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Two-Dimensional Divisor Problems Related to Symmetric *L*-Functions

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Abstract: In this paper, we study two-dimensional divisor problems of the Fourier coefficients of some automorphic product *L*-functions attached to the primitive holomorphic cusp form $f(z)$ with weight k for the full modular group $SL_2(\mathbb{Z})$. Additionally, we establish the upper bound and the asymptotic formula for these divisor problems on average, respectively.

Keywords: Hecke *L*-function; symmetric *L*-function; divisor problem; Fourier coefficients

1. Introduction

As usual, let $\mathbb{H}_k$ denote the set of primitive holomorphic cusp forms with even integral weight $k \geq 2$ for the full modular group $SL_2(\mathbb{Z})$; then $\mathbb{H}_k$ is made up of the common eigenfunctions of all Hecke operators T_n. Then the Fourier series expansion of Hecke eigenfunction f at the cusp ∞ has the following form.

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\text{Im } z > 0),$$

where the coefficient $\lambda_f(n)$ denotes the n-th normalized eigenvalue, which is the coefficient divided by $n^{\frac{k-1}{2}}$ of the Hecke operator T_n. Note that $\lambda_f(n)$ is real valued and also multiplicative. Let n be an integer greater than one, Deligne [1] proved that

$$|\lambda_f(n)| \leq \tau(n),$$

where $\tau(n)$ denotes the number of n's positive divisors. For prime p, we have

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p) \quad \text{and} \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{1}$$

Studying the properties and average behaviors of various sums concerning $\lambda_f(n)$ and $\lambda_{f \times f}(n)$ is a meaningful and interesting problem. In number theory, classical problems are investigate mean value estimates of these Fourier coefficients and related problems with the corresponding automorphic *L*-functions (for examples, see [1–23], etc.). In particular, we give a brief introduction for the general divisor problem.

Let $\omega \geq 1$ is an integer, and

$$\lambda_{\omega,f}(n) = \sum_{n = n_1 n_2 \cdots n_\omega} \lambda_f(n_1)\lambda_f(n_2) \cdots \lambda_f(n_\omega),$$

$$\lambda_{\omega,f \times f}(n) = \sum_{n = n_1 n_2 \cdots n_\omega} \lambda_{f \times f}(n_1)\lambda_{f \times f}(n_2) \cdots \lambda_{f \times f}(n_\omega).$$

when $\omega = 1$, we actually have $\lambda_{1,f}(n) = \lambda_f(n)$ and $\lambda_{1,f \times f}(n) = \lambda_{f \times f}(n)$. Hecke [24] proved

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{2}}.$$

Later, the above upper bound was improved by many authors (see [1,6,15]). Additionally, the best result up to now was due to Wu [18]:

$$\sum_{n \leq x} \lambda_f(n) \ll x^{\frac{1}{3}} \log^{\rho} x,$$

where

$$\rho = \frac{102 + 7\sqrt{21}}{210} \left(\frac{6 - \sqrt{21}}{5} \right)^{\frac{1}{2}} + \frac{102 - 7\sqrt{21}}{210} \left(\frac{6 + \sqrt{21}}{5} \right)^{\frac{1}{2}} - \frac{33}{35} = -0.118 \cdots.$$

Rankin [14] and Selberg [16] showed that

$$\sum_{n \leq x} \lambda_{f \times f}(n) = C_f x + O(x^{\frac{3}{5}}),$$

where C_f is a positive constant depending on f. Kanemitsu, Sankaranarayanan and Tanigawa [25] considered a general divisor problem and established

$$\sum_{n \leq x} \lambda_{\omega, f}(n) \quad \ll \quad x^{1 - \frac{3}{2k+2} + \epsilon},$$

$$\sum_{n \leq x} \lambda_{\omega, f \times f}(n) \quad = \quad M_{\omega}(x) + O(x^{1 - \frac{1}{2k} + \epsilon}),$$

where $\omega \geq 2$ is an integer; $M_{\omega}(x)$ derives from a residue and has the form $x P_{\omega-1}(\log x)$; $P_{\omega-1}(t)$ represents a polynomial of t with degree $\omega - 1$. Fomenko [26] also showed the same result for the sum $\sum_{n \leq x} \lambda_{\omega, f}(n)$. Later, Kanemitsu, Sankaranarayanan and Tanigawa's result was improved by Lü [27], and in this direction, many scholars have obtained a series of results (see [28–30], etc.).

In this paper, we consider the two-dimensional divisor problems related to the Fourier coefficients $\lambda_f(n)$, $\lambda_{f \times f}(n)$. To state our results, we first introduce some notation. For any fixed integers $1 < a < b$, we write

$$\lambda_f^{a,b}(n) = \sum_{n = n_1^a n_2^b} \lambda_f(n_1) \lambda_f(n_2) \tag{2}$$

and

$$\lambda_{f \times f}^{a,b}(n) = \sum_{n = n_1^a n_2^b} \lambda_{f \times f}(n_1) \lambda_{f \times f}(n_2). \tag{3}$$

The two-dimensional divisor problems can be considered as the average behaviors of the coefficients $\lambda_f^{a,b}(n)$ and $\lambda_{f \times f}^{a,b}(n)$. We set

$$S_f(a, b; x) := \sum_{n \leq x} \lambda_f^{a,b}(n) \tag{4}$$

and

$$S_{f \times f}(a, b; x) := \sum_{n \leq x} \lambda_{f \times f}^{a,b}(n). \tag{5}$$

For these sums (4) and (5), we establish the upper bound and the asymptotic formula by considering the sizes of a and b, respectively. We refer to Section 3 for our detailed results.

In the following Section 2, we first introduce some specific automorphic L-functions and quote some lemmas. We state our results in Section 3, and show their proofs in Sections 4 and 5. In Sections 6 and 7, we state an application of our results and give a conclusion, respectively.

2. Some Lemmas

In this section, to prove Theorems 1 and 2 we first introduce some specific automorphic *L*-functions, which is important for the proof of our results and also help us understand the Fourier coefficients in another way. For Re $s > 1$, we define the Hecke *L*-function $L(s, f)$ attached to f as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}.$$

Moreover, the Rankin–Selberg *L*-function attached to f could be defined as

$$L(s, f \times f) = \prod_p \left(1 - \frac{\alpha_f^2(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\beta_f^2(p)}{p^s}\right)^{-1}.$$

Then $L(s, f \times f)$ can be rewritten in the following form:

$$L(s, f \times f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)}{n^s} := \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}(n)}{n^s}.$$

The *j*-th symmetric power *L*-function attached to f could be defined as follows.

$$L(s, \mathrm{sym}^j f) := \prod_p \prod_{m=0}^{j} \left(1 - \frac{\alpha_f(p)^{j-m} \beta_f(p)^m}{p^s}\right), \quad \text{Re } s > 1. \tag{6}$$

Additionally, the *j*-th symmetric power *L*-function attached to f could be expressed in the following Dirichlet series:

$$
\begin{aligned}
L(s, \mathrm{sym}^j f) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\
&= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \frac{\lambda_{\mathrm{sym}^j f}(p^2)}{p^{2s}} + \frac{\lambda_{\mathrm{sym}^j f}(p^3)}{p^{3s}} + \cdots \right), \quad \text{Re } s > 1. \tag{7}
\end{aligned}
$$

The *j*-th symmetric *L*-function $L(s, \mathrm{sym}^j f)$, $j = 1, 2, 3, 4$ could be analytic continued to an entire function over the whole complex plane $\mathbb{C}$ and has a confirmed functional equation. We refer to papers of Hecke [31], Gelbert and Jacquet [32], Kim [33] and Kim and Shahidi [34,35] for these properties of $L(s, \mathrm{sym}^j f)$, $j = 1, 2, 3, 4$. Therefore, we can note that $L(s, \mathrm{sym}^j f)$, $j = 1, 2, 3, 4$ could be recognized as general *L*-functions in the sense of Perelli [36].

With the help of these automorphic *L*-functions, we then quote the following lemmas, which include the individual and averaged subconvexity bounds for Riemann zeta-function $\zeta(s)$, symmetric square *L*-function $L(s, \mathrm{sym}^2 f)$ and corresponding Rankin–Selberg *L*-function $L(s, f \times f)$. From the following Lemma 1 we know that the Rankin–Selberg *L*-function $L(s, f \times f)$ could be decomposed into the product of Riemann zeta-function $\zeta(s)$ and corresponding symmetric square *L*-function $L(s, \mathrm{sym}^2 f)$.

Lemma 1. *For Re $s > 1$, one has*

$$L(s, f \times f) = \zeta(s) L(s, \mathrm{sym}^2 f). \tag{8}$$

Proof. By the comparison of Euler products of two sides of (8) and applying Deligne's result (1), we could easily get this lemma. This lemma can also be found in [27,28]. □

Lemma 2. *For any $\epsilon > 0$, one has the mean value estimate*

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{12} dt \ll T^{2+\epsilon} \tag{9}$$

uniformly for $T \geq 1$ and the upper bounds

$$\zeta(\sigma + it) \ll \begin{cases} (1 + |t|)^{\frac{13}{42}(1-\sigma)+\epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \tag{10}$$

where $|t| \geq 1$.

Proof. The mean value result (9) is due to Heath-Brown [37]. The upper bounds (10) are due to Bourgain [38]. □

Lemma 3. *For any $\epsilon > 0$, one has the mean value estimate*

$$\int_1^T |L(\sigma + it, \mathrm{sym}^2 f)|^2 dt \ll T^{3(1-\sigma)+\epsilon} \tag{11}$$

uniformly for $T \geq 1$ and the upper bounds

$$L(\sigma + it, \mathrm{sym}^2 f) \ll \begin{cases} (1 + |t|)^{\frac{5}{4}(1-\sigma)+\epsilon}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \tag{12}$$

where $|t| \geq 1$.

Proof. The mean value result (11) follows from analytic properties of $L(s, \mathrm{sym}^2 f)$ and standard arguments in number theory. The upper bounds (12) are due to Nunes [13]. □

Lemma 4. *For any $\epsilon > 0$, one has the mean value estimate*

$$\int_T^{2T} \left| L\left(\frac{1}{2} + it, f\right) \right|^2 dt \sim CT \log T \tag{13}$$

uniformly for $T \geq 1$ and the upper bounds

$$L(\sigma + it, f) \ll \begin{cases} (1 + |t|)^{\frac{2}{3}(1-\sigma)}, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \\ 1, & \text{if } \sigma > 1, \end{cases} \tag{14}$$

where $|t| \geq 1$.

Proof. The results in this lemma were established by Good [5]. □

3. Main Theorems

In this paper, we consider the two-dimensional divisor problems related to the Fourier coefficients $\lambda_f(n)$, $\lambda_{f \times f}(n)$ and establish the following two theorems. To establish these two theorems, we apply some classical methods and instruments, such as Perron's formula, Cauchy's residue theorem, decomposition of the Rankin–Selberg L-function, upper bounds and mean values of specific functions.

Theorem 1. *Suppose that a and b are any fixed integers with $1 < a < b$. Then for any $\epsilon > 0$, one has*

$$S_f(a, b; x) = \sum_{n \leq x} \lambda_f^{a,b}(n) \ll \begin{cases} x^{1 - \frac{3(2a-1)}{2(5a-b)}+\epsilon}, & \text{if } b \leq 2a, \\ x^{\frac{1}{2a}+\epsilon}, & \text{if } b > 2a. \end{cases}$$

Theorem 2. *Suppose that a and b are any fixed integers with $1 < a < b$. Then for any $\epsilon > 0$, one has*

$$S_{f \times f}(a, b; x) = \sum_{n \leq x} \lambda_{f \times f}^{a,b}(n)$$

$$= \begin{cases} L(1, \text{sym}^2 f)L\left(\frac{b}{a}, f \times f\right)x^{\frac{1}{a}} + O\left(x^{1 - \frac{84(2a-1)}{486a - 131b} + \epsilon}\right), & \text{if } 318a^2 - 131ab - 402a + 131b < 0, b \leq 2a, \\ O\left(x^{1 - \frac{84(2a-1)}{486a - 131b} + \epsilon}\right), & \text{if } 318a^2 - 131ab - 402a + 131b > 0, b \leq 2a, \\ L(1, \text{sym}^2 f)L\left(\frac{b}{2}, f \times f\right)x^{\frac{1}{2}} + O\left(x^{\frac{7}{16} + \epsilon}\right), & \text{if } a = 2, b > 2a, \\ O\left(x^{1 - \frac{3(2a-1)}{8a} + \epsilon}\right), & \text{if } a \geq 3, b > 2a. \end{cases}$$

4. Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. Let $s = \sigma + it$ and $\eta = 1 + \epsilon$. We have

$$L(as, f)L(bs, f) = \sum_{n=1}^{\infty} \frac{\lambda_f^{a,b}(n)}{n^s}. \tag{15}$$

Then, by applying Perron's formula (see the Proposition 5.54 in [39]), we can obtain

$$S_f(a, b; x) = \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} L(as, f)L(bs, f)\frac{x^s}{s}ds + O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{16}$$

where T is a parameter which will be decided later.

We shift the line of the integral of (16) to the line $\text{Re } s = \frac{1}{2a}$. Then Cauchy's residue theorem shows that

$$S_f(a, b; x) = G_1 + G_2 + G_3 + O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{17}$$

where

$$G_1 = \frac{1}{2\pi i} \int_{\frac{1}{2a} - iT}^{\frac{1}{2a} + iT} L(as, f)L(bs, f)\frac{x^s}{s}ds,$$

$$G_2 = \frac{1}{2\pi i} \int_{\frac{1}{2a} + iT}^{\eta + iT} L(as, f)L(bs, f)\frac{x^s}{s}ds,$$

$$G_3 = \frac{1}{2\pi i} \int_{\eta - iT}^{\frac{1}{2a} - iT} L(as, f)L(bs, f)\frac{x^s}{s}ds.$$

The following work is to estimate these three terms, G_1, G_2 and G_3. The estimates of these integrals on the horizontal parts are analogous; thus, we always consider G_2 and G_3 firstly in the following parts. To get this goal, we consider two cases $b \leq 2a$ and $b > 2a$.

We first consider the case $b \leq 2a$. To estimate G_2 and G_3, we divide the integral interval into the following four short intervals $I_1, \cdots, I_4$ and apply Lemma 4.

Interval 1. $I_1 := \left\{s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, \frac{a}{2b} \leq b\sigma \leq 1\right\} = \left\{s = \sigma + iT : \frac{1}{2a} \leq \sigma \leq \frac{1}{b}\right\}$. In this interval, we have

$$T^{-1} \times \int_{I_1} x^{\sigma}|L(a\sigma + iat, f)L(b\sigma + ibt, f)|d\sigma$$

$$\ll \max_{\frac{1}{2a} \leq \sigma \leq \frac{1}{b}} x^{\sigma}T^{\frac{2}{3}(1-a\sigma) + \frac{2}{3}(1-b\sigma) + \epsilon}T^{-1}$$

$$\ll \max_{\frac{1}{2a} \leq \sigma \leq \frac{1}{b}} T^{\frac{1}{3} + \epsilon}\left(\frac{x}{T^{\frac{2}{3}(b+a)}}\right)^{\sigma}$$

$$\ll x^{\frac{1}{2a}}T^{-\frac{b}{3a} + \epsilon} + x^{\frac{1}{b}}T^{-\frac{1}{3} - \frac{2a}{3b} + \epsilon}. \tag{18}$$

Interval 2. $I_2 := \left\{ s = \sigma + iT : \frac{1}{2} \le a\sigma \le 1, \ 1 < b\sigma \le b\eta \right\} = \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \le \frac{1}{a} \right\}$.
In this interval, we have

$$
\begin{aligned}
T^{-1} \quad & \times \quad \int_{I_2} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)| d\sigma \\
& \ll \quad \max_{\frac{1}{b} < \sigma \le \frac{1}{a}} x^\sigma T^{\frac{2}{3}(1-a\sigma)+\epsilon} T^{-1} \\
& \ll \quad \max_{\frac{1}{b} < \sigma \le \frac{1}{a}} T^{-\frac{1}{3}+\epsilon} \left(\frac{x}{T^{\frac{2a}{3}}} \right)^\sigma \\
& \ll \quad x^{\frac{1}{b}} T^{-\frac{1}{3} - \frac{2a}{3b} + \epsilon} + x^{\frac{1}{a}} T^{-1+\epsilon}.
\end{aligned}
\tag{19}
$$

Interval 3. $I_3 := \left\{ s = \sigma + iT : 1 < a\sigma \le a\eta, \ \frac{b}{2a} \le b\sigma \le 1 \right\}$.
This interval is an empty set noting that $\frac{1}{b} < \frac{1}{a}$.

Interval 4. $I_4 := \{ s = \sigma + iT : 1 < a\sigma \le a\eta, \ 1 < b\sigma \le b\eta \} = \left\{ s = \sigma + iT : \frac{1}{a} < \sigma \le \eta \right\}$.
In this interval, we have

$$
\begin{aligned}
T^{-1} \quad & \times \quad \int_{I_4} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)| d\sigma \\
& \ll \quad \max_{\frac{1}{a} < \sigma \le \eta} x^\sigma T^{-1} \\
& \ll \quad x^{\frac{1}{a}} T^{-1+\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\end{aligned}
\tag{20}
$$

Therefore, from (18)–(20) we have

$$
\begin{aligned}
G_2 + G_3 \quad & \ll \quad T^{-1} \int_{\frac{1}{2a}}^{\eta} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)| d\sigma \\
& = \quad T^{-1} \int_{I_1 \cup \cdots \cup I_4} x^\sigma |L(a\sigma + iat, f)L(b\sigma + ibt, f)| d\sigma \\
& \ll \quad x^{\frac{1}{2a}} T^{-\frac{b}{3a}+\epsilon} + x^{\frac{1}{b}} T^{-\frac{1}{3} - \frac{2a}{3b} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\end{aligned}
\tag{21}
$$

Now we turn to estimate G_1. We have

$$
\begin{aligned}
G_1 \quad & \ll \quad x^{\frac{1}{2a}} \int_1^T \left| L\left(\frac{1}{2} + iat, f \right) L\left(\frac{b}{2a} + ibt, f \right) \right| t^{-1} dt + x^{\frac{1}{2a}} \\
& \ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \le T} T_1^{-1} \int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + iat, f \right) L\left(\frac{b}{2a} + ibt, f \right) \right| dt + x^{\frac{1}{2a}}.
\end{aligned}
$$

Then, by Lemma 4 and applying Cauchy's inequality, we can deduce

$$
\begin{aligned}
G_1 \quad & \ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \le T} T_1^{-1} T_1^{\frac{2}{3} \times (1 - \frac{b}{2a})} \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + iat, f \right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} 1 dt \right)^{\frac{1}{2}} + x^{\frac{1}{2a}} \\
& \ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \le T} T_1^{\frac{2}{3} - \frac{b}{3a} + \epsilon} \\
& \ll \quad x^{\frac{1}{2a}} T^{\frac{2}{3} - \frac{b}{3a} + \epsilon}.
\end{aligned}
\tag{22}
$$

From (17), (21) and (22), we have

$$
S_f(a, b; x) \ll x^{\frac{1}{b}} T^{-\frac{1}{3} - \frac{2a}{3b} + \epsilon} + x^{\frac{1}{2a}} T^{\frac{2}{3} - \frac{b}{3a} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\tag{23}
$$

Taking $T = x^{\frac{3(2a-1)}{2(5a-b)}}$ in (23), we can get

$$
S_f(a, b; x) \ll x^{1 - \frac{3(2a-1)}{2(5a-b)} + \epsilon},
$$

which proves the first result of Theorem 1.

For the case $b > 2a$, to estimate $G_2 + G_3$ we also divide the integral interval into four short intervals $I_1^*, \cdots I_4^*$, which are different from ones for the case $b \leq 2a$. In fact, the corresponding short intervals I_1^* and I_3^* become empty sets at the current case. However, we still can estimate $G_2 + G_3$ by following a similar argument to the corresponding parts of the case $b \leq 2a$ and get

$$G_2 + G_3 \ll x^{\frac{1}{2a}} T^{-\frac{2}{3}+\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.$$

The estimate of G_1 becomes the following at the current case by noting $\frac{b}{2a} > 1$.

$$
\begin{aligned}
G_1 \quad &\ll \quad x^{\frac{1}{2a}} \int_1^T \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) \right| t^{-1} dt + x^{\frac{1}{2a}} \\
&\ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \leq T} T_1^{-1} \int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) L\left(\frac{b}{2a} + ibt, f\right) \right| dt + x^{\frac{1}{2a}} \\
&\ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \leq T} T_1^{-1} \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + iat, f\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} 1 dt \right)^{\frac{1}{2}} + x^{\frac{1}{2a}} \\
&\ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \leq T} T_1^{-1+\frac{1}{2}+\frac{1}{2}+\epsilon} + x^{\frac{1}{2a}} \\
&\ll \quad x^{\frac{1}{2a}} T^{\epsilon}.
\end{aligned}
$$

Thus, we have, recalling (17),

$$S_f(a, b; x) \ll x^{\frac{1}{2a}} T^{\epsilon} + x^{1+\epsilon} T^{-1+\epsilon}. \tag{24}$$

Taking $T = x^{1-\frac{1}{2a}}$ in (24), we can obtain

$$S_f(a, b; x) \ll x^{\frac{1}{2a}+\epsilon},$$

which proves the second result of Theorem 1.

5. Proof of Theorem 2

We shall prove Theorem 2, the process of which is more complicated than Theorem 1, in this section. Let also $s = \sigma + it$ and $\eta = 1 + \epsilon$. Note that

$$L(as, f \times f) L(bs, f \times f) = \sum_{n=1}^{\infty} \frac{\lambda_{f \times f}^{a,b}(n)}{n^s}. \tag{25}$$

Then, by applying Perron's formula (see the Proposition 5.54 in [11]), we have

$$S_{f \times f}(a, b; x) = \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} ds + O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{26}$$

where T is a parameter which will be decided later. Then we shift the line of the integral of (26) to the line Re $s = \frac{1}{2a}$. In view of (8), we know that at the point $s = 1$, $L(s, \text{sym}^2 f)$ is holomorphic, which was proved by Gelbart–Jacquet [32]. Thus, the points $s = \frac{1}{a}$ and $s = \frac{1}{b}$ are the only two possible simple poles of the integrand of (26) in the range $R_T := \{s = \sigma + it : \frac{1}{2a} \leq \sigma \leq 1 + \epsilon, |t| \leq T\}$ depending on the size difference between b and $2a$. Thus, we consider two cases, $b \leq 2a$ and $b > 2a$.

We first consider the case $b \leq 2a$. In this situation, the points $s = \frac{1}{a}$ and $s = \frac{1}{b}$ are all simple poles of the integrand of (26) in the range R_T. Then, Cauchy's residue theorem gives

$$
\begin{aligned}
S_{f \times f}(a, b; x) &= \left\{ \mathrm{Res}_{s=\frac{1}{a}} + \mathrm{Res}_{s=\frac{1}{b}} \right\} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} \\
&\quad + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\epsilon}}{T} \right) \\
&= L(1, \mathrm{sym}^2 f) L\left(\frac{b}{a}, f \times f \right) x^{\frac{1}{a}} + L\left(\frac{a}{b}, f \times f \right) L(1, \mathrm{sym}^2 f) x^{\frac{1}{b}} \\
&\quad + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\epsilon}}{T} \right),
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
J_1 &= \frac{1}{2\pi i} \int_{\frac{1}{2a} - iT}^{\frac{1}{2a} + iT} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} ds, \\
J_2 &= \frac{1}{2\pi i} \int_{\frac{1}{2a} + iT}^{\eta + iT} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} ds, \\
J_3 &= \frac{1}{2\pi i} \int_{\eta - iT}^{\frac{1}{2a} - iT} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} ds,
\end{aligned}
$$

and the main terms $L(1, \mathrm{sym}^2 f) L\left(\frac{b}{a}, f \times f \right) x^{\frac{1}{a}}$ and $L\left(\frac{a}{b}, f \times f \right) L(1, \mathrm{sym}^2 f) x^{\frac{1}{b}}$ derive from the residues of $L(as, f \times f) L(bs, f \times f) \frac{x^s}{s}$ at the simple poles $s = \frac{1}{a}$ and $s = \frac{1}{b}$, respectively.

Now the remaining work is to handle these three terms: J_1, J_2 and J_3. Additionally, the estimates of these integrals on the horizontal parts are analogous, and thus we deal with J_2 and J_3 firstly. To estimate J_2 and J_3, similarly to the method of estimating G_2 and G_3, we also divide the integral interval into the following four short intervals $I'_1, \cdots, I'_4$ and apply Lemmas 2 and 3.

Interval 1. $I'_1 := \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, \frac{b}{2a} \leq b\sigma \leq 1 \right\} = \left\{ s = \sigma + iT : \frac{1}{2a} \leq \sigma \leq \frac{1}{b} \right\}$. In this interval, we have

$$
\begin{aligned}
T^{-1} &\times \int_{I'_1} x^\sigma \mid \zeta(a\sigma + iat) L(a\sigma + iat, \mathrm{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \mathrm{sym}^2 f) \mid d\sigma \\
&\ll \max_{\frac{1}{2a} \leq \sigma \leq \frac{1}{b}} x^\sigma T^{(\frac{13}{42} + \frac{5}{4})(1 - a\sigma) + (\frac{13}{42} + \frac{5}{4})(1 - b\sigma)} T^{-1+\epsilon} \\
&\ll \max_{\frac{1}{2a} \leq \sigma \leq \frac{1}{b}} T^{\frac{89}{42} + \epsilon} \left(\frac{x}{T^{\frac{131}{84}(a+b)}} \right)^\sigma \\
&\ll x^{\frac{1}{2a}} T^{\frac{225}{168} - \frac{131b}{168a} + \epsilon} + x^{\frac{1}{b}} T^{\frac{47}{84} - \frac{131a}{84b} + \epsilon}.
\end{aligned}
\tag{28}
$$

Interval 2. $I'_2 := \left\{ s = \sigma + iT : \frac{1}{2} \leq a\sigma \leq 1, 1 < b\sigma \leq b\eta \right\} = \left\{ s = \sigma + iT : \frac{1}{b} < \sigma \leq \frac{1}{a} \right\}$. In this interval, we have

$$
\begin{aligned}
T^{-1} &\times \int_{I'_2} x^\sigma \mid \zeta(a\sigma + iat) L(a\sigma + iat, \mathrm{sym}^2 f) \zeta(b\sigma + ibt) L(b\sigma + ibt, \mathrm{sym}^2 f) \mid d\sigma \\
&\ll \max_{\frac{1}{b} < \sigma \leq \frac{1}{a}} x^\sigma T^{(\frac{13}{42} + \frac{5}{4})(1 - a\sigma)} T^{-1+\epsilon} \\
&\ll \max_{\frac{1}{b} < \sigma \leq \frac{1}{a}} T^{\frac{47}{84}} \left(\frac{x}{T^{\frac{131}{84}a}} \right)^\sigma \\
&\ll x^{\frac{1}{a}} T^{-1+\epsilon} + x^{\frac{1}{b}} T^{\frac{47}{84} - \frac{131a}{84b} + \epsilon}.
\end{aligned}
\tag{29}
$$

Interval 3. $I_3' := \left\{ s = \sigma + iT : 1 < a\sigma \le a\eta, \ \frac{b}{2a} \le b\sigma \le 1 \right\}$.

This interval is an empty set noting that $\frac{1}{b} < \frac{1}{a}$.

Interval 4. $I_4' := \{ s = \sigma + iT : 1 < a\sigma \le a\eta, \ 1 < b\sigma \le b\eta \} = \left\{ s = \sigma + iT : \frac{1}{a} < \sigma \le \eta \right\}$.

In this interval, we have

$$
\begin{aligned}
T^{-1} \quad &\times \quad \int_{I_4'} x^\sigma \mid \zeta(a\sigma + iat)L(a\sigma + iat, \mathrm{sym}^2 f)\zeta(b\sigma + ibt)L(b\sigma + ibt, \mathrm{sym}^2 f) \mid d\sigma \\
&\ll \quad \max_{\frac{1}{a} < \sigma \le \eta} x^\sigma T^{-1+\epsilon} \\
&\ll \quad x^{1+\epsilon} T^{-1+\epsilon}.
\end{aligned}
\tag{30}
$$

From (28)–(30) we can obtain

$$
\begin{aligned}
&J_2 + J_3 \\
&\ll \quad T^{-1} \int_{\frac{1}{2a}}^{\eta} x^\sigma \mid \zeta(a\sigma + iat)L(a\sigma + iat, \mathrm{sym}^2 f)\zeta(b\sigma + ibt)L(b\sigma + ibt, \mathrm{sym}^2 f) \mid d\sigma \\
&= \quad T^{-1} \int_{I_1' \cup \cdots \cup I_4'} x^\sigma \mid \zeta(a\sigma + iat)L(a\sigma + iat, \mathrm{sym}^2 f)\zeta(b\sigma + ibt)L(b\sigma + ibt, \mathrm{sym}^2 f) \mid d\sigma \\
&\ll \quad x^{\frac{1}{2a}} T^{\frac{225}{168} - \frac{131b}{168a} + \epsilon} + x^{\frac{1}{b}} T^{\frac{47}{84} - \frac{131a}{84b} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.
\end{aligned}
\tag{31}
$$

For J_1, we have

$$
\begin{aligned}
J_1 \quad &\ll \quad x^{\frac{1}{2a}} \int_1^T \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \mathrm{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \mathrm{sym}^2 f\right) \right| t^{-1} dt \\
&\quad + x^{\frac{1}{2a} + \epsilon} \\
&\ll \quad x^{\frac{1}{2a} + \epsilon} + x^{\frac{1}{2a}} \log T \max_{1 \le T_1 \le T} T_1^{-1} \\
&\quad \times \int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \mathrm{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \mathrm{sym}^2 f\right) \right| dt.
\end{aligned}
\tag{32}
$$

Then from Lemmas 2 and 3, and using Hölder's inequality, we can obtain

$$
\begin{aligned}
J_1 \quad &\ll \quad x^{\frac{1}{2a}} \log T \max_{T_1 \le T} T_1^{-1} T_1^{(\frac{13}{42} + \frac{5}{4})(1 - \frac{b}{2a})} \int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \mathrm{sym}^2 f\right) \right| dt + x^{\frac{1}{2a} + \epsilon} \\
&\ll \quad x^{\frac{1}{2a} + \epsilon} + x^{\frac{1}{2a}} \log T \max_{T_1 \le T} T_1^{(\frac{13}{42} + \frac{5}{4})(1 - \frac{b}{2a}) - 1} \\
&\quad \times \left(\int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{\frac{1}{12}} \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + iat, \mathrm{sym}^2 f\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} 1 dt \right)^{\frac{5}{12}} \\
&\ll \quad x^{\frac{1}{2a}} \max_{T_1 \le T} T_1^{\frac{159}{84} - \frac{131b}{168a} + \epsilon} \\
&\ll \quad x^{\frac{1}{2a}} T^{\frac{159}{84} - \frac{131b}{168a} + \epsilon}.
\end{aligned}
\tag{33}
$$

Thus, by putting (27), (31) and (33) together, we have

$$
\begin{aligned}
S_{f \times f}(a, b; x) \quad &= \quad L(1, \mathrm{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}} + L\left(\frac{a}{b}, f \times f\right) L(1, \mathrm{sym}^2 f) x^{\frac{1}{b}} \\
&\quad + O\left(x^{\frac{1}{2a}} T^{\frac{159}{84} - \frac{131b}{168a} + \epsilon} + x^{\frac{1}{b}} T^{\frac{47}{84} - \frac{131a}{84b} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon} \right).
\end{aligned}
\tag{34}
$$

Taking $T = x^{\frac{84(2a-1)}{486a - 131b}}$ in (34), we can obtain

$$
\begin{aligned}
S_{f \times f}(a, b; x) \quad &= \quad L(1, \mathrm{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}} + L\left(\frac{a}{b}, f \times f\right) L(1, \mathrm{sym}^2 f) x^{\frac{1}{b}} \\
&\quad + O\left(x^{1 - \frac{84(2a-1)}{486a - 131b} + \epsilon} \right).
\end{aligned}
\tag{35}
$$

Note that $1 - \frac{84(2a-1)}{486a-131b} > \frac{1}{b}$ always holds. Then, comparing the first term and the error term in (35), we have, recalling $b \leq 2a$,

$$S_{f \times f}(a, b; x) = \begin{cases} L(1, \text{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}} + O\left(x^{1 - \frac{84(2a-1)}{486a-131b} + \epsilon}\right), & \text{if } 318a^2 - 131ab - 402a + 131b < 0, b \leq 2a, \\ O\left(x^{1 - \frac{84(2a-1)}{486a-131b} + \epsilon}\right), & \text{if } 318a^2 - 131ab - 402a + 131b > 0, b \leq 2a, \end{cases}$$

which implies the first and second results of Theorem 2.

For the case $b > 2a$, we use a similar argument to the corresponding case of Theorem 1. In this situation, the point $s = \frac{1}{a}$ is the only simple pole of the integrand of (26) in the range R_T by noting $\frac{1}{b} < \frac{1}{2a}$. Then Cauchy's residue theorem shows

$$\begin{aligned} S_{f \times f}(a, b; x) &= \text{Res}_{s = \frac{1}{a}} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} + O\left(\frac{x^{1+\epsilon}}{T}\right) \\ &\quad + \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2a} - iT}^{\frac{1}{2a} + iT} + \int_{\frac{1}{2a} + iT}^{\eta + iT} + \int_{\eta - iT}^{\frac{1}{2a} - iT} \right\} L(as, f \times f) L(bs, f \times f) \frac{x^s}{s} ds \\ &:= L(1, \text{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}} + J_1' + J_2' + J_3' + O\left(\frac{x^{1+\epsilon}}{T}\right), \end{aligned} \tag{36}$$

where the main term $L(1, \text{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}}$ derives from the residue of $L(as, f \times f) L(bs, f \times f) \frac{x^s}{s}$ at the simple pole $s = \frac{1}{a}$.

To estimate $J_2' + J_3'$ we also divide the integral interval into four short intervals $I_1'^*, \cdots I_4'^*$, which are different from ones for the case $b \leq 2a$. In fact, the corresponding short intervals $I_1'^*$ and $I_3'^*$ become empty sets in this situation. However, we still can estimate $J_2' + J_3'$ by following a similar argument to the corresponding parts of the case $b \leq 2a$ and get

$$J_2' + J_3' \ll x^{\frac{1}{2a}} T^{-\frac{37}{168} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}.$$

The estimate of J_1' becomes the following at the current case by noting $\frac{b}{2a} > 1$.

$$\begin{aligned} J_1' &\ll x^{\frac{1}{2a}} \int_1^T \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \zeta\left(\frac{b}{2a} + ibt\right) L\left(\frac{b}{2a} + ibt, \text{sym}^2 f\right) \right| t^{-1} dt + x^{\frac{1}{2a} + \epsilon} \\ &\ll x^{\frac{1}{2a}} \log T \max_{1 \leq T_1 \leq T} T_1^{-1} \int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right| dt + x^{\frac{1}{2a} + \epsilon} \\ &\ll x^{\frac{1}{2a}} \log T \max_{T_1 \leq T} T_1^{-1} \left(\int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{1}{2} + iat\right) \right|^{12} dt \right)^{\frac{1}{12}} \left(\int_{\frac{T_1}{2}}^{T_1} \left| L\left(\frac{1}{2} + iat, \text{sym}^2 f\right) \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{T_1}{2}}^{T_1} 1 dt \right)^{\frac{5}{12}} \\ &\quad + x^{\frac{1}{2a} + \epsilon} \\ &\ll x^{\frac{1}{2a}} T^{\frac{1}{3} + \epsilon}. \end{aligned}$$

Thus, recalling (36) we have

$$S_{f \times f}(a, b; x) = L(1, \text{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}} + O\left(x^{\frac{1}{2a}} T^{\frac{1}{3} + \epsilon} + x^{1+\epsilon} T^{-1+\epsilon}\right). \tag{37}$$

Taking $T = x^{\frac{3(2a-1)}{8a}}$ in (37), we can obtain

$$S_{f \times f}(a, b; x) = L(1, \text{sym}^2 f) L\left(\frac{b}{a}, f \times f\right) x^{\frac{1}{a}} + O\left(x^{1 - \frac{3(2a-1)}{8a} + \epsilon}\right). \tag{38}$$

Note that when $a \geq 3$, we have $1 - \frac{3(2a-1)}{8a} > \frac{1}{a}$. Therefore, we have, recalling $b > 2a$,

$$S_{f \times f}(a, b; x) = \begin{cases} L(1, \text{sym}^2 f) L\left(\frac{b}{2}, f \times f\right) x^{\frac{1}{2}} + O\left(x^{\frac{7}{16} + \epsilon}\right), & \text{if } a = 2, b > 2a, \\ O\left(x^{1 - \frac{3(2a-1)}{8a} + \epsilon}\right), & \text{if } a \geq 3, b > 2a, \end{cases} \tag{39}$$

which implies the third and fourth results of Theorem 2.

6. Application

As an application of Theorems 1 and 2, we may consider detecting the sign changes of $\lambda_f^{a,b}(n)$ and $\lambda_{f \times f}^{a,b}(n)$, i.e., estimating the following two quantities:

$$\mathcal{N}_f(x) = \sum_{\substack{n \leq x \\ \lambda_f^{a,b}(n) \geq 0}} 1 \quad \text{and} \quad \mathcal{N}_{f \times f}(x) = \sum_{\substack{n \leq x \\ \lambda_{f \times f}^{a,b}(n) \geq 0}} 1.$$

To estimate these quantities $\mathcal{N}_f(x)$ and $\mathcal{N}_{f \times f}(x)$, we need to establish the lower and upper bounds to the sums

$$\sum_{n \leq x} |\lambda_f^{a,b}(n)| \quad \text{and} \quad \sum_{n \leq x} |\lambda_{f \times f}^{a,b}(n)|,$$

respectively. Then, comparing these bounds with Theorems 1 and 2, we can get the estimates of these quantities $\mathcal{N}_f(x)$ and $\mathcal{N}_{f \times f}(x)$.

7. Conclusions

In this paper, we studied the mean value estimates of two-dimensional divisor problems related to some automorphic L-functions. We focused on the average behaviors of $\lambda_f^{a,b}(n)$ and $\lambda_{f \times f}^{a,b}(n)$, respectively. In this research, we established the upper bounds for the sum $\sum_{n \leq x} \lambda_f^{a,b}(n)$ and the asymptotic formulas for the sum $\sum_{n \leq x} \lambda_{f \times f}^{a,b}(n)$. The conditions of the integers a, b satisfy $1 < a < b$, because of the complexities and difficulties. To overcome these complexities and difficulties, we need to estimate the integrals of the horizontal parts more carefully. Some classical methods and instruments, such as Perron's formula and Cauchy's residue theorem; the decomposition of the Rankin–Selberg L-function; upper bounds and mean values of the Riemann zeta-function; the Hecke L-function and the symmetric square L-function, are also indispensable. With the results of this paper, we can further understand the properties of the Fourier coefficients $\lambda_f^{a,b}(n)$ and $\lambda_{f \times f}^{a,b}(n)$.

References

1. Deligne, P. La conjecture de Weil. *Publ. Math. l'IHÉS* **1974**, *43*, 273–307. [CrossRef]
2. Barthel, L.; Ramakrishnan, D. A nonvanishing result for twists of L-functions of $GL(n)$. *Duke Math. J.* **1994**, *74*, 681–700. [CrossRef]
3. Chandrasekharan, K.; Narasimhan, R. Functional equations with multiple gamma factors and the average order of arithmetical functions. *Ann. Math.* **1962**, *76*, 93–136. [CrossRef]
4. Dong, L.L.; Liu, H.F.; Zhang, D.Y. Zero density estimates for automorphic L-functions of $SL(2, \mathbb{Z})$. *Acta Math. Hung.* **2016**, *148*, 191–210. [CrossRef]
5. Good, A. The square mean of Dirichlet series associated with cusp forms. *Mathematika* **1982**, *29*, 278–295. [CrossRef]
6. Hafner, J.L.; Ivić, A. On sums of Fourier coefficients of cusp forms. *L'Enseignement Math.* **1989**, *35*, 375–382.

7. Jiang, Y.J.; Lü, G.S.; Yan, X.F. Mean value theorem connected with Fourier coefficients of Hecke-Maass forms for $SL(m, \mathbb{Z})$. *Math. Proc. Camb. Philos. Soc.* **2016**, *161*, 339–356 [CrossRef]

8. Lao, H.X. On the fourth moment of coefficients of symmetric square *L*-function. *Chin. Ann. Math. Ser. B* **2012**, *33*, 877–888. [CrossRef]

9. Lao, H.X. The cancellation of Fourier coefficients of cusp forms over different sparse sequences. *Acta Math. Sin. (Engl. Ser.)* **2013**, *29*, 1963–1972. [CrossRef]

10. Lao, H.X. Mean value of Dirichlet series coefficients of Rankin-Selberg *L*-functions. *Lith. Math. J.* **2017**, *57*, 351–358. [CrossRef]

11. Lao, H.X. On comparing Hecke eigenvalues of cusp forms. *Acta Math. Hung.* **2020**, *160*, 58–71. [CrossRef]

12. Lao, H.X.; Wei, H.B. Ω-result on coefficients of automorphic *L*-functions over sparse sequences. *J. Korean Math. Soc.* **2015**, *52*, 945–954. [CrossRef]

13. Nunes, R.M. On the subconvexity estimate for self-dual *GL*(3) *L*-functions in the *t*-aspect. *arXiv* **2017**, arXiv:1703.04424vl.

14. Rankin, R.A. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithemtical functions II. The order of the Fourier coefficients of the integral modular forms. *Proc. Camb. Philos. Soc.* **1939**, *35*, 357–372. [CrossRef]

15. Rankin, R.A. Sums of cusp form coefficients. In *Automorphic Forms and Analytic Number Theory (Montreal, PQ, 1989)*; University Montreal: Montreal, QC, Canada, 1990; pp. 115–121.

16. Selberg, A. Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist. *Arch. Math. Naturvid.* **1940**, *43*, 47–50.

17. Song, P.; Zhai, W.G.; Zhang, D.Y. Power moments of Hecke eigenvalues for congruence group. *J. Number Theory* **2019**, *198*, 139–158. [CrossRef]

18. Wu, J. Power sums of Hecke eigenvalues and application. *Acta Arith.* **2009**, *137*, 333–344. [CrossRef]

19. Ye, Y.B.; Zhang, D.Y. Zero density for automorphic *L*-functions. *J. Number Theory* **2013**, *133*, 3877–3901. [CrossRef]

20. Zhang, D.Y.; Lau, Y.K.; Wang, Y.N. Remark on the paper "On products of Fourier coefficients of cusp forms". *Arch. Math.* **2017**, *108*, 263–269. [CrossRef]

21. Zhang, D.Y.; Wang, Y.N. Ternary quadratic form with prime variables attached to Fourier coefficients of primitive holomorphic cusp form. *J. Number Theory* **2017**, *176*, 211–225. [CrossRef]

22. Zhang, D.Y.; Wang, Y.N. Higher-power moments of Fourier coefficients of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$. *Ramanujan J.* **2018**, *47*, 685–700. [CrossRef]

23. Zhang, R.; Han, X.; Zhang, D.Y. Power moments of the Riesz mean error term of symmetric square *L*-function in short intervals. *Symmetry* **2020**, *12*, 2036. [CrossRef]

24. Hecke, E. Theorie der Eisensteinsche Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik. *Abh. Math. Sem. Univ. Hambg.* **1927**, *5*, 199–224. [CrossRef]

25. Kanemitsu, S.; Sankaranarayanan, A.; Tanigawa, Y. A mean value theorem for Dirichlet series and a general divisor problem. *Monatshefte Math.* **2002**, *136*, 17–34. [CrossRef]

26. Fomenko, O.M. Mean value theorems for a class of Dirichlet series. *J. Math. Sci.* **1999**, *157*, 659–672.

27. Lü, G.S. On general divisor problems involving Hecke eigenvalues. *Acta. Math. Hung.* **2012**, *135*, 148–159. [CrossRef]

28. Liu, H.F. Mean value estimates of the coefficients of product *L*-functions. *Acta Math. Hung.* **2018**, *156*, 102–111. [CrossRef]

29. Liu, H.F.; Zhang, R. Some problems involving Hecke eigenvalues. *Acta Math. Hung.* **2019**, *159*, 287–298. [CrossRef]

30. Lü, G.S.; Sankaranarayanan, A. On the coefficients of triple product *L*-functions. *Rocky Mt. J. Math.* **2017**, *47*, 553–570.

31. Iwaniec, H. *Topics in Classical Automorphic Forms, Graduate Studies in Mathematics*; American Mathematical Society: Providence, RI, USA, 1997; Volume 17.

32. Gelbart, S.; Jacquet, H. A relation between automorphic representations of *GL*(2) and *GL*(3). *Ann. Sci. l'École Norm. Supér.* **1978**, *11*, 471–542. [CrossRef]

33. Kim, H.H. Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2. *J. Am. Math. Soc.* **2003**, *16*, 139–183. [CrossRef]

34. Kim, H.H.; Shahidi, F. Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2. *Ann. Math.* **2002**, *155*, 837–893. [CrossRef]

35. Kim, H.H.; Shahidi, F. Cuspidality of symmetric powers with applications. *Duke Math. J.* **2002**, *112*, 177–197.

36. Perelli, A. General *L*-functions. *Ann. Mat. Pura Appl.* **1982**, *130*, 287–306. [CrossRef]

37. Heath-Brown, D.R. The twelfth power moment of the Riemann zeta-function. *Q. J. Math.* **1978**, *29*, 443–462. [CrossRef]

38. Bourgain, J. Decoupling, exponential sums and the Riemann zeta function. *Am. Math. Soc.* **2017**, *30*, 205–224. [CrossRef]

39. Iwaniec, H.; Kowalski, E. *Analytic Number Theory*; American Mathematical Society Colloquium Publications; American Mathematical Society: Providence, RI, USA, 2004; Volume 53.