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Exact Solutions and Conservation Laws of the Time-Fractional Gardner Equation with Time-Dependent Coefficients

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Abstract: In this paper, we employ the certain theory of Lie symmetry analysis to discuss the time-fractional Gardner equation with time-dependent coefficients. The Lie point symmetry is applied to realize the symmetry reduction of the equation, and then the power series solutions in some specific cases are obtained. By virtue of the fractional conservation theorem, the conservation laws are constructed.

Keywords: time-fractional Gardner equation; Lie symmetry analysis; power series solutions; conservation laws

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1. Introduction

The conversant Korteweg–de Vries(KdV) equation, as the prototype of an integrable nonlinear partial differential equation (NPDE), is an active subject in the area of mathematical physics. The Gardner equation

$$u_t + 6uu_x + 6u^2u_x + u_{xxx} = 0, \tag{1}$$

also called the combined KdV-mKdV equation, which appears in applications of describing various significant phenomena in fluid mechanics, plasma physics and quantum field theory, has been widely studied by various methods, including numerical methods and analytical methods. Taking into account the inhomogeneity of the medium and boundary, variable-coefficient nonlinear partial differential equations (NPDEs) always reflect certain nonlinear physical phenomena more truly than constant-coefficient NPDEs. There exists a vast amount of materials on various types of variable-coefficients Gardner equations [1–7]. In [8], the following form of a Gardner equation with time-dependent coefficients

$$u_t + a(t)uu_x + b(t)u^2u_x + c(t)u_{xxx} + d(t)u_x + f(t)u = 0 \tag{2}$$

is discussed, where $u(x,t)$ is the amplitude of the relevant wave model, x is the horizontal coordinate, t is the time and the time-dependent coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $f(t)$ are all analytic functions, which are related to the background density and shear flow stratification. By combining with symbolic computation, the author has deduced the Painlevé integrability condition, lax pair and Bäcklund transformation, as well as the N-soliton-like solutions of two special equations. S. Kumar et al. [9] have determined the Painlevé property and the exact solutions by Painlevé analysis and Lie group analysis. In addition, the solutions of (2) are constructed by applying the general solution of the Riccati equation in [10]. Setting $a(t) = \alpha t^m$, $b(t) = \beta t^n$, $c(t) = \gamma t^p$, $d(t) = \delta t^q$ and $f(t) = \mu t^r$, Equation (2) is reduced as below:

$$u_t + \alpha t^m uu_x + \beta t^n u^2u_x + \gamma t^p u_{xxx} + \delta t^q u_x + \mu t^r u = 0. \tag{3}$$

Liu et al. [11] have given the explicit solution via the Painlevé analysis and Lie group analysis. In particular, for $r = -1$, Equation (3) is simplified as

$$u_t + \alpha t^m u u_x + \beta t^n u^2 u_x + \gamma t^p u_{xxx} + \delta t^q u_x + \frac{\mu}{t} u = 0. \tag{4}$$

Taking $m = \beta = p = \delta = 0$, we get its special case of the generalized cylindrical KdV type equation

$$u_t + \alpha u u_x + \gamma u_{xxx} + \frac{\mu}{t} u = 0. \tag{5}$$

Similarly, they have produced the exact solutions of (4) and (5) by Painlevé analysis and Lie symmetry [12]. Fractional NPDEs possessing nonlocality can more succinctly and accurately describe the mechanical and physical processes with historical memory and spatial global correlation than integer order NPDEs [13–15]. In this paper, we consider the extended time-fractional Gardner equation with time-dependent coefficients

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\mu}{t} u + a t^l u u_x + b t^m u^2 u_x + \beta t^p u_{xxx} + \gamma t^q u_x = 0. \tag{6}$$

Lie symmetry is one of the most versatile and effective methods to derive the analytical solutions of fractional NPDEs [16–23]. The famous Noether theorem points out that each symmetry corresponds to a conservation law. Lukashchuk proposed the fractional Noether operator and went on to construct conservation laws of time-fractional diffusion waves and sub-diffusion equations [24–27]. Conservation laws have important applications in the integrability of partial differential equations, stability and global behavior of solutions, reliability of numerical solutions, construction of nonlocal systems and extension of generalized symmetric methods. Symmetry reflects the structural characteristics of NPDEs, solutions reveal the laws of physical behavior of NPDEs, and conservation laws reflect the motion characteristics of NPDEs. The primary purpose of this paper is to obtain new exact solutions and construct the conservation laws of (6).

The arrangement of the paper is as follows. In Section 2, the idea of Lie symmetry acting on the NPDE with Riemann–Liouville (RL) fractional derivatives are given. In Section 3, we introduce the infinitesimal transformation of a one parameter Lie group into (6) and derive the vector fields in appropriate cases. Symmetry reduction is realized by means of the definition of the RL derivative. In Section 4, some new explicit solutions are obtained by the power series method. In Section 5, the conserved vectors of (6) are constructed. Section 6 makes some conclusions.

2. Preliminaries

The basic idea of Lie symmetry is to identify the similar variables of NPDE. Combining the definition of the RL derivative can reduce an independent variable and get the corresponding simplified equation.

First, we give the definition of RL fractional derivative [28]

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \begin{cases} \frac{\partial^n f}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} f(x,s) ds, & n-1 < \alpha < n, \end{cases} \tag{7}$$

where $\Gamma(n-\alpha)$ is the gamma function. Note an important property of the RL fractional derivative

$$D_t^\alpha t^r = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} t^{r-\alpha},\ r > 0. \tag{8}$$

The Erdélyi–Kober(EK) fractional differential operator is defined as [29]

$$\left(\mathcal{P}_{\delta}^{\tau,\alpha}h\right)(z) := \prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\delta}z\frac{d}{dz}\right)\left(\mathcal{K}_{\delta}^{\tau+\alpha,n-\alpha}h\right)(z),$$
$$n = \begin{cases}[\alpha]+1, & \alpha \notin \mathbb{N},\\ \alpha, & \alpha \in \mathbb{N},\end{cases} \tag{9}$$

where

$$\left(\mathcal{K}_{\delta}^{\tau,\alpha}h\right)(z) := \begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^{\infty}(\omega-1)^{\alpha-1}\omega^{-(\tau+\alpha)}h\left(z\omega^{\frac{1}{\delta}}\right)d\omega, & \alpha > 0,\\ h(z), & \alpha = 0\end{cases} \tag{10}$$

is the EK fractional integral operator. Then, we consider the time-fractional NPDE of order α ($\alpha \in (0,1)$)

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = F(x,t,u,u_x,u_{xx},u_{xxx}). \tag{11}$$

The one-parameter Lie group of infinitesimal transformations is as follows:

$$\begin{aligned}
x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2),\\
t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2),\\
u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2),\\
\frac{\partial^{\alpha}u^*}{\partial t^{*\alpha}} &= \frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \epsilon\eta^{\alpha,t}(x,t,u) + O(\epsilon^2),\\
\frac{\partial u^*}{\partial x^*} &= \frac{\partial u}{\partial x} + \epsilon\eta^{x}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^2 u^*}{\partial x^{*2}} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon^2),\\
\frac{\partial^3 u^*}{\partial x^{*3}} &= \frac{\partial^3 u}{\partial x^3} + \epsilon\eta^{xxx}(x,t,u) + O(\epsilon^2),
\end{aligned} \tag{12}$$

where ϵ is the group parameter, ξ, τ, η are infinitesimal operators, η^x, η^{xx}, and η^{xxx} are extended infinitesimal functions of integer order and $\eta^{\alpha,t}$ is the extended infinitesimal function of order α, which is defined by using the generalized Leibnitz rule and generalized chain rule

$$\begin{aligned}
\eta^{\alpha,t} =& D_t^{\alpha}\eta + (\eta_u - \alpha D_t(\tau))D_t^{\alpha}u - uD_t^{\alpha}\eta_u + \mu + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}D_t^{n}\eta_u\right.\\
&\left.-\binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]\mathrm{D}_t^{\alpha-n}(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^{n}(\xi)\mathrm{D}_t^{\alpha-n}(u_x),
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
\mu =& \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r\\
&\times\frac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.
\end{aligned} \tag{14}$$

The corresponding Lie algebra is given in the following form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{15}$$

The invariance criterion must be adapted if the vector field can generate Lie group symmetry of (6)

$$\mathrm{pr}^{(\alpha,3)}V(\Delta)|_{\Delta=0}=0, \tag{16}$$

where $\mathrm{pr}^{(\alpha,3)}$ is the third prolongation operator, expanded as

$$\mathrm{pr}^{(\alpha,3)}=V+\eta^{\alpha,t}\partial_{\partial_t^\alpha u}+\eta^x\partial_{u_x}+\eta^{xx}\partial_{u_{xx}}+\eta^{xxx}\partial_{u_{xxx}}. \tag{17}$$

Under the invariance condition, we get

$$\tau(x,t,u)|_{t=0}=0. \tag{18}$$

3. Symmetry Reduction

In this section, Lie symmetry is applied to (6) to achieve its similarity reductions. After inserting the Lie group transformation, the expansion of (16) is obtained:

$$\begin{aligned}&\eta^{\alpha,t}+\frac{\mu}{t}\eta-\frac{\mu}{t^2}\tau u+al\tau t^{l-1}uu_x+at^lu\eta^x+bm\tau t^{m-1}u^2u_x+2bt^mu\eta u_x\\&+bt^mu^2\eta_x+\beta p\tau t^pu_{xxx}+\beta t^p\eta^{xxx}+\gamma q\tau t^{q-1}u_x+\gamma t^p\eta^x=0.\end{aligned} \tag{19}$$

By replacing the extended infinitesimal functions and making the coefficients of the derivatives of u equal to 0, the determining equations of the vector fields are derived as

$$\begin{aligned}&\xi=(m-l-1)(p-q)c_1x+c_2,\\&\tau=2(m-l-1)c_1t,\\&\eta=2\alpha(m-l)c_1u.\end{aligned} \tag{20}$$

Case 1. The parameter variables of (6) take any value. The vector field represents

$$V_1=\frac{\partial}{\partial x}. \tag{21}$$

Case 2. $m-l-1=0$, p and q are arbitrary constants. The symmetry Lie algebra is two-dimensional, spanned by

$$V_1=\frac{\partial}{\partial x},\ V_2=2\alpha u\frac{\partial}{\partial u}. \tag{22}$$

Case 3. For $m-l=0$ and $p-q=0$, we get the two vector fields

$$V_1=\frac{\partial}{\partial x},\ V_3=-2t\frac{\partial}{\partial t}. \tag{23}$$

For symmetry V_3, this gives rise to the group-invariant solution $u=f(z)$, where $z=x$ is the similarity variable. Replacing this solution in (6) produces the nonlinear ordinary differential equation (NODE)

$$\frac{\mu}{t}f(z)+at^lf(z)f'(z)+bt^mf^2(z)f'(z)+\beta t^pf'''(z)+\gamma t^qf'(z)=0. \tag{24}$$

Case 4. For $m-l=0$ and $p-q\neq 0$, the Lie algebra extends by the following Lie point symmetry generators:

$$V_1=\frac{\partial}{\partial x},\ V_4=-2t\frac{\partial}{\partial t}+(q-p)x\frac{\partial}{\partial x}. \tag{25}$$

For symmetry V_4, there emerges the corresponding group-invariant solution $u=f(z)$ with $z=xt^{\frac{q-p}{2}}$. Subsequently, we implement the symmetry reduction that makes (6) reduce to a fractional NODE in the EK sense. Let $n-1<\alpha<n$, $n=1,\ 2,\ 3,\ldots$; then, the RL fractional derivative with respect to t exerts

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f(xs^{\frac{q-p}{2}})ds\right). \tag{26}$$

Taking $\rho=\frac{t}{s}$, Equation (26) is transformed into the following form:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\left(t^{n-\alpha}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty(\rho-1)^{n-\alpha-1}\rho^{-(n-\alpha+1)}f(zs^{\frac{p-q}{2}})d\rho\right)\\&=\frac{\partial^n}{\partial t^n}\left(t^{n-\alpha}\left(\mathcal{K}^{1,n-\alpha}_{\frac{2}{p-q}}f\right)(z)\right).\end{aligned} \tag{27}$$

Since $t\frac{\partial}{\partial t}\varphi(z)=\frac{q-p}{2}z\frac{d}{dz}\varphi(z)$, we get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^{n-1}}{\partial t^{n-1}}\left(t^{n-\alpha-1}\left(n-\alpha-\frac{p-q}{2}z\frac{d}{dz}\right)\left(\mathcal{K}^{1,n-\alpha}_{\frac{2}{p-q}}f\right)(z)\right). \tag{28}$$

Repeating the operation $n-1$ times, there appears the EK fractional differential operator

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=t^{-\alpha}\prod_{j=0}^{n-1}\left(1+j-\alpha-\frac{p-q}{2}z\frac{d}{dz}\right)\left(\mathcal{K}^{1,n-\alpha}_{\frac{2}{p-q}}f\right)(z)\\&=t^{-\alpha}\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{2}{p-q}}f\right)(z).\end{aligned} \tag{29}$$

Inserting the group-instant solution $u=f(z)$ and (29) into (6) allows the NODE to be obtained as

$$\begin{aligned}&\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{2}{p-q}}f\right)(z)+\mu t^{\alpha-1}f(z)+at^{l+\frac{q-p}{2}+\alpha}f(z)f'(z)+bt^{m+\frac{q-p}{2}+\alpha}f^2(z)f'(z)\\&+\beta t^{\frac{3q-p}{2}+\alpha}f'''(z)+\gamma t^{\frac{3q-p}{2}+\alpha}f'(z)=0.\end{aligned} \tag{30}$$

Case 5. For $m-l-1\neq 0$, $m-l\neq 0$ and $p-q=0$, we obtain the following 2 vector fields

$$V_1=\frac{\partial}{\partial x},\ V_5=2(m-l-1)t\frac{\partial}{\partial t}+2\alpha(m-l)u\frac{\partial}{\partial u}. \tag{31}$$

The vector field V_5 arouses the group-invariant solution $u=t^{\frac{\alpha(m-l)}{m-l-1}}f(z)$ with $z=x$. In view of the property of (8) and the above group-invariant solution, Equation (6) is simplified as

$$\begin{aligned}&\frac{\Gamma(\frac{\alpha(m-l)}{m-l-1}+1)}{\Gamma(\frac{\alpha}{m-l-1}+1)}f(z)+\mu t^{\alpha-1}f(z)+at^{l+\frac{2\alpha(m-l)}{m-l-1}}f(z)f'(z)\\&+bt^{m+\frac{\alpha(3m-3l-1)}{m-l-1}}f^2(z)f'(z)+\beta t^{p+\alpha}f'''(z)+\gamma t^{q+\alpha}f'(z)=0.\end{aligned} \tag{32}$$

Case 6. For $m-l-1\neq 0$, $m-l\neq 0$ and $p-q\neq 0$, the symmetry Lie algebra is spanned by the two forms, respectively

$$V_1=\frac{\partial}{\partial x},\ \ V_6=(m-l-1)(p-q)x\frac{\partial}{\partial x}+2(m-l-1)t\frac{\partial}{\partial t}+2\alpha(m-l)u\frac{\partial}{\partial u}. \tag{33}$$

For vector field V_6, we have the group-invariant solution $u = t^{\frac{\alpha(m-l)}{m-l-1}} f(z)$ with the similarity variable $z = xt^{\frac{q-p}{2}}$. Imitating (26)–(29) to export the following results

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{\frac{\alpha}{m-l-1}} \left(\mathcal{P}_{\frac{2}{p-q}}^{1+\frac{\alpha}{m-l-1},\alpha} f \right)(z), \tag{34}$$

Equation (6) is reduced to the NODE in the sense of the EK fractional derivative

$$\begin{aligned} &\left(\mathcal{P}_{\frac{2}{p-q}}^{1+\frac{\alpha}{m-l-1},\alpha} f \right)(z) + \mu t^{\alpha-1} f(z) + a t^{l+\frac{q-p}{2}+\frac{\alpha(2m-2l-1)}{m-l-1}} f(z) f'(z) \\ &+ b t^{l+\frac{q-p}{2}+\frac{\alpha(3m-3l-1)}{m-l-1}} f^2(z) f'(z) + \beta t^{\frac{3q-p}{2}+\alpha} f'''(z) + \gamma t^{\frac{3q-p}{2}+\alpha} f'(z) = 0. \end{aligned} \tag{35}$$

4. Power Series Solutions

The solutions of nonlinear ordinary differential equations obtained by symmetry reduction of (6) cannot be expressed by elementary functions or their integral formulas, though it is feasible to provide this solutions by the power series method. The power series solutions can reflect the amplitude of the relevant wave model. Now, we determine the power series solutions of (24), (30), (32) and (35).

We define (24) having the form of power series solution

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{36}$$

so that

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n, \quad f'''(z) = \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} z^n. \tag{37}$$

We proceed to transform (24) into the following form

$$\begin{aligned} &\frac{\mu}{t} \sum_{n=0}^{\infty} a_n z^n + a \sum_{n=0}^{\infty} \sum_{i=0}^{n} (n+1-i) a_i a_{n+1-i} z^n t^l + b \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{i} (j+1) a_{j+1} a_{n-j} \\ &\times z^n t^m + \beta \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} z^n t^p + \gamma \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n t^q = 0. \end{aligned} \tag{38}$$

From (38), we get for $n = 0$

$$a_3 = -\frac{1}{6\beta t^p} \left(\frac{\mu}{t} a_0 + a a_0 a_1 t^l + b a_0 a_1 t^m + \gamma a_1 t^q \right), \tag{39}$$

and for $n \geq 1$

$$\begin{aligned} a_{n+3} = &-\frac{1}{\beta t^p (n+3)(n+2)(n+1)} \left(\frac{\mu}{t} a_n + a \sum_{i=0}^{n} (n+1-i) a_i a_{n+1-i} t^l \right. \\ &\left. + b \sum_{i=0}^{n} \sum_{j=0}^{i} (j+1) a_{j+1} a_{n-j} t^m + \gamma (n+1) a_{n+1} t^q \right). \end{aligned} \tag{40}$$

It can be seen that the coefficients of the power series (36) are completely determined by the constants a_0, a_1, a_2, μ, a, b, β and γ. This shows that (24) has a power series solution with the coefficients determined by (39) and (40). Introducing the coefficient functions into (36) and combining with the group-invariant solution $u = f(x)$, the power series solution of (24) presents

$$\begin{aligned} u =& a_0 + a_1 x + a_2 x^2 - \frac{1}{6\beta}\Big(\frac{\mu}{t}a_0 + a a_0 a_1 t^l + b a_0 a_1 t^m + \gamma a_1 t^q\Big) x^3 t^{-p} \\ &+ \sum_{n=1}^{\infty}\Big(-\frac{1}{\beta(n+3)(n+2)(n+1)}\Big(\frac{\mu}{t}a_n + a\sum_{i=0}^{n}(n+1-i)a_i a_{n+1-i}t^l \\ &+ b\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)a_{j+1}a_{n-j}t^m + \gamma(n+1)a_{n+1}t^q\Big)\Big) x^{n+3} t^{-p}. \end{aligned} \tag{41}$$

Subsequently, considering (30), one gets

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{\Gamma(2+\frac{(p-q)n}{2})}{\Gamma(2-\alpha+\frac{(p-q)n}{2})}a_n z^n + \mu\sum_{n=0}^{\infty}a_n z^n t^{\alpha-1} + a\sum_{n=0}^{\infty}\sum_{i=0}^{n}(n+1-i)a_i \\ &\times a_{n+1-i}z^n t^{l+\alpha+\frac{q-p}{2}} + b\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)a_{j+1}a_{n-j}a_{i-j}z^n t^{m+\alpha+\frac{q-p}{2}} \\ &+ \beta\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}z^n t^{\alpha+\frac{3q-p}{2}} + \gamma\sum_{n=0}^{\infty}(n+1)z^n t^{\alpha+\frac{3q-p}{2}} = 0. \end{aligned} \tag{42}$$

There we omit the expression of the coefficient functions of (36), which can similarly be obtained by making $n = 0$ and $n \geq 1$. Connecting the group-invariant solution $u = f(xt^{\frac{q-p}{2}})$, the power series solution of (30) is received

$$\begin{aligned} u =& a_0 + a_1 x t^{\frac{q-p}{2}} + a_2 x^2 t^{q-p} - \frac{1}{6\beta}\Big(\frac{\Gamma(2)}{\Gamma(2-\alpha)}a_0 + \mu a_0 t^{\alpha-1} + a a_0 a_1 t^{l+\alpha+\frac{q-p}{2}} \\ &+ b a_0^2 a_1 t^{m+\alpha+\frac{q-p}{2}} + \gamma a_1 t^{\alpha+\frac{3q-p}{2}}\Big) x^3 t^{-p-\alpha} \\ &+ \sum_{n=1}^{\infty}\Bigg(-\frac{1}{\beta(n+3)(n+2)(n+1)t^{\alpha+\frac{3q-p}{2}}}\Bigg(\frac{\Gamma(2+\frac{(p-q)n}{2})}{\Gamma(2-\alpha+\frac{(p-q)n}{2})}a_n \\ &+ \mu a_n t^{\alpha-1} + a\sum_{i=0}^{n}(n+1-i)a_i a_{n+1-i}t^{l+\alpha+\frac{q-p}{2}} + b\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1) \\ &\times a_{j+1}a_{n-j}a_{i-j}t^{m+\alpha+\frac{q-p}{2}} + \gamma(n+1)a_{n+1}\Bigg)\Bigg) x^{n+3} t^{-p-\alpha+\frac{n(q-p)}{2}}, \end{aligned} \tag{43}$$

where the coefficients are determined by the constants a_0, a_1, a_2, μ, a, b, β and γ.

In the light of the previous procedures, Equation (32) can be rewritten

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\alpha(m-l)}{m-l-1}+1)}{\Gamma(\frac{\alpha}{m-l-1}+1)}a_n z^n + \mu\sum_{n=0}^{\infty}a_n z^n t^{\alpha-1} + a\sum_{n=0}^{\infty}\sum_{i=0}^{n}(n+1-i)a_i a_{n+1-i} \\ &\times z^n t^{l+\frac{\alpha(2m-2l-1)}{m-l-1}} + b\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)a_{j+1}a_{n-j}a_{i-j}z^n t^{m+\frac{\alpha(3m-3l-1)}{m-l-1}} \\ &+ \beta\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}z^n t^{p+\alpha} + \gamma\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n t^{q+\alpha} = 0. \end{aligned} \tag{44}$$

Omitting the calculation of coefficient functions and concerning group-invariant solution $u = t^{\frac{\alpha(m-l)}{m-l-1}} f(x)$, the power series solution of (32) is as below:

$$
\begin{aligned}
u =& a_0 t^{\frac{\alpha(m-l)}{m-l-1}} + a_1 x t^{\frac{\alpha(m-l)}{m-l-1}} + a_2 x^2 t^{\frac{\alpha(m-l)}{m-l-1}} - \frac{1}{6\beta}\left(\frac{\Gamma\left(\frac{\alpha(m-l)}{m-l-1}+1\right)}{\Gamma\left(\frac{\alpha}{m-l-1}+1\right)} a_0 + \mu a_0 t^{\alpha-1}\right. \\
&\left. + a a_0 a_1 t^{l+\frac{\alpha(2m-2l-1)}{m-l-1}} + b a_0^2 a_1 t^{m+\frac{\alpha(3m-3l-1)}{m-l-1}} + \gamma a_1 t^{p+\alpha}\right) x^3 t^{-p+\frac{\alpha}{m-l-1}} \\
&+ \sum_{n=1}^{\infty}\left(-\frac{1}{\beta(n+3)(n+2)(n+1)}\left(\frac{\Gamma\left(\frac{\alpha(m-l)}{m-l-1}+1\right)}{\Gamma\left(\frac{\alpha}{m-l-1}+1\right)} a_n + \mu a_n t^{\alpha-1}\right.\right. \\
&+ a\sum_{i=0}^{n}(n+1-i) a_i a_{n+1-i} t^{l+\frac{\alpha(2m-2l-1)}{m-l-1}} + b\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1) a_{j+1} \\
&\left.\left.\times a_{n-j} a_{i-j} t^{m+\frac{\alpha(3m-3l-1)}{m-l-1}} + \gamma(n+1) a_{n+1} t^{q+\alpha}\right)\right) x^{n+3} t^{-p+\frac{\alpha}{m-l-1}},
\end{aligned}
\tag{45}
$$

where the coefficients are determined by the constants a_0, a_1, a_2, μ, a, b, β and γ.

Likewise, Equation (35) can be expressed as the following:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\Gamma\left(2+\frac{\alpha(m-l)}{m-l-1}+\frac{(p-q)n}{2}\right)}{\Gamma\left(2+\frac{\alpha}{m-l-1}+\frac{(p-q)n}{2}\right)} a_n z^n + \mu\sum_{n=0}^{\infty} a_n z^n t^{\alpha-1} + a\sum_{n=0}^{\infty}\sum_{i=0}^{n}(n+1-i) \\
&\times a_i a_{n+1-i} z^n t^{l+\frac{q-p}{2}+\frac{\alpha(2m-2l-1)}{m-l-1}} + b\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1) a_{j+1} a_{n-j} a_{i-j} \\
&\times z^n t^{m+\frac{q-p}{2}+\frac{\alpha(3m-3l-1)}{m-l-1}} + \beta\sum_{n=0}^{\infty}(n+3)(n+2)(n+1) a_{n+3} z^n t^{\frac{3q-p}{2}+\alpha} \\
&+ \gamma\sum_{n=0}^{\infty}(n+1) a_{n+1} z^n t^{\frac{3q-p}{2}+\alpha} = 0.
\end{aligned}
\tag{46}
$$

We naturally omit the calculation of coefficient functions and link the group-invariant solution $u = t^{\frac{\alpha(m-l)}{m-l-1}} f(z)$ with the similar variable $z = x t^{\frac{q-p}{2}}$, so that the power series solution of (35) is as following:

$$
\begin{aligned}
u =& a_0 t^{\frac{\alpha(m-l)}{m-l-1}} + a_1 x t^{\frac{q-p}{2}+\frac{\alpha(m-l)}{m-l-1}} + a_2 x^2 t^{q-p+\frac{\alpha(m-l)}{m-l-1}} - \frac{1}{6\beta}\left(\frac{\Gamma\left(\frac{2+\alpha(m-l)}{m-l-1}\right)}{\Gamma\left(2+\frac{\alpha}{m-l-1}\right)} a_0\right. \\
&+ \mu a_0 t^{\alpha-1} + a a_0 a_1 t^{l+\frac{q-p}{2}+\frac{\alpha(2m-2l-1)}{m-l-1}} + b a_0^2 a_1 t^{m+\frac{q-p}{2}+\frac{\alpha(3m-3l-1)}{m-l-1}} \\
&\left. + \gamma a_1 t^{\alpha+\frac{3q-p}{2}}\right) x^3 t^{-p-\alpha} + \sum_{n=1}^{\infty}\left(-\frac{1}{\beta(n+3)(n+2)(n+1)}\right. \\
&\times\left(\frac{\Gamma\left(2+\frac{\alpha(m-l)}{m-l-1}+\frac{(p-q)n}{2}\right)}{\Gamma\left(2+\frac{\alpha}{m-l-1}+\frac{(p-q)n}{2}\right)} a_n + \mu a_n t^{\alpha-1} + a\sum_{i=0}^{n}(n+1-i) a_i a_{n+1-i}\right. \\
&\times t^{l+\frac{q-p}{2}+\frac{\alpha(2m-2l-1)}{m-l-1}} + b\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1) a_{j+1} a_{n-j} a_{i-j} \\
&\left.\left.\times t^{m+\frac{q-p}{2}+\frac{\alpha(3m-3l-1)}{m-l-1}} + \gamma(n+1) a_{n+1} t^{\alpha+\frac{3q-p}{2}}\right)\right) x^{n+3} t^{-p-\alpha+\frac{(q-p)n}{2}},
\end{aligned}
\tag{47}
$$

where the coefficients are determined by the constants a_0, a_1, a_2, μ, a, b, β and γ.

5. Conservation Laws

By combining Lie point symmetry and the adjoint equation, we derive the conservation laws of (6). For the vector field (15), the conservation law is determined by the following formula [25]

$$D_x(C^x) + D_t(C^t) = 0, \tag{48}$$

where (C^x, C^t) is the conserved vector. To construct the conservation laws of (6) means to get the conserved vectors.

Given the formal Lagrangian,

$$\mathcal{L} = v(x,t)\Big(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\mu}{t}u + at^l uu_x + bt^m u^2 u_x + \beta t^p u_{xxx} + \gamma t^q u_x\Big). \tag{49}$$

New dependent variables $v(x,t)$ is the solution of the adjoint equation. The adjoint equation is as below:

$$\frac{\delta \mathcal{L}}{\delta u} = 0, \tag{50}$$

which is, with the Euler–Lagrangian operator, defined by

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} + (D_t^\alpha)^* \frac{\partial \mathcal{L}}{\partial D_t^\alpha u} - D_x \frac{\partial \mathcal{L}}{\partial u_x} - D_x^3 \frac{\partial \mathcal{L}}{\partial u_{xxx}}, \tag{51}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α. The components C^x, C^t of the conserved vector, which are generated by each symmetry generator, are composed of the following formulas:

$$\begin{aligned} C^x =& W\Big(\frac{\partial \mathcal{L}}{\partial u_x} - D_x\Big(\frac{\partial \mathcal{L}}{\partial u_{xx}}\Big) + D_x^2\Big(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\Big)\Big) + D_x(W)\Big(\frac{\partial \mathcal{L}}{\partial u_{xx}} \\ & - D_x\Big(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\Big)\Big) + D_x^2(W)\Big(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\Big), \\ C^t =& \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial \mathcal{L}}{\partial D_t^\alpha u} - (-1)^n J\Big(W, D_t^n \frac{\partial \mathcal{L}}{\partial D_t^\alpha u}\Big), \end{aligned} \tag{52}$$

where W is the Lie characteristic function, namely $W = \eta - \tau u_t - \xi u_x$, and J is the integral operator given by $J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T f(x,r)g(x,s)(r-s)^{n-1-\alpha} dr ds$.

There produce the specific components of the conserved vectors related to the Lie symmetry of (6).

Case 1. For $V_1 = \frac{\partial}{\partial x}$, we have $W_1 = -u_x$. The components of the conserved vector are derived as follows:

$$\begin{aligned} C^x =& (-u_x)(at^l uv + bt^m u^2 v + \gamma t^q v + \beta t^p v_{xx}) + \beta t^p u_{xx} v_x - \beta t^p u_{xxx} v, \\ C^t =& vD_t^{\alpha-1}(-u_x) + J(-u_x, v_t). \end{aligned} \tag{53}$$

Case 2. For $V_2 = 2au\frac{\partial}{\partial u}$, we have $W_2 = 2au$. The components of the conserved vector are derived as follows:

$$\begin{aligned} C^x =& (2au)(at^l uv + bt^m u^2 v + \gamma t^q v + \beta t^p v_{xx}) - 2\alpha\beta t^p u_x v_x + 2\alpha\beta t^p u_{xx} v, \\ C^t =& vD_t^{\alpha-1}(2au) + J(2au, v_t). \end{aligned} \tag{54}$$

Case 3. For $V_3 = -2t\frac{\partial}{\partial t}$, we have $W_3 = 2tu_t$. The components of the conserved vector are derived as follows:

$$\begin{aligned} C^x =& (2tu_t)(at^l uv + bt^m u^2 v + \gamma t^q v + \beta t^p v_{xx}) - 2\beta t^{p+1} u_{xt} v_x + 2\beta t^{p+1} u_{xxt} v, \\ C^t =& vD_t^{\alpha-1}(2tu_t) + J(2tu_t, v_t). \end{aligned} \tag{55}$$

Case 4. For $V_4 = -2t\frac{\partial}{\partial t} + (q-p)x\frac{\partial}{\partial x}$, we have $W_4 = 2tu_t + (p-q)xu_x$. The components of the conserved vector are derived as follows:

$$\begin{aligned} C^x =&(2tu_t + (p-q)xu_x)(at^l uv + bt^m u^2 v + \gamma t^q v + \beta t^p v_{xx}) + (2tu_{xt} + (p-q) \\ &\times (u_x + xu_{xx}))(-\beta t^p v_x) + (2tu_{xxt} + (p-q)(2u_{xx} + xu_{xxx}))(\beta t^p v), \\ C^t =&vD_t^{\alpha-1}(2tu_t + (p-q)xu_x) + J(2tu_t + (p-q)xu_x, v_t). \end{aligned} \tag{56}$$

Case 5. For $V_5 = 2(m-l-1)t\frac{\partial}{\partial t} + 2\alpha(m-l)u\frac{\partial}{\partial u}$, we have $W_5 = 2\alpha(m-l)u - 2(m-l-1)tu_t$. The components of the conserved vector are derived as follows:

$$\begin{aligned} C^x =&(2\alpha(m-l)u - 2(m-l-1)tu_t)(at^l uv + bt^m u^2 v + \gamma t^q v + \beta t^p v_{xx}) \\ &+ (2\alpha(m-l)u_x - 2(m-l-1)tu_{xt})(-\beta t^p v_x) \\ &+ (2\alpha(m-l)u_{xx} - 2(m-l-1)tu_{xxt})(\beta t^p v), \\ C^t =&vD_t^{\alpha-1}(2\alpha(m-l)u - 2(m-l-1)tu_t) \\ &+ J(2\alpha(m-l)u - 2(m-l-1)tu_t, v_t). \end{aligned} \tag{57}$$

Case 6. For $V_6 = (m-l-1)(p-q)x\frac{\partial}{\partial x} + 2(m-l-1)t\frac{\partial}{\partial t} + 2\alpha(m-l)u\frac{\partial}{\partial u}$, we have $W_5 = 2\alpha(m-l)u - 2(m-l-1)tu_t - (m-l-1)(p-q)xu_x$. The components of the conserved vector are derived as follows:

$$\begin{aligned} C^x =&(2\alpha(m-l)u - 2(m-l-1)tu_t - (m-l-1)(p-q)xu_x)(at^l uv \\ &+ bt^m u^2 v + \gamma t^q v + \beta t^p v_{xx}) + (2\alpha(m-l)u_x - 2(m-l-1)tu_{xt} \\ &- (m-l-1)(p-q)(u_x + xu_{xx}))(-\beta t^p v_x) + (2\alpha(m-l)u_{xx} \\ &- 2(m-l-1)tu_{xxt} - (m-l-1)(p-q)(2u_{xx} + xu_{xxx}))(\beta t^p v), \\ C^t =&vD_t^{\alpha-1}(2\alpha(m-l)u - 2(m-l-1)tu_t - (m-l-1)(p-q)xu_x) \\ &+ J(2\alpha(m-l)u - 2(m-l-1)tu_t - (m-l-1)(p-q)xu_x, v_t). \end{aligned} \tag{58}$$

6. Conclusions

We have shown feasible ways to determine the exact solutions and conservation laws of the time-fractional Gardner equation with time-dependent coefficients. The advantage of Lie symmetry is to reduce the equation into a NODE which is easy to solve. The power series method is convenient and effective for solving obtained NODEs, and can be further extended to solve other NPDEs by the similar routines. The exact solutions in the form of a power series can be used to test the accuracy of the numerical solutions, that is, to determine whether the numerical method for obtaining the numerical solution is reasonable by comparing the images of the two types of solutions. On the basis of symmetry and new conservation theorem, the conserved vectors are constructed in certain situations.

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Abbreviations

The following abbreviations are used in this manuscript:

KdV	Korteweg–de Vries
NPDE	Nonlinear partial differential equation
NPDEs	Nonlinear partial differential equations
RL	Riemann–Liouville
EK	Erdélyi–Kober
NODE	Nonlinear ordinary differential equation

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