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Quantum Fisher Information and Bures Distance Correlations of Coupled Two Charge-Qubits Inside a Coherent Cavity with the Intrinsic Decoherence

Abdel-Baset A. Mohamed ^{1,2,*} , Eied. M. Khalil ^{3,4}, Mahmoud M. Selim ¹  and Hichem Eleuch ^{5,6,7}

¹ Department of Mathematics, College of Science and Humanities in Al-Aflaj, Prince Sattam Bin Abdulaziz University, Al-Aflaj 11942, Saudi Arabia; m.selim@psau.edu.sa

² Faculty of Science, Assiut University, Assiut 71515, Egypt

³ Department of Mathematics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; eiedkhalil@tu.edu.sa

⁴ Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt

⁵ Department of Applied Physics and Astronomy, University of Sharjah, Sharjah 27272, United Arab Emirates; hichem.eleuch@adu.ac.ae

⁶ Department of Applied Sciences and Mathematics, College of Arts and Sciences, Abu Dhabi University, Abu Dhabi 59911, United Arab Emirates

⁷ Institute for Quantum Science and Engineering, Texas A&M University, College Station, TX 77843, USA

* Correspondence: abdelbastm@aun.edu.eg



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Abstract: The dynamics of two charged qubits containing Josephson Junctions inside a cavity are investigated under the intrinsic decoherence effect. New types of quantum correlations via local quantum Fisher information and Bures distance norm are explored. We show that we can control the quantum correlations robustness by the intrinsic decoherence rate, the qubit-qubit coupling as well as by the initial coherent states superposition. The phenomenon of sudden changes and the freezing behavior for the local quantum Fisher information are sensitive to the initial coherent state superposition and the intrinsic decoherence.

Keywords: local quantum Fisher information; two-qubit coupling; intrinsic decoherence

1. Introduction

Quantum correlation (QC) is substantial for quantum technologies [1–3]. It is shown that some quantum correlated states have no quantum entanglement [4,5]. Consequently, more quantifiers were introduced to detect different QCs beyond entanglement as: measurement-induced disturbance [6], quantum discord [7] and geometrical measures which based on p -norm [8,9], Bures distance norm [10] and skew information quantity [11]. QCs are the basic concept for the quantum engineering, quantum cryptography and quantum information [12–16]. The entanglement quantifiers as concurrence, negativity, entropy, and Bures distance entanglement (BDE) present the same behavior. These measures however, have ordering difficulties [17]. The hierarchy among the quantum correlations (Bell nonlocal, entanglement and non-entanglement quantum correlations (e.g., quantum discord or local quantum Fisher information) must be satisfied by all correlation quantifiers [18].

Quantum Fisher information (QFI) is useful quantifier for multipartite entanglement [19], quantum speedup limittime [20], uncertainty relations [21], and quantum phase transition [22]. The multipartite entanglement which witnessed by QFI is useful for quantum metrology [23], complex structures of topological states [24,25] and quantum phase-transition [26]. QFI is also useful to describe the optimal accuracy in parameter estimation protocols [27–29]. QFI is employed to establish the link between quantum correlation and quantum metrology [30,31]. Recently, local quantum Fisher information (LQFI) is introduced as a quantifier of the nonclassical correlation beyond entanglement [32–34].

It is well-known that quantum phenomena are very sensitive to the decoherence and dissipation resources [35–38], they are the main cause for the transition between the quantum and classical dynamics. For closed systems the coherence can be lost, this phenomenon is identified as intrinsic decoherence (ID) [39]. The ID manifests without interaction with the environment, and the system does not dissipate and its energy is preserved. Therefore, it is of great interest to investigate the intrinsic decoherence effect on the correlations.

The goal of this paper is to explore the quantum correlation generation via the LQFI and Bures distance entanglement between the two charged qubits that are placed into a cavity as well as how these correlations could be further enhanced by exploiting two-qubit coupling and the decoherence. It is worth mentioning that the two-charge-qubit system has been realized experimentally [40,41] to serve as a unit information for the quantum computing [42–45].

Our main motivation here is to establish a comprehensive comparison between the behavior of both the BDE and the LQFI in the case of a coupled two charge-qubits system. We will focus on: (1) confirming that these real qubit-cavity interactions have ability to generate quantum correlation via the LQFI beyond the Bures distance entanglement, (2) studying the effect of the initial coherent field cavity.

In this paper, the generation of new types of quantum correlations is analyzed in a two qubit system that has several potential applications and realizations [46,47]. The intrinsic decoherence model is used for the cavity prepared initially in a superpositions of coherent states. The analytical results allow the study of robustness of the generated BDE and LQFI correlations.

In Section 2, we derive the time evolution of the physical model. The definition of the QFI and BDE quantifiers are presented in Section 3. We analyze the dynamics of the QFI and BDE correlations in Section 4. We end-up by a conclusion.

2. Time Evolution of the Physical Model

Our considered system is formed by two charged qubits, each one is realized via Cooper-pair box containing two identical Josephson junctions with the coupling energies E_J and capacitors C_J . The two qubits are placed into a cavity (see Figure 1).

The Cooper-pair box works as qubit in the microwave region in which (i) it is placed in the middle of the cavity, the qubit-cavity interaction is optimal in the microwave region. (ii) the electromagnetic field is not very strong such that all higher orders are neglected except the first order of $\frac{\pi\eta}{\Phi_0}$. The constant η has units of the magnetic flux and depends on the geometrical design of the SC cavity field, and Φ_0 is the applied quantum flux.

Consequently, the Hamiltonian of the qubit-cavity system can be written as [48]

$$\hat{H} = \omega \hat{\psi}^\dagger \hat{\psi} + \sum_{k=A,B} \omega_z^k \hat{\sigma}_z^k + \frac{\pi\eta E_J}{\Phi_0} (\hat{\psi} + \hat{\psi}^\dagger) \hat{\sigma}_x^k. \quad (1)$$

where ω represents the frequency of the SC-cavity field that has the lowering $\hat{\psi}$ and raising $\hat{\psi}^\dagger$ operators. ω_z^k denotes to the k -qubit frequency.

The $\hat{\sigma}_z^k$ and $\hat{\sigma}_x^k$ are the k -qubit Pauli operators.

The interaction between the two qubits and the SC-cavity field cannot be neglected when the distance between the two SC-qubits is less than the field wavelength.

After adding the two-qubit coupling term, the Hamiltonian of the resonant system ($\omega = \omega_z^A = \omega_z^B$), in the rotating-wave approximation, is given by

$$\begin{aligned} \hat{H} = & \omega (\hat{\psi}^\dagger \hat{\psi} + \hat{\sigma}_A^z + \hat{\sigma}_B^z) + \lambda (\hat{\psi} \sigma_A^+ + \hat{\psi}^\dagger \sigma_A^-) \\ & + \lambda (\hat{\psi} \sigma_B^+ + \hat{\psi}^\dagger \sigma_B^-) + K (\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+), \end{aligned} \quad (2)$$

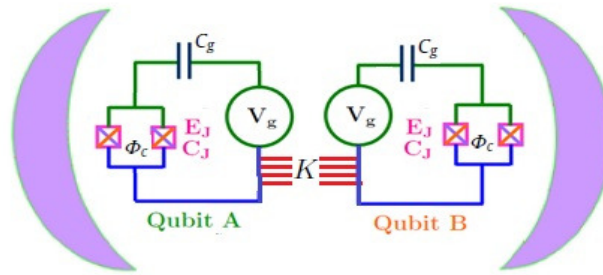


Figure 1. Schematic diagram of two identical coupled charge qubits with coupling strength K , are placed in a single-mode SC-cavity. Each qubit is characterized by a junction capacitance C_J and a coupling energy E_J which is tuned by applying a Φ_c to the Cooper-pair box that containing two identical Josephson junctions with the coupling energies E_J and capacitors C_J .

The artificial atoms and the single-mode cavity are taken in the resonant case where their frequencies are equal to ω , and their interaction coupling constant is presented by $\lambda = \frac{\pi\eta E_J}{\Phi_0}$. $\hat{\sigma}_k^+ = |1_k\rangle\langle 0_k|$ and $\hat{\sigma}_k^- = |0_k\rangle\langle 1_k|$ stand the raising and lowering k -qubit operators. K represents the qubit-qubit coupling strength.

The first term of Equation (2) describes the free Hamiltonian of the cavity field and the qubits while the second and third term describe the qubit-cavity interactions with the coupling constant λ . The last term represents the qubit-qubit coupling by with the interaction constant K .

In the space qubit-cavity states $\{|D_1\rangle = |1_A 1_B, n\rangle, |D_2\rangle = |1_A 0_B, n+1\rangle, |D_3\rangle = |0_A 1_B, n+1\rangle, |D_4\rangle = |0_A 0_B, n+2\rangle\}$ where $|n\rangle$ represent the number state with $n = 0, 1, 2, \dots$, the eigenstates $|S_k^n\rangle$ ($k = 1-4$) of the Hamiltonian (2) are given by:

$$\begin{aligned} |S_1^n\rangle &= d_{11}|D_1\rangle + d_{14}|D_4\rangle, \\ |S_2^n\rangle &= d_{22}|D_2\rangle + d_{23}|D_3\rangle, \\ |S_3^n\rangle &= d_{31}|D_1\rangle + d_{32}|D_2\rangle - d_{33}|D_3\rangle + d_{34}|D_4\rangle, \\ |S_4^n\rangle &= d_{41}|D_1\rangle + d_{42}|D_2\rangle + d_{43}|D_3\rangle + d_{44}|D_4\rangle. \end{aligned} \quad (3)$$

The d_{ij} satisfy the eigenvalue-problem: $\hat{H}|S_k^l\rangle = E_k^l|S_k^l\rangle$, where E_k^l ($k = 1-4$) are the corresponding eigenvalues,

$$\begin{aligned} E_1^l &= \omega(l+1), \\ E_2^l &= \omega(l+1) - K, \\ E_3^l &= \omega(l+1) + \frac{1}{2}K - \frac{1}{2}\lambda\sqrt{(K/\lambda)^2 + 16l + 24}, \\ E_4^l &= \omega(l+1) + \frac{1}{2}K + \frac{1}{2}\lambda\sqrt{(K/\lambda)^2 + 16l + 24}. \end{aligned} \quad (4)$$

Here, the intrinsic decoherence effect is examined by using the Milburn's equation, which is a generalized version of the Schrödinger equation where the qubit-cavity system evolves stochastically under sequences of identical unitary transformations [39]. The intrinsic decoherence is generated in absence of the environment-system interactions. These interactions lead to other irreversible effects [49] such as the dissipation. The Milburn's equation [39] that governs the system with intrinsic decoherence is given by

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] - \frac{1}{2}\gamma[H, [H, \rho(t)]], \quad (5)$$

where γ designs the intrinsic decoherence rate.

Here, we assume that the initial reduced density matrix $\rho(0)$ is in the uncorrelated state,

$$\rho(0) = |\varphi_c(0)\rangle\langle\varphi_c(0)| \otimes |1_A 1_B\rangle\langle 1_A 1_B|, \quad (6)$$

where the two SC-qubits are initially assumed in their upper states, and the cavity-field is started initially in the superposition coherent states as

$$|\varphi_c(0)\rangle = (|\mu\rangle + r|-\mu\rangle)/\sqrt{A}, \quad (7)$$

where $A = 1 + r^2 + 2re^{-2|\mu|^2}$, $|\mu\rangle$ presents the coherent state with the intensity coherence μ . It is defined by

$$|\mu\rangle = \sum_{n=1}^{\infty} F_n |n\rangle = e^{-|\mu|^2/2} \sum_{n=1}^{\infty} \frac{\mu^n}{\sqrt{n!}} |n\rangle, \quad (8)$$

For $r = 0$ we have the coherent state and $r = 1$ represents the even coherent state. From Equations (2) and (6), we get

$$\begin{aligned} \hat{\rho}(t) = \sum_{l,k=0} \frac{1}{A} [1 + r(-1)^l][1 + r(-1)^k] F_l F_k^* \bigg\{ & d_{11}d_{11}\hat{\chi}_{11} + d_{11}d_{31}\hat{\chi}_{13} \\ & + d_{11}d_{41}\hat{\chi}_{14} + d_{31}d_{11}\hat{\chi}_{31} + d_{41}d_{11}\hat{\chi}_{41} + d_{31}d_{31}\hat{\chi}_{33} + d_{41}d_{31}\hat{\chi}_{43} \\ & + d_{31}d_{41}\hat{\chi}_{34} + d_{41}d_{41}\hat{\chi}_{44} \bigg\} \end{aligned} \quad (9)$$

where $\hat{\chi}_{ij}$ are given by

$$\hat{\chi}_{ij} = D_f e^{-i\lambda(E_i^l - E_j^k)t} |S_i^l\rangle\langle S_j^k|, \quad (10)$$

$D_f = e^{-\frac{\gamma}{2}(E_i^l - E_j^k)^2 t}$ designs the decoherence term.

3. Quantum Fisher Information and Bures Distance Entanglement

3.1. Local Quantum Fisher Information

QFI is used to build the bridge between the quantum correlations and quantum metrology [30,31]. For a given bipartite quantum state (say A and B) ρ^{AB} the quantum Fisher information associated with the local evolution generated by $I_A \otimes H_B$ (H_B is the local Hamiltonian acting on B . The operator I_A represents the identity acting on A) can be written as

$$\begin{aligned} F(\rho^{AB}, H_B) &= 4 \sum_{m,n:\pi_m+\pi_n>0} \frac{(\pi_m - \pi_n)^2}{\pi_m + \pi_n} |\langle\psi_m|I_A \otimes H_B|\psi_n\rangle|^2. \end{aligned} \quad (11)$$

where $\rho^{AB} = \sum_m \pi_m |\psi_m\rangle\langle\psi_m|$ is the spectral decomposition of ρ^{AB} , where $\{\pi_m\}$ and $\{|\psi_m\rangle\}$ represent the eigenvalues and the eigenvectors of the bipartite state ρ^{AB} with $\pi_m \geq 0$ and $\sum_m \pi_m = 1$. The Equation (11) is also equivalent to the following expression [32],

$$\begin{aligned} F(\rho^{AB}, H_B) &= 4\text{Tr}\{\rho^{AB} H_B^2\} \\ &- \sum_{m,n:\pi_m+\pi_n>0} \frac{8\pi_m\pi_n}{\pi_m + \pi_n} |\langle\psi_m|I_A \otimes H_B|\psi_n\rangle|^2. \end{aligned} \quad (12)$$

The local quantum Fisher information (LQFI) quantifies the quantum correlation. It is the minimum of the quantum Fisher information over all local Hamiltonians H_B of a fixed spectral class [32],

$$L(\rho^{AB}) = \frac{1}{4} \inf_{H_B} F(\rho^{AB}, H_B), \quad (13)$$

In our case of the two qubits, the general form of the reduced local Hamiltonian for the qubit B is: $H_B = \vec{r} \cdot \vec{\sigma}$, where $|\vec{r}| = 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ represents the vector formed by the Pauli matrices. The LQFI is given by the expression [32],

$$L(t) = 1 - \pi_W^{\max}, \quad (14)$$

where $\pi_W^{\max}$ is the biggest eigenvalue of the 3×3 symmetric matrix $W = [w_{ij}]$,

$$w_{ij} = \sum_{m,n:\pi_m+\pi_n=0} \frac{2\pi_m\pi_n}{\pi_m+\pi_n} \langle \psi_m | \Pi_i | \psi_n \rangle \langle \psi_n | \Pi_j | \psi_m \rangle.$$

with $\Pi_k = I_A \otimes \sigma_B^k$ ($k = i, j$). The dynamics of the local quantum Fisher information for the two-qubit reduced density matrix $\rho^{AB}(t)$, can be calculated by using Equation (9) as

$$\rho^{AB}(t) = \text{Tr}_f\{\hat{\rho}(t)\}. \quad (15)$$

where the operation Tr_f traces tracing out the cavity states. The eigenvalues $\{\pi_m\}$ and the eigenvectors $\{|\psi_m\rangle\}$ of the two-qubit state ρ^{AB} are calculated numerically.

3.2. Bures Distance Entanglement

The two-qubit entanglement $\rho^{AB}(t)$ can be quantified by the Bures distance [10]. The Bures distance entanglement (BDE) is defined as

$$B(t) = \sqrt{2 - \sqrt{2 + 2\sqrt{1 - C(t)^2}}}, \quad (16)$$

$C(t)$ is the concurrence [50],

$$C(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \quad (17)$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ design the enginevalues of the matrix: $R = \rho^{AB}(\sigma_y \otimes \sigma_y) \rho^{AB*}(\sigma_y \otimes \sigma_y)$. The BDE function $B(t)$ is bounded by two values; Zero which represent the case of unentangled states and $\sqrt{2} - \sqrt{2}$ for a maximal entangled state.

4. Dynamics of the Correlation Quantifiers

By using Bures entanglement measure and local quantum Fisher information, the quantum correlation robustness is shown for different values of the qubit-qubit coupling, the initial coherent states, and the intrinsic decoherence.

In Figure 2a, the Bures distance entanglement and the LQFI when the field is initially in the coherent state $r = 0$ with $\mu = 2\sqrt{2}$ and in the absence of the intrinsic decoherence and the dipole coupling. We observe due to the qubit-cavity interaction, increase of the LQFI oscillation frequency as the time evolves. Dashed curve of Figure 2a shows that the dynamics of the Bures distance entanglement differs form the LQFI correlation. The amplitudes of the LQFI always greater than those of the BDE. The phenomena of sudden death and growth entanglement [51,52] is appeared in the behavior of the Bures distance entanglement, where the two qubits have non-zero LQFI correlation.

Figure 2b shows that the generated LQFI correlation and the BDE can be enhanced in the presence of the two-qubit interaction. The amplitudes of the LQFI and the BDE

increase whereas their oscillation frequencies are reduced. LQFI correlation stabilizes for a particular time interval, which means that the LQFI correlation is frozen for certain time. The LQFI correlation is more stable than the Bures distance entanglement.

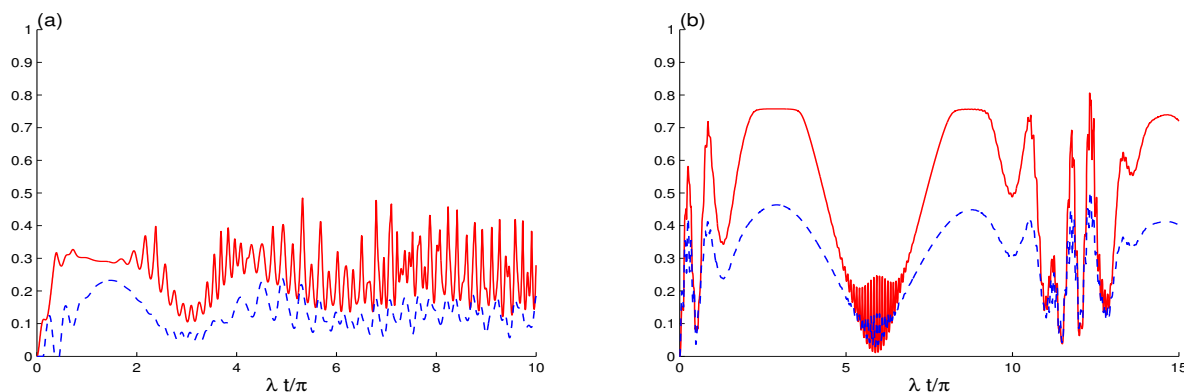


Figure 2. The local quantum Fisher information $L(t)$ (solid curve) and Bures distance entanglement $B(t)$ (dashed curve) for $\mu = 2\sqrt{2}$, $\gamma = 0.0$, $r = 0$ for different two-qubit coupling values: $K/\lambda = 0.0$ in (a) and $K/\lambda = 20$ in (b).

Figure 3 illustrates the intrinsic decoherence effect on the dynamics of the LQFI and the Bures distance quantifiers. In the absence of the two-qubit interaction, the extreme values of the LQFI and BDE are closer to their frozen correlations. The oscillatory correlations between the two qubits are disappeared due to the decoherence. The LQFI reaches its frozen quantum correlation which is more stable than the entanglement. The frozen LQFI and BDE shows a robust feature against the intrinsic decoherence. Figure 3b exalts the decoherence effect after considering the two-qubit interaction, the oscillatory correlations of both the LQFI and the Bures distance quantifiers are reduced. The stability of the generated correlations is enhanced.

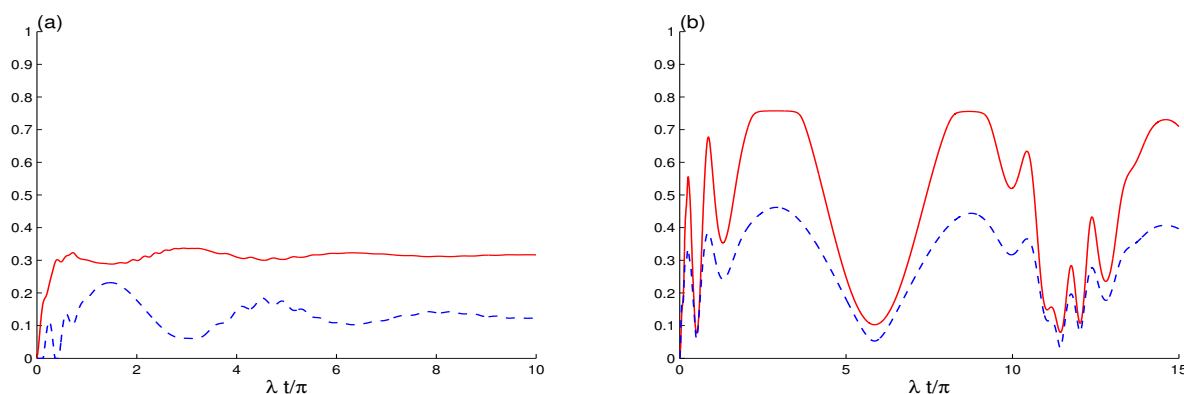


Figure 3. As Figure 2, but with $\gamma/\lambda = 0.005$.

In Figure 4, the LQFI and the Bures distance quantifiers show the amount of the generated correlation for the case $r = 1$. From Figure 4a, we observe that the increase of the initial coherent intensity leads to the amplitude enhancement of the correlation oscillations. The dynamical behavior of LQFI and BDE depend on the parameter r . The Bures distance entanglement vanishes instantly at certain particular times. In these times, the disentangled two qubits have non-zero LQFI correlation. The effect of the increase of the initial field intensity is clearly observable on the amplitudes and the frequency of the oscillatory correlations when the two-qubit interaction is considered (see Figure 4b). By comparing the results of Figure 2b of the case $r = 0$ with Figure 4b of the case $r = 1$, we observe that the increase of the initial intensity enhances the correlations as well as their robustness against the decoherence. In the absence of the ID effect, the two-qubit state does

not possess frozen LQFI for an initial even coherent state ($r = 1$). This is unlike the case of the initial coherent state ($r = 0$). The existence of the frozen LQFI phenomenon depends on the initial field cavity states.

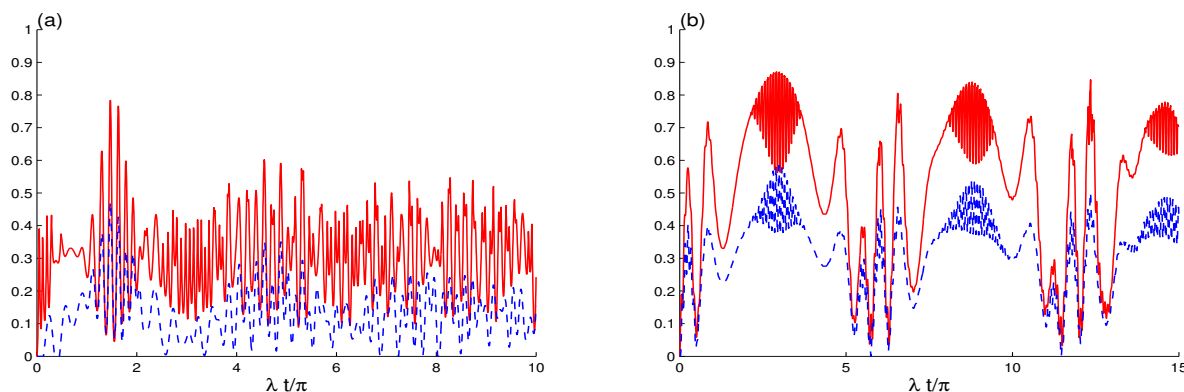


Figure 4. As Figure 2, but when the cavity is initially in the even coherent state.

In Figure 5, we observe that the LQFI correlation reaches a constant value at certain time and after that the two-qubit state exhibits frozen dynamics for LQFI correlation. On the other hand, the oscillatory behavior of the Bures distance entanglement is almost disappeared. BDE exhibits after a short time a pseudo-stationary entanglement due to the intrinsic decoherence effect. From Figure 5b, we deduce that the two-qubit interaction enhances the LQFI and the Bures distance correlations. We also note the emergence of the frozen correlation and the sudden quantum changed for the LQFI correlation [53,54].

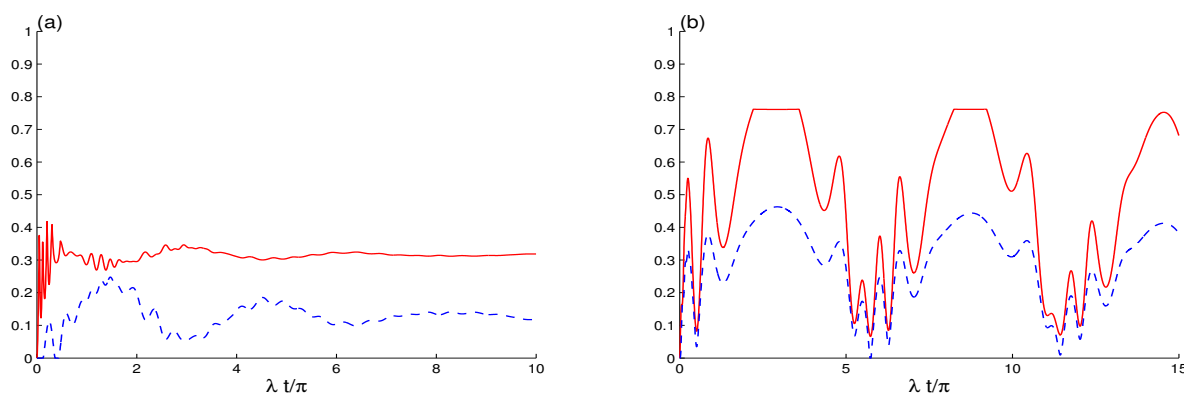


Figure 5. As Figure 4, but with $\gamma/\lambda = 0.005$.

In Table 1, we summarize the generated phenomena, oscillation frequency (OF), oscillation amplitude (OAs), Sudden death (SD), and Sudden change (SC), frozen correlation (FC), and stable correlation (SC) observed in the plots of LQFI and BDE. Not that “Yes” and “No” refer to the existence of the considered phenomenon.

Table 1. The comparison between the LQFI and the BDE.

	LQFI	BDE	The Observations	Figs.
OF	Yes	Yes	OF of $L(t)$ is more than of $B(t)$	Figures 2 and 4
OA	Yes	Yes	OA of $L(t)$ are larger than of $B(t)$	All
SD	No	Yes	SD is only in $B(t)$	Figures 4a and 5a
SC	Yes	No	SC are only in $L(t)$	Figure 5b
FC	Yes	No	FC is only in $L(t)$	Figures 2b, 3b and 5b
SC	Yes	Yes	SC of $L(t)$ is more than of $B(t)$	Figures 3a and 5a

5. Conclusions

We consider two coupled charged qubits, each one is realized via Cooper-pair box containing two identical Josephson junctions. The two qubits are placed into a cavity. For the qubit-cavity system, an analytical solution of the intrinsic decoherence model is introduced. The robustness of the generated local quantum Fisher information correlation and Bures distance entanglement are investigated. The local quantum Fisher information and Bures distance norm correlations can be enhanced by increasing the two-qubit coupling for an initial even coherent state. The two-qubit coupling leads to the manifestation of the phenomena of frozen quantum LQFI correlation and sudden changes in the behavior of LQFI. The intrinsic decoherence destroys the oscillations of the correlation between the two qubits. The LQFI frozen correlation is more stable than the stationary entanglement.

These results offer practical applications in the quantum information since the LQFI and Bures distance entanglement present stable correlations. It was shown theoretically [55–57] and experimentally [58,59] that quantum correlations can be frozen over defined time. It is also proven experimentally that the coupled charge qubits are more appropriate to build a practical quantum computer due to their potential suitability for integrated devices [42,60]. The generation of correlations across multiple qubits with the decoherence was implemented experimentally using different protocols on superconducting and quantum-dot-molecules systems [61,62]. One of them was proposed and experimentally validated for two superconducting qubits coupled to a shared engineered noise source [62], which allows the simultaneous reconstruction of all the single-qubit and two-qubit cross-correlation spectra. Experimentally, some interesting two-qubit entangled operations were implemented in photonic systems [63]. Via three separate cavities, each containing a semiconductor quantum dot molecule, connected by optical fibers, a further protocol was proposed to produce a 3-qubit entangled state [64]. Quantum simulations are needed to manipulate the quantum information processing in various two-qubit systems such as quantum dots and quantum superconducting circuits. Some recent examples can be cited as the preparation of entanglement via Pauli-spin blockade [65], the realization of two-qubit quantum Fourier, circulant symmetry-protected entanglement [66], and the design of a scalable qubit-coupled dispersive communication architecture [43].

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