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About the Cauchy–Bunyakovsky–Schwarz Inequality for Hilbert Space Operators

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Abstract: The symmetric shape of some inequalities between two sequences of real numbers generates inequalities of the same shape in operator theory. In this paper, we study a new refinement of the Cauchy–Bunyakovsky–Schwarz inequality for Euclidean spaces and several inequalities for two bounded linear operators on a Hilbert space, where we mention Bohr's inequality and Bergström's inequality for operators. We present an inequality of the Cauchy–Bunyakovsky–Schwarz type for bounded linear operators, by the technique of the monotony of a sequence. We also prove a refinement of the Aczél inequality for bounded linear operators on a Hilbert space. Finally, we present several applications of some identities for Hermitian operators.

Keywords: Cauchy–Bunyakovsky–Schwarz inequality; Bohr's inequality; Bergström's inequality; Aczél inequality

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1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. $\mathbb{B}(H)$ is the set of all bounded linear operators on the Hilbert space H. $\mathbb{B}(H)_{sa}$ is a convex domain of self-adjoint (or Hermitian) operators in $\mathbb{B}(H)$ ($T \in \mathbb{B}(H)_{sa}$ if $T = T^*$). $\mathbb{B}^+(H)$ is the set of positive operators on H ($T \in \mathbb{B}^+(H)$ if $\langle Tv, v\rangle \geq 0$ for every $v \in H$, we write $T \geq 0$), and $\mathbb{B}^{++}(H)$ is the set of all bounded positive invertible operators on H. The following condition: $T \in \mathbb{B}(H)$: $\langle Tv, v\rangle \geq 0$ for every $v \in H$ is equivalent to the following condition: T is self-adjoint, and $\sigma(T) \subset [0, \infty)$, where $\sigma(T) = \{\lambda : T - \lambda I \text{ is not invertible}\}$; and I is the identity operator [1]. If $T \geq 0$, then there exists a unique $T_0 \geq 0$ such that $T = T_0^2$. The absolute value or modulus of the operator $T \in \mathbb{B}(H)$ is given by $|T| = (T^*T)^{1/2}$, so $|T|^2 = T^*T$. It is easy to see that $|T|$ is always positive and $|T| = 0$ if only if $T = \mathbf{0}$. We write $T_1 \geq T_2$ if T_1 and T_2 are self-adjoint operators and if $T_1 - T_2 \geq 0$. In [2], we found several inequalities for absolute value operators.

Many results in the theory of inequalities, probability and statistics, Hilbert spaces theory, and numerical and complex analysis are given by using the Cauchy–Bunyakovsky–Schwarz inequality (the C-B-S inequality).

Extensions, refinements, or generalizations of this inequality have been presented in many papers (see [3–8]).

The C-B-S inequality is defined as follows: let $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ be two sequences of real numbers, then:

$$\left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2, \tag{1}$$

with equality if and only if sequences $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are proportional. In [7], for arbitrary complex sequences $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$, we have:

$$\left(\sum_{i=1}^{n} |a_i|^2\right) \cdot \left(\sum_{i=1}^{n} |b_i|^2\right) \geq \left|\sum_{i=1}^{n} a_i b_i\right|^2, \tag{2}$$

with equality if and only if sequences **a** and **b** are proportional.

We remark that the symmetric shape of some inequalities for real numbers indicates ideas for extending these inequalities in operator theory.

In Inequality (2), for $n = 2, p, q > 1, 1/p + 1/q = 1, a_1 = \sqrt{p}a, a_2 = \sqrt{p}b, b_1 = \dfrac{1}{\sqrt{p}}, b_2 = \dfrac{1}{\sqrt{q}}$, we obtain the classical Bohr inequality [9], given by the following:

$$|a + b|^2 \leq p|a|^2 + q|b|^2, \tag{3}$$

where $a, b \in \mathbb{C}$, with equality if and only if $(p - 1)a = b$. In [10], Hirzallah established an extention of Bohr's inequality to $\mathbb{B}(H)$; thus:

$$|T_1 - T_2|^2 + |(p - 1)T_1 + T_2|^2 \leq p|T_1|^2 + q|T_2|^2, \tag{4}$$

with $T_1, T_2 \in \mathbb{B}(H)$ and $q \geq p > 1$ with $1/p + 1/q = 1$. Zhang, in [11], studied the operator inequalities of the Bohr type.

An important consequence of the C-B-S inequality is Aczél's inequality.

Several methods in the theory of functional equations in one variable were studied in [12] by Aczél and showed the following inequality: let A and B be two positive real numbers, and let $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ be two sequences of positive real numbers such that:

$$A^2 - a_1^2 - ... - a_n^2 > 0 \text{ and: } B^2 - b_1^2 - ... - b_n^2 > 0.$$

Then:

$$\left(A^2 - a_1^2 - ... - a_n^2\right)\left(B^2 - b_1^2 - ... - b_n^2\right) \leq (AB - a_1 b_1 - ... - a_n b_n)^2. \tag{5}$$

Equality holds if and only if the sequences **a** and **b** are proportional. This inequality has many applications in non-Euclidean geometry, in the theory of functional equations, and in operators theory (see [13–16]).

Popoviciu [17] presented a generalized form of the inequality of Aczél, as follows: let A and B be two positive real numbers, and let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\mathbf{a} = (a_1, a_2, ..., a_n)$, $\mathbf{b} = (b_1, b_2, ..., b_n)$ be two sequences of positive real numbers such that:

$$A^p - a_1^p - ... - a_n^p > 0 \text{ and: } B^q - b_1^q - ... - b_n^q > 0.$$

Then:

$$\left(A^p - a_1^p - ... - a_n^p\right)^{1/p}\left(B^q - b_1^q - ... - b_n^q\right)^{1/q} \leq AB - a_1 b_1 - ... - a_n b_n, \tag{6}$$

Equality holds if and only if sequences **a** and **b** are proportional.

In the special case $p = q = 2$, we deduce the classical Aczél inequality. In [18], we found an approach of some bounds for several statistical indicators with the Aczél inequality, and in [19], we found a proof of the Aczél inequality given with tools of the Lorentz–Finsler geometry.

Motivated by the above results, in Section 2, we study a new refinement of the C-B-S inequality for the Euclidean space and several inequalities for two bounded linear operators on the Hilbert space H, where we mention Bohr's inequality and Bergström's inequality for operators. We also show an inequality of the Cauchy–Bunyakovsky–Schwarz type for bounded linear operators, by the technique of the monotony of a sequence. Finally, we prove a refinement of the Aczél inequality for bounded linear operators on the Hilbert space H. In Section 3, we present some identities for real numbers obtained from some identities for Hermitian operators.

This work is important because it extends a series of inequalities for real numbers to inequalities that are true for different classes of operators. This development is not easy in most cases. We also obtain new inequalities between operators, which by choosing a particular case, can generate new inequalities for real numbers and for matrices.

2. Results on the Cauchy–Bunyakovsky–Schwarz inequality and on the Aczél Inequality for Operators

The symmetric shape of some inequalities between two sequences of real numbers suggests inequalities of the same shape in operator theory.

Theorem 1. *For any vectors $x_1, x_2, ..., x_n$ in a Euclidean space H and for arbitrary real numbers $\lambda_1, \lambda_2, ..., \lambda_n$, with $\lambda_i \neq 0$, $i = \overline{1, n}$, we have:*

$$\sum_{i=1}^{n} \lambda_i^2 \sum_{i=1}^{n} \|x_i\|^2 - \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|^2 \geq \max_{i,j \in \{1,...,n\}} \frac{\|\lambda_i x_j - \lambda_j x_i\|^2}{\lambda_i^2 + \lambda_j^2} \sum_{i=1}^{n} \lambda_i^2, \tag{7}$$

for any $n \geq 2$.

Proof. We use the technique of the monotony of a sequence given in [20]. This technique is given for real numbers, but we study its application in broader contexts. Therefore, we consider the sequence:

$$S_n = \sum_{i=1}^{n} \|x_i\|^2 - \frac{\left\| \sum_{i=1}^{n} \lambda_i x_i \right\|^2}{\sum_{i=1}^{n} \lambda_i^2}, n \geq 1.$$

To study the monotony of the sequence $S_k, k \leq n$, we evaluate the difference of two consecutive terms of the sequence. Therefore, we have:

$$S_{k+1} - S_k = \|x_{k+1}\|^2 + \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i \right\|^2}{\sum_{i=1}^{k} \lambda_i^2} - \frac{\left\| \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1} \right\|^2}{\sum_{i=1}^{k} \lambda_i^2 + \lambda_{k+1}^2}.$$

For two vectors $x_1, x_2 \in H$ and for real numbers $a_1, a_2 \neq 0$, with $a_1 + a_2 \neq 0$, the following equality holds:

$$\frac{\|x_1\|^2}{a_1} + \frac{\|x_2\|^2}{a_2} - \frac{\|x_1 + x_2\|^2}{a_1 + a_2} = \frac{\|a_2 x_1 - a_1 a_2\|^2}{a_1 a_2 (a_1 + a_2)}.$$

If $a_1 a_2 (a_1 + a_2) > 0$ in the above equality, then we have:

$$\frac{\|x_1\|^2}{a_1} + \frac{\|x_2\|^2}{a_2} \geq \frac{\|x_1 + x_2\|^2}{a_1 + a_2}. \tag{8}$$

Using the inequality from Relation (8), we have:

$$\|x_{k+1}\|^2 + \frac{\left\|\sum\limits_{i=1}^{k} \lambda_i x_i\right\|^2}{\sum\limits_{i=1}^{k} \lambda_i^2} = \frac{\left\|\lambda_{k+1} x_{k+1}\right\|^2}{\lambda_{k+1}^2} + \frac{\left\|\sum\limits_{i=1}^{k} \lambda_i x_i\right\|^2}{\sum\limits_{i=1}^{k} \lambda_i^2} \geq \frac{\left\|\sum\limits_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1}\right\|^2}{\sum\limits_{i=1}^{k} \lambda_i^2 + \lambda_{k+1}^2}$$

This means that $S_{k+1} - S_k \geq 0$, so sequence S_k is increasing.
Therefore, we obtain $S_n \geq S_{n-1} \geq ... \geq S_2 \geq S_1 = 0$. However,

$$S_2 = \|x_1\|^2 + \|x_2\|^2 - \frac{\|\lambda_1 x_1 + \lambda_2 x_2\|^2}{\lambda_1^2 + \lambda_2^2} = \frac{\|\lambda_1 x_2 - \lambda_2 x_1\|^2}{\lambda_1^2 + \lambda_2^2}$$

and taking into account that we can rearrange the terms of the sequence S_n, we have the inequality:

$$S_n = \sum_{i=1}^{n} \|x_i\|^2 - \frac{\left\|\sum\limits_{i=1}^{n} \lambda_i x_i\right\|^2}{\sum\limits_{i=1}^{n} \lambda_i^2} \geq S_2 = \frac{\|\lambda_1 x_2 - \lambda_2 x_1\|^2}{\lambda_1^2 + \lambda_2^2}.$$

Multiplying the above inequality by $\sum\limits_{i=1}^{n} \lambda_i^2 > 0$, we deduce the inequality of the statement. □

Remark 1. *Since $S_n \geq S_1 = 0$, $n \geq 1$, in the proof of Theorem 1, we obtain:*

$$\sum_{i=1}^{n} \lambda_i^2 \sum_{i=1}^{n} \|x_i\|^2 - \left\|\sum_{i=1}^{n} \lambda_i x_i\right\|^2 \geq 0, \tag{9}$$

for any vectors $x_1, x_2, ..., x_n$ in a Euclidean space H and for every n-tuple of real numbers $\lambda_1, \lambda_2, ..., \lambda_n$. This inequality is an inequality of the Cauchy–Bunyakovsky–Schwarz type for Euclidean spaces.

Corollary 1. *For any vectors $x_1, x_2, ..., x_n$ in a Euclidean space H, $n \geq 2$, we have:*

$$n \sum_{i=1}^{n} \|x_i\|^2 - \left\|\sum_{i=1}^{n} x_i\right\|^2 \geq \frac{n}{2} \cdot \max_{i,j \in \{1,...,n\}} \|x_i - x_j\|^2. \tag{10}$$

Proof. If in Inequality (7), we take $\lambda_1 = \lambda_2 = ... = \lambda_n \neq 0$, then we obtain Inequality (10). □

Remark 2. *If in Inequality (7), we take $x_1 = x_2 = ... = x_n \neq 0$, then we deduce the inequality:*

$$n \sum_{i=1}^{n} \lambda_i^2 - \left(\sum_{i=1}^{n} \lambda_i\right)^2 \geq \max_{i,j \in \{1,...,n\}} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i^2 + \lambda_j^2} \sum_{i=1}^{n} \lambda_i^2, \tag{11}$$

for $\lambda_1, \lambda_2, ..., \lambda_n$ real numbers with $\lambda_i \neq 0$, $i = \overline{1, n}$, $n \geq 2$. This inequality can be found in [20].

Next, we study the problem of the existence of such relations, as above, for the bounded linear operators on a Hilbert space. Next, we present several results related to the bounded linear operators on a Hilbert space.

Lemma 1. *Let $T_1, T_2 \in \mathbb{B}(H)$. Then, for real numbers a, b, the following identity holds:*

$$(a-b)(a|T_1|^2 - b|T_2|^2) + ab|T_1 + T_2|^2 = |aT_1 + bT_2|^2. \tag{12}$$

Proof. In the left part of the identity, we have:

$$
\begin{aligned}
&(a^2-ab)|T_1|^2+(b^2-ab)|T_2|^2)+ab(T_1^*+T_2^*)(T_1+T_2)\\
&=(a^2-ab)|T_1|^2+(b^2-ab)|T_2|^2)+ab|T_1|^2+abT_1^*T_2+abT_2^*T_1+ab|T_2|^2\\
&=a^2|T_1|^2+b^2|T_2|^2)+abT_1^*T_2+abT_2^*T_1=|aT_1+bT_2|^2.
\end{aligned}
$$

Therefore, the statement is true. □

Remark 3. *In Relation (12), if we replace b by $-b$, we deduce the equality:*

$$(a+b)(a|T_1|^2+b|T_2|^2)-ab|T_1+T_2|^2=|aT_1-bT_2|^2, \tag{13}$$

for all $T_1,T_2\in\mathbb{B}(H)$ and for every $a,b\in\mathbb{R}$.

Replacing T_2 by $-T_2$ in Relation (13), we obtain:

$$(a+b)(a|T_1|^2+b|T_2|^2)-ab|T_1-T_2|^2=|aT_1+bT_2|^2, \tag{14}$$

for all $T_1,T_2\in\mathbb{B}(H)$ and for every $a,b\in\mathbb{R}$.

If in Equality (14), we choose $a=1+t$ and $b=1+\frac{1}{t}$, where $t\in\mathbb{R}^*=\mathbb{R}-\{0\}$, we deduce an identity given by Fuji and Zuo [21]:

$$(1+t)|T_1|^2+(1+\frac{1}{t})|T_2|^2-|T_1-T_2|^2=\frac{1}{t}|tT_1+T_2|^2. \tag{15}$$

for all $T_1,T_2\in\mathbb{B}(H)$ and for every $t\in\mathbb{R}^*$.

Proposition 1. *Let $T_1,T_2\in\mathbb{B}(H)$. We assume that $p,q>1$, with $\frac{1}{p}+\frac{1}{q}=1$, then the following identity holds:*

$$p|T_1|^2+q|T_2|^2-|T_1+T_2|^2=\frac{1}{pq}|pT_1-qT_2|^2=\frac{1}{p-1}|(p-1)T_1-T_2|^2. \tag{16}$$

Proof. For $a=\frac{1}{q}$ and $b=\frac{1}{p}$ in Relation (13), we obtain the statement. □

Remark 4. *For $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, from Equality (16) and taking into account that $|pT_1-qT_2|^2\geq 0$, we deduce the Bohr inequality for operators [22]:*

$$p|T_1|^2+q|T_2|^2\geq|T_1+T_2|^2, \tag{17}$$

for all $T_1,T_2\in\mathbb{B}(H)$.

Using Equality (16), we obtain a reverse and an improvement of Bohr's inequality for operators given in [23,24]; thus, for $p\geq 2$, we have:

$$0\leq|T_1+T_2|^2\leq p|T_1|^2+q|T_2|^2\leq|T_1+T_2|^2+|(p-1)T_1-T_2|^2 \tag{18}$$

and for $1<p\leq 2$, we deduce:

$$p|T_1|^2+q|T_2|^2\geq|T_1+T_2|^2+|(p-1)T_1-T_2|^2. \tag{19}$$

Replacing in Relation (14) b by $1-a$, we find an identity from [23], given by:

$$a|T_1|^2+(1-a)|T_2|^2-|aT_1+(1-a)T_2|^2=a(1-a)|T_1-T_2|^2, \tag{20}$$

for all $T_1,T_2\in\mathbb{B}(H)$ and for every $a\in\mathbb{R}$.

If $a \in [0,1]$, then $a(1-a)|T_1 - T_2|^2 \geq 0$, so we obtain:

$$|aT_1 + (1-a)T_2|^2 \leq a|T_1|^2 + (1-a)|T_2|^2, \tag{21}$$

for all $T_1, T_2 \in \mathbb{B}(H)$. This implies the fact that application $|\cdot|^2$ is convex. Other extensions, generalizations, and improvement of Bohr's inequality can be found in [22,25,26].

Proposition 2. *Let $T_1, T_2 \in \mathbb{B}(H)$ and $a \in [0,1]$. Then, the following inequality holds:*

$$\frac{1}{2}min\{a, 1-a\}|T_1 - T_2|^2 \leq a|T_1|^2 + (1-a)|T_2|^2 - |aT_1 + (1-a)T_2|^2$$
$$\leq \frac{1}{2}max\{a, 1-a\}|T_1 - T_2|^2. \tag{22}$$

Equality holds when $a = \dfrac{1}{2}$ or $T_1 = T_2$.

Proof. Since we have the inequality:

$$\frac{1}{2}min\{a, 1-a\} \leq a(1-a) \leq \frac{1}{2}max\{a, 1-a\} \tag{23}$$

for any $a \in [0,1]$, using Identity (20), the inequality of the statement follows. Because, if $a \leq b \leq c$ and $T \in \mathbb{B}^+(H)$, then $aT \leq bT \leq cT$ $(T = |T_1 - T_2|^2 \geq 0)$, it is obvious that for $a = \dfrac{1}{2}$ we obtain the equality of the statement. For $T_1 = \lambda T_2$, we deduce:

$$\frac{1}{2}min\{a, 1-a\}(\lambda - 1)^2 \leq a(1-a)(\lambda - 1)^2 \leq \frac{1}{2}max\{a, 1-a\}(\lambda - 1)^2,$$

so we need to have $\lambda = 1$. □

Corollary 2. *Let $T_1, T_2 \in \mathbb{B}(H)$ and $a, b > 0$. Then, we have:*

$$\frac{a+b}{2}min\{a, b\}|T_1 - T_2|^2 \leq (a+b)(a|T_1|^2 + b|T_2|^2) - |aT_1 + bT_2|^2$$
$$\leq \frac{a+b}{2}max\{a, b\}|T_1 - T_2|^2. \tag{24}$$

Proof. Because $a, b > 0$ implies $1 > \dfrac{a}{a+b} > 0$, therefore, replacing a by $\dfrac{a}{a+b}$ in Inequality (22), we obtain the inequality of the statement. □

Next, we obtain another improvement of Bohr's inequality.

Corollary 3. *Let $T_1, T_2 \in \mathbb{B}(H)$ and $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequality:*

$$\frac{1}{2max\{p, q\}}|pT_1 - qT_2|^2 \leq p|T_1|^2 + q|T_2|^2 - |T_1 + T_2|^2 \leq \frac{1}{2min\{p, q\}}|pT_1 - qT_2|^2, \tag{25}$$

is true.

Proof. We replace in Relation (22) T_1 by pT_1, T_2 by pT_2, and a by $\frac{1}{p}$. Thus, by simple calculations, we obtain the inequality of the statement. □

Let $T_1, T_2 \in \mathbb{B}^{++}(H)$; we have the following operators:

$$T_1 \nabla_\lambda T_2 := (1-\lambda)T_1 + \lambda T_2,$$

called the arithmetic mean of operators T_1 and T_2 (see [27]).

Proposition 3. *If $T_1, T_2 \in \mathbb{B}^{++}(H)$ and $\lambda \in [0,1]$, then the following inequality holds:*

$$\frac{1}{2}min\{\lambda, 1-\lambda\}|T_1 - T_2|^2 \leq (1-\lambda)|T_1|^2 + \lambda|T_2|^2 - |T_1 \nabla_\lambda T_2|^2 \leq \frac{1}{2}max\{\lambda, 1-\lambda\}|T_1 - T_2|^2. \tag{26}$$

Equality holds when $\lambda = \frac{1}{2}$ and $T_1 = T_2$.

Let $T_1, T_2 \in \mathbb{B}^{++}(H)$; we have the following operators:

$$T_1 \sharp_\lambda T_2 := T_1^{1/2}\left(T_1^{-1/2} T_2 T_1^{-1/2}\right)^\lambda T_1^{1/2}$$

called the geometric mean of operators T_1 and T_2 (see [27]).

Proposition 4. *Let $T_1, T_2 \in \mathbb{B}^{++}(H)$ and $a \in [0,1]$. Then, the following inequality holds:*

$$\begin{aligned}\frac{1}{2}min\{\lambda, 1-\lambda\}|T_1 \nabla_\lambda T_2 - T_1 \sharp_\lambda T_2|^2 &\leq \lambda|T_1 \nabla_\lambda T_2|^2 + (1-\lambda)|T_1 \sharp_\lambda T_2|^2 - |(T_1 \nabla_\lambda T_2)\nabla_\lambda (T_1 \sharp_\lambda T_2)|^2 \\ &\leq \frac{1}{2}max\{\lambda, 1-\lambda\}|T_1 \nabla_\lambda T_2 - T_1 \sharp_\lambda T_2|^2. \end{aligned} \tag{27}$$

Equality holds when $\lambda = \frac{1}{2}$ or $T_1 = T_2$.

Proof. Since we replace in Relation (22) T_1 by $T_1 \nabla_\lambda T_2$ and T_2 by $T_1 \sharp_\lambda T_2$, the inequality of the statement follows. □

Proposition 5. *If $T_1, T_2 \in \mathbb{B}(H)$, then for real numbers $p_1, p_2 \neq 0$, with $p_1 + p_2 \neq 0$, the following equality holds:*

$$\frac{|T_1|^2}{p_1} + \frac{|T_2|^2}{p_2} - \frac{|T_1 + T_2|^2}{p_1 + p_2} = \frac{|p_2 T_1 - p_1 T_2|^2}{p_1 p_2 (p_1 + p_2)}. \tag{28}$$

Proof. In Relation (13), we take $a = p_2$ and $b = p_1$, and dividing by $p_1 p_2 (p_1 + p_2) \neq 0$, we obtain the equality of the statement. □

The variant for complex numbers of Relation (28) was given in [8,28,29].

Corollary 4. *Let $T_1, T_2 \in \mathbb{B}(H)$ and $p_1, p_2 \in \mathbb{R}$ such that $p_1 p_2 (p_1 + p_2) > 0$. Then, we have:*

$$\frac{|T_1|^2}{p_1} + \frac{|T_2|^2}{p_2} \geq \frac{|T_1 + T_2|^2}{p_1 + p_2}. \tag{29}$$

Equality holds if and only if $p_2 T_1 = p_1 T_2$.

Proof. Since the term $\frac{|p_2 T_1 - p_1 T_2|^2}{p_1 p_2 (p_1 + p_2)}$ is positive and uses Inequality (28), we obtain the inequality of the statement. □

This inequality can be viewed as Bergström's inequality for operators, the classical Bergström inequality can be found in [8].

Theorem 2. *For any operators $T_1, T_2, ..., T_n$ in $\mathbb{B}(H)$ and for arbitrary real numbers $\lambda_1, \lambda_2, ..., \lambda_n$, with $\lambda_i \neq 0$, $i = \overline{1,n}$, $n \geq 2$, we have:*

$$\sum_{i=1}^n \lambda_i^2 \sum_{i=1}^n |T_i|^2 - \Big|\sum_{i=1}^n \lambda_i T_i\Big|^2 \geq \frac{|\lambda_i T_j - \lambda_j T_i|^2}{\lambda_i^2 + \lambda_j^2} \sum_{i=1}^n \lambda_i^2, \tag{30}$$

for all $i, j \in \{1, ..., n\}$.

Proof. We consider sequence F_n, $n \geq 1$, given by:

$$F_n = \sum_{i=1}^{n} |T_i|^2 - \frac{\left| \sum\limits_{i=1}^{n} \lambda_i T_i \right|^2}{\sum\limits_{i=1}^{n} \lambda_i^2}.$$

We study the monotony of sequence $F_k, k \leq n$. Therefore, we have the difference between two consecutive terms of the sequence:

$$F_{k+1} - F_k = |T_{k+1}|^2 + \frac{\left| \sum\limits_{i=1}^{k} \lambda_i T_i \right|^2}{\sum\limits_{i=1}^{k} \lambda_i^2} - \frac{\left| \sum\limits_{i=1}^{k} \lambda_i T_i + \lambda_{k+1} T_{k+1} \right|^2}{\sum\limits_{i=1}^{k} \lambda_i^2 + \lambda_{k+1}^2}.$$

Using Bergström's inequality for operators, for two terms, we have:

$$|T_{k+1}|^2 + \frac{\left| \sum\limits_{i=1}^{k} \lambda_i T_i \right|^2}{\sum\limits_{i=1}^{k} \lambda_i^2} = \frac{\left| \lambda_{k+1} T_{k+1} \right|^2}{\lambda_{k+1}^2} + \frac{\left| \sum\limits_{i=1}^{k} \lambda_i T_i \right|^2}{\sum\limits_{i=1}^{k} \lambda_i^2} \geq \frac{\left| \sum\limits_{i=1}^{k} \lambda_i T_i + \lambda_{k+1} T_{k+1} \right|^2}{\sum\limits_{i=1}^{k} \lambda_i^2 + \lambda_{k+1}^2}$$

This means that $F_{k+1} - F_k \geq 0$, so sequence F_k is increasing.
Therefore, we obtain $F_n \geq F_{n-1} \geq ... \geq F_2 \geq F_1 = 0$. However,

$$F_2 = |T_1|^2 + |T_2|^2 - \frac{|\lambda_1 T_1 + \lambda_2 T_2|^2}{\lambda_1^2 + \lambda_2^2} = \frac{|\lambda_1 T_2 - \lambda_2 T_1|^2}{\lambda_1^2 + \lambda_2^2}$$

and taking into account that we can rearrange the terms of the sequences, we have the inequality:

$$F_n = \sum_{i=1}^{n} |T_i|^2 - \frac{\left| \sum\limits_{i=1}^{n} \lambda_i T_i \right|^2}{\sum\limits_{i=1}^{n} \lambda_i^2} \geq \frac{|\lambda_i T_j - \lambda_j T_i|^2}{\lambda_i^2 + \lambda_j^2},$$

for all $i, j \in \{1, ..., n\}$, $n \geq 2$. Multiplying the above inequality by $\sum\limits_{i=1}^{n} \lambda_i^2 > 0$, we deduce the inequality of the statement. $\square$

Remark 5. *This inequality represents an improvement of the C-B-S inequality for operators:*

$$\sum_{i=1}^{n} \lambda_i^2 \sum_{i=1}^{n} |T_i|^2 \geq \left| \sum_{i=1}^{n} \lambda_i T_i \right|^2, \tag{31}$$

for $n \geq 1$ *and for any operators* $T_1, T_2, ..., T_n$ *in* $\mathbb{B}(H)$ *and for arbitrary real numbers* $\lambda_1, \lambda_2, ..., \lambda_n$.

Theorem 3. *For any operators* $T_1, T_2, ..., T_n$ *in* $\mathbb{B}(H)$ *and for complex numbers* $b_1, b_2, ..., b_n \in \mathbb{C}$, *such that there is at least one* $b_i \neq 0$, *we have:*

$$\left(\sum_{i=1}^{n} |b_i|^2 \right) \sum_{i=1}^{n} |T_i|^2 - |\sum_{i=1}^{n} b_i T_i|^2 = \left(\sum_{i=1}^{n} |b_i|^2 \right) \sum_{i=1}^{n} |T_i - \overline{b}_i V|^2, \tag{32}$$

where $V = \left(\sum_{i=1}^{n} b_i T_i\right)\left(\sum_{i=1}^{n} |b_i|^2\right)^{-1}$.

Proof. For $\sum_{i=1}^{n} |b_i|^2 \neq 0$, we have the following:

$$\sum_{i=1}^{n} |T_i - \overline{b}_i V|^2 = \sum_{i=1}^{n} (T_i^* - b_i V^*)(T_i - \overline{b}_i V)$$

$$= \sum_{i=1}^{n} |T_i|^2 - \sum_{i=1}^{n} \overline{b}_i T_i^* V - \sum_{i=1}^{n} b_i V^* T_i + \left(\sum_{i=1}^{n} |b_i|^2 \right) |V|^2. \tag{33}$$

If we take $V = (\sum_{i=1}^{n} b_i T_i)\left(\sum_{i=1}^{n} |b_i|^2\right)^{-1}$ in Relation (33), then we deduce the relation:

$$\sum_{i=1}^{n} |T_i - \overline{b}_k V|^2 = \sum_{i=1}^{n} |T_i|^2 - \frac{|\sum_{i=1}^{n} b_i T_i|^2}{\sum_{i=1}^{n} |b_i|^2},$$

which proves the equality of the statement. □

Lemma 2. *Let T_1, T_2, T_3 in $\mathbb{B}(H)$. Then, the equality holds:*

$$|T_1|^2 + |T_2|^2 + |T_3|^2 + |T_1 + T_2 + T_3|^2 = |T_1 + T_2|^2 + |T_2 + T_3|^2 + |T_1 + T_3|^2. \tag{34}$$

Proof. Using the properties of the modulus operator, we have the following calculations:

$$|T_1 + T_2|^2 + |T_2 + T_3|^2 + |T_1 + T_3|^2 = (T_1^* + T_2^*)(T_1 + T_2) + (T_2^* + T_3^*)(T_2 + T_3)$$
$$+(T_1^* + T_3^*)(T_1 + T_3) = 2(|T_1|^2 + |T_2|^2 + |T_3|^2) + T_1^* T_2 + T_2^* T_1 + T_2^* T_3 + T_3^* T_2$$
$$+T_1^* T_3 + T_3^* T_1 = |T_1|^2 + |T_2|^2 + |T_3|^2 + |T_1 + T_2 + T_3|^2.$$

Consequently, the proof is complete. □

Proposition 6. *Let $T_1, T_2, T_3 \in \mathbb{B}(H)$. Then, for real numbers $b_1, b_2, b_3 \neq 0$, the following equality holds:*

$$(b_1 + b_2 + b_3)\left(\frac{|T_1|^2}{b_1} + \frac{|T_2|^2}{b_2} + \frac{|T_3|^2}{b_3} \right) - |T_1 + T_2 + T_3|^2$$

$$= \frac{|b_2 T_1 - b_1 T_2|^2}{b_1 b_2} + \frac{|b_3 T_1 - b_1 T_2|^2}{b_1 b_2} + \frac{|b_2 T_1 - b_1 T_2|^2}{b_1 b_2}. \tag{35}$$

Proof. From Equality (28), we obtain the following three equalities:

$$(b_1 + b_2)\left(\frac{|T_1|^2}{b_1} + \frac{|T_2|^2}{b_2} \right) = |T_1 + T_2|^2 + \frac{|b_2 T_1 - b_1 T_2|^2}{b_1 b_2},$$

$$(b_2 + b_3)\left(\frac{|T_2|^2}{b_2} + \frac{|T_3|^2}{b_3} \right) = |T_2 + T_3|^2 + \frac{|b_3 T_2 - b_2 T_3|^2}{b_2 b_3},$$

$$(b_3 + b_1)\left(\frac{|T_3|^2}{b_3} + \frac{|T_1|^2}{b_1} \right) = |T_3 + T_1|^2 + \frac{|b_1 T_3 - b_3 T_1|^2}{b_1 b_3}. \tag{36}$$

Adding the above relations, we deduce:

$$|T_1|^2 + |T_2|^2 + |T_3|^2 + (b_1 + b_2 + b_3)\left(\frac{|T_1|^2}{b_1} + \frac{|T_2|^2}{b_2} + \frac{|T_3|^2}{b_3} \right)$$

$$= |T_1 + T_2|^2 + |T_2 + T_3|^2 + |T_1 + T_3|^2 + \sum_{1 \leq i < j \leq 3} \frac{|b_i T_j - b_j T_i|^2}{b_i b_j}. \tag{37}$$

Therefore, using Lemma 17, we obtain the equality of the statement. □

The identity from Proposition 18 suggests a general result for n operators, namely:

Theorem 4. *For any operators $T_1, T_2, ..., T_n$ in $\mathbb{B}(H)$ and for real numbers $b_1, b_2, ..., b_n \in \mathbb{R}^*$, $n \geq 2$, we have:*

$$\left(\sum_{i=1}^{n} b_i\right) \sum_{i=1}^{n} \frac{|T_i|^2}{b_i} - |\sum_{i=1}^{n} T_i|^2 = \sum_{1\leq i<j\leq n} \frac{|b_i T_j - b_j T_i|^2}{b_i b_j}. \tag{38}$$

Proof. We use the mathematical induction to prove the relation of the statement. We consider the following proposition:

$$P(n) : \left(\sum_{i=1}^{n} b_i\right) \sum_{i=1}^{n} \frac{|T_i|^2}{b_i} - |\sum_{i=1}^{n} T_i|^2 = \sum_{1\leq i<j\leq n} \frac{|b_i T_j - b_j T_i|^2}{b_i b_j}, n \geq 2.$$

For $n = 2$, the proposition is true, taking into account the relation from Proposition 12. Assume that $P(n)$ is true; we will prove that $P(n+1)$ is true.

$$\left(\sum_{i=1}^{n+1} b_i\right) \sum_{i=1}^{n+1} \frac{|T_i|^2}{b_i} - |\sum_{i=1}^{n+1} T_i|^2 =$$

$$\left(\sum_{i=1}^{n} b_i\right) \sum_{i=1}^{n} \frac{|T_i|^2}{b_i} + b_{n+1}\frac{|T_1|^2}{b_1} + ... + b_{n+1}\frac{|T_n|^2}{b_n} + |T_{n+1}|^2 - |\sum_{i=1}^{n} T_i|^2$$

$$-(T_1^* + ... + T_n^*)T_{n+1} - (T_1 + ... + T_n)T_{n+1}^* - |T_{n+1}|^2$$

$$= \sum_{1\leq i<j\leq n} \frac{|b_i T_j - b_j T_i|^2}{b_i b_j} + \sum_{i=1}^{n} \frac{|b_i T_{n+1} - b_{n+1} T_i|^2}{b_i b_{n+1}} = \sum_{1\leq i<j\leq n+1} \frac{|b_i T_j - b_j T_i|^2}{b_i b_j}.$$

Therefore, from the principle of mathematical induction, we deduce the statement. □

The above results were given in [28] for the commuting Gramian normal operators, and Equality (38) was given in [8] for complex numbers.

Remark 6. *If $b_1 = b_2 = ... = b_n$, $n \geq 2$, then Relation (38), becomes:*

$$n\left(\sum_{i=1}^{n} |T_i|^2\right) - |\sum_{i=1}^{n} T_i|^2 = \sum_{1\leq i<j\leq n} |T_i - T_j|^2, \tag{39}$$

for any operators $T_1, T_2, ..., T_n$ in $\mathbb{B}(H)$.

Corollary 5. *Let $T_1, T_2, ..., T_n \in \mathbb{B}(H)$, $n \geq 2$, and real numbers $b_1, b_2, ..., b_n \in \mathbb{R}$. Then, the following equality holds:*

$$\left(\sum_{i=1}^{n} b_i^2\right) \sum_{i=1}^{n} |T_i|^2 - |\sum_{i=1}^{n} b_i T_i|^2 = \sum_{1\leq i<j\leq n} |b_i T_j - b_j T_i|^2. \tag{40}$$

Proof. If $b_i \in \mathbb{R}^*$, for every $i \in \{1, ..., n\}$, then replacing b_i by b_i^2 and T_i by $b_i T_i$, for all $i \in \{1, ..., n\}$, in Relation (38), we deduce the statement. Assume that $b_i \neq 0$ for $i \in \{1, ..., k\}$ and $b_i = 0$ for $i \in \{k+1, ..., n\}$; we proved Relation (40) for k terms. □

Remark 7. *If $T_i \in \mathbb{B}(H)$ and $r_i > 1$, $i \in \{1, ..., n\}$, with $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, then for $b_i = \frac{1}{r_i}$ in the equality from Theorem 19, we obtain:*

$$\sum_{i=1}^{n} r_i |T_i|^2 - |\sum_{i=1}^{n} T_i|^2 = \sum_{1 \le i < j \le n} \frac{|r_i T_i - r_j T_j|^2}{r_i r_j} = \sum_{1 \le i < j \le n} \left| \sqrt{\frac{r_i}{rj}} T_i - \sqrt{\frac{r_j}{r_i}} T_j \right|^2, \quad (41)$$

the identity given by Fuji and Zuo in [21].

Now, we will focus on a general result related to Aczél's inequality for operators, namely:

Theorem 5. *For any operators $T, T_1, T_2, ..., T_n$ in $\mathbb{B}(H)$ and for positive real numbers $b, b_1, b_2, ..., b_n$ such that $b^2 - b_1^2 - ... - b_n^2 > 0$, $n \ge 1$, we have:*

$$\begin{aligned} \left| bT - \sum_{i=1}^{n} b_i T_i \right|^2 - \left(|T|^2 - \sum_{i=1}^{n} |T_i|^2 \right) \left(b^2 - \sum_{i=1}^{n} b_i^2 \right) \ge \\ \ge \frac{|b_j T - b T_j|^2}{b^2 - b_j^2} \left(b^2 - \sum_{i=1}^{n} b_i^2 \right) \ge 0. \end{aligned} \quad (42)$$

for all $j \in \{1, ..., n\}$.

Proof. We use the technique of monotony to sequence A_n, which is defined as follows:

$$A_n = \frac{\left| aT - \sum_{i=1}^{n} b_i T_i \right|^2}{b^2 - \sum_{i=1}^{n} b_i^2} - |A|^2 + \sum_{i=1}^{n} |T_i|^2, n \ge 1.$$

For $k \le n$, we have:

$$A_{k+1} - A_k = \frac{\left| bT - \sum_{i=1}^{k} b_i T_i - b_{k+1} T_{k+1} \right|^2}{b^2 - \sum_{i=1}^{k} b_i^2 - b_{k+1}^2} + |T_{k+1}|^2 - \frac{\left| bT - \sum_{i=1}^{k} b_i T_i \right|^2}{b^2 - \sum_{i=1}^{k} b_i^2}.$$

Using Bergström's inequality for operators, for two terms, we have:

$$\frac{\left| bT - \sum_{i=1}^{k} b_i T_i - b_{k+1} T_{k+1} \right|^2}{b^2 - \sum_{i=1}^{k} b_i^2 - b_{k+1}^2} + \frac{|b_{k+1} T_{k+1}|^2}{b_{k+1}^2} \ge \frac{\left| bT - \sum_{i=1}^{k} b_i T_i \right|^2}{b^2 - \sum_{i=1}^{k} b_i^2}.$$

This means that $A_{k+1} - A_k \ge 0$, so sequence A_k is increasing.
Therefore, we obtain $A_n \ge A_{n-1} \ge ... \ge A_2 \ge A_1$. However,

$$A_1 = \frac{|bT - b_1 T_1|^2}{b^2 - b_1^2} - |T|^2 + |T_1|^2 = \frac{|b_1 T - b T_1|^2}{b^2 - b_1^2}$$

and taking into account that we can rearrange the terms of the two sequences, we have the inequality:

$$A_n \ge \frac{|b_j T - b T_j|^2}{b^2 - b_j^2},$$

for all $j \in \{1, ..., n\}$. Multiplying by $b^2 - \sum\limits_{i=1}^{n} b_i^2 > 0$, we deduce the inequality of the statement. □

Remark 8. *Inequality (42) gives an inequality of the Aczél type for operators; thus:*

$$\left(|T|^2 - \sum_{i=1}^{n} |T_i|^2\right)\left(b^2 - \sum_{i=1}^{n} b_i^2\right) \leq \left|bT - \sum_{i=1}^{n} b_i T_i\right|^2. \tag{43}$$

3. Applications of Some Identities of Hermitian Operators

If we choose various particular cases for different classes of operators, then we deduce a series of known identities. Therefore, we have the following:

(1) If we take $T_i = a_i I$, where I is the identity operator and $a_i \in \mathbb{R}$, then using Relation (40), we find:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) I - \left(\sum_{i=1}^{n} a_i b_i\right)^2 I = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2 I,$$

which means that:

$$\left(\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 - \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2\right) I = \mathbf{0}.$$

However, when $qI = \mathbf{0}$, $q \in \mathbb{R}$ implies $q = 0$, because $0 = \langle 0, x \rangle = \langle qIx, x \rangle = q\langle x, x \rangle = q\|x\|^2$, for every $x \in H$, so $q = 0$. Therefore, we obtain the Lagrange identity:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2. \tag{44}$$

(2) If we take $T_i = T - a_i I$ in Relation (40), where T is the Hermitian operator, $T \neq 0$, and $a_i \in \mathbb{R}$, then we deduce:

$$\left(n\sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} b_i\right)^2 - \sum_{1 \leq i < j \leq n} (b_i - b_j)^2\right)|T|^2$$

$$-2\left(\sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i b_i - \sum_{1 \leq i < j \leq n} (b_i - b_j)(b_i a_j - b_j a_i)\right) T$$

$$+\left(\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2\right) I = \mathbf{0}.$$

For all $i \in \{1, 2, ..., n\}$, we have:

$$n\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} b_i\right)^2 = \sum_{1 \leq i < j \leq n} (b_j - b_i)^2. \tag{45}$$

From the above relations, we deduce that:

$$-2\left(\sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i b_i - \sum_{1 \leq i < j \leq n} (b_i - b_j)(b_i a_j - b_j a_i)\right) T = 0,$$

and it follows that:

$$\sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i b_i = \sum_{1\le i<j\le n} (b_i - b_j)(b_i a_j - b_j a_i), \tag{46}$$

because from $qT = \mathbf{0}$, for a nonzero Hermitian operator T and $q \in \mathbb{R}$, we deduce $q = 0$. In the same way, if in Relation (40), we take $T_i = c_i T - a_i I$, we obtain the following identity:

$$\sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} a_i c_i - \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} b_i c_i = \sum_{1\le i<j\le n} (b_i c_j - b_j c_i)(b_i a_j - b_j a_i). \tag{47}$$

In Relation (47) for $b_i = 1$, for all $i \in \{1, 2, ..., n\}$, we have:

$$n\sum_{i=1}^{n} a_i c_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} c_i = \sum_{1\le i<j\le n} (a_i - a_j)(c_i - c_j). \tag{48}$$

This proves Chebyshev's inequality [7]; thus:

(a) If $a_1 \ge a_2 \ge ... \ge a_n$ and $c_1 \ge c_2 \ge ... \ge c_n$, then we deduce:

$$n\sum_{i=1}^{n} a_i c_i \ge \sum_{i=1}^{n} a_i \sum_{i=1}^{n} c_i; \tag{49}$$

(b) If $a_1 \ge a_2 \ge ... \ge a_n$ and $c_1 \le c_2 \le ... \le c_n$, then we have:

$$n\sum_{i=1}^{n} a_i c_i \le \sum_{i=1}^{n} a_i \sum_{i=1}^{n} c_i. \tag{50}$$

In Equality (47), if $b_i > 0$, for all $i \in \{1, 2, ..., n\}$, then we replace b_i by $\sqrt{b_i}$, c_i by $\sqrt{b_i} c_i$, and a_i by $\sqrt{b_i} a_i$, and then, we have:

$$\sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i b_i c_i - \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} b_i c_i = \sum_{1\le i<j\le n} (b_i c_j - b_j c_i)(b_i a_j - b_j a_i). \tag{51}$$

In Equality (47), if $\alpha_i, \gamma_i > 0$, for all $i \in \{1, 2, ..., n\}$, then we take $b_i = \sqrt{\alpha_i \gamma_i}$, $c_i = \sqrt{\frac{\gamma_i}{\alpha_i}} \beta_i$, $a_i = \sqrt{\frac{\alpha_i}{\gamma_i}} \delta_i$, $\beta_i, \delta_i \in \mathbb{R}$, and then, we have:

$$\sum_{i=1}^{n} \alpha_i \gamma_i \sum_{i=1}^{n} \beta_i \delta_i - \sum_{i=1}^{n} \alpha_i \delta_i \sum_{i=1}^{n} \beta_i \gamma_i = \sum_{1\le i<j\le n} (\alpha_i \beta_j - \alpha_j \beta_i)(\gamma_i \delta_j - \gamma_j \delta_i), \tag{52}$$

which is in fact the Cauchy–Binet formula [7].

In this paper, some inequalities that characterize the bounds of the variance of a random variable in the discrete case can be identified, where the variance is an important statistical indicator that measures the degree of data dispersion. In this sense, Inequalities (10) and (11) can be seen.

Consequently, for a random variable in the discrete case X, with $P(X = \lambda_i) = \frac{1}{n}$, for all $i \in \{1, ..., n\}$, we deduce:

$$Var(X) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i\right)^2 \ge \frac{1}{n^2} \max_{i,j\in\{1,...,n\}} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i^2 + \lambda_j^2} \sum_{i=1}^{n} \lambda_i^2, \tag{53}$$

for $\lambda_1, \lambda_2, ..., \lambda_n$ are real numbers with $\lambda_i \ne 0, i = \overline{1, n}$, $n \ge 2$.

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