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A Hilbert-Type Integral Inequality in the Whole Plane Related to the Arc Tangent Function

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Abstract: In this work we establish a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the kernel of the arc tangent function. We prove that the constant factor, which is associated with the cosine function, is optimal. Some special cases as well as some operator expressions are also presented.

Keywords: Hilbert-type integral inequality; weight function; equivalent statement; operator; cosine function

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1. Introduction

If

$$0 < \int_0^\infty f^2(x)dx < \infty \ \ and \ \ 0 < \int_0^\infty g^2(y)dy < \infty,$$

then we have the following well-known Hilbert integral inequality (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy\right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Recently, using weight functions, some extensions of (1) were established in Yang's two books (see [2,3]) and the papers [4–9]. Most of them are constructed in the quarter plane of the first quadrant.

In 2007, Yang [10] proved the following Hilbert-type integral inequality in the whole plane (namely (x, y)-plane) involving the exponential function:

$$\begin{aligned}&\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda}dxdy \\ < \ & B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_{-\infty}^\infty e^{-\lambda x}f^2(x)dx \int_{-\infty}^\infty e^{-\lambda y}g^2(y)dy\right)^{\frac{1}{2}},\end{aligned} \tag{2}$$

with the best possible constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$, $\lambda > 0$, where by $B(u,v)$ we denote the beta function). In the papers [11–22], the authors have presented some new Hilbert-type integral inequalities in the whole plane for which they have established optimal constant factors.

In 2017, Hong [23] proved two equivalent statements between a Hilbert-type inequality with the general homogenous kernel and a few parameters. This domain of research is very vibrant with many authors investigating other types of integral inequalities (cf. [24–38]).

In this paper, we follow the idea of Hong's work in [23] and using techniques of real analysis as well as weight functions, we prove a few equivalent statements of a Hilbert-type integral inequality in the whole plane related to the kernel of the arc tangent function. The constant factor which is related to the cosine function is proved to be the best possible. Within this work, we also consider some particular cases of interest as well as operator expressions.

2. Some Lemmas

For $\rho > 0, 0 < \sigma < \gamma$, setting $h(u) := \arctan \frac{\rho}{u^\gamma}$ $(u > 0)$, we obtain

$$\begin{aligned} k_\rho^{(\gamma)}(\sigma) &:= \int_0^\infty h(u) u^{\sigma-1} du \\ &= \int_0^\infty \left(\arctan \frac{\rho}{u^\gamma}\right) u^{\sigma-1} du \ (v = \rho^2 u^{-2\gamma}) \\ &= \frac{\rho^{\sigma/\gamma}}{2\gamma} \int_0^\infty (\arctan v^{\frac{1}{2}}) v^{\frac{-\sigma}{2\gamma}-1} dv \\ &= \frac{-\rho^{\sigma/\gamma}}{\sigma} \int_0^\infty (\arctan v^{\frac{1}{2}}) dv^{\frac{-\sigma}{2\gamma}} \\ &= \frac{\rho^{\sigma/\gamma}}{2\sigma} \int_0^\infty \frac{v^{\frac{\gamma-\sigma}{2\gamma}-1}}{1+v} dv = \frac{\rho^{\sigma/\gamma}\pi}{2\sigma \sin \frac{\pi(\gamma-\sigma)}{2\gamma}} \\ &= \frac{\rho^{\sigma/\gamma}\pi}{2\sigma \cos \frac{\pi\sigma}{2\gamma}} \in \mathbf{R}_+ = (0, \infty). \end{aligned} \tag{3}$$

For $\mathbf{R} := (-\infty, \infty), \delta \in \{-1, 1\}, \alpha, \beta \in (-1, 1)$, we set

$$\begin{aligned} x_\alpha &: = |x| + \alpha x, y_\beta := |y| + \beta y \ \ (x, y \in \mathbf{R}), \\ E_\delta &: = \{t \in \mathbf{R}; |t|^\delta \geq 1\}, E_{-\delta} = \{t \in \mathbf{R}; |t|^\delta \leq 1\}. \end{aligned}$$

Lemma 1. *For $c > 0, \theta = \alpha, \beta$, we have*

$$\int_{E_\delta} t_\theta^{-c\delta-1} dt = \frac{1}{c}\left[\frac{1}{(1+\theta)^{c\delta+1}} + \frac{1}{(1-\theta)^{c\delta+1}}\right], \tag{4}$$

$$\int_{E_{-\delta}} t_\theta^{c\delta-1} dt = \frac{1}{c}\left[\frac{1}{(1+\theta)^{-c\delta+1}} + \frac{1}{(1-\theta)^{-c\delta+1}}\right]. \tag{5}$$

For $c \leq 0$, it follows that

$$\int_{E_\delta} t_\theta^{-c\delta-1} dt = \int_{E_{-\delta}} t_\theta^{c\delta-1} dt = \infty.$$

Proof. Setting

$$E_\delta^+ := \{t \in \mathbf{R}_+; t^\delta \geq 1\}, \ E_\delta^- := \{-t \in \mathbf{R}_+; (-t)^\delta \geq 1\},$$

it follows that $E_\delta = E_\delta^+ \cup E_\delta^-$ and

$$\begin{aligned} \int_{E_\delta} t_\theta^{-c\delta-1} dt &= \int_{E_\delta^+} [(1+\theta)t]^{-c\delta-1} dt + \int_{E_\delta^-} [(1-\theta)(-t)]^{-c\delta-1} dt \\ &= \left[\frac{1}{(1+\theta)^{c\delta+1}} + \frac{1}{(1-\theta)^{c\delta+1}}\right] \int_{E_\delta^+} t^{-c\delta-1} dt. \end{aligned}$$

Setting $u = t^{\delta}$ (or $t = u^{\frac{1}{\delta}}$), we obtain

$$\int_{E_{\delta}^{+}} t^{-c\delta-1}dt = \frac{1}{|\delta|}\int_{1}^{\infty} u^{\frac{1}{\delta}(-c\delta-1)}u^{\frac{1}{\delta}-1}du = \int_{1}^{\infty} u^{-c-1}du.$$

Hence, for $c > 0$, Formula (4) follows and for $c \leq 0$, we get that

$$\int_{E_{\delta}} t_{\theta}^{-c\delta-1}dt = \infty.$$

Since

$$\begin{aligned}\int_{E_{-\delta}} t_{\theta}^{c\delta-1}dt &= \int_{E_{(-\delta)}} t_{\theta}^{-c(-\delta)-1}dt \\ &= \left[\frac{1}{(1+\theta)^{-c\delta+1}} + \frac{1}{(1-\theta)^{-c\delta+1}}\right]\int_{0}^{1} u^{c-1}du,\end{aligned}$$

for $c > 0$, Equation (5) follows and for $c \leq 0$, we have

$$\int_{E_{-\delta}} t_{\theta}^{c\delta-1}dt = \infty.$$

This completes the proof of the lemma. □

In what follows, we assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\delta \in \{-1, 1\}$, $\alpha, \beta \in (-1, 1)$, $\rho > 0$, $0 < \sigma < \gamma$, $\sigma_1 \in \mathbf{R}$,

$$h(u) = \arctan\frac{\rho}{u^{\gamma}} \ \ (u > 0),$$

$k_{\rho}^{(\gamma)}(\sigma)$ is indicated by (3) and

$$K_{\alpha,\beta}^{(\gamma)}(\sigma) := \frac{2k_{\rho}^{(\gamma)}(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}}. \tag{6}$$

For $n \in \mathbf{N} = \{1, 2, \cdots\}$, $E_{-1} = [-1, 1]$, $x \in E_{\delta}$, we define the following expressions:

$$\begin{aligned}I^{(-)}(x) &:= \int_{-1}^{0}\left[\arctan\frac{\rho}{(x_{\alpha}^{\delta}y_{\beta})^{\gamma}}\right]y_{\beta}^{\sigma+\frac{1}{qn}-1}dy, \\ I^{(+)}(x) &:= \int_{0}^{1}\left[\arctan\frac{\rho}{(x_{\alpha}^{\delta}y_{\beta})^{\gamma}}\right]y_{\beta}^{\sigma+\frac{1}{qn}-1}dy, \\ I(x) &:= \int_{E_{-1}}\left[\arctan\frac{\rho}{(x_{\alpha}^{\delta}y_{\beta})^{\gamma}}\right]y_{\beta}^{\sigma+\frac{1}{qn}-1}dy = I^{(-)}(x) + I^{(+)}(x).\end{aligned}$$

Since $y_{\beta} = |y| + \beta y = (sgn(y) + \beta)y$, where

$$sgn(y) := \begin{cases} -1, y < 0 \\ 0, y = 0 \\ 1, y > 0 \end{cases},$$

$$x_{\alpha}^{\delta} = (1 + \alpha \cdot sgn(x))^{\delta}|x|^{\delta} \geq \min_{\delta\in\{-1,1\}}(1 \pm |\alpha|)^{\delta} \ \ (x \in E_{\delta}),$$

and $1 - |\alpha| \leq (1 + |\alpha|)^{-1} \leq 1 + |\alpha| \leq (1 - |\alpha|)^{-1}$, we have

$$(1 \pm \beta)x_{\alpha}^{\delta} \geq m_{\alpha,\beta} := (1 - |\beta|)(1 - |\alpha|) > 0 \ \ (x \in E_{\delta}). \tag{7}$$

For fixed $x \in E_\delta$, setting $u = x_\alpha^\delta y_\beta$, we obtain

$$\begin{aligned}
I^{(-)}(x) &= \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1-\beta}\int_0^{(1-\beta)x_\alpha^\delta}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du \\
&\geq \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1-\beta}\int_0^{m_{\alpha,\beta}}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du, \\
I^{(+)}(x) &= \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{(1+\beta)x_\alpha^\delta}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du \\
&\geq \frac{x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{m_{\alpha,\beta}}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du, \\
I(x) &= x_\alpha^{-\delta(\sigma+\frac{1}{qn})}\left[\frac{1}{1-\beta}\int_0^{(1-\beta)x_\alpha^\delta}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du\right. \\
&\quad \left.+\frac{1}{1+\beta}\int_0^{(1+\beta)x_\alpha^\delta}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du\right] \\
&\geq \frac{2x_\alpha^{-\delta(\sigma+\frac{1}{qn})}}{1-\beta^2}\int_0^{m_{\alpha,\beta}}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du. \qquad (8)
\end{aligned}$$

For $n \in \mathbf{N}$, $x \in F_\delta$, we define the following expressions:

$$\begin{aligned}
J^{(-)}(x) &: = \int_{-\infty}^{-1} y_\beta^{\sigma+\frac{1}{qn}-1}\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}dy, \\
J^{(+)}(x) &: = \int_1^{\infty} y_\beta^{\sigma+\frac{1}{qn}-1}\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}dy, \\
J(x) &: = \int_{E_1} y_\beta^{\sigma+\frac{1}{qn}-1}\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}dy = J^{(-)}(x)+J^{(+)}(x).
\end{aligned}$$

Since for $x \in E_{-\delta}$,

$$x_\alpha^\delta = (1+\alpha\cdot sgn(x))^\delta|x|^\delta \leq \max_{\delta\in\{-1,1\}}\{(1\pm|\alpha|)^\delta\} = (1-|\alpha|)^{-1},$$

we have

$$M_{\alpha,\beta} := (1+|\beta|)(1-|\alpha|)^{-1} \geq (1\pm\beta)x_\alpha^\delta \ \ (x \in E_{-\delta}). \qquad (9)$$

For fixed $x \in E_{-\delta}$, setting $u = x_\alpha^\delta y_\beta$, we obtain

$$\begin{aligned} J^{(-)}(x) &= \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1-\beta}\int_{(1-\beta)x_\alpha^\delta}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du \\ &\geq \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1-\beta}\int_{M_{\alpha,\beta}}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du, \\ J^{(+)}(x) &= \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1+\beta}\int_{(1+\beta)x_\alpha^\delta}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du \\ &\geq \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1+\beta}\int_{M_{\alpha,\beta}}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du, \\ J(x) &= x_\alpha^{-\delta(\sigma-\frac{1}{qn})}\left[\frac{1}{1-\beta}\int_{(1-\beta)x_\alpha^\delta}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du\right. \\ &\quad \left.+\frac{1}{1+\beta}\int_{(1+\beta)x_\alpha^\delta}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du\right] \\ &\geq \frac{2x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1-\beta^2}\int_{M_{\alpha,\beta}}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du. \end{aligned} \tag{10}$$

In view of (8) and (10), we derive the lemma below:

Lemma 2. *We have the following inequalities:*

$$\begin{aligned} I_1 &:= \int_{E_\delta} I(x)x_\alpha^{\delta(\sigma_1-\frac{1}{pn})-1}dx \\ &\geq \frac{2}{1-\beta^2}\int_{E_\delta} x_\alpha^{-\delta(\sigma-\sigma_1+\frac{1}{n})-1}dx\int_0^{m_{\alpha,\beta}}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du, \end{aligned} \tag{11}$$

$$\begin{aligned} J_1 &:= \int_{E_{-\delta}} J(x)x_\alpha^{\delta(\sigma_1+\frac{1}{pn})-1}dx \\ &\geq \frac{2}{1-\beta^2}\int_{E_{-\delta}} x_\alpha^{\delta(\sigma_1-\sigma+\frac{1}{n})-1}dx\int_{M_{\alpha,\beta}}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du. \end{aligned} \tag{12}$$

Lemma 3. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned} I &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right]f(x)g(y)dxdy \\ &\leq M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}} \end{aligned} \tag{13}$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N})$, we consider the functions

$$f_n(x) := \begin{cases} x_\alpha^{\delta(\sigma_1-\frac{1}{pn})-1}, x \in E_\delta \\ 0, x \in \mathbf{R}\backslash E_\delta \end{cases}, \quad g_n(y) := \begin{cases} y_\beta^{\sigma+\frac{1}{qn}-1}, y \in E_{-1} \\ 0, y \in \mathbf{R}\backslash E_{-1} \end{cases},$$

and by (4) and (5), we obtain

$$
\begin{aligned}
\widetilde{J}_1 \quad : \quad &= \left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g_n^q(y)dy\right]^{\frac{1}{q}} \\
&= \left(\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{E_{-1}} y_\beta^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} \\
&= n\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right]^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{(1+\beta)^{-\frac{1}{n}+1}} + \frac{1}{(1-\beta)^{-\frac{1}{n}+1}}\right]^{\frac{1}{q}} < \infty.
\end{aligned}
$$

By (11) and (13) (for $f = f_n, g = g_n$), we have

$$
\begin{aligned}
&\frac{2}{1-\beta^2}\int_{E_\delta} x_\alpha^{-\delta(\sigma-\sigma_1+\frac{1}{n})-1}dx \int_0^{m_{\alpha,\beta}} (\arctan \frac{\rho}{u^\gamma}) u^{\sigma+\frac{1}{qn}-1}du \\
\leq\ & I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan \frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] f_n(x)g_n(y)dxdy \\
\leq\ & M\widetilde{J}_1 < \infty.
\end{aligned}
$$

Since for any $n \geq \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N}), \sigma - \sigma_1 + \frac{1}{n} \leq 0$, by Lemma 1 it follows that

$$
\int_{E_\delta} x_\alpha^{-\delta(\sigma-\sigma_1+\frac{1}{n})-1}dx = \infty.
$$

In view of

$$
\int_0^{m_{\alpha,\beta}} (\arctan \frac{\rho}{u^\gamma}) u^{\sigma+\frac{1}{qn}-1}du > 0,
$$

we derive that $\infty \leq M\widetilde{J}_1 < \infty$, which is a contradiction.

If $\sigma > \sigma_1$, then for $n \geq \frac{1}{\sigma-\sigma_1}$ $(n \in \mathbf{N})$, we consider the functions

$$
\widetilde{f}_n(x) := \begin{cases} x_\alpha^{\delta(\sigma_1+\frac{1}{pn})-1}, x \in F_\delta \\ 0, x \in \mathbf{R}\backslash F_\delta \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} y_\beta^{\sigma-\frac{1}{qn}-1}, y \in E_1 \\ 0, y \in \mathbf{R}\backslash E_1 \end{cases},
$$

and by (4) and (5), we obtain

$$
\begin{aligned}
\widetilde{J}_2 \quad : \quad &= \left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\
&= \left(\int_{E_{-\delta}} x_\alpha^{\frac{\delta}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{E_1} y_\beta^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}} \\
&= n\left[\frac{1}{(1+\alpha)^{-\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{-\frac{\delta}{n}+1}}\right]^{\frac{1}{p}} \\
&\quad \times \left[\frac{1}{(1+\beta)^{\frac{1}{n}+1}} + \frac{1}{(1-\beta)^{\frac{1}{n}+1}}\right]^{\frac{1}{q}}.
\end{aligned}
$$

By (12) and (13) (for $f = \widetilde{f}_n, g = \widetilde{g}_n$), we have

$$\begin{aligned} & \frac{2}{1-\beta^2}\int_{E_{-\delta}} x_\alpha^{\delta(\sigma_1-\sigma+\frac{1}{n})-1}dx\int_{M_{\alpha,\beta}}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma-\frac{1}{qn}-1}du \\ \leq\ & J_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right]\widetilde{f}_n(x)\widetilde{g}_n(y)dxdy \\ \leq\ & M\widetilde{J}_2 < \infty. \end{aligned}$$

Since for $n \geq \frac{1}{\sigma-\sigma_1}$ $(n \in \mathbf{N})$, $\sigma_1 - \sigma + \frac{1}{n} \leq 0$, by Lemma 1 it follows that

$$\int_{F_\delta} x_\alpha^{\delta(\sigma_1-\sigma+\frac{1}{n})-1}dx = \infty.$$

In view of

$$\int_{M_{\alpha,\beta}}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma-\frac{1}{qn}-1}du > 0,$$

we have $\infty \leq M\widetilde{J}_2 < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

This completes the proof of the lemma. □

For $\sigma_1 = \sigma$, we also get the lemma below:

Lemma 4. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right]f(x)g(y)dxdy \\ \leq\ & M\left[\int_{-\infty}^{\infty}x_\alpha^{p(1-\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}} \end{aligned} \tag{14}$$

holds true, then we have $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

Proof. For $\sigma_1 = \sigma$, by (8), we have

$$\begin{aligned} I_1 &= \int_{E_\delta} I(x)x_\alpha^{\delta(\sigma-\frac{1}{pn})-1}dx = I_1^{(-)} + I_1^{(+)}, \\ I_1^{(-)} &:= \int_{E_\delta} I^{(-)}(x)x_\alpha^{\delta(\sigma-\frac{1}{pn})-1}dx, I_1^{(+)} := \int_{E_\delta} I^{(+)}(x)x_\alpha^{\delta(\sigma-\frac{1}{pn})-1}dx. \end{aligned}$$

In view of the presented results, for $n > \frac{1}{q(\gamma-\sigma)}$, we obtain

$$\begin{aligned} I_1^{(-)} &= \frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\int_0^{(1-\beta)x_\alpha^\delta}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}dudx \\ &= \frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\left[\int_0^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du\right. \\ &\quad \left.-\int_{(1-\beta)x_\alpha^\delta}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}du\right]dx \\ &= \frac{n}{1-\beta}\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right]k_\rho^{(\gamma)}(\sigma+\frac{1}{qn}) \\ &\quad -\frac{1}{1-\beta}\int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1}\int_{(1-\beta)x_\alpha^\delta}^{\infty}(\arctan\frac{\rho}{u^\gamma})u^{\sigma+\frac{1}{qn}-1}dudx. \end{aligned} \tag{15}$$

For $\gamma > \sigma + d$ $(d > 0)$, we have that $(\arctan \frac{\rho}{u^\gamma})u^{\sigma+d}$ is continuous in $(0, \infty)$, and

$$(\arctan \frac{\rho}{u^\gamma})u^{\sigma+d} \to 0 \ \ (u \to \infty).$$

There exists a positive constant M_1, such that

$$(\arctan \frac{\rho}{u^\gamma})u^{\sigma+d} \le M_1 \ \ (u \in [m_{\alpha,\beta}, \infty)).$$

By (4), it follows that

$$\begin{aligned}
0 &< \int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1} \int_{(1-\beta)x_\alpha^\delta}^{\infty} (\arctan \frac{\rho}{u^\gamma}) u^{\sigma+\frac{1}{qn}-1} du dx \\
&\le M_1 \int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1} \left(\int_{(1-\beta)x_\alpha^\delta}^{\infty} u^{-d+\frac{1}{qn}-1} du \right) dx = \frac{M_1 \int_{E_\delta} x_\alpha^{-\delta(d+\frac{1}{pn})-1} dx}{(1-\beta)^{\sigma-\frac{1}{qn}}} \\
&= \frac{(d+\frac{1}{pn})^{-1} M_1}{(1-\beta)^{\sigma-\frac{1}{qn}}} \left[\frac{1}{(1+\alpha)^{\delta(d+\frac{1}{pn})+1}} + \frac{1}{(1-\alpha)^{\delta(d+\frac{1}{pn})+1}} \right],
\end{aligned}$$

namely

$$\frac{1}{1-\beta} \int_{E_\delta} x_\alpha^{-\frac{\delta}{n}-1} \int_{(1-\beta)x_\alpha^\delta}^{\infty} (\arctan \frac{\rho}{u^\gamma}) u^{\sigma+\frac{1}{qn}-1} du dx = O(1),$$

and then by (15), it follows that

$$\frac{1}{n} I_1^{(-)} = \frac{k_\rho^{(\gamma)}(\sigma + \frac{1}{qn})}{1-\beta} \left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}} \right] - \frac{O(1)}{n}. \tag{16}$$

Similarly, we have

$$\frac{1}{n} I_1^{(+)} = \frac{k_\rho^{(\gamma)}(\sigma + \frac{1}{qn})}{1+\beta} \left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}} \right] - \frac{\widetilde{O}(1)}{n}. \tag{17}$$

By (14) (for $f = f_n, g = g_n$), we have

$$\frac{1}{n} I_1 = \frac{1}{n} \left(I_1^{(-)} + I_1^{(+)} \right) \le \frac{1}{n} M \widetilde{J}_1.$$

For $n \to \infty$, by Fatou's lemma (cf. [39]), (16) and (17), we obtain

$$\frac{2}{1-\beta^2} \cdot \frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2} \le M \left(\frac{2}{1-\alpha^2} \right)^{\frac{1}{p}} \left(\frac{2}{1-\beta^2} \right)^{\frac{1}{q}},$$

namely

$$K_{\alpha,\beta}^{(\gamma)}(\sigma) = \frac{2k_\rho^{(\gamma)}(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}} \le M.$$

This completes the proof of the lemma. □

Lemma 5. *We define the following weight functions:*

$$\omega_\delta(\sigma, y) \ \ : \ \ = y_\beta^\sigma \int_{-\infty}^{\infty} \left[\arctan \frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma} \right] x_\alpha^{\delta\sigma-1} dx \ (y \in \mathbf{R}), \tag{18}$$

$$\varpi_\delta(\sigma, x) \ \ : \ \ = x_\alpha^{\delta\sigma} \int_{-\infty}^{\infty} \left[\arctan \frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma} \right] y_\beta^{\sigma-1} dy \ (x \in \mathbf{R}). \tag{19}$$

Then we have

$$\frac{1-\alpha^2}{2}\omega_\delta(\sigma,y) = \frac{1-\beta^2}{2}\varpi_\delta(\sigma,x) = k_\rho^{(\gamma)}(\sigma) \quad (x,y \in \mathbf{R}\backslash\{0\}). \tag{20}$$

Proof. For fixed $y \in (-\infty,0)\cup(0,\infty)$, setting $u = x_\alpha^\delta y_\beta$, we obtain

$$\begin{aligned}
\omega_\delta(\sigma,y) &= y_\beta^\sigma \int_{-\infty}^0 \left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] x_\alpha^{\delta\sigma-1}dx \\
&+ y_\beta^\sigma \int_0^\infty \left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] x_\alpha^{\delta\sigma-1}dx \\
&= \frac{2}{1-\alpha^2}\int_0^\infty (\arctan\frac{\rho}{u^\gamma})u^{\sigma-1}du = \frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2};
\end{aligned}$$

for fixed $x \in (-\infty,0)\cup(0,\infty)$, setting $u = x_\alpha^\delta y_\beta$, it follows that

$$\begin{aligned}
\varpi_\delta(\sigma,x) &= x_\alpha^{\delta\sigma} \int_{-\infty}^0 \left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] y_\beta^{\sigma-1}dy \\
&+ x_\alpha^{\delta\sigma} \int_0^\infty \left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] y_\beta^{\sigma-1}dy \\
&= \frac{2}{1-\beta^2}\int_0^\infty (\arctan\frac{\rho}{u^\gamma})u^{\sigma-1}du = \frac{2k_\rho^{(\gamma)}(\sigma)}{1-\beta^2}.
\end{aligned}$$

Hence, we derive (20).

This completes the proof of the lemma. □

3. Main Results and Some Particular Cases

Theorem 1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) For any $f(x) \geq 0$, we have the following inequality:

$$\begin{aligned}
J &:= \left[\int_{-\infty}^\infty y_\beta^{p\sigma-1}\left(\int_{-\infty}^\infty [\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}]f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\
&\leq M\left[\int_{-\infty}^\infty x_\alpha^{p(1-\delta\sigma_1)-1}f^p(x)dx\right]^{\frac{1}{p}};
\end{aligned} \tag{21}$$

(ii) for any $f(x), g(y) \geq 0$, we have the following inequality:

$$\begin{aligned}
I &= \int_{-\infty}^\infty\int_{-\infty}^\infty \left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] f(x)g(y)dxdy \\
&\leq M\left[\int_{-\infty}^\infty x_\alpha^{p(1-\delta\sigma_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^\infty y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}};
\end{aligned} \tag{22}$$

(iii) $\sigma_1 = \sigma$, and $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

Proof. $(i) \Rightarrow (ii)$. By Hölder's inequality (cf. [40]), we get

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\left(y_\beta^{\sigma-\frac{1}{p}}\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}]f(x)dx\right)\left(y_\beta^{-\sigma+\frac{1}{p}}g(y)\right)dy \\ &\leq J\left[\int_{-\infty}^{\infty}y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{23}$$

Then by (21), we have (22).

$(ii) \Rightarrow (iii)$. By Lemma 1, we have $\sigma_1 = \sigma$. Then by Lemma 2, we get $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

$(iii) \Rightarrow (i)$. For $\sigma_1 = \sigma$, by Hölder's inequality with weight (see [40]) and (18), we have

$$\begin{aligned} &\left(\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}]f(x)dx\right)^p \\ &= \left\{\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}]\left[\frac{y_\beta^{(\sigma-1)/p}}{x_\alpha^{(\delta\sigma-1)/q}}f(x)\right]\left[\frac{x_\alpha^{(\delta\sigma-1)/q}}{y_\beta^{(\sigma-1)/p}}\right]dx\right\}^p \\ &\leq \int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right]\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}dx \\ &\quad\times\left[\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}]\frac{x^{\delta\sigma-1}}{y_\beta^{(\sigma-1)q/p}}dx\right]^{p/q} \\ &= \left[\omega_\delta(\sigma,y)y_\beta^{q(1-\sigma)-1}\right]^{p-1}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right]\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}dx \\ &= \left(\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2}\right)^{p-1}y_\beta^{-p\sigma+1}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right]\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}dx. \end{aligned} \tag{24}$$

By Fubini's theorem, (24) and (19), we derive that

$$\begin{aligned} J &\leq \left(\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2}\right)^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}]\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(\delta\sigma-1)p/q}}dxdy\right]^{\frac{1}{p}} \\ &= \left(\frac{2k_\rho^{(\gamma)}(\sigma)}{1-\alpha^2}\right)^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\varpi_\delta(\sigma,x)x_\delta^{p(1-\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}} \\ &= K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty}x_\delta^{p(1-\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned}$$

For $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$, we have (21) (for $\sigma_1 = \sigma$).
Therefore, Statements (i), (ii) and (iii) are equivalent.
This completes the proof of the theorem. □

For $\sigma_1 = \sigma$, we deduce the theorem below:

Theorem 2. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty}x_\alpha^{p(1-\delta\sigma)-1}f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty} y_{\beta}^{p\sigma-1}\left(\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_{\alpha}^{\delta}y_{\beta})^{\gamma}}]f(x)dx\right)^{p}dy\right]^{\frac{1}{p}} < M\left[\int_{-\infty}^{\infty} x_{\alpha}^{p(1-\delta\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}; \quad (25)$$

(ii) for any $f(x) \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty} x_{\alpha}^{p(1-\delta\sigma)-1}f^{p}(x)dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty} y_{\beta}^{q(1-\sigma)-1}g^{q}(y)dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_{\alpha}^{\delta}y_{\beta})^{\gamma}}\right]f(x)g(y)dxdy < M\left[\int_{-\infty}^{\infty} x_{\alpha}^{p(1-\delta\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_{\beta}^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}; \quad (26)$$

(iii) $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$.

Moreover, if the statement (iii) holds true, then the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (25) and (26) is the best possible.

In particular:

(1) for $\delta = 1$, we have the following equivalent inequalities with the nonhomogeneous kernel:

$$\left[\int_{-\infty}^{\infty} y_{\beta}^{p\sigma-1}\left(\int_{-\infty}^{\infty}[\arctan\frac{\rho}{(x_{\alpha}y_{\beta})^{\gamma}}]f(x)dx\right)^{p}dy\right]^{\frac{1}{p}} < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_{\alpha}^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \quad (27)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan\frac{\rho}{(x_{\alpha}y_{\beta})^{\gamma}}\right]f(x)g(y)dxdy < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_{\alpha}^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_{\beta}^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \quad (28)$$

where $K_{\alpha,\beta}^{(\gamma)}(\sigma)$ is the best possible constant factor;

(2) for $\delta = -1$, we have the following equivalent inequalities with the homogeneous kernel of degree 0:

$$\left[\int_{-\infty}^{\infty} y_{\beta}^{p\sigma-1}\left(\int_{-\infty}^{\infty}[\arctan\rho(\frac{x_{\alpha}}{y_{\beta}})^{\gamma}]f(x)dx\right)^{p}dy\right]^{\frac{1}{p}} < K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty} x_{\alpha}^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \quad (29)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\arctan\rho(\frac{x_\alpha}{y_\beta})^\gamma\right]f(x)g(y)dxdy$$
$$< \quad K_{\alpha,\beta}^{(\gamma)}(\sigma)\left[\int_{-\infty}^{\infty}x_\alpha^{p(1+\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \tag{30}$$

where $K_{\alpha,\beta}^{(\gamma)}(\sigma)$ is the best possible constant factor.

Proof. For $\sigma_1 = \sigma$ and the assumption of statement (i), if (24) assumes the form of equality for some $y \in (-\infty, 0) \cup (0, \infty)$, then (see [40]) there exist constants A and B, such that they are not both zero, and

$$A\frac{y_\beta^{\sigma-1}}{x_\alpha^{(\delta\sigma-1)p/q}}f^p(x) = B\frac{x^{\delta\sigma-1}}{y_\beta^{(\sigma-1)q/p}} \quad a.e. \ in \ \mathbf{R}.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$x_\alpha^{p(1-\delta\sigma)-1}f^p(x) = y_\beta^{q(1-\sigma)}\frac{B}{Ax_\alpha} \quad a.e. \ in \ \mathbf{R}.$$

Since

$$\int_{-\infty}^{\infty}x_\alpha^{-1}dx = \infty,$$

it contradicts the fact that

$$0 < \int_{-\infty}^{\infty}x_\alpha^{p(1-\delta\sigma)-1}f^p(x)dx < \infty.$$

Hence, (24) takes the form of strict inequality, and so does (21). Hence, (25) and (26) are true.

In view of Theorem 1, we can establish the equivalency between the statements (i), (ii) and (iii) in Theorem 2.

In case the statement (iii) is valid, namely $K_{\alpha,\beta}^{(\gamma)}(\sigma) \leq M$, if there exists a constant $M \leq K_{\alpha,\beta}^{(\gamma)}(\sigma)$, such that (26) is satisfied, then we can derive that the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (26) is optimal.

The constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (25) remains the best possible. Otherwise, by (23) (for $\sigma_1 = \sigma$), we would reach a contradiction that the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (26) is not optimal.

This completes the proof of the theorem. □

4. Operator Expressions

We set the following functions: $\varphi(x) := x_\alpha^{p(1-\delta\sigma)-1}(x \in \mathbf{R})$ and $\psi(y) := y_\beta^{q(1-\sigma)-1}$, wherefrom $\psi^{1-p}(y) = y_\beta^{p\sigma-1}$ $(y \in \mathbf{R})$. Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}) := \left\{f : ||f||_{p,\varphi} := \left(\int_{-\infty}^{\infty}\varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$

$$L_{q,\psi}(\mathbf{R}) = \left\{g : ||g||_{q,\psi} = \left(\int_{-\infty}^{\infty}\psi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty\right\},$$
$$L_{p,\psi^{1-p}}(\mathbf{R}) = \left\{h : ||h||_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty}\psi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty\right\}.$$

In view of Theorem 2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$h_1(y) := \int_{-\infty}^{\infty} \left[\arctan \frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}\right] f(x)dx \; (y \in \mathbf{R}),$$

by (25), we have

$$||h_1||_{p,\psi^{1-p}} = \left[\int_{-\infty}^{\infty} \psi^{1-p}(y) h_1^p(y) dy\right]^{\frac{1}{p}} < M||f||_{p,\varphi} < \infty. \tag{31}$$

Definition 1. *Define a Hilbert-type integral operator with the nonhomogeneous kernel $T : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $Tf = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying $Tf(y) = h_1(y)$, for any $y \in \mathbf{R}$.*

In view of (31), it follows that

$$||Tf||_{p,\psi^{1-p}} = ||h_1||_{p,\psi^{1-p}} < M||f||_{p,\varphi},$$

and thus the operator T is bounded satisfying

$$||T|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})} \frac{||Tf||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \le M.$$

If we define the formal inner product of Tf and g as follows:

$$(Tf, g) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [\arctan \frac{\rho}{(x_\alpha^\delta y_\beta)^\gamma}] f(x)dx \right) g(y)dy,$$

then we can rewrite Theorem 2 as follows:

Theorem 3. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) For any $f(x) \ge 0, f \in L_{p,\varphi}(\mathbf{R}), ||f||_{p,\varphi} > 0$, the following inequality holds true:

$$||Tf||_{p,\psi^{1-p}} < M||f||_{p,\varphi}; \tag{32}$$

(ii) for any $f(x), g(y) \ge 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\psi}(\mathbf{R}), ||f||_{p,\varphi}, ||g||_{q,\psi} > 0$, the following inequality holds true:

$$(Tf, g) < M||f||_{p,\varphi}||g||_{q,\psi}; \tag{33}$$

(iii) $K_{\alpha,\beta}^{(\gamma)}(\sigma) \le M$.

Moreover, if the statement (iii) holds true, then the constant factor $M = K_{\alpha,\beta}^{(\gamma)}(\sigma)$ in (32) and (33) is optimal, i.e., $||T|| = K_{\alpha,\beta}^{(\gamma)}(\sigma)$.

Remark 1. *(1) In particular, for $\alpha = \beta = 0$ in (27) and (28) we have the following equivalent inequalities*:

$$\left[\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} (\arctan \frac{\rho}{|xy|^\gamma}) f(x)dx \right)^p dy\right]^{\frac{1}{p}}$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{\sigma \cos \frac{\pi\sigma}{2\gamma}} \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}, \tag{34}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\arctan\frac{\rho}{|xy|^{\gamma}}\right)f(x)g(y)dxdy$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \tag{35}$$

where $\frac{\rho^{\sigma/\gamma}\pi}{\sigma\cos\frac{\pi\sigma}{2\gamma}}$ *is the optimal constant factor. If* $f(-x)=f(x), g(-y)=g(y)$ $(x,y\in\mathbf{R}_{+})$, *then we have the following equivalent inequalities:*

$$\left[\int_{0}^{\infty}y^{p\sigma-1}\left(\int_{0}^{\infty}[\arctan\frac{\rho}{(xy)^{\gamma}}]f(x)dx\right)^{p}dy\right]^{\frac{1}{p}}$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{2\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{0}^{\infty}x^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \tag{36}$$

$$\int_{0}^{\infty}\int_{0}^{\infty}\left[\arctan\frac{\rho}{(xy)^{\gamma}}\right]f(x)g(y)dxdy$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{2\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{0}^{\infty}x^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{0}^{\infty}y^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \tag{37}$$

where $\frac{\rho^{\sigma/\gamma}\pi}{2\sigma\cos\frac{\pi\sigma}{2\gamma}}$ *is the best possible constant factor.*

(2) For $\alpha=\beta=0$ in (29) and (30) we have the following equivalent inequalities:

$$\left[\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left(\int_{-\infty}^{\infty}(\arctan\rho|\frac{x}{y}|^{\gamma})f(x)dx\right)^{p}dy\right]^{\frac{1}{p}}$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{-\infty}^{\infty}|x|^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \tag{38}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\arctan\rho|\frac{x}{y}|^{\gamma}\right)f(x)g(y)dxdy$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{-\infty}^{\infty}|x|^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \tag{39}$$

where $\frac{\rho^{\sigma/\gamma}\pi}{\sigma\cos\frac{\pi\sigma}{2\gamma}}$ is the best possible constant factor. If $f(-x)=f(x), g(-y)=g(y)$ $(x,y\in\mathbf{R}_{+})$, then we have the following equivalent inequalities:

$$\left[\int_{0}^{\infty}y^{p\sigma-1}\left(\int_{0}^{\infty}[\arctan\rho(\frac{x}{y})^{\gamma}]f(x)dx\right)^{p}dy\right]^{\frac{1}{p}}$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{2\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{0}^{\infty}x^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \tag{40}$$

$$\int_{0}^{\infty}\int_{0}^{\infty}\left[\arctan\rho(\frac{x}{y})^{\gamma}\right]f(x)g(y)dxdy$$
$$< \frac{\rho^{\sigma/\gamma}\pi}{2\sigma\cos\frac{\pi\sigma}{2\gamma}}\left[\int_{0}^{\infty}x^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{0}^{\infty}y^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \tag{41}$$

where $\frac{\rho^{\sigma/\gamma}\pi}{2\sigma\cos\frac{\pi\sigma}{2\gamma}}$ is the best possible constant factor.

5. Conclusions

In this paper, making use of ideas of Hong [23], and by employing techniques of real analysis as well as weight functions, we obtain in Theorem 1 a few equivalent statements of a Hilbert-type integral inequality in the whole plane associated with the kernel of the arc tangent function. In Theorem 2, the constant factor associated with the cosine function is proved to be optimal. Furthermore, in Theorem 3 and Remark 1 we also consider some particular cases and operator expressions. The lemmas and theorems within this work provide an extensive account of this type of inequalities.

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