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Hilfer-Polya, ψ-Hilfer Ostrowski and ψ-Hilfer-Hilbert-Pachpatte Fractional Inequalities

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Abstract: Here we present Hilfer-Polya, ψ-Hilfer Ostrowski and ψ-Hilfer-Hilbert-Pachpatte types fractional inequalities. They are univariate inequalities involving left and right Hilfer and ψ-Hilfer fractional derivatives. All estimates are with respect to norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. At the end we provide applications.

Keywords: fractional integral inequalities; right and left ψ-Hilfer and Hilfer fractional derivatives

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1. Introduction

We are motivated by the following famous Polya's integral inequality, see [1], (p. 62, [2]), [3] and (p. 83, [4]).

Theorem 1. *Let $f(x)$ be a differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a,b]$ such that*

$$\left|f'(\xi)\right| > \frac{4}{(b-a)^2} \int_a^b f(x)dx. \tag{1}$$

We are inspired also by the related first fractional Polya Inequality, see Chapter 2, p. 9, [5].

In this article, we establish fractional integral inequalities using the Hilfer and ψ-Hilfer fractional derivatives. These are of Polya, Ostrowski and Hilbert-Pachpatte types.

2. Background

Let $-\infty < a < b < \infty$, the left and right Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ ($\mathcal{R}(\alpha) > 0$) are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{2}$$

$x > a$; where Γ stands for the gamma function,

And

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \tag{3}$$

$x < b$.

The Riemann-Liouville left and right fractional derivatives of order $\alpha \in \mathbb{C}$ ($\mathcal{R}(\alpha) \geq 0$) are defined by

$$\left(\Delta_{a+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} y(t)dt \tag{4}$$

($n = \lceil \mathcal{R}(\alpha) \rceil$, $\lceil \cdot \rceil$ means ceiling of the number; $x > a$)

$$\left(\Delta_{b-}^{\alpha} y\right)(x) = (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{b-}^{n-\alpha} y\right)(x) =$$

$$\frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b (t-x)^{n-\alpha-1} y(t)dt \tag{5}$$

($n = \lceil \mathcal{R}(\alpha) \rceil$; $x < b$), respectively, where $\mathcal{R}(\alpha)$ is the real part of α.

In particular, when $\alpha = n \in \mathbb{Z}_+$, then

$$\left(\Delta_{a+}^{0} y\right)(x) = \left(\Delta_{b-}^{0} y\right)(x) = y(x);$$

$$(\Delta_{a+}^{n} y)(x) = y^{(n)}(x), \text{ and } (\Delta_{b-}^{n} y)(x) = (-1)^n y^{(n)}(x), \ n \in \mathbb{N},$$

see [6].

Let $\alpha > 0$, $I = [a, b] \subset \mathbb{R}$, f an integrable function defined on I and $\psi \in C^1(I)$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in I$. Left fractional integrals and left Riemann-Liouville fractional derivatives of a function f with respect to another function ψ are defined as ([6,7])

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha-1} f(t)dt, \tag{6}$$

and

$$\Delta_{a+}^{\alpha,\psi} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n I_{a+}^{n-\alpha,\psi} f(x) = \tag{7}$$

$$\frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} f(t)dt,$$

respectively, where $n = \lceil \alpha \rceil$.

Similarly, we define the right ones:

$$I_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} f(t)dt, \tag{8}$$

and

$$\Delta_{b-}^{\alpha,\psi} f(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n I_{b-}^{n-\alpha,\psi} f(x) =$$

$$\frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n \int_x^b \psi'(t)(\psi(t) - \psi(x))^{n-\alpha-1} f(t)dt. \tag{9}$$

The following semigroup property holds; if $\alpha, \beta > 0$, $f \in C(I)$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f = I_{a+}^{\alpha+\beta,\psi} f \ \text{ and } \ I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f = I_{b-}^{\alpha+\beta,\psi} f.$$

Next let again $\alpha > 0$, $n = \lceil \alpha \rceil$, $I = [a, b]$, $f, \psi \in C^n(I)$: ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left ψ-Caputo fractional derivative of f of order α is given by ([8])

$${}^{C}D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x), \tag{10}$$

and the right ψ-Caputo fractional derivative ([8])

$${}^{C}D_{b-}^{\alpha,\psi} f(x) = I_{b-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x). \tag{11}$$

We set

$$f_{\psi}^{[n]}(x) := f_{\psi}^{(n)} f(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^n f(x). \tag{12}$$

Clearly, when $\alpha = m \in \mathbb{N}$ we have

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = f_{\psi}^{[m]}(x) \text{ and } {}^{C}D_{b-}^{\alpha,\psi} f(x) = (-1)^{m} f_{\psi}^{[m]}(x), $$

and if $\alpha \notin \mathbb{N}$, then

$$ {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t)dt, \tag{13} $$

and

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1} f_{\psi}^{[n]}(t)dt. \tag{14} $$

If $\psi(x) = x$, then we get the usual left and right Caputo fractional derivatives

$$ {}^{C}D_{a+}^{m} f(x) = f^{(m)}(x), \ {}^{C}D_{b-}^{m} f(x) = (-1)^{m} f^{(m)}(x), $$

for $m \in \mathbb{N}$, and ($\alpha \notin \mathbb{N}$)

$$ D_{*a}^{\alpha} f(x) = {}^{C}D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \tag{15} $$

$$ D_{b-}^{\alpha}(x) = {}^{C}D_{b-}^{\alpha} f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} (t-x)^{n-\alpha-1} f^{(n)}(t)dt. \tag{16} $$

Also we set

$$ {}^{C}D_{a+}^{0,\psi} f(x) = {}^{C}D_{b-}^{0,\psi} f(x) = f(x). $$

Next we will deal with the ψ-Hilfer fractional derivative.

Definition 1. *([9]) Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $I = [a,b] \subset \mathbb{R}$ and $f, \psi \in C^{n}([a,b])$, ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The ψ-Hilfer fractional derivative (left-sided and right-sided) ${}^{H}\mathbb{D}_{a+(b-)}^{\alpha,\beta;\psi} f$ of order α and type $0 \leq \beta \leq 1$, respectively, are defined by*

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x), \tag{17} $$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x), \ \ x \in [a,b]. \tag{18} $$

The original Hilfer fractional derivatives ([10]) come from $\psi(x) = x$, and are denoted by ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f(x)$ and ${}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f(x)$.

When $\beta = 0$, we get Riemann-Liouville fractional derivatives, while when $\beta = 1$ we have Caputo type fractional derivatives.

We define $\gamma = \alpha + \beta(n-\alpha)$. We notice that $n-1 < \alpha \leq \alpha + \beta(n-\alpha) \leq \alpha + n - \alpha = n$, hence $\lceil \gamma \rceil = n$. We can easily write that ([9])

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\gamma-\alpha;\psi} \Delta_{a+}^{\gamma;\psi} f(x), \tag{19} $$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\gamma-\alpha;\psi} \Delta_{b-}^{\gamma;\psi} f(x), \ \ x \in [a,b]. \tag{20} $$

We have that ([9])

$$ \Delta_{a+}^{\gamma,\psi} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x), \tag{21} $$

and

$$\Delta_{b-}^{\gamma,\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x). \tag{22}$$

In particular, when $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$; $\gamma = \alpha + \beta(1-\alpha)$, *we have that*

$$^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\gamma-\alpha-1} \Delta_{a+}^{\gamma;\psi} f(t)dt, \tag{23}$$

and

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\gamma-\alpha-1} \Delta_{b-}^{\gamma;\psi} f(t)dt, \tag{24}$$

$x \in [a,b]$.

Remark 1. *([9]) Let* $\mu = n(1-\beta) + \beta\alpha$, *then* $\lceil \mu \rceil = n$.
Assume that $g(x) = I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) \in C^n([a,b])$, *we have that*

$$^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{n-\mu;\psi} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n g(x). \tag{25}$$

Thus,

$$^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f = {}^{C}D_{a+}^{\mu;\psi} g(x) = {}^{C}D_{a+}^{\mu;\psi} \left[I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) \right]. \tag{26}$$

Assume that $w(x) = I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \in C^n([a,b])$. *Hence*

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n w(x) = I_{b-}^{n-\mu;\psi} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n w(x). \tag{27}$$

Thus,

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f = {}^{C}D_{b-}^{\mu;\psi} w(x) = {}^{C}D_{b-}^{\mu;\psi} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \right). \tag{28}$$

We mention the simplified ψ-Hilfer fractional Taylor formulae:

Theorem 2. *(see also [9]) Let* $\psi, f \in C^n([a,b])$, *with* ψ *being increasing such that* $\psi'(x) \neq 0$ *over* $[a,b]$, *where* $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, *and* $\gamma = \alpha + \beta(n-\alpha)$, $x \in [a,b]$. *Then*

$$f(x) - \sum_{k=1}^{n-1} \frac{(\psi(x)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f \right)(a) =$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \, {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t)dt, \tag{29}$$

and

$$f(x) - \sum_{k=1}^{n-1} \frac{(-1)^k(\psi(b)-\psi(x))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f \right)(b) =$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} \, {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t)dt. \tag{30}$$

Here notice that $\left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f \right)(a) = \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f \right)(b) = 0$.
We also mention the following alternative ψ-Hilfer fractional Taylor formulae:

Theorem 3. *([11]) Let* $f, \psi \in C^n([a,b])$, *with* ψ *being increasing,* $\psi'(x) \neq 0$ *over* $[a,b] \subset \mathbb{R}$, $\alpha > 0 : \lceil \alpha \rceil = n$, $0 \leq \beta \leq 1$, $\mu = n(1-\beta) + \beta\alpha$. *Assume that* $g(x) = I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x)$, $w(x) = I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \in C^n([a,b])$.
Then

(1)

$$I_{a+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x)=g(x)-\sum_{k=0}^{n-1}\frac{g_{\psi}^{[k]}(a)}{k!}(\psi(x)-\psi(a))^{k}, \tag{31}$$

where

$$g_{\psi}^{[k]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{k}g(x),\ \ k=0,1,...,n-1,$$

and

(2)

$$I_{b-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x)=w(x)-\sum_{k=0}^{n-1}\frac{(-1)^{k}w_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^{k}, \tag{32}$$

where

$$w_{\psi}^{[k]}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{k}w(x),\ \ k=0,1,...,n-1;\ x\in[a,b].$$

Next we list two Hilfer fractional derivatives representation formulae:

Theorem 4. *([11]) Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $\lceil\alpha\rceil=n$, $0<\beta<1$; $f\in C^{n}([a,b])$, $[a,b]\subset\mathbb{R}$; and set $\gamma=\alpha+\beta(n-\alpha)$. Assume further that $\Delta_{a+}^{\gamma}f\in C([a,b]):\Delta_{a+}^{\gamma-j}f(a)=0$, for $j=1,...,n$. Let also $\overline{\alpha}>0:\lceil\overline{\alpha}\rceil=\overline{n}$, with $\overline{\gamma}=\overline{\alpha}+\beta(\overline{n}-\overline{\alpha})$, and assume that $\alpha>\overline{\alpha}$ and $\gamma>\overline{\gamma}$. Then*

$${}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f(x)=\frac{1}{\Gamma(\alpha-\overline{\alpha})}\int_{a}^{x}(x-t)^{\alpha-\overline{\alpha}-1}\ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta}f(t)dt, \tag{33}$$

$\forall\ x\in[a,b]$,

Furthermore, ${}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f\in AC([a,b])$ (absolutely continuous functions) if $\alpha-\overline{\alpha}\geq1$ and ${}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f\in C([a,b])$ if $\alpha-\overline{\alpha}\in(0,1)$.

Theorem 5. *([11]) Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $\lceil\alpha\rceil=n$, $0<\beta<1$; $f\in C^{n}([a,b])$, $[a,b]\subset\mathbb{R}$; and set $\gamma=\alpha+\beta(n-\alpha)$. Assume further that $\Delta_{b-}^{\gamma}f\in C([a,b]):\Delta_{b-}^{\gamma-j}f(b)=0$, $j=1,...,n$. Let also $\overline{\alpha}>0:\lceil\overline{\alpha}\rceil=\overline{n}$, with $\overline{\gamma}=\overline{\alpha}+\beta(\overline{n}-\overline{\alpha})$, and assume that $\alpha>\overline{\alpha}$ and $\gamma>\overline{\gamma}$. Then*

$${}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta}f(x)=\frac{1}{\Gamma(\alpha-\overline{\alpha})}\int_{x}^{b}(t-x)^{\alpha-\overline{\alpha}-1}\ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta}f(t)dt, \tag{34}$$

$\forall\ x\in[a,b]$,

Furthermore, ${}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta}f\in AC([a,b])$ if $\alpha-\overline{\alpha}\geq1$ and ${}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta}f\in C([a,b])$ if $\alpha-\overline{\alpha}\in(0,1)$.

3. Main Results

We present the following Hilfer-Polya type fractional inequalities:

Theorem 6. *Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $\lceil\alpha\rceil=n$, $0<\beta<1$; $f\in C^{n}([a,b])$, $[a,b]\subset\mathbb{R}$; and set $\gamma=\alpha+\beta(n-\alpha)$. Assume further that $\Delta_{a+}^{\gamma}f\in C([a,b]):\Delta_{a+}^{\gamma-j}f(a)=0$, for $j=1,...,n$; and $\Delta_{b-}^{\gamma}f\in C([a,b]):\Delta_{b-}^{\gamma-j}f(b)=0$, $j=1,...,n$. Let also $\overline{\alpha}>0:\lceil\overline{\alpha}\rceil=\overline{n}$, with $\overline{\gamma}=\overline{\alpha}+\beta(\overline{n}-\overline{\alpha})$, and assume that $\alpha>\overline{\alpha}$ and $\gamma>\overline{\gamma}$.*

Set

$${}^{H}\mathbb{D}^{\overline{\alpha},\beta}f(x):=\begin{cases}{}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta}f(x), & x\in\left[a,\frac{a+b}{2}\right],\\ \\ {}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta}f(x), & x\in\left(\frac{a+b}{2},b\right],\end{cases} \tag{35}$$

and

$$M_{1}:=\max\left\{\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]},\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}. \tag{36}$$

Then

$$\left|\int_a^b {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x)dx\right| \le \int_a^b \left|{}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x)\right| dx \le \frac{M_1(b-a)^{\alpha-\overline{\alpha}+1}}{2^{\alpha-\overline{\alpha}}\Gamma(\alpha-\overline{\alpha}+2)}. \tag{37}$$

Proof. From (33) we have

$$ {}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{a+} f(x) = \frac{1}{\Gamma(\alpha-\overline{\alpha})}\int_a^x (x-t)^{\alpha-\overline{\alpha}-1}\, {}^{H}\mathbb{D}^{\alpha,\beta}_{a+} f(t)dt, \tag{38}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$.

By (34), we get

$$ {}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{b-} f(x) = \frac{1}{\Gamma(\alpha-\overline{\alpha})}\int_x^b (t-x)^{\alpha-\overline{\alpha}-1}\, {}^{H}\mathbb{D}^{\alpha,\beta}_{b-} f(t)dt, \tag{39}$$

$\forall\, x \in \left[\frac{a+b}{2}, b\right]$.

We derive that

$$\left|{}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{a+} f(x)\right| \le \frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{a+} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+1)}(x-a)^{\alpha-\overline{\alpha}}, \tag{40}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$, and similarly,

$$\left|{}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{b-} f(x)\right| \le \frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{b-} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha-\overline{\alpha}+1)}(b-x)^{\alpha-\overline{\alpha}}, \tag{41}$$

$\forall\, x \in \left[\frac{a+b}{2}, b\right]$.

We notice that:

$$\int_a^b {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x)dx = \int_a^{\frac{a+b}{2}} {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x)dx + \int_{\frac{a+b}{2}}^b {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x)dx =$$

$$\int_a^{\frac{a+b}{2}} {}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{a+} f(x)dx + \int_{\frac{a+b}{2}}^b {}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{b-} f(x)dx. \tag{42}$$

We further derive that

$$\int_a^{\frac{a+b}{2}} \left|{}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{a+} f(x)\right| dx \le \frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{a+} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+1)}\int_a^{\frac{a+b}{2}} (x-a)^{\alpha-\overline{\alpha}}dx = \tag{43}$$

$$\frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{a+} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+2)}\left(\frac{a+b}{2}-a\right)^{\alpha-\overline{\alpha}+1} = \frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{a+} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+2)}\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+1}.$$

That is, it holds

$$\int_a^{\frac{a+b}{2}} \left|{}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{a+} f(x)\right| dx \le \frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{a+} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+2)}\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+1}. \tag{44}$$

Similarly, it holds

$$\int_{\frac{a+b}{2}}^b \left|{}^{H}\mathbb{D}^{\overline{\alpha},\beta}_{b-} f(x)\right| dx \le \frac{\left\|{}^{H}\mathbb{D}^{\alpha,\beta}_{b-} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha-\overline{\alpha}+2)}\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+1}. \tag{45}$$

Therefore, we obtain

$$\left|\int_a^b {}^H\mathbb{D}^{\overline{\alpha},\beta} f(x)dx\right| \leq \int_a^b \left|{}^H\mathbb{D}^{\overline{\alpha},\beta} f(x)\right|dx =$$

$$\int_a^{\frac{a+b}{2}} \left|{}^H\mathbb{D}_{a+}^{\overline{\alpha},\beta} f(x)\right|dx + \int_{\frac{a+b}{2}}^b \left|{}^H\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x)\right|dx \leq$$

$$\frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+2)}\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+1} + \frac{\left\|{}^H\mathbb{D}_{b-}^{\alpha,\beta} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha-\overline{\alpha}+2)}\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+1} = \tag{46}$$

$$\frac{\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+1}}{\Gamma(\alpha-\overline{\alpha}+2)}\left[\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|{}^H\mathbb{D}_{b-}^{\alpha,\beta} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right] \leq$$

$$\frac{2M_1(b-a)^{\alpha-\overline{\alpha}+1}}{2^{\alpha-\overline{\alpha}+1}\Gamma(\alpha-\overline{\alpha}+2)} = \frac{M_1(b-a)^{\alpha-\overline{\alpha}+1}}{2^{\alpha-\overline{\alpha}}\Gamma(\alpha-\overline{\alpha}+2)}.$$

□

We continue with the L_1-variant:

Theorem 7. *All as in Theorem 6 with $\alpha-\overline{\alpha}>1$ (i.e., $\alpha>\overline{\alpha}+1$). Call*

$$M_2 := \max\left\{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f\right\|_{1,\left[a,\frac{a+b}{2}\right]}, \left\|{}^H\mathbb{D}_{b-}^{\alpha,\beta} f\right\|_{1,\left[\frac{a+b}{2},b\right]}\right\}. \tag{47}$$

Then

$$\left|\int_a^b {}^H\mathbb{D}^{\overline{\alpha},\beta} f(x)dx\right| \leq \int_a^b \left|{}^H\mathbb{D}^{\overline{\alpha},\beta} f(x)\right|dx \leq \frac{M_2(b-a)^{\alpha-\overline{\alpha}}}{2^{\alpha-\overline{\alpha}-1}\Gamma(\alpha-\overline{\alpha}+1)}. \tag{48}$$

Proof. By (38) we have

$$\left|{}^H\mathbb{D}_{a+}^{\overline{\alpha},\beta} f(x)\right| \leq \frac{1}{\Gamma(\alpha-\overline{\alpha})}\int_a^x (x-t)^{\alpha-\overline{\alpha}-1}\left|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f(t)\right|dt \leq$$

$$\frac{(x-a)^{\alpha-\overline{\alpha}-1}}{\Gamma(\alpha-\overline{\alpha})}\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f\right\|_{1,\left[a,\frac{a+b}{2}\right]}, \tag{49}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$.

Similarly, from (39) we find that

$$\left|{}^H\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x)\right| \leq \frac{(b-x)^{\alpha-\overline{\alpha}-1}}{\Gamma(\alpha-\overline{\alpha})}\left\|{}^H\mathbb{D}_{b-}^{\alpha,\beta} f\right\|_{1,\left[\frac{a+b}{2},b\right]}, \tag{50}$$

$\forall\, x \in \left[\frac{a+b}{2}, b\right]$.

Furthermore, we obtain

$$\int_a^{\frac{a+b}{2}} \left|{}^H\mathbb{D}_{a+}^{\overline{\alpha},\beta} f(x)\right|dx \leq \frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f\right\|_{1,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha})}\int_a^{\frac{a+b}{2}} (x-a)^{\alpha-\overline{\alpha}-1}dx =$$

$$\frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta} f\right\|_{1,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha}+1)}\left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}}. \tag{51}$$

Similarly, we derive

$$\int_{\frac{a+b}{2}}^{b} \left| {}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x) \right| dx \leq \frac{\left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f \right\|_{1,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha-\overline{\alpha}+1)} \left(\frac{b-a}{2} \right)^{\alpha-\overline{\alpha}}. \tag{52}$$

Therefore we obtain

$$\begin{aligned} &\left| \int_a^b {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x) dx \right| \leq \int_a^b \left| {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x) \right| dx = \\ &\int_a^{\frac{a+b}{2}} \left| {}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta} f(x) \right| dx + \int_{\frac{a+b}{2}}^{b} \left| {}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x) \right| dx \leq \\ &\frac{\left[\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{1,\left[a,\frac{a+b}{2}\right]} + \left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f \right\|_{1,\left[\frac{a+b}{2},b\right]} \right]}{\Gamma(\alpha-\overline{\alpha}+1)} \left(\frac{b-a}{2} \right)^{\alpha-\overline{\alpha}} \leq \\ &\frac{2M_2}{\Gamma(\alpha-\overline{\alpha}+1)} \frac{(b-a)^{\alpha-\overline{\alpha}}}{2^{\alpha-\overline{\alpha}}} = \frac{M_2}{\Gamma(\alpha-\overline{\alpha}+1)} \frac{(b-a)^{\alpha-\overline{\alpha}}}{2^{\alpha-\overline{\alpha}-1}}. \end{aligned} \tag{53}$$

□

Next comes the L_q-variant of Hilfer-Polya fractional inequality:

Theorem 8. *All as in Theorem 6 with $\alpha - \overline{\alpha} > \frac{1}{q}$, where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Call*

$$M_3 := \max\left\{ \left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{q,\left[a,\frac{a+b}{2}\right]}, \left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f \right\|_{q,\left[\frac{a+b}{2},b\right]} \right\}. \tag{54}$$

Then

$$\begin{aligned} &\left| \int_a^b {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x) dx \right| \leq \int_a^b \left| {}^{H}\mathbb{D}^{\overline{\alpha},\beta} f(x) \right| dx \leq \\ &\frac{M_3}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}} \left(\alpha-\overline{\alpha}+\frac{1}{p} \right)} \frac{(b-a)^{\alpha-\overline{\alpha}+\frac{1}{p}}}{2^{\alpha-\overline{\alpha}-\frac{1}{q}}}. \end{aligned} \tag{55}$$

Proof. By (38) we have

$$\begin{aligned} &\left| {}^{H}\mathbb{D}_{a+}^{\overline{\alpha},\beta} f(x) \right| \leq \frac{1}{\Gamma(\alpha-\overline{\alpha})} \int_a^x (x-t)^{\alpha-\overline{\alpha}-1} \left| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f(t) \right| dt \leq \\ &\frac{1}{\Gamma(\alpha-\overline{\alpha})} \left(\int_a^x (x-t)^{p(\alpha-\overline{\alpha}-1)} dt \right)^{\frac{1}{p}} \left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{q,\left[a,\frac{a+b}{2}\right]} = \\ &\frac{(x-a)^{\left(\alpha-\overline{\alpha}-\frac{1}{q}\right)}}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}} \left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{q,\left[a,\frac{a+b}{2}\right]}, \end{aligned} \tag{56}$$

$\forall\, x \in \left[a, \frac{a+b}{2}\right]$, with $\alpha - \overline{\alpha} > \frac{1}{q}$.

And, by (39), similarly we derive

$$\left| {}^{H}\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x) \right| \leq \frac{(b-x)^{\left(\alpha-\overline{\alpha}-\frac{1}{q}\right)}}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}} \left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta} f \right\|_{q,\left[\frac{a+b}{2},b\right]}, \tag{57}$$

$\forall\, x \in \left[\frac{a+b}{2}, b\right]$, with $\alpha - \overline{\alpha} > \frac{1}{q}$.

Consequently, we obtain that

$$\int_a^{\frac{a+b}{2}} \left| {}^H\mathbb{D}_{a+}^{\overline{\alpha},\beta} f(x) \right| dx \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{q,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}} \int_a^{\frac{a+b}{2}} (x-a)^{\left(\alpha-\overline{\alpha}-\frac{1}{q}\right)} dx =$$

$$\frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{q,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}\left(\alpha-\overline{\alpha}+\frac{1}{p}\right)} \left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+\frac{1}{p}}. \tag{58}$$

Similarly, we derive

$$\int_{\frac{a+b}{2}}^{b} \left| {}^H\mathbb{D}_{b-}^{\overline{\alpha},\beta} f(x) \right| dx \leq \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta} f \right\|_{q,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}\left(\alpha-\overline{\alpha}+\frac{1}{p}\right)} \left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+\frac{1}{p}}. \tag{59}$$

Therefore, we obtain

$$\left| \int_a^b {}^H\mathbb{D}^{\overline{\alpha},\beta} f(x) dx \right| \leq \int_a^b \left| {}^H\mathbb{D}^{\overline{\alpha},\beta} f(x) \right| dx =$$

$$\frac{\left(\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{q,\left[a,\frac{a+b}{2}\right]} + \left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta} f \right\|_{q,\left[\frac{a+b}{2},b\right]} \right)}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}\left(\alpha-\overline{\alpha}+\frac{1}{p}\right)} \left(\frac{b-a}{2}\right)^{\alpha-\overline{\alpha}+\frac{1}{p}} \leq \tag{60}$$

$$\frac{M_3}{\Gamma(\alpha-\overline{\alpha})(p(\alpha-\overline{\alpha}-1)+1)^{\frac{1}{p}}\left(\alpha-\overline{\alpha}+\frac{1}{p}\right)} \frac{(b-a)^{\alpha-\overline{\alpha}+\frac{1}{p}}}{2^{\alpha-\overline{\alpha}-\frac{1}{q}}},$$

proving the claim. □

Next come ψ-Hilfer-Ostrowski type inequalities for several functions involved. For basic ψ-Hilfer-Ostrowski type inequalities involving one function see [11]. We make

Remark 2. *Our setting here follows: Let $f_i \in C^n([a,b])$, $\alpha \notin \mathbb{N}$, $n = \lceil \alpha \rceil$, $\alpha > 0$; $i = 1,...,r \in \mathbb{N} - \{1\}$, $x_0 \in [a,b]$. Assume that $g_{1i}(x) = I_{x_0+}^{(1-\beta)(n-\alpha);\psi} f_i(x) \in C^n([x_0,b])$ and $w_{1i}(x) = I_{x_0-}^{(1-\beta)(n-\alpha);\psi} f_i(x) \in C^n([a,x_0])$, for all $i = 1,...,r$.*
Define

$$\varphi_{ix_0}(x) := \left\{ \begin{array}{l} g_{1i}(x), \;\; x \in [x_0,b] \\ \\ w_{1i}(x), \;\; x \in [a,x_0) \end{array} \right\}. \tag{61}$$

Notice that if $\beta = 1$, we get $g_{1i}(x_0) = w_{1i}(x_0) = \varphi_{ix_0}(x_0) = f_i(x_0)$, all $i = 1,...,r$.
In general, for $f \in C([a,b])$ we have

$$\left| I_{a+}^{\alpha,\psi} f(x) \right| \leq \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} |f(t)| dt \leq$$

$$\frac{\|f\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} (\psi(x)-\psi(a))^{\alpha}, \;\; \forall\, x \in [a,b]. \tag{62}$$

Hence $I_{a+}^{\alpha,\psi} f(a) = 0$.
Similalry, we have

$$\left| I_{b-}^{\alpha,\psi} f(x) \right| \leq \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} |f(t)| dt \leq$$

$$\frac{\|f\|_{\infty,[a,b]}}{\Gamma(\alpha+1)}(\psi(b)-\psi(x))^{\alpha},\ \forall\, x\in[a,b]. \tag{63}$$

That is $I_{b-}^{\alpha,\psi}f(b)=0$.

So when $0\le\beta<1$*, by the above we obtain* $g_{1i}(x_0)=w_{1i}(x_0)=\varphi_{ix_0}(x_0)=0$*, for all* $i=1,...,r$.

Thus, it is always true that $g_{1i}(x_0)=w_{1i}(x_0)$*,* $i=1,...,r$.

We present

Theorem 9. *Let* $\psi, f_i\in C^n([a,b])$*,* $\alpha\notin\mathbb{N}$*,* $n=\lceil\alpha\rceil$*,* $\alpha>0$*;* $i=1,...,r\in\mathbb{N}-\{1\}$*,* $x_0\in[a,b]$*. Here* ψ *is increasing,* $\psi'(x)\neq 0$ *over* $[a,b]\subset\mathbb{R}$*,* $0\le\beta\le1$*,* $\mu=n(1-\beta)+\beta\alpha$*. Assume that* $g_{1i}(x)=I_{x_0+}^{(1-\beta)(n-\alpha);\psi}f_i(x)\in C^n([x_0,b])$ *and* $w_{1i}(x)=I_{x_0-}^{(1-\beta)(n-\alpha);\psi}f_i(x)\in C^n([a,x_0])$*, for all* $i=1,...,r$*, and* $\varphi_{ix_0}(x)$ *is as in (61). Assume also that* $g_{1i\psi}^{[k]}(x_0)=w_{1i\psi}^{[k]}(x_0)=0$*, for all* $k=1,...,n-1$.

Then

(1)

$$\theta^{\psi}(f_1,...,f_r)(x_0):= \tag{64}$$

$$r\int_a^b\left(\prod_{\lambda=1}^r\varphi_{\lambda x_0}(x)\right)dx-\sum_{i=1}^r\left[\varphi_{ix_0}(x_0)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^r\varphi_{jx_0}(x)\right)dx\right]=$$

$$\sum_{i=1}^r\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r\varphi_{jx_0}(x)\right)\left(I_{x_0-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i(x)\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r\varphi_{jx_0}(x)\right)\left(I_{x_0+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i(x)\right)dx\right]\right],$$

and in case of $0\le\beta<1$*, we have that*

$$\theta^{\psi}(f_1,...,f_r)(x_0)=r\int_a^b\left(\prod_{\lambda=1}^r\varphi_{\lambda x_0}(x)\right)dx, \tag{65}$$

(2) furthermore, it holds

$$\left|\theta^{\psi}(f_1,...,f_r)(x_0)\right|\le\frac{1}{\Gamma(\mu+1)}$$

$$\left\{\left(\sum_{i=1}^r\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[a,x_0]}\left\|\prod_{\substack{j=1\\j\neq i}}^r\varphi_{jx_0}\right\|_{1,[a,x_0]}\right)(\psi(x_0)-\psi(a))^{\mu}+\right.$$

$$\left.\left(\sum_{i=1}^r\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[x_0,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^r\varphi_{jx_0}\right\|_{1,[x_0,b]}\right)(\psi(b)-\psi(x_0))^{\mu}\right\}. \tag{66}$$

It follows the L_1-variant.

Theorem 10. *All as in Theorem 9, with $\alpha > 1$. Then*

$$\left|\theta^{\psi}(f_1,...,f_r)(x_0)\right| \leq \frac{1}{\Gamma(\mu)}$$

$$\left\{\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_1([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right)(\psi(x_0)-\psi(a))^{\mu-1}+\right.$$

$$\left.\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_1([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right)(\psi(b)-\psi(x_0))^{\mu-1}\right\}. \tag{67}$$

Next we have the L_q-variant.

Theorem 11. *All as in Theorem 9. Let also $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$ with $\alpha > \frac{1}{q}$. Then*

$$\left|\theta^{\psi}(f_1,...,f_r)(x_0)\right| \leq \frac{1}{\Gamma(\mu)(p(\mu-1)+1)^{\frac{1}{p}}}$$

$$\left\{\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_q([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right)(\psi(x_0)-\psi(a))^{\mu-\frac{1}{q}}+\right.$$

$$\left.\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_q([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right)(\psi(b)-\psi(x_0))^{\mu-\frac{1}{q}}\right\}. \tag{68}$$

Proof of Theorems 9–11.

By Theorem 3 we have

$$\begin{gathered} g_{1i}(x)-g_{1i}(x_0) = I_{x_0+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i(x),\ \ \forall\, x\in[x_0,b], \\ \text{and} \\ w_{1i}(x)-w_{1i}(x_0) = I_{x_0-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i(x),\ \ \forall\, x\in[a,x_0], \end{gathered} \tag{69}$$

for all $i = 1,...,r$.

That is

$$\begin{gathered} \varphi_{ix_0}(x)-\varphi_{ix_0}(x_0) = I_{x_0+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i(x),\ \ \forall\, x\in[x_0,b], \\ \text{and} \\ \varphi_{ix_0}(x)-\varphi_{ix_0}(x_0) = I_{x_0-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i(x),\ \ \forall\, x\in[a,x_0), \end{gathered} \tag{70}$$

for all $i = 1,...,r$.

Multiplying (70) by $\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}(x)\right)$ we get, respectively,

$$\prod_{\lambda=1}^{r}\varphi_{\lambda x_0}(x)-\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}(x)\right)\varphi_{ix_0}(x_0) = \left(\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}(x)\right)I_{x_0+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i(x), \tag{71}$$

$\forall\, x \in [x_0, b]$,

And

$$\prod_{\lambda=1}^{r} \varphi_{\lambda x_0}(x) - \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) \varphi_{ix_0}(x_0) = \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) I_{x_0-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi} f_i(x), \tag{72}$$

$\forall\, x \in [a, x_0)$, for all $i = 1, \ldots, r$.

Adding (71) and (72), separately, we obtain

$$r\left(\prod_{\lambda=1}^{r} \varphi_{\lambda x_0}(x) \right) - \sum_{i=1}^{r} \left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) \varphi_{ix_0}(x_0) \right] = \tag{73}$$

$$\sum_{i=1}^{r} \left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) I_{x_0+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi} f_i(x) \right],$$

$\forall\, x \in [x_0, b]$,

In addition,

$$r\left(\prod_{\lambda=1}^{r} \varphi_{\lambda x_0}(x) \right) - \sum_{i=1}^{r} \left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) \varphi_{ix_0}(x_0) \right] = \tag{74}$$

$$\sum_{i=1}^{r} \left[\left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) I_{x_0-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi} f_i(x) \right],$$

$\forall\, x \in [a, x_0)$.

Next integrate (73) and (74) with respect to $x \in [a, b]$. We have

$$r\int_{x_0}^{b} \left(\prod_{\lambda=1}^{r} \varphi_{\lambda x_0}(x) \right) dx - \sum_{i=1}^{r} \left[\varphi_{ix_0}(x_0) \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) dx \right] =$$

$$\sum_{i=1}^{r} \left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) \left(I_{x_0+}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi} f_i(x) \right) dx \right], \tag{75}$$

and

$$r\int_{a}^{x_0} \left(\prod_{\lambda=1}^{r} \varphi_{\lambda x_0}(x) \right) dx - \sum_{i=1}^{r} \left[\varphi_{ix_0}(x_0) \int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) dx \right] =$$

$$\sum_{i=1}^{r} \left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \varphi_{jx_0}(x) \right) \left(I_{x_0-}^{\mu;\psi}\ {}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi} f_i(x) \right) dx \right]. \tag{76}$$

Finally adding (75) and (76) we obtain the useful and nice identity (64).

Identity (64) implies

$$\left|\theta^{\psi}(f_1,\ldots,f_r)(x_0)\right| \le \sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\left(I_{x_0-}^{\mu;\psi}\left|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right|(x)\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\left(I_{x_0+}^{\mu;\psi}\left|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right|(x)\right)dx\right]\right]= \tag{77}$$

$$\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\frac{1}{\Gamma(\mu)}\right.\right.$$

$$\left.\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\mu-1}\left|\left({}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right)(t)\right|dt\right)dx\right]+$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\frac{1}{\Gamma(\mu)}\right.$$

$$\left.\left.\left(\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\mu-1}\left|\left({}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right)(t)\right|dt\right)dx\right]\right]\le$$

$$\frac{1}{\Gamma(\mu+1)}\sum_{i=1}^{r}\left[\left[\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[a,x_0]}\int_a^{x_0}(\psi(x_0)-\psi(x))^{\mu}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)dx\right]\right. \tag{78}$$

$$\left.+\left[\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}(\psi(x)-\psi(x_0))^{\mu}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)dx\right]\right]\le$$

$$\frac{1}{\Gamma(\mu+1)}\sum_{i=1}^{r}\left[\left[\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[a,x_0]}(\psi(x_0)-\psi(a))^{\mu}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right]+\right.$$

$$\left.\left[\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[x_0,b]}(\psi(b)-\psi(x_0))^{\mu}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right]\right]= \tag{79}$$

$$\frac{1}{\Gamma(\mu+1)}\left\{\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[a,x_0]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right)(\psi(x_0)-\psi(a))^{\mu}+\right.$$

$$\left.\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{\infty,[x_0,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right)(\psi(b)-\psi(x_0))^{\mu}\right\}, \tag{80}$$

proving (66).

If $\alpha \notin \mathbb{N}$ and $\alpha > 1$, then $n = \lceil \alpha \rceil > 1$, and $n - 1 \geq 1 > \beta(n-\alpha)$. Hence $n - \beta(n-\alpha) > 1$ and $\mu > 1$. So we have

$$\left|\theta^{\psi}(f_1,...,f_r)(x_0)\right| \leq \sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\frac{1}{\Gamma(\mu)}\right.\right.$$
$$\left.\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\mu-1}\left|\left({}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right)(t)\right|dt\right)dx\right]+$$
$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\frac{1}{\Gamma(\mu)}\right.$$
$$\left.\left.\left(\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\mu-1}\left|\left({}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right)(t)\right|dt\right)dx\right]\right] \leq \tag{81}$$
$$\frac{1}{\Gamma(\mu)}\sum_{i=1}^{r}\left[\left[\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_1([a,x_0],\psi)}\int_a^{x_0}(\psi(x_0)-\psi(x))^{\mu-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)dx\right]\right.$$
$$\left.+\left[\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_1([x_0,b],\psi)}\int_{x_0}^{b}(\psi(x)-\psi(x_0))^{\mu-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)dx\right]\right] \leq$$
$$\frac{1}{\Gamma(\mu)}\sum_{i=1}^{r}\left[\left[\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_1([a,x_0],\psi)}(\psi(x_0)-\psi(a))^{\mu-1}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right]+\right. \tag{82}$$
$$\left.\left[\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_1([x_0,b],\psi)}(\psi(b)-\psi(x_0))^{\mu-1}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right]\right] =$$
$$\frac{1}{\Gamma(\mu)}\left\{\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_1([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right)(\psi(x_0)-\psi(a))^{\mu-1}+\right. \tag{83}$$
$$\left.\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_1([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right)(\psi(b)-\psi(x_0))^{\mu-1}\right\},$$

proving (67).

Let $\alpha > 0$ with $\lceil \alpha \rceil = n \in \mathbb{N}$, and let $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Clearly $n > \frac{1}{q}$. Let $0 < \beta \leq 1$, then $\alpha\beta > \frac{\beta}{q}$, furthermore $\mu = n(1-\beta)+\beta\alpha > \frac{\beta}{q}+n(1-\beta) \geq \frac{\beta}{q}+1-\beta = \frac{\beta}{q}+\frac{1}{p}+\frac{1}{q}-\frac{\beta}{p}-\frac{\beta}{q} = \frac{1}{q}+\frac{1}{p}(1-\beta) \geq \frac{1}{q}$. That is $\mu > \frac{1}{q}$.

From (81), by using Hölder's inequality twice, we have

$$\left|\theta^{\psi}(f_1,...,f_r)(x_0)\right| \leq \frac{1}{\Gamma(\mu)}$$

$$\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\frac{(\psi(x_0)-\psi(x))^{\frac{p(\mu-1)+1}{p}}}{(p(\mu-1)+1)^{\frac{1}{p}}}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_q([a,x_0],\psi)}dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)\frac{(\psi(x)-\psi(x_0))^{\frac{p(\mu-1)+1}{p}}}{(p(\mu-1)+1)^{\frac{1}{p}}}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_q([x_0,b],\psi)}dx\right]\right]= \tag{84}$$

$$\frac{1}{\Gamma(\mu)(p(\mu-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)(\psi(x_0)-\psi(x))^{\mu-\frac{1}{q}}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_q([a,x_0],\psi)}dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|\varphi_{jx_0}(x)\right|\right)(\psi(x)-\psi(x_0))^{\mu-\frac{1}{q}}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_q([x_0,b],\psi)}dx\right]\right]\leq$$

$$\frac{1}{\Gamma(\mu)(p(\mu-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_q([a,x_0],\psi)}(\psi(x_0)-\psi(a))^{\mu-\frac{1}{q}}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right]+\right. \tag{85}$$

$$\left.\left[\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_q([x_0,b],\psi)}(\psi(b)-\psi(x_0))^{\mu-\frac{1}{q}}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right]\right]=$$

$$\frac{1}{\Gamma(\mu)(p(\mu-1)+1)^{\frac{1}{p}}}$$

$$\left\{\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0-}^{\alpha,\beta;\psi}f_i\right\|_{L_q([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[a,x_0]}\right)(\psi(x_0)-\psi(a))^{\mu-\frac{1}{q}}+\right. \tag{86}$$

$$\left.\left(\sum_{i=1}^{r}\left\|{}^{H}\mathbb{D}_{x_0+}^{\alpha,\beta;\psi}f_i\right\|_{L_q([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\varphi_{jx_0}\right\|_{1,[x_0,b]}\right)(\psi(b)-\psi(x_0))^{\mu-\frac{1}{q}}\right\},$$

proving (68). □

Next we present a ψ-Hilfer-Hilbert-Pachpatte left fractional inequality:

Theorem 12. *Let $i=1,2$; ψ_i, $f_i\in C^{n_i}([a_i,b_i])$, with ψ_i being strictly increasing over $[a_i,b_i]$, where $n_i-1<\alpha_i<n_i$, $0\leq\beta_i\leq 1$, and $\gamma_i=\alpha_i+\beta_i(n_i-\alpha_i)$, $x_i\in[a_i,b_i]$. Assume that*

$f_{i\psi_i}^{[n_i-k_i]}\left(I_{a_i+}^{(1-\beta_i)(n_i-\alpha_i);\psi_i}f_i\right)(a_i)=0$, *for* $k_i=1,\ldots,n_i-1$. *Let also* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *such that* $\alpha_1>\frac{1}{q}$ *and* $\alpha_2>\frac{1}{p}$. *Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|dx_1dx_2}{\left(\frac{(\psi_1(x_1)-\psi_1(a_1))^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)|+1)}+\frac{(\psi_2(x_2)-\psi_2(a_2))^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1;\psi_1}f_1\right\|_{L_q([a_1,b_1],\psi_1)}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2;\psi_2}f_2\right\|_{L_p([a_2,b_2],\psi_2)}. \tag{87}$$

Proof. By Theorem 2 we have

$$f_i(x_i)=\frac{1}{\Gamma(\alpha_i)}\int_{a_i}^{x_i}\psi_i'(t_i)(\psi_i(x_i)-\psi_i(t_i))^{\alpha_i-1}\,{}^{H}\mathbb{D}_{a_i+}^{\alpha_i,\beta_i;\psi_i}f_i(t_i)dt_i, \tag{88}$$

$\forall\ x_i\in[a_i,b_i]$, $i=1,2$.

Then

$$|f_i(x_i)|\leq\frac{1}{\Gamma(\alpha_i)}\int_{a_i}^{x_i}\psi_i'(t_i)(\psi_i(x_i)-\psi_i(t_i))^{\alpha_i-1}\left|{}^{H}\mathbb{D}_{a_i+}^{\alpha_i,\beta_i;\psi_i}f_i(t_i)\right|dt_i, \tag{89}$$

$i=1,2$, $\forall\ x_i\in[a_i,b_i]$.

By Hölder's inequality we obtain

$$|f_1(x_1)|\leq\frac{1}{\Gamma(\alpha_1)}\frac{(\psi_1(x_1)-\psi_1(a_1))^{\frac{p(\alpha_1-1)+1}{p}}}{(p(\alpha_1-1)+1)^{\frac{1}{p}}}\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1;\psi_1}f_1\right\|_{L_q([a_1,b_1],\psi_1)}, \tag{90}$$

$\forall\ x_1\in[a_1,b_1]$,

and

$$|f_2(x_2)|\leq\frac{1}{\Gamma(\alpha_2)}\frac{(\psi_2(x_2)-\psi_2(a_2))^{\frac{q(\alpha_2-1)+1}{q}}}{(q(\alpha_2-1)+1)^{\frac{1}{q}}}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2;\psi_2}f_2\right\|_{L_p([a_2,b_2],\psi_2)}, \tag{91}$$

$\forall\ x_2\in[a_2,b_2]$.

Hence we have

$$|f_1(x_1)||f_2(x_2)|\leq\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)(p(\alpha_1-1)+1)^{\frac{1}{p}}(q(\alpha_2-1)+1)^{\frac{1}{q}}}$$

$$(\psi_1(x_1)-\psi_1(a_1))^{\frac{p(\alpha_1-1)+1}{p}}(\psi_2(x_2)-\psi_2(a_2))^{\frac{q(\alpha_2-1)+1}{q}} \tag{92}$$

$$\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1;\psi_1}f_1\right\|_{L_q([a_1,b_1],\psi_1)}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2;\psi_2}f_2\right\|_{L_p([a_2,b_2],\psi_2)}\leq$$

(using Young's inequality for $a,b\geq0$, $a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left(\frac{(\psi_1(x_1)-\psi_1(a_1))^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)}+\frac{(\psi_2(x_2)-\psi_2(a_2))^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right) \tag{93}$$

$$\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1;\psi_1}f_1\right\|_{L_q([a_1,b_1],\psi_1)}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2;\psi_2}f_2\right\|_{L_p([a_2,b_2],\psi_2)},$$

$\forall\ x_i\in[a_i,b_i]$; $i=1,2$.

So far we have

$$\frac{|f_1(x_1)||f_2(x_2)|}{\left(\frac{(\psi_1(x_1)-\psi_1(a_1))^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)}+\frac{(\psi_2(x_2)-\psi_2(a_2))^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)}\leq \tag{94}$$

$$\frac{\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1;\psi_1}f_1\right\|_{L_q([a_1,b_1],\psi_1)}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2;\psi_2}f_2\right\|_{L_p([a_2,b_2],\psi_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)},$$

$\forall\ x_i \in [a_i,b_i]$; $i=1,2$.

The denominator in (94) can be zero only when $x_1=a_1$ and $x_2=a_2$.

Therefore we obtain (87), by integrating (94) over $[a_1,b_1]\times[a_2,b_2]$. $\square$

It follows the right side analog of last theorem.

Theorem 13. *Let $i=1,2$; ψ_i, $f_i\in C^{n_i}([a_i,b_i])$, with ψ_i being strictly increasing over $[a_i,b_i]$, where $n_i-1<\alpha_i<n_i$, $0\leq\beta_i\leq 1$, and $\gamma_i=\alpha_i+\beta_i(n_i-\alpha_i)$, $x_i\in[a_i,b_i]$. Let also $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, such that $\alpha_1>\frac{1}{q}$ and $\alpha_2>\frac{1}{p}$. Assume that $f_{i\psi_i}^{[n_i-k_i]}\left(I_{b_i-}^{(1-\beta_i)(n_i-\alpha_i);\psi_i}f_i\right)(b_i)=0$, for $k_i=1,\ldots,n_i-1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|dx_1dx_2}{\left(\frac{(\psi_1(b_1)-\psi_1(x_1))^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)|+1)}+\frac{(\psi_2(b_2)-\psi_2(x_2))^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\|{}^{H}\mathbb{D}_{b_1-}^{\alpha_1,\beta_1;\psi_1}f_1\right\|_{L_q([a_1,b_1],\psi_1)}\left\|{}^{H}\mathbb{D}_{b_2-}^{\alpha_2,\beta_2;\psi_2}f_2\right\|_{L_p([a_2,b_2],\psi_2)}. \tag{95}$$

Proof. Similar to Theorem 12, by the use of (30). $\square$

We continue with other Hilfer-Hilbert-Pachpatte fractional inequalities.

Theorem 14. *Let $i=1,2$; $\alpha_i>0$, $\alpha_i\notin\mathbb{N}$, $\lceil\alpha_i\rceil=n_i$, $0<\beta_i<1$, $f_i\in C^{n_i}([a_i,b_i])$, $[a_i,b_i]\subset\mathbb{R}$ and set $\gamma_i=\alpha_i+\beta_i(n_i-\alpha_i)$. Assume further that $\Delta_{a_i+}^{\gamma_i}f_i\in C([a_i,b_i]):\Delta_{a_i+}^{\gamma_i-j_i}f_i(a_i)=0$, for $j_i=1,\ldots,n_i$. Let also $\overline{\alpha}_i>0:\lceil\overline{\alpha}_i\rceil=\overline{n}_i$, with $\overline{\gamma}_i=\overline{\alpha}_i+\beta_i(\overline{n}_i-\overline{\alpha}_i)$, and assume that $\alpha_i>\overline{\alpha}_i$ and $\gamma_i>\overline{\gamma}_i$. Furthermore, let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, such that $\alpha_1>\frac{1}{q}$ and $\alpha_2>\frac{1}{p}$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|{}^{H}\mathbb{D}_{a_1+}^{\overline{\alpha}_1,\beta_1}f_1(x_1)\right|\left|{}^{H}\mathbb{D}_{a_2+}^{\overline{\alpha}_2,\beta_2}f_2(x_2)\right|dx_1dx_2}{\left(\frac{(x_1-a_1)^{p(\alpha_1-\overline{\alpha}_1-1)+1}}{p(p(\alpha_1-\overline{\alpha}_1-1)|+1)}+\frac{(x_2-a_2)^{q(\alpha_2-\overline{\alpha}_2-1)+1}}{q(q(\alpha_2-\overline{\alpha}_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1-\overline{\alpha}_1)\Gamma(\alpha_2-\overline{\alpha}_2)}\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1}f_1\right\|_{L_q([a_1,b_1])}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2}f_2\right\|_{L_p([a_2,b_2])}. \tag{96}$$

Proof. Similar to Theorem 12, by the use of Theorem 4. $\square$

It follows

Theorem 15. *Let $i=1,2$; $\alpha_i>0$, $\alpha_i\notin\mathbb{N}$, $\lceil\alpha_i\rceil=n_i$, $0<\beta_i<1$, $f_i\in C^{n_i}([a_i,b_i])$, $[a_i,b_i]\subset\mathbb{R}$ and set $\gamma_i=\alpha_i+\beta_i(n_i-\alpha_i)$. Assume further that $\Delta_{b_i-}^{\gamma_i}f_i\in C([a_i,b_i]):\Delta_{b_i-}^{\gamma_i-j_i}f_i(b_i)=0$, for $j_i=1,\ldots,n_i$. Let also $\overline{\alpha}_i>0:\lceil\overline{\alpha}_i\rceil=\overline{n}_i$, with $\overline{\gamma}_i=\overline{\alpha}_i+\beta_i(\overline{n}_i-\overline{\alpha}_i)$, and assume that $\alpha_i>\overline{\alpha}_i$ and $\gamma_i>\overline{\gamma}_i$. Furthermore, let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, such that $\alpha_1>\frac{1}{q}$ and $\alpha_2>\frac{1}{p}$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|{}^{H}\mathbb{D}_{b_1-}^{\overline{\alpha}_1,\beta_1}f_1(x_1)\right|\left|{}^{H}\mathbb{D}_{b_2-}^{\overline{\alpha}_2,\beta_2}f_2(x_2)\right|dx_1dx_2}{\left(\frac{(b_1-x_1)^{p(\alpha_1-\overline{\alpha}_1-1)+1}}{p(p(\alpha_1-\overline{\alpha}_1-1)|+1)}+\frac{(b_2-x_2)^{q(\alpha_2-\overline{\alpha}_2-1)+1}}{q(q(\alpha_2-\overline{\alpha}_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1-\overline{\alpha}_1)\Gamma(\alpha_2-\overline{\alpha}_2)}\left\|{}^{H}\mathbb{D}_{b_1-}^{\alpha_1,\beta_1}f_1\right\|_{L_q([a_1,b_1])}\left\|{}^{H}\mathbb{D}_{b_2-}^{\alpha_2,\beta_2}f_2\right\|_{L_p([a_2,b_2])}. \tag{97}$$

Proof. Similar to Theorem 12, by the use of Theorem 5. □

We finish with two applications:

Corollary 1. *All as in Theorem 12, with $\psi_1(x_1)=e^{x_1}$, $\psi_2(x_2)=e^{x_2}$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|dx_1dx_2}{\left(\frac{(e^{x_1}-e^{a_1})^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)|+1)}+\frac{(e^{x_2}-e^{a_2})^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\|{}^{H}\mathbb{D}_{a_1+}^{\alpha_1,\beta_1;e^{x_1}}f_1\right\|_{L_q([a_1,b_1],e^{x_1})}\left\|{}^{H}\mathbb{D}_{a_2+}^{\alpha_2,\beta_2;e^{x_2}}f_2\right\|_{L_p([a_2,b_2],e^{x_2})}. \tag{98}$$

Proof. By Theorem 12. □

Corollary 2. *All as in Theorem 13, with $[a_i,b_i]\subset(0,+\infty)$, $i=1,2$; and $\psi_1(x_1)=\ln x_1$, $\psi_2(x_2)=\ln x_2$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|dx_1dx_2}{\left(\frac{\left(\ln\frac{b_1}{x_1}\right)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)|+1)}+\frac{\left(\ln\frac{b_2}{x_2}\right)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\left\|{}^{H}\mathbb{D}_{b_1-}^{\alpha_1,\beta_1;\ln x_1}f_1\right\|_{L_q([a_1,b_1],\ln x_1)}\left\|{}^{H}\mathbb{D}_{b_2-}^{\alpha_2,\beta_2;\ln x_2}f_2\right\|_{L_p([a_2,b_2],\ln x_2)}. \tag{99}$$

Proof. By Theorem 13. □

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