbol{B}_T|\left(\frac{T}{p}\right)^{\alpha-1} + \max|\boldsymbol{A}_T|\left(\frac{T}{p}\right)^{\alpha-2}\right], \tag{29}$$

$$\bar{\Theta}_T := \frac{1}{\min|\Lambda|}\left[\max|\boldsymbol{B}_T|\frac{\left(\frac{T}{p}\right)^{\alpha-\nu-1}}{\Gamma_{p,q}(\alpha-\nu)} + \max|\boldsymbol{A}_T|\frac{\left(\frac{T}{p}\right)^{\alpha-\nu-2}}{\Gamma_{p,q}(\alpha-\nu-1)}\right], \tag{30}$$

$$\Theta_\eta := \frac{1}{\min|\Lambda|}\left[\max|\boldsymbol{B}_\eta|\left(\frac{T}{p}\right)^{\alpha-1} + \max|\boldsymbol{A}_\eta|\left(\frac{T}{p}\right)^{\alpha-2}\right], \tag{31}$$

$$\bar{\Theta}_\eta := \frac{1}{\min|\Lambda|}\left[\max|\boldsymbol{B}_\eta|\frac{\left(\frac{T}{p}\right)^{\alpha-\nu-1}}{\Gamma_{p,q}(\alpha-\nu)} + \max|\boldsymbol{A}_\eta|\frac{\left(\frac{T}{p}\right)^{\alpha-\nu-2}}{\Gamma_{p,q}(\alpha-\nu-1)}\right]. \tag{32}$$

Then, the boundary value problem (4) *has a unique solution on* $I_{p,q}^T$.

Proof. For each $t \in I_{p,q}^T$ and $u, v \in \mathcal{C}$, we have

$$\begin{aligned}
\left|(\Psi_{p,q}^{\gamma}u)(t) - (\Psi_{p,q}^{\gamma}v)(t)\right| &\le \frac{\varphi_0}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)}\int_0^t (t-qs)_{p,q}^{\underline{\gamma-1}}\left|u\left(\frac{s}{p^{\gamma-1}}\right) - v\left(\frac{s}{p^{\gamma-1}}\right)\right| d_{p,q}s. \\
&\le \frac{\varphi_0}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)}\|u-v\|\int_0^{\frac{T}{p}}\left(\frac{T}{p} - qs\right)_{p,q}^{\underline{\gamma-1}} d_{p,q}s. \\
&= \frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}\|u-v\|.
\end{aligned}$$

Similarly, we have $\left|(\Upsilon_{p,q}^{\nu}u)(t) - (\Upsilon_{p,q}^{\nu}v)(t)\right| \le \frac{\psi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\|u-v\|$.
We set

$$\begin{aligned}
\mathcal{F}|u-v|(t) &:= \left|F\left[t, u(t), (\Psi_{p,q}^{\gamma}u)(t)\right] - F\left[t, v(t), (\Psi_{p,q}^{\gamma}v)(t)\right]\right|, \\
\mathcal{H}|u-v|(t) &:= \left|H\left[t, u(t), (\Upsilon_{p,q}^{\nu}u)(t)\right] - H\left[t, v(t), (\Upsilon_{p,q}^{\nu}v)(t)\right]\right|.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left|\mathcal{O}_\eta^*[\phi_1, F_u + H_u] - \mathcal{O}_\eta^*[\phi_1, F_v + H_v]\right| \\
\le\ &|\phi_1(u) - \phi_1(v)| + \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\eta}\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qx)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} \\
&g_1\left(\frac{x}{p^{\beta-1}}\right)\left[\lambda\mathcal{F}|u-v|\left(\frac{s}{p^{\alpha-1}}\right) + \mu\mathcal{H}|u-v|\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s\, d_{p,q}x
\end{aligned}$$

$$\begin{aligned}
&\leq \quad \omega_1\|u-v\|_{\mathcal{C}} + \left(\lambda\left[M_1 + M_2\varphi_0\frac{\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}\right] + \mu\left[N_1 + N_2\psi_0\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right]\right) \times \\
&\qquad \frac{G_1\eta^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\|u-v\|_{\mathcal{C}} \\
&\leq \quad \left[\omega_1 + \frac{\mathcal{L}G_1\eta^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\right]\|u-v\|_{\mathcal{C}}.
\end{aligned}$$

Similarly, we obtain

$$\left|\mathcal{O}_T^*[\phi_1, F_u + H_u] - \mathcal{O}_T^*[\phi_1, F_v + H_v]\right| \leq \left[\omega_2 + \frac{\mathcal{L}G_2\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\right]\|u-v\|_{\mathcal{C}},$$

and

$$\begin{aligned}
&\left|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)\right| \\
\leq \quad & \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^{\frac{T}{p}}\left(\frac{T}{p} - qs\right)_{p,q}^{\alpha-1}\left[\lambda\mathcal{F}|u-v|\left(\frac{s}{p^{\alpha-1}}\right) + \mu\mathcal{H}|u-v|\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s \\
& + \frac{\left(\frac{T}{p}\right)^{\alpha-1}}{|\Lambda|}\Big\{|\mathbf{B}_T|\left|\mathcal{O}_\eta^*[\phi_1, F_u + H_u] - \mathcal{O}_\eta^*[\phi_1, F_v + H_v]\right| \\
& + |\mathbf{B}_\eta|\left|\mathcal{O}_T^*[\phi_2, F_u + H_u] - \mathcal{O}_T^*[\phi_2, F_v + H_v]\right|\Big\} \\
& + \frac{\left(\frac{T}{p}\right)^{\alpha-2}}{|\Lambda|}\Big\{|\mathbf{A}_T|\left|\mathcal{O}_\eta^*[\phi_1, F_u + H_u] - \mathcal{O}_\eta^*[\phi_1, F_v + H_v]\right| \\
& + |\mathbf{A}_\eta|\left|\mathcal{O}_T^*[\phi_2, F_u + H_u] - \mathcal{O}_T^*[\phi_2, F_v + H_v]\right|\Big\} \\
\leq \quad & \left[\mathcal{L}\left(\frac{\left(\frac{T}{p}\right)^{\alpha}}{\Gamma_{p,q}(\alpha+1)} + \frac{G_1\eta^{\alpha+\beta}\Theta_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\Theta_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right) + \omega_1\Theta_T + \omega_2\Theta_\eta\right]\|u-v\|_{\mathcal{C}}. \qquad (33)
\end{aligned}$$

Next, we consider $(D_{p,q}^{\nu}\mathcal{A}u)$. We have

$$\begin{aligned}
&(D_{p,q}^{\nu}\mathcal{A}u)(t) \\
&= \frac{1}{p^{\binom{\alpha}{2}+\binom{-\nu}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(-\nu)}\int_0^t\int_0^{\frac{x}{p^{-\nu-1}}}(t-qx)_{p,q}^{\underline{-\nu-1}}\left(\frac{x}{p^{-\nu-1}} - qs\right)_{p,q}^{\alpha-1} \times \qquad (34) \\
&\quad \left(\lambda F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), (\Psi_{p,q}^{\gamma}u)\left(\frac{s}{p^{\alpha-1}}\right)\right] + \mu H\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right),\right.\right. \\
&\quad \left.\left.\left(\Upsilon_{p,q}^{\nu}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]\right) d_{p,q}s\, d_{p,q}x - \frac{\left\{\mathbf{B}_T\mathcal{O}_\eta^*[\phi_1, F_u + H_u] - \mathbf{B}_\eta\mathcal{O}_T^*[\phi_2, F_u + H_u]\right\}}{\Lambda\, p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)} \times \\
&\quad \int_0^t (t-qs)_{p,q}^{\underline{-\nu-1}}\left(\frac{s}{p^{-\nu-1}}\right)^{\alpha-1} d_{p,q}s + \frac{\left\{\mathbf{A}_T\mathcal{O}_\eta^*[\phi_1, F_u + H_u] - \mathbf{A}_\eta\mathcal{O}_T^*[\phi_2, F_u + H_u]\right\}}{\Lambda\, p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)} \times \\
&\quad \int_0^t (t-qs)_{p,q}^{\underline{-\nu-1}}\left(\frac{s}{p^{-\nu-1}}\right)^{\alpha-2} d_{p,q}s.
\end{aligned}$$

Similarly as above, we have

$$\begin{aligned}
&\left|(D_{p,q}^{\nu}\mathcal{F}u)(t) - (D_{p,q}^{\nu}\mathcal{F}v)(t)\right| \\
&\leq \left[\mathcal{L}\left(\frac{\left(\frac{T}{p}\right)^{\alpha-\nu}}{\Gamma_{p,q}(\alpha-\nu+1)} + \frac{G_1\eta^{\alpha+\beta}\bar{\Theta}_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\bar{\Theta}_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right) + \omega_1\bar{\Theta}_T + \omega_2\bar{\Theta}_\eta\right]\|u-v\|_{\mathcal{C}}. \qquad (35)
\end{aligned}$$

From (33) and (35), we obtain

$$\|\mathcal{F}u - \mathcal{F}v\|_{\mathcal{C}} \leq \mathcal{X}\|u - v\|_{\mathcal{C}}.$$

Thus, by (H_5) the operator $\mathcal{A}$ is a contraction. By the Banach contraction mapping principle we deduce that $\mathcal{A}$ has a fixed point which is the unique solution of problem (4) on $I_{p,q}^{T}$. The proof is finished. □

4. An Existence Result

In this section, we present an existence result for the boundary value problem (4) by using the Schaefer's fixed-point theorem [24].

Theorem 2. *Let us assume that $F, H : I_{p,q}^{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\varphi_1, \varphi_2 : C(I_{p,q}^{T}, \mathbb{R}) \to \mathbb{R}$ are given functionals. Let us suppose that the following conditions hold:*

(H_6) *There exist positive constants F, H such that, for each $t \in I_{p,q}^{T}$ and $u_i \in \mathbb{R}$, $i = 1, 2$,*

$$\Big|F[t, u_1, u_2]\Big| \leq F \ \text{ and } \ \Big|H[t, u_1, u_2]\Big| \leq H.$$

(H_7) *There exist positive constants O_1, O_2 such, that for each $u \in \mathcal{C}$,*

$$|\varphi_1(u)| \leq O_1 \ \text{ and } \ |\varphi_2(u)| \leq O_2.$$

Then, the boundary problem (4) has at least one solution on $I_{p,q}^{T}$.

Proof. We need to show that the operator $\mathcal{A}$ is compact by applying the well-known Arzelá–Ascoli theorem. So, we show that the operator $\mathcal{A}(B_R)$ is a uniformly bounded set, where $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq R, \ R > 0\}$ and an equicontinuous set.

(i) For each $t \in I_{p,q}^{T}$ and $u \in B_R$, we have

$$\begin{aligned}
&\Big|\mathcal{O}_{\eta}^{*}[\phi_1, F_u + H_u]\Big| \\
&\leq \ O_1 + \frac{[\lambda F + \mu H]}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \int_0^{\eta} \int_0^{\frac{x}{p^{\beta-1}}} (\eta - qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} g_1\left(\frac{x}{p^{\beta-1}}\right) d_{p,q}s \, d_{p,q}x \\
&\leq \ O_1 + \frac{G_1\eta^{\alpha+\beta}[\lambda F + \mu H]}{\Gamma_{p,q}(\alpha+\beta+1)}.
\end{aligned} \tag{36}$$

Similarly, we have

$$\Big|\mathcal{O}_{T}^{*}[\phi_2, F_u + H_u]\Big| \leq O_2 + \frac{G_2\left(\frac{T}{p}\right)^{\alpha+\beta}[\lambda F + \mu H]}{\Gamma_{p,q}(\alpha+\beta+1)}. \tag{37}$$

From (36) and (37) and for each $t \in I_{p,q}^{T}$, we find that

$$\begin{aligned}
&|(\mathcal{A}u)(t)| \\
&\leq \ \frac{[\lambda F + \mu H]}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s \\
&\quad + \frac{\left(\frac{T}{p}\right)^{\alpha-1}}{|\Lambda|}\left\{|\mathbf{B}_T|\Big|\mathcal{O}_{\eta}^{*}[\phi_1, F_u + H_u]\Big| + |\mathbf{B}_{\eta}|\Big|\mathcal{O}_{T}^{*}[\phi_2, F_u + H_u]\Big|\right\} \\
&\quad + \frac{\left(\frac{T}{p}\right)^{\alpha-2}}{|\Lambda|}\left\{|\mathbf{A}_T|\Big|\mathcal{O}_{\eta}^{*}[\phi_1, F_u + H_u]\Big| + |\mathbf{A}_{\eta}|\Big|\mathcal{O}_{T}^{*}[\phi_2, F_u + H_u]\Big|\right\} \\
&\leq \ [\lambda F + \mu H]\left[\frac{\left(\frac{T}{p}\right)^{\alpha}}{\Gamma_{p,q}(\alpha+1)} + \frac{G_1\eta^{\alpha+\beta}\Theta_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\Theta_{\eta}}{\Gamma_{p,q}(\alpha+\beta+1)}\right] + O_1\Theta_T + O_2\Theta_{\eta}.
\end{aligned} \tag{38}$$

In addition, we obtain

$$\left|\left(D^{\nu}_{p,q}\mathcal{A}u\right)(t)\right| \le [\lambda F + \mu H]\left[\frac{\left(\frac{T}{p}\right)^{\alpha-\nu}}{\Gamma_{p,q}(\alpha-\nu+1)} + \frac{G_1\eta^{\alpha+\beta}\bar{\Theta}_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\bar{\Theta}_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right] + O_1\bar{\Theta}_T + O_2\bar{\Theta}_\eta. \tag{39}$$

Form (38) and (39) we obtain

$$\|\mathcal{A}u\|_{\mathcal{C}} \le [\lambda F + \mu H]\left[\Phi + \frac{G_1\eta^{\alpha+\beta}\Theta^*_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\Theta^*_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right] + O_1\Theta^*_T + O_2\Theta^*_\eta < \infty,$$

which implies that $\mathcal{A}(B_R)$ is uniformly bounded.

(ii) We show that $\mathcal{A}(B_R)$ is equicontinuous. For any $t_1, t_2 \in I^T_{p,q}$ with $t_1 < t_2$, we have

$$\begin{aligned}\left|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)\right| \le\ & \frac{[\lambda F + \mu H]}{\Gamma_{p,q}(\alpha+1)}\left|t_2^{\alpha} - t_1^{\alpha}\right| \\ & + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Lambda|}\Big\{|\mathbf{B}_T|\mathcal{O}^*_\eta[\phi_1, F_u + H_u] + |\mathbf{B}_\eta|\mathcal{O}^*_T[\phi_2, F_u + H_u]\Big\} \\ & + \frac{\left|t_2^{\alpha-2} - t_1^{\alpha-2}\right|}{|\Lambda|}\Big\{|\mathbf{A}_T|\mathcal{O}^*_\eta[\phi_1, F_u + H_u] + |\mathbf{A}_\eta|\mathcal{O}^*_T[\phi_2, F_u + H_u]\Big\},\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\left|(D^{\nu}_{p,q}\mathcal{A}u)(t_2) - (D^{\nu}_{p,q}\mathcal{A}u)(t_1)\right| \\ &\le \frac{[\lambda F + \mu H]}{\Gamma_{p,q}(\alpha-\nu+1)}\left|t_2^{\alpha-\nu} - t_1^{\alpha-\nu}\right| \\ &+ \frac{\Gamma_{p,q}(\alpha)\left|t_2^{\alpha-\nu-1} - t_1^{\alpha-\nu-1}\right|}{|\Lambda|\Gamma_{p,q}(\alpha-\nu)}\Big\{|\mathbf{B}_T|\mathcal{O}^*_\eta[\phi_1, F_u + H_u] + |\mathbf{B}_\eta|\mathcal{O}^*_T[\phi_2, F_u + H_u]\Big\} \\ &+ \frac{\Gamma_{p,q}(\alpha-1)\left|t_2^{\alpha-\nu-2} - t_1^{\alpha-\nu-2}\right|}{|\Lambda|\Gamma_{p,q}(\alpha-\nu-1)}\Big\{|\mathbf{A}_T|\mathcal{O}^*_\eta[\phi_1, F_u + H_u] + |\mathbf{A}_\eta|\mathcal{O}^*_T[\phi_2, F_u + H_u]\Big\}.\end{aligned} \tag{41}$$

The right-hand side of (40) and (41) tends to zero as $t_1 \to t_2$, independently of u, which implies that $\mathcal{A}(B_R)$ is an equicontinuous set. By using the Arzelá–Ascoli theorem, the set $\mathcal{A}(B_R)$ is compact.

(iii) Finally, we show that $\mathcal{W} = \{u \in \mathcal{C} :\ u = \xi\mathcal{A}u, 0 < \xi < 1\}$ is a bounded set. Let $u \in \mathcal{W}$. Then, as in (i), we have

$$\begin{aligned}|u(t)| &\le \xi\|\mathcal{A}x\|_{\mathcal{C}} \\ &\le [\lambda F + \mu H]\left[\Phi + \frac{G_1\eta^{\alpha+\beta}\Theta^*_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\Theta^*_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right] + O_1\Theta^*_T + O_2\Theta^*_\eta,\end{aligned} \tag{42}$$

which yields

$$\|\mathcal{A}u\|_{\mathcal{C}} \le [\lambda F + \mu H]\left[\Phi + \frac{G_1\eta^{\alpha+\beta}\Theta^*_T + G_2\left(\frac{T}{p}\right)^{\alpha+\beta}\Theta^*_\eta}{\Gamma_{p,q}(\alpha+\beta+1)}\right] + O_1\Theta^*_T + O_2\Theta^*_\eta.$$

Therefore, $\mathcal{W}$ is bounded.

Hence, by Schaefer's fixed-point theorem, we deduce that the operator $\mathcal{A}$ has a fixed point, which is a solution of boundary value problem (4). The proof is finished. □

5. Examples

Example 1. *Let us consider the fractional (p,q)-integrodifference equation*

$$\begin{aligned} D^{\frac{4}{3}}_{\frac{2}{3},\frac{1}{2}} u(t) &= \left[\frac{e^{-[\cos^2(2\pi t)+\pi]}}{200+e^{\sin^2(2\pi t)}}\right] \frac{|u(t)| + e^{-(10t+\pi)}|\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}} u(t)|}{1+|u(t)|} \\ &\quad + \left[\frac{e^{-[\sin^2(2\pi t)+10]}}{(t+20)^2}\right] \frac{|u(t)| + e^{-(5t+\frac{\pi}{2})}|\Upsilon^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}} u(t)|}{1+|u(t)|}, \ t \in I^{10}_{\frac{2}{3},\frac{1}{2}} \end{aligned} \tag{43}$$

subject to fractional (p,q)-integral boundary condition

$$\begin{aligned} \mathcal{I}^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}} \left(e + \cos\frac{1215}{256}\right)^2 u\left(\frac{1215}{256}\right) &= \sum_{i=0}^{\infty} \frac{C_i|u(t_i)|}{1+|u(t_i)|}, \ \ t_i = \sigma^i_{\frac{2}{3},\frac{1}{2}}(10), \\ \mathcal{I}^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}} (\pi + \sin 15)^2 u(15) &= \sum_{i=0}^{\infty} \frac{D_i|u(t_i)|}{1+|u(t_i)|}, \ \ t_i = \sigma^i_{\frac{2}{3},\frac{1}{2}}(10), \end{aligned} \tag{44}$$

where $\varphi(t,s) = \dfrac{e^{-|t-s|}}{(2t+\pi)^3}$, $\psi(t,s) = \dfrac{e^{-|t-s|}}{(t+e)^3}$ *and* C_i, D_i *are given constants, with* $\dfrac{1}{1000} \le \sum_{i=0}^{\infty} C_i \le \dfrac{e}{1000}$ *and* $\dfrac{1}{500} \le \sum_{i=0}^{\infty} D_i \le \dfrac{\pi}{500}$.

Here, $p = \frac{2}{3}$, $q = \frac{1}{2}$, $\alpha = \frac{4}{3}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{3}$, $\nu = \frac{1}{4}$, $T = 10$, $\eta = \sigma^4_{\frac{2}{3},\frac{1}{2}}(10) = \frac{1215}{256}$, $\lambda = e^{-\pi}$, $\mu = e^{-10}$, $\phi_1(u) = \sum_{i=0}^{\infty} \dfrac{C_i|u(t_i)|}{1+|u(t_i)|}$, $\phi_2 = \sum_{i=0}^{\infty} \dfrac{D_i|u(t_i)|}{1+|u(t_i)|}$, $g_1(t) = (e+\cos t)^2$, $g_2(t) = (\pi + \sin t)^2$,

$$F\big[t, u(t), \Psi^{\gamma}_{p,q} u(t)\big] = \left[\frac{e^{-\cos^2(2\pi t)}}{200+e^{\sin^2(2\pi t)}}\right] \frac{|u(t)| + e^{-(10t+\pi)}|\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}} u(t)|}{1+|u(t)|}$$

and

$$H\Big[t, u(t), \Upsilon^{\nu}_{p,q} u(t)\Big] = \left[\frac{e^{-\sin^2(2\pi t)}}{(t+20)^2}\right] \frac{|u(t)| + e^{-(5t+\frac{\pi}{2})}|\Upsilon^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}} u(t)|}{1+|u(t)|}.$$

For all $t \in I^{10}_{\frac{2}{3},\frac{1}{2}}$ *and* $u, v \in \mathbb{R}$, *we have*

$$\begin{aligned} \big|F\big[t,u,\Psi^{\gamma}_{p,q}u\big] - F\big[t,v,\Psi^{\gamma}_{p,q}v\big]\big| &\le \frac{1}{201}|u-v| + \frac{1}{201e^{\pi}}\big|\Psi^{\gamma}_{p,q}u - \Psi^{\gamma}_{p,q}v\big|, \\ \Big|H\Big[t,u,\Upsilon^{\nu}_{p,q}u\Big] - F\Big[t,v,\Upsilon^{\nu}_{p,q}v\Big]\Big| &\le \frac{1}{400}|u-v| + \frac{1}{400e^{\frac{\pi}{2}}}\Big|\Upsilon^{\nu}_{p,q}u - \Upsilon^{\nu}_{p,q}v\Big|. \end{aligned}$$

Thus, (H_1) *and* (H_2) *hold with* $M_1 = 0.004975$, $M_2 = 0.0002145$, $N_1 = 0.0025$ *and* $N_2 = 0.0005197$.

For all $u, v \in \mathcal{C}$,

$$\begin{aligned} |\phi_1(u) - \phi_1(v)| &\le \frac{e}{1000}\|u-v\|_{\mathcal{C}}, \\ |\phi_2(u) - \phi_2(v)| &\le \frac{\pi}{500}\|u-v\|_{\mathcal{C}}. \end{aligned}$$

So, (H_3) *holds with* $\omega_1 = 0.00272$ *and* $\omega_2 = 0.00628$.

Moreover, (H_4) *holds with* $g_1 = 2.9525$, $G_1 = 13.8256$, $g_2 = 4.5864$ *and* $G_2 = 17.1528$. *After calculating, we find that*

$$|\boldsymbol{A}_\eta| \leq 20.9307, \quad |\boldsymbol{A}_T| \approx 32.7815, \quad |\boldsymbol{B}_\eta| \approx 39.839, \quad |\boldsymbol{B}_T| \approx 62.3957,$$
$$|\Lambda| \geq 410.0706, \quad \varphi_0 = 0.0323 \text{ and } \psi_0 = 0.04979.$$

We can show that

$$\mathcal{L} \approx 0.0002159, \quad \Theta_T \approx 0.38884, \quad \Theta_\eta \approx 0.24799, \quad \bar{\Theta}_T \approx 0.19883, \quad \bar{\Theta}_\eta \approx 0.12504,$$
$$\Theta_T^* \approx 1.5769 \text{ and } \Theta_\eta^* \approx 1.0066.$$

So, (H_5) *holds with*

$$\mathcal{X} \approx 0.64592 < 1.$$

Hence, by Theorem 1, the boundary value problem (43) and (44) has a unique solution on $I^{10}_{\frac{2}{3},\frac{1}{2}}$.

Example 2. *Let us consider the fractional* (p,q)*-integrodifference equation*

$$\begin{aligned} D^{\frac{4}{3}}_{\frac{2}{3},\frac{1}{2}} u(t) &= \frac{1}{10}\left(t+\frac{1}{3}\right)e^{-(t+\pi)\left[u(t)+\left|\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}} u(t)\right|\right]} \\ &\quad + \frac{1}{15}\left(t+\frac{2}{5}\right)e^{-(t+10)\left[u(t)+\left|\Upsilon^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}} u(t)\right|\right]}, \ t \in I^{10}_{\frac{2}{3},\frac{1}{2}} \end{aligned} \tag{45}$$

with fractional (p,q)*-integral boundary condition*

$$\begin{aligned} \mathcal{I}^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}}\left(e+\cos\frac{1215}{256}\right)^2 u\left(\frac{1215}{256}\right) &= \sum_{i=0}^{\infty} C_i e^{-|u(t_i)|}, \ t_i = \sigma^i_{\frac{2}{3},\frac{1}{2}}(10), \\ \mathcal{I}^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}}(\pi+\sin 15)^2 u(15) &= \sum_{i=0}^{\infty} D_i e^{-|u(t_i)|}, \ t_i = \sigma^i_{\frac{2}{3},\frac{1}{2}}(10), \end{aligned} \tag{46}$$

where C_i *and* D_i *are given constants with* $\frac{1}{1000} \leq \sum_{i=0}^{\infty} C_i \leq \frac{\pi}{1000}$ *and* $\frac{1}{500} \leq \sum_{i=0}^{\infty} D_i \leq \frac{e}{500}$.

Here, $p = \frac{2}{3}$, $q = \frac{1}{2}$, $\alpha = \frac{4}{3}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{3}$, $\nu = \frac{1}{4}$, $T = 10$, $\eta = \sigma^4_{\frac{2}{3},\frac{1}{2}}(10) = \frac{1215}{256}$, $\lambda = e^{-\pi}$, $\mu = e^{-10}$.

It is clear that $\left|F\left[t,u,\Psi^{\gamma}_{p,q}u\right]\right| \leq \frac{23}{15} = F$, $\left|H\left[t,u,\Upsilon^{\nu}_{p,q}u\right]\right| \leq \frac{77}{75} = H$ *for* $t \in I^{10}_{\frac{2}{3},\frac{1}{2}}$ *and* $|\varphi_1(u)| \leq \frac{\pi}{1000} = O_1$, $|\varphi_2(u)| \leq \frac{e}{500} = O_2$ *for* $u \in \mathcal{C}$.

Hence, (H_5) *and* (H_6) *hold. Therefore, the boundary value problem (45) and (46) has at least one solution on* $I^{10}_{\frac{2}{3},\frac{1}{2}}$ *by Theorem 2.*

6. Conclusions

A nonlocal fractional (p,q)-integral boundary value problem for separate fractional (p,q)-integrodifference Equation (4) is studied. Our problem contains two fractional (p,q)-difference operators and two fractional (p,q)-integral operators. The existence of a unique solution is established via the Banach contraction mapping principle, while the existence result is proved using the Schaefer's fixed-point theorem. In addition, some properties of the (p,q)-integral are also studied. It is imperative to mention that our results are new in the given configuration and enrich the literature on boundary value problems involving (p,q)-integrodifference equations. In the future, we plan to extend this work by considering new boundary value problems.

Author Contributions: Conceptualization, T.D.; methodology, T.D., S.K.N. and T.S.; formal analysis, T.D., S.K.N. and T.S.; funding acquisition, T.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research study was funded by the Science, Research and Innovation Promotion Fund under Basic Research Plan—Saun Dusit University.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This research study was supported by Chiang Mai University.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jackson, F.H. On q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
2. Jackson, F.H. On q-difference integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
3. Chakrabarti, R.; Jagannathan, R. A (p, q)-oscillator realization of two-parameter quantum algebras. *J. Math. Phys. Math. Gen.* **1991**, *24*, 5683–5701. [CrossRef]
4. Mursaleen, M.; Ansari, K.J.; Khan, A. Some approximation results by (p, q)-analogue of Bernstein-Stancu operators. *Appl. Math. Comput.* **2015**, *264*, 392–402; Corrigendum in *Appl. Math. Comput.* **2015**, *269*, 744–746. [CrossRef]
5. Mursaleen, M.; Nasiruzzaman, M.; Khan, A.; Ansari, K.J. Some approximation results on Bleimann-Butzer-Hahn operators defined by (p, q)-integers. *Filomat* **2016**, *30*, 639–648. [CrossRef]
6. Mursaleen, M.; Khan, F.; Khan, A. Approximation by (p, q)-Lorentz polynomials on a compact disk. *Complex Anal. Oper. Theory* **2016**, *10*, 1725–1740. [CrossRef]
7. Rahman, S.; Mursaleen, M.; Alkhaldi, A.H. Convergence of iterates of q-Bernstein and (p, q)-Bernstein operators and the Kelisky-Rivlin type theorem. *Filomat* **2018**, *32*, 4351–4364. [CrossRef]
8. Khan, K.; Lobiyal, D.K. Bézier curves based on Lupaș (p, q)-analogue of Bernstein functions in CAGD. *J. Comput. Appl. Math.* **2017**, *317*, 458–477. [CrossRef]
9. Jagannathan, R.; Rao, K.S. Two-parameter quantum algebras, twin-basic number, and associated generalized hypergeometric series. *arXiv* **2006**, arXiv:math/0602613.
10. Sahai, V.; Yadav, S. Representations of two parameter quantum algebras and (p, q)-special functions. *J. Math. Anal. Appl.* **2007**, *335*, 268–279. [CrossRef]
11. Sadjang, P.N. On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. *arXiv* **2018**, arXiv:1309.3934.
12. Duran, U. Post Quantum Calculus. Master's Thesis, University of Gaziantep, Şehitkamil/Gaziantep, Turkey, 2016.
13. Milovanovic, G.V.; Gupta, V.; Malik, N. (p, q)-Beta functions and applications in approximation. *Bol. Soc. Mat. Mex.* **2018**, *24*, 219–237. [CrossRef]
14. Cheng, W.T.; Zhang, W.H.; Cai, Q.B. (p, q)-gamma operators which preserve x^2. *J. Inequal. Appl.* **2019**, *2019*, 108. [CrossRef]
15. Soontharanon, J.; Sitthiwirattham, T. On fractional (p, q)-Calculus. *Adv. Differ. Equ.* **2020**, *2020*, 35. [CrossRef]
16. Pheak, N.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Fractional (p, q)-calculus on finite intervals and some Integral inequalities. *Symmetry* **2021**, *3*, 504.
17. Pheak, N.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Praveen Agarwal, Some trapezoid and midpoint type inequalities via fractional (p, q)-calculus. *Adv. Differ. Equ.* **2021**, *2021*, 333.
18. Promsakon, C.; Kamsrisuk, N.; Ntouyas, S.K.; Tariboon, J. On the second-order (p, q)-difference equations with separated boundary conditions. *Adv. Math. Phys.* **2018**, *2018*, 9089865. [CrossRef]
19. Kamsrisuk, N.; Promsakon, C.; Ntouyas, S.K.; Tariboon, J. Nonlocal boundary value problems for (p, q)-difference equations. *Differ. Equ. Appl.* **2018**, *10*, 183–195. [CrossRef]
20. Nuntigrangjana, T.; Putjuso, S.; Ntouyas, S.K.; Tariboon, J. Impulsive quantum (p, q)-difference equations. *Adv. Differ. Equ.* **2020**, *2020*, 98. [CrossRef]
21. Qin, Z.; Sun, S. Positive solutions for fractional (p, q)-difference boundary value problems. *J. Appl. Math. Comput.* **2021**, 1–8. [CrossRef]
22. Qin, Z.; Sun, S. On a nonlinear fractional (p, q)-difference Schrödinger equation. *J. Appl. Math. Comput.* **2021**. [CrossRef]
23. Soontharanon, J.; Sitthiwirattham, T. Existence results of nonlocal Robin boundary value problems for fractional (p, q)-integrodifference equations. *Adv. Differ. Equ.* **2020**, *2020*, 342. [CrossRef]
24. Smart, D.R. *Fixed Point Theorems*; Cambridge University Press: Cambridge, UK, 1980.