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A New Formula for Calculating Uncertainty Distribution of Function of Uncertain Variables

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Abstract: As a mathematical tool to rationally handle degrees of belief in human beings, uncertainty theory has been widely applied in the research and development of various domains, including science and engineering. As a fundamental part of uncertainty theory, uncertainty distribution is the key approach in the characterization of an uncertain variable. This paper shows a new formula to calculate the uncertainty distribution of strictly monotone function of uncertain variables, which breaks the habitual thinking that only the former formula can be used. In particular, the new formula is symmetrical to the former formula, which shows that when it is too intricate to deal with a problem using the former formula, the problem can be observed from another perspective by using the new formula. New ideas may be obtained from the combination of uncertainty theory and symmetry.

Keywords: uncertainty theory; uncertainty distribution; uncertain variable

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1. Introduction

In practice, the estimated distribution function is usually not close enough to the real frequency. According to Liu [1], in this case, if we applied probability theory to modeling degrees of belief, it would lead to counterintuitive results. To overcome this problem, we apply uncertainty theory, which was established by Liu [2] in 2007 and completed by Liu [3] in 2009. Nowadays, uncertainty theory has been successfully employed in various fields of science and has spawned numerous theoretical branches.

One of the most essential concepts in uncertainty theory is uncertain variables, which were defined by Liu [2] in 2007. It is used to represent indeterminate quantities, such as stock price, market demand, and product lifetime. Furthermore, in order to characterize uncertain variables, Liu [2] presented the concept of uncertainty distribution, and Liu [4] developed the definition of inverse uncertainty distribution. In addition, some operational laws were proposed by Liu [4] to calculate the uncertainty distribution and inverse uncertainty distribution of strictly monotone functions of independent uncertain variables. Meanwhile, for the purpose of ranking uncertain variables, Liu [2] gave the definition of the expected value of uncertain variables. Based on the expected value operator, Liu [2] proposed variance, distance and moments between uncertain variables. After that, Yao [5] presented a formula for calculating the variance of an uncertain variable. Until now, substantial work has been done grounded in uncertain variables, such as Chen and Dai [6], Ma, Yang and Yao [7], Zhang [8] , etc.

As a fundamental part of uncertainty theory, uncertainty distribution is the key approach in the characterization of uncertain variables. In fact, numerous research studies show that it is sufficient to know the uncertainty distribution rather than the uncertain variable itself. Therefore, uncertainty distribution is crucial to the development of uncertainty theory, and many scholars have made significant progress in it. For instance, Peng and Iwamura [9] proved a sufficient and necessary condition of a function's uncertainty distribution. Liu and Lio [10] presented a review of sufficient and necessary condition of uncertainty distribution. They thoroughly solved the problem of what an uncertainty distributioin is.

This paper aims to provide a new formula for calculating the uncertainty distribution of strictly monotone function of independent uncertain variables. The rest of the paper is organized as follows: Section 2 shows some foundational definitions in uncertainty theory. Section 3 presents a new formula and some examples. Finally, Section 4 provides a concise conclusion.

2. Preliminaries

The key to uncertainty theory is the uncertain measure defined by Liu [2] and Liu [3] with the normality axiom, duality axiom, subadditivity axiom, and product axiom. Among them, the product axiom is the most essential difference between uncertainty theory and probability theory. Furthermore, according to Liu [2], an uncertain variable is a measurable function from an uncertainty space to the set of real numbers, and the uncertainty distribution Θ of an uncertain variable κ is defined by

$$\Theta(z) = \mathrm{M}\{\kappa \le z\}, \qquad \forall z \in \Re.$$

Moreover, Liu [4] declared that the inverse uncertainty distribution is the inverse function of uncertainty distribution. Furthermore, Liu [3] proclaimed that the uncertain variables $\kappa_1, \kappa_2, \cdots, \kappa_n$ are said to be independent if

$$\mathrm{M}\left\{\bigcap_{i=1}^{n}(\kappa_i \in A_i)\right\} = \bigwedge_{i=1}^{n} \mathrm{M}\{\kappa_i \in A_i\}$$

for any Borel sets $A_1, A_2, \cdots, A_n$ of real numbers.

Finally, some theorems are given as follows.

Theorem 1 (Liu [4])**.** *Assume that $\kappa_1, \kappa_2, \cdots, \kappa_n$ are independent uncertain variables with regular uncertainty distributions $\Theta_1, \Theta_2, \cdots, \Theta_n$, respectively. If $h(z_1, z_2, \cdots, z_n)$ is continuous, strictly increasing with respect to $z_1, z_2, \cdots, z_m$ and strictly decreasing with respect to $z_{m+1}, z_{m+2}, \cdots, z_n$, then $\kappa = h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Upsilon^{-1}(\rho) = h(\Theta_1^{-1}(\rho), \cdots, \Theta_m^{-1}(\rho), \Theta_{m+1}^{-1}(1-\rho), \cdots, \Theta_n^{-1}(1-\rho)).$$

Theorem 2 (Liu [4])**.** *Assume $\kappa_1, \kappa_2, \cdots$, and$\kappa_n$ are independent uncertain variables with regular uncertainty distributions $\Theta_1, \Theta_2, \cdots, \Theta_n$, respectively. If $h(z_1, z_2, \cdots, z_n)$ is continuous, strictly increasing with respect to $z_1, z_2, \cdots, z_m$ and strictly decreasing with respect to $z_{m+1}, z_{m+2}, \cdots, z_n$, then $\kappa = h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ is an uncertain variable with an uncertainty distribution*

$$\Upsilon(z) = \sup_{h(z_1, z_2, \cdots, z_n) = z} \left(\min_{1 \le i \le m} \Theta_i(z_i) \wedge \min_{m+1 \le i \le n} (1 - \Theta_i(z_i)) \right).$$

3. A New Formula for Calculating Uncertainty Distribution

In this section, we obtain a new formula for calculating the uncertainty distribution of strictly monotone function of independent uncertain variables, and three examples are given for illustrating the formula. The formula is obtained in a symmetrical form as follows:

Theorem 3. *Assume $\kappa_1, \kappa_2, \cdots, \kappa_n$ are independent uncertain variables with regular uncertainty distributions $\Theta_1, \Theta_2, \cdots, \Theta_n$, respectively. If $h(z_1, z_2, \cdots, z_n)$ is continuous, strictly increasing with respect to $z_1, z_2, \cdots, z_m$ and strictly decreasing with respect to $z_{m+1}, z_{m+2}, \cdots, z_n$, then $\kappa = h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ is an uncertain variable with an uncertainty distribution*

$$\Upsilon(z) = \inf_{h(z_1,z_2,\cdots,z_n)=z} \left(\max_{1\le i\le m} \Theta_i(z_i) \vee \max_{m+1\le i\le n} (1-\Theta_i(z_i)) \right).$$

Proof. Without loss of generality, let us prove the case of $m = 1$ and $n = 2$. It follows from Theorem 1 that $\kappa = h(\kappa_1, \kappa_2)$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(\rho) = h(\Theta_1^{-1}(\rho), \Theta_2^{-1}(1-\rho)).$$

Write

$$z = \Upsilon^{-1}(\rho).$$

Then

$$\Upsilon(z) = \rho.$$

On the one hand, for any z_1 and z_2 with $h(z_1, z_2) = z$, since h is strictly increasing with respect to z_1 and strictly decreasing with respect to z_2, and $h(z_1, z_2) = z = h(\Theta_1^{-1}(\rho), \Theta_2^{-1}(1-\rho))$, we have

$$z_1 \ge \Theta_1^{-1}(\rho) \ \text{ or } \ z_2 \le \Theta_2^{-1}(1-\rho).$$

That is,

$$\rho \le \Theta_1(z_1) \ \text{ or } \ \rho \le 1 - \Theta_2(z_2).$$

Thus

$$\Upsilon(z) = \rho \le \Theta_1(z_1) \vee (1 - \Theta_2(z_2)).$$

By the arbitrariness of z_1 and z_2 with $h(z_1, z_2) = z$, we obtain

$$\Upsilon(z) \le \inf_{h(z_1,z_2)=z} (\Theta_1(z_1) \vee (1 - \Theta_2(z_2)). \tag{1}$$

On the other hand, take

$$z_1' = \Theta_1^{-1}(\rho), \ \ z_2' = \Theta_2^{-1}(1-\rho),$$

i.e.,

$$h(z_1', z_2') = h(\Theta_1^{-1}(\rho), \Theta_2^{-1}(1-\rho)) = \Upsilon^{-1}(\rho) = z,$$
$$\Upsilon(z) = \rho = \rho \vee \rho = \Theta_1(z_1') \vee (1 - \Theta_2(z_2')).$$

Thus

$$\Upsilon(z) \ge \inf_{h(z_1,z_2)=z} \Theta_1(z_1) \vee (1 - \Theta_2(z_2)). \tag{2}$$

It follows from (1) and (2) that

$$\Upsilon(z) = \inf_{h(z_1,z_2)=z} \Theta_1(z_1) \vee (1 - \Theta_2(z_2)).$$

That is, $h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ has an uncertainty distribution

$$\Upsilon(z) = \inf_{h(z_1,z_2,\cdots,z_n)=z} \left(\max_{1\le i\le m} \Theta_i(z_i) \vee \max_{m+1\le i\le n} (1-\Theta_i(z_i)) \right).$$

The theorem is verified.

Furthermore, it follows from Theorem 2 that $h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ has an uncertainty distribution

$$\Upsilon(z) = \sup_{h(z_1,z_2,\cdots,z_n)=z} \left(\min_{1\le i\le m} \Theta_i(z_i) \wedge \min_{m+1\le i\le n} (1-\Theta_i(z_i)) \right).$$

Thus

$$\begin{aligned}\Upsilon(z) &= \inf_{h(z_1,z_2,\cdots,z_n)=z}\left(\max_{1\le i\le m}\Theta_i(z_i)\vee\max_{m+1\le i\le n}(1-\Theta_i(z_i))\right)\\ &= \sup_{h(z_1,z_2,\cdots,z_n)=z}\left(\min_{1\le i\le m}\Theta_i(z_i)\wedge\min_{m+1\le i\le n}(1-\Theta_i(z_i))\right).\end{aligned}$$

□

Remark 1. *If the equation $h(z_1,z_2,\cdots,z_n)=z$ does not have a root for some values of z, then we set*

$$\Upsilon(z)=\begin{cases}0, & \text{if } h(z_1,z_2,\cdots,z_n)>z\\ 1, & \text{if } h(z_1,z_2,\cdots,z_n)<z.\end{cases}$$

Example 1. *Assume $\kappa_1,\kappa_2,\cdots,\kappa_n$ are independent uncertain variables with regular uncertainty distributions $\Theta_1,\Theta_2,\cdots,\Theta_n$, respectively. It follows from Theorem 3 that*

$$\kappa=\kappa_1+\kappa_2+\cdots+\kappa_m-\kappa_{m+1}-\kappa_{m+2}-\cdots-\kappa_n$$

has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{z_1+\cdots+z_m-z_{m+1}-\cdots-z_n=z}\left(\max_{1\le i\le m}\Theta_i(z_i)\wedge\max_{m+1\le i\le n}(1-\Theta_i(z_i))\right)\\ &= \sup_{z_1+\cdots+z_m-z_{m+1}-\cdots-z_n=z}\left(\min_{1\le i\le m}\Theta_i(z_i)\vee\min_{m+1\le i\le n}(1-\Theta_i(z_i))\right).\end{aligned}$$

In particular, if $m=1$ and $n=2$, then $\kappa_1-\kappa_2$ has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{q\in\Re}\Theta_1(z+q)\vee(1-\Theta_2(q))\\ &= \sup_{q\in\Re}\Theta_1(z+q)\wedge(1-\Theta_2(q)).\end{aligned}$$

if $m=n=2$, then $\kappa_1+\kappa_2$ has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{q\in\Re}\Theta_1(z-q)\vee\Theta_2(q).\\ &= \sup_{q\in\Re}\Theta_1(z-q)\wedge\Theta_2(q).\end{aligned}$$

Example 2. *Assume $\kappa_1,\kappa_2,\cdots,\kappa_n$ are independent positive uncertain variables with regular uncertainty distributions $\Theta_1,\Theta_2,\cdots,\Theta_n$, respectively. It follows from Theorem 3 that*

$$\kappa=\frac{\kappa_1\kappa_2\cdots\kappa_m}{\kappa_{m+1}\kappa_{m+2}\cdots\kappa_n}$$

has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{\frac{z_1z_2\cdots z_m}{z_{m+1}z_{m+2}\cdots z_n}=z}\left(\max_{1\le i\le m}\Theta_i(z_i)\vee\max_{m+1\le i\le n}(1-\Theta_i(z_i))\right)\\ &= \sup_{\frac{z_1z_2\cdots z_m}{z_{m+1}z_{m+2}\cdots z_n}=z}\left(\min_{1\le i\le m}\Theta_i(z_i)\wedge\min_{m+1\le i\le n}(1-\Theta_i(z_i))\right).\end{aligned}$$

In particular, if $m = 1$ and $n = 2$, then κ_1/κ_2 has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{q>0} \Theta_1(zq) \vee (1 - \Theta_2(q)) \\ &= \sup_{q>0} \Theta_1(zq) \wedge (1 - \Theta_2(q)).\end{aligned}$$

If $m = n = 2$, then $\kappa_1 * \kappa_2$ has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{q>0} \Theta_1(z/q) \vee \Theta_2(q) \\ &= \sup_{q>0} \Theta_1(z/q) \wedge \Theta_2(q).\end{aligned}$$

Theorem 4. *Assume $\kappa_1, \kappa_2, \cdots, \kappa_n$ are independent uncertain variables with regular uncertainty distributions $\Theta_1, \Theta_2, \cdots, \Theta_n$, respectively. Let $h(z_1, z_2, \cdots, z_n)$ be continuous, strictly increasing with respect to $z_1, z_2, \cdots, z_m$ and strictly decreasing with respect to $z_{m+1}, z_{m+2}, \cdots, z_n$, and $\kappa = h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ be an uncertain variable with an uncertainty distribution Υ. For any real number z, $z_1, z_2, \cdots, z_n$ satisfying $h(z_1, z_2, \cdots, z_n) = z$, the equations*

$$\max_{1 \le i \le m} \Theta_i(z_i) \vee \max_{m+1 \le i \le n} (1 - \Theta_i(z_i))$$

and

$$\min_{1 \le i \le m} \Theta_i(z_i) \wedge \min_{m+1 \le i \le n} (1 - \Theta_i(z_i))$$

have an intersection. In particular, if

$$0 < \Upsilon(z) < 1,$$

then the intersection is unique. Assume the intersection is $(z_1^*, z_2^*, \cdots, z_n^*)$. Then

$$\begin{aligned}\Upsilon(z) &= \Theta_1(z_1^*) = \Theta_2(z_2^*) = \cdots = \Theta_m(z_m^*) \\ &= 1 - \Theta_{m+1}(z_{m+1}^*) = 1 - \Theta_{m+2}(z_{m+2}^*) = \cdots = 1 - \Theta_n(z_n^*).\end{aligned} \tag{3}$$

Proof. It follows from Theorem 1 that $\kappa = h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(\rho) = h(\Theta_1^{-1}(\rho), \cdots, \Theta_m^{-1}(\rho), \Theta_{m+1}^{-1}(1-\rho), \cdots, \Theta_n^{-1}(1-\rho)).$$

Write

$$z = \Upsilon^{-1}(\rho).$$

Then

$$\rho = \Upsilon(z), \ \ h(\Theta_1^{-1}(\rho), \cdots, \Theta_m^{-1}(\rho), \Theta_{m+1}^{-1}(1-\rho), \cdots, \Theta_n^{-1}(1-\rho)) = z.$$

Take

$$z_i^* = \Theta_i^{-1}(\rho), \ \ 1 \le i \le m,$$

$$z_j^* = \Theta_j^{-1}(1-\rho), \ \ m+1 \le j \le n,$$

i.e.,

$$h(z_1^*, z_2^*, \cdots, z_n^*) = h(\Theta_1^{-1}(\rho), \cdots, \Theta_m^{-1}(\rho), \Theta_{m+1}^{-1}(1-\rho), \cdots, \Theta_n^{-1}(1-\rho)) = z,$$

$$\max_{1 \le i \le m} \Theta_i(z_i^*) \vee \max_{m+1 \le i \le n} (1 - \Theta_i(z_i^*)) = \rho \vee \rho = \rho,$$

$$\min_{1 \le i \le m} \Theta_i(z_i^*) \wedge \min_{m+1 \le i \le n} (1 - \Theta_i(z_i^*)) = \rho \wedge \rho = \rho.$$

Thus, $(z_1^*, z_2^*, \cdots, z_n^*)$ is an intersection. If there is another intersection $(y_1, y_2, \cdots, y_n)$, then it follows from Theorem 3 that $h(\kappa_1, \kappa_2, \cdots, \kappa_n)$ has an uncertainty distribution

$$\begin{aligned}\Upsilon(z) &= \inf_{h(z_1,z_2,\cdots,z_n)=z} \left(\max_{1\le i\le m} \Theta_i(z_i) \vee \max_{m+1\le i\le n} (1-\Theta_i(z_i)) \right) \\ &= \sup_{h(z_1,z_2,\cdots,z_n)=z} \left(\min_{1\le i\le m} \Theta_i(z_i) \wedge \min_{m+1\le i\le n} (1-\Theta_i(z_i)) \right).\end{aligned}$$

And

$$0 < \rho = \Upsilon(z) < 1.$$

Take

$$\beta = \max_{1\le i\le m} \Theta_i(y_i) \vee \max_{m+1\le i\le n} (1-\Theta_i(y_i)) = \min_{1\le i\le m} \Theta_i(y_i) \wedge \min_{m+1\le i\le n} (1-\Theta_i(y_i)).$$

That is,

$$\beta = \Theta_1(y_1) = \cdots = \Theta_m(y_m) = 1 - \Theta_{m+1}(y_{m+1}) = \cdots = 1 - \Theta_n(y_n).$$

If $\beta > \rho$, then

$$y_i > z_i^*, \ \ 1 \le i \le m,$$
$$y_j < z_j^*, \ \ m+1 \le j \le n.$$

Thus

$$h(y_1, y_2, \cdots, y_n) > h(z_1, z_2, \cdots, z_n) = z. \tag{4}$$

It is in contradiction with $h(y_1, y_2, \cdots, y_n) = z$. Similarly, if $\beta < \rho$, then

$$y_i < z_i^*, \ \ 1 \le i \le m,$$
$$y_j > z_j^*, \ \ m+1 \le j \le n.$$

Thus,

$$h(y_1, y_2, \cdots, y_n) < h(z_1, z_2, \cdots, z_n) = z. \tag{5}$$

It is in contradiction with $h(y_1, y_2, \cdots, y_n) = z$. That is, the intersection is unique. The theorem is verified. □

Corollary 1. *Assume $\kappa_1, \kappa_2, \cdots, \kappa_n$ are iid uncertain variables with a common regular uncertainty distribution Θ. The uncertain variable*

$$\kappa = \kappa_1 + \kappa_2 + \cdots + \kappa_m - \kappa_{m+1} - \kappa_{m+2} - \cdots - \kappa_n$$

has an uncertainty distribution

$$\Upsilon(z) = \begin{cases} \Theta\left(\dfrac{z}{n}\right), & \text{if } m = n \\ 1 - \Theta\left(-\dfrac{z}{n}\right), & \text{if } m = 0 \\ \Theta(z^*), & \text{otherwise} \end{cases}$$

where z^* is the unique root of

$$\Theta(z^*) + \Theta\left(\frac{mz^* - z}{n - m}\right) = 1.$$

Proof. It follows from Theorem 3 that

$$\begin{aligned}\Upsilon(z) &= \inf_{z_1+\cdots+z_m-z_{m+1}-\cdots-z_n=z}\left(\max_{1\le i\le m}\Theta(z_i)\vee\max_{m+1\le j\le n}(1-\Theta(z_j))\right)\\ &= \sup_{z_1+\cdots+z_m-z_{m+1}-\cdots-z_n=z}\left(\min_{1\le i\le m}\Theta(z_i)\wedge\min_{m+1\le j\le n}(1-\Theta(z_j))\right).\end{aligned}$$

and it follows from Theorem 4 that

$$\begin{aligned}\Upsilon(z) &= \Theta(z_1)=\Theta(z_2)=\cdots=\Theta(z_m)\\ &= 1-\Theta(z_{m+1})=1-\Theta(z_{m+2})=\cdots=1-\Theta(z_n).\end{aligned}$$

Assume $0<\Upsilon(z)<1$, and take

$$z_1=z_2=\cdots=z_m=a,$$

$$z_{m+1}=z_{m+2}=\cdots=z_n=b.$$

Then

$$ma-(n-m)b=z,$$
$$\Theta(a)+\Theta(b)=1.$$

If $n=m$, then

$$a=\frac{z}{n},$$

and

$$\Upsilon(z)=\Theta(a)=\Theta(\frac{z}{n}).$$

Otherwise, we have

$$b=\frac{ma-z}{n-m},$$

and

$$\Theta(a)+\Theta(\frac{ma-z}{n-m})=\Theta(a)+\Theta(b)=1.$$

Write

$$z^*=a.$$

Then

$$\Upsilon(z)=\Theta(a)=\Theta(z^*),$$

and z^* is a root of

$$\Theta(z^*)+\Theta(\frac{mz^*-z}{n-m})=1.$$

Futhermore, the fuction

$$\Theta(z^*)+\Theta(\frac{mz^*-z}{n-m})$$

is a strictly increasing function with respect to z^*. Thus z^* is the unique root. In particular, if $m=0$, then

$$\Upsilon(z)=\Theta(z^*)=1-\Theta(\frac{mz^*-z}{n-m})=1-\Theta(-\frac{z}{n}).$$

Moreover, if $\Upsilon(z)=0$ or $\Upsilon(z)=1$, then the above conclusion still holds in light of the continuity of the regular uncertainty distribution Θ. The corollary is verified. □

4. Conclusions

In summary, this paper proved a new formula for calculating uncertainty distribution of strictly monotone function of uncertain variables. In addition, three cases were given to illustrate the formula. Furthermore, in the sense of uncertain measure, Liu [11] proposed the ruin index, Lio and Liu [12] proposed the shortage index, and Yao [13] proposed the busy period based on the former formula. In future research, some new results can be obtained using the new formula and may be applied in practice better.

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