symmetry

Article

New Hermite–Hadamard Inequalities in Fuzzy-Interval Fractional Calculus and Related Inequalities

Muhammad Bilal Khan [1], Pshtiwan Othman Mohammed [2,*], Muhammad Aslam Noor [1] and Y. S. Hamed [3]

1 Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, Pakistan; bilal42742@gmail.com (M.B.K.); aslamnoor@comsats.edu.pk (M.A.N.)
2 Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, 46001, Kurdistan Region, Iraq
3 Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; yasersalah@tu.edu.sa
* Correspondence: pshtiwan.muhammad@univsul.edu.iq

Abstract: It is a familiar fact that inequalities have become a very popular method using fractional integrals, and that this method has been the driving force behind many studies in recent years. Many forms of inequality have been studied, resulting in the introduction of new trend in inequality theory. The aim of this paper is to use a fuzzy order relation to introduce various types of inequalities. On the fuzzy interval space, this fuzzy order relation is defined level by level. With the help of this relation, firstly, we derive some discrete Jensen and Schur inequalities for convex fuzzy interval-valued functions (convex fuzzy-IVF), and then, we present Hermite–Hadamard inequalities (*HH*-inequalities) for convex fuzzy-IVF via fuzzy interval Riemann–Liouville fractional integrals. These outcomes are a generalization of a number of previously known results, and many new outcomes can be deduced as a result of appropriate parameter "γ" and real valued function "Ω" selections. We hope that our fuzzy order relations results can be used to evaluate a number of mathematical problems related to real-world applications.

Keywords: convex fuzzy interval-valued function; fuzzy fractional integral operator; Jensen inequality; Schur inequality; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

check for updates

Citation: Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Hamed, Y.S. New Hermite–Hadamard Inequalities in Fuzzy-Interval Fractional Calculus and Related Inequalities. *Symmetry* **2021**, *13*, 673. https://doi.org/10.3390/sym13040673

Academic Editor: Alina Alb Lupas

Received: 13 March 2021
Accepted: 7 April 2021
Published: 13 April 2021

1. Introduction

A real valued function $\mathcal{Q} : \mathfrak{T} \to \mathbb{R}$ is called convex on the interval $\mathfrak{T}$ if the condition

$$\mathcal{Q}(\varrho \ddagger + (1-\varrho)y) \leq \varrho \mathcal{Q}(\ddagger) + (1-\varrho)\mathcal{Q}(y), \tag{1}$$

holds for all $\ddagger,\ y \in \mathfrak{T}$ and $\varrho \in [0,\ 1]$. We say that $\mathcal{Q}$ is concave if $-\mathcal{Q}$ is convex.

The concept of convexity in the sense of integral problems is a fascinating area for research. Therefore, many inequalities have been introduced as applications of convex functions such as Gagliardo–Nirenberg-type inequality [1], Ostrowski-type inequality [2], Olsen-type inequality [3], midpoint-type inequality [4] Hardy-type inequality [5], and trapezoidal-type inequality [6]. Among those, the Hermite–Hadamard inequality is an interesting outcome in convex analysis. The *HH*-inequality [7,8] for convex function $\mathcal{Q} : \mathfrak{T} \to \mathbb{R}$ on an interval $\mathfrak{T} = [u,\ \nu]$ is defined in the following way:

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu-u}\int_u^\nu \mathcal{Q}(\ddagger)d\ddagger \leq \frac{\mathcal{Q}(u)\ +\ \mathcal{Q}(\nu)}{2}, \tag{2}$$

for all $u,\ \nu \in \mathfrak{T}$. If $\mathcal{Q}$ is concave, then inequality (2) is reversed. For more useful details, see [9,10] and the references therein.

In 2013, Sarika et al. [11] introduced the following fractional HH-inequality for convex function:

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}(\nu)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}(u)\right] \leq \frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}. \tag{3}$$

where $\mathcal{Q} : \mathfrak{T} = [u,\ \nu] \to \mathbb{R}$ assumed to be a positive function on $[u,\ \nu]$, $\mathcal{Q} \in L_1([u,\ \nu])$ with $u \leq\ \nu$, and $\mathcal{I}_{u^+}^{\alpha}$ and $\mathcal{I}_{\nu^-}^{\alpha}$ are the left sided and right sided Riemann–Liouville fractional of order $0 \leq \alpha$, and, respectively, are defined as follows:

$$\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}(\ddagger) = \frac{1}{\Gamma(\alpha)}\int_u^{\ddagger}(\ddagger-\varrho)^{\alpha-1}\mathcal{Q}(\varrho)d(\varrho) \quad (\ddagger > u), \tag{4}$$

$$\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}(\ddagger) = \frac{1}{\Gamma(\alpha)}\int_{\ddagger}^{\nu}(\varrho-\ddagger)^{\alpha-1}\mathcal{Q}(\varrho)d(\varrho) \quad (\ddagger < \nu). \tag{5}$$

If $\alpha = 1$, then from Equation (3), we obtain Equation (2). We can easily say that inequality (3) is a generalization of inequality (2). Many fractional inequalities have been introduced by several authors in the view of inequality (3) for different convex and nonconvex functions.

In 1966, the concept of interval analysis was first introduced by the late American mathematician Ramon E. Moore in [9]. Since its inception, various authors in the mathematical community have paid close attention to this area of research. Interval analysis has been found to be useful in global optimization and constraint solution algorithms, according to experts. It has slowly risen in popularity over the last few decades. Scientists and engineers engaged in scientific computation have discovered that interval analysis is useful, especially in terms of accuracy, round-off error affects, and automatic validation of results. After the invention of interval analysis, the researchers working in the area of inequalities wants to know whether the inequalities in the abovementioned results can be found substituted with the inclusions relation. In certain cases, the question is answered correctively. In light of this, Zhao et al. [12] arrived at the following conclusion for an interval-valued function (IVF).

Let $\mathcal{Q} : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ be a convex IVF given by $\mathcal{Q}(\ddagger) = [\mathcal{Q}_*(\ddagger),\ \mathcal{Q}^*(\ddagger)]$ for all $\ddagger \in [u,\ \nu]$, where $\mathcal{Q}_*(\ddagger)$ is a convex function and $\mathcal{Q}^*(\ddagger)$ is a concave function. If $\mathcal{Q}$ is Riemann integrable, then

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \supseteq \frac{1}{\nu-u}\ (IR)\int_u^{\nu}\ \mathcal{Q}(\ddagger)d\ddagger \supseteq \frac{\mathcal{Q}(u)+\ \mathcal{Q}(\nu)}{2}. \tag{6}$$

After that, several authors developed a strong relationship between inequalities and IVFs by means of an inclusion relation via different integral operators, as one can see in Costa [13], Costa and Roman-Flores [14], Roman-Flores et al. [15,16], and Chalco-Cano et al. [17,18], but also in more general set valued maps by Nikodem et al. [19], and Matkowski and Nikodem [20]. In particular, Zhang et al. [21] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation, and proved that these Jensen's inequalities are a generalized form of Costa Jensen's inequalities [13].

In the last two decades, in the development of pure and applied mathematics, fractional calculus has played a key role. Yet, it attains magnificent deliberation in the ongoing research work; this is because of its applications in various directions such as image processing, signal processing, physics, biology, control theory, computer networking, and fluid dynamics [22–25].

For inclusion relation, Budek [26] established the following strong relation between convex IVF and interval HH-inequality as a counter part of (6)

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}(\nu)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}(u)\right] \supseteq \frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}. \tag{7}$$

where $\mathcal{Q} : \mathfrak{T} = [u,\ \nu] \to \mathcal{K}_C{}^+$ is assumed to be a positive IVF on $[u,\ \nu]$, $\mathcal{Q} \in L_1([u,\ \nu],\ \mathcal{K}_C{}^+)$ with $u \leq\ \nu$, and $\mathcal{I}^{\alpha}_{u^+}$ and $\mathcal{I}^{\alpha}_{\nu^-}$ are the left sided and right sided Riemann–Liouville fractional of order $0 \leq \alpha$.

If $\alpha = 1$, then from (7), we obtain (6). We can easily say that inequality (7) is a generalization of inequality (6). Many fractional inequalities have been introduced by several authors in the view of inequality (7) for different convex- and nonconvex-IVFs.

Due to the vast applications of convexity and fractional HH-inequality in mathematical analysis and optimization, many authors have discussed the applications, refinements, generalizations, and extensions, see [27–38] and the references therein.

Recently, fuzzy interval analysis and fuzzy interval differential equations have been put forward to deal the ambiguity originating from insufficient data in some mathematical or computer models that find out real-world phenomena [39–50]. There are some integrals to deal with fuzzy-IVFs, where the integrands are fuzzy-IVFs for instance; recently, Costa et al. [13] derived some Jensen's integral inequality for IVFs and fuzzy-IVFs through Kulisch–Miranker and fuzzy order relation, see [47]. Recently, Allahviranloo et al. [48] introduced the following fuzzy interval Riemann–Liouville fractional integral operators:

Let $\alpha > 0$ and $L([u,\ \nu], \mathbb{F}_0)$ be the collection of all Lebesgue measurable fuzzy-IVFs on $[u, \nu]$. Then, the fuzzy interval left and right Riemann–Liouville fractional integral of $\mathcal{Q} \in L([u,\ \nu], \mathbb{F}_0)$ with order $\alpha > 0$ are defined by

$$\mathcal{I}^{\alpha}_{u^+}\ \mathcal{Q}(\ddagger) = \frac{1}{\Gamma(\alpha)} \int_u^{\ddagger} (\ddagger - \varrho)^{\alpha-1} \mathcal{Q}(\varrho) d(\varrho), \quad (\ddagger > u), \tag{8}$$

and

$$\mathcal{I}^{\alpha}_{\nu^-}\ \mathcal{Q}(\ddagger) = \frac{1}{\Gamma(\alpha)} \int_{\ddagger}^{\nu} (\varrho - \ddagger)^{\alpha-1} \mathcal{Q}(\varrho) d(\varrho), \quad (\ddagger < \nu), \tag{9}$$

respectively, where $\Gamma(\ddagger) = \int_0^{\infty} \varrho^{\ddagger-1} u^{-\varrho} d(\varrho)$ is the Euler γ function. The fuzzy interval left and right Riemann–Liouville fractional integral $\ddagger$ based on left and right end point functions can be defined as

$$\begin{aligned} \left[\mathcal{I}^{\alpha}_{u^+}\ \mathcal{Q}(\ddagger)\right]^{\gamma} &= \tfrac{1}{\Gamma(\alpha)} \textstyle\int_u^{\ddagger} (\ddagger - \varrho)^{\alpha-1} \mathcal{Q}_{\gamma}(\varrho) d(\varrho) \\ &= \tfrac{1}{\Gamma(\alpha)} \textstyle\int_u^{\ddagger} (\ddagger - \varrho)^{\alpha-1} [\mathcal{Q}_*(\varrho,\ \gamma), \mathcal{Q}^*(\varrho,\ \gamma)] d(\varrho),\ (\ddagger > u), \end{aligned} \tag{10}$$

where

$$\mathcal{I}^{\alpha}_{u^+}\ \mathcal{Q}_*(\ddagger,\ \gamma) = \frac{1}{\Gamma(\alpha)} \int_u^{\ddagger} (\ddagger - \varrho)^{\alpha-1} \mathcal{Q}_*(\varrho,\ \gamma) d(\varrho), \quad (\ddagger > u), \tag{11}$$

and

$$\mathcal{I}^{\alpha}_{u^+}\ \mathcal{Q}^*(\ddagger,\ \gamma) = \frac{1}{\Gamma(\alpha)} \int_u^{\ddagger} (\ddagger - \varrho)^{\alpha-1} \mathcal{Q}^*(\varrho,\ \gamma) d(\varrho),\ \ (\ddagger > u). \tag{12}$$

Similarly, we can define right Riemann–Liouville fractional integral $\mathcal{Q}$ of $\ddagger$ based on left and right end point functions.

Motivated by the ongoing research work and by the importance of the concept of fuzzy convexity, in Section 2, we have discussed the preliminary notions, definitions and some related results. Section 3, introduces discrete Jensen and Schur inequalities for convex fuzzy-IVF by means of fuzzy order relation. Basically, these inequalities are generalizations of convex fuzzy IVFs. Section 4 derives the most important class of HH-inequalities for convex fuzzy IVFs, known as fuzzy interval fractional HH-inequalities, and establishes some related inequalities.

2. Preliminaries

In this section, we first give some definitions, preliminary notations, and results which will be helpful for further study. Then, we define new definitions and discuss some properties of convex fuzzy-IVFs.

Let $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\eta \in \mathcal{K}_C$ be defined by

$$\eta = [\eta_*,\ \eta^*] = \{\ddagger \in \mathbb{R} |\ \eta_* \leq \ddagger \leq \eta^*\},(\eta_*,\ \eta^* \in \mathbb{R}).$$

If $\eta_* = \eta^*$, then η is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\eta_* \geq 0$, then $[\eta_*,\ \eta^*]$ is called a positive interval. The set of all positive intervals is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\eta_*,\ \eta^*] : [\eta_*,\ \eta^*] \in \mathcal{K}_C \text{ and } \eta_* \geq 0\}$.

Let $\varrho \in \mathbb{R}$ and $\varrho\eta$ be defined by

$$\varrho.\eta \ = \begin{cases} [\varrho\eta_*,\ \varrho\eta^*] \text{ if } \varrho \geq 0, \\ [\varrho\eta^*, \varrho\eta_*] \ \text{ if } \varrho < 0\,. \end{cases} \tag{13}$$

Then, the Minkowski difference $\xi - \eta$, addition $\eta + \xi$ and $\eta \times \xi$ for $\eta, \xi \in \mathcal{K}_C$ are defined by

$$\begin{aligned} [\xi_*,\ \xi^*] - [\eta_*,\ \eta^*] \ &= [\xi_* - \eta_*,\ \ \xi^* - \eta^*], \\ [\xi_*,\ \xi^*] + [\eta_*,\ \eta^*] \ &= [\xi_* + \eta_*,\ \ \xi^* + \eta^*], \end{aligned} \tag{14}$$

and

The inclusion "$\subseteq$" means that

$$\xi \subseteq \eta \text{ if and only if}, [\xi_*,\ \xi^*] \subseteq [\eta_*,\ \eta^*] \text{ if and only if}, \ \eta_* \leq \xi_*,\ \xi^* \leq \eta^*. \tag{15}$$

Remark 1. *[47] The relation "$\leq_I$" defined on $\mathcal{K}_C$ by*

$$[\nabla_*,\ \nabla^*] \leq_I [\eta_*,\ \eta^*] \text{ if and only if } \nabla_* \leq \eta_*,\ \ \nabla^* \leq \eta^*, \tag{16}$$

for all $[\nabla_*,\ \nabla^*], [\eta_*,\ \eta^*] \in \mathcal{K}_C$, it is an order relation. For given $[\nabla_*,\ \nabla^*], [\eta_*,\ \eta^*] \in \mathcal{K}_C$, we say that $[\nabla_*,\ \nabla^*] \leq_I [\eta_*,\ \eta^*]$ if and only if $\nabla_* \leq \eta_*, \nabla^* \leq \eta^*$ or $\nabla_* \leq \eta_*, \nabla^* < \eta^*$.

For $[\xi_*,\ \xi^*], [\eta_*,\ \eta^*] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $[\xi_*,\ \xi^*]$ and $[\eta_*,\ \eta^*]$ is defined by

$$d([\xi_*,\ \xi^*],\ [\eta_*,\ \eta^*]) = max\{[\xi_*,\ \xi^*],\ [\eta_*,\ \eta^*]\}. \tag{17}$$

It is a familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.

Let $\mathbb{R}$ be the set of real numbers. A fuzzy subset A of $\mathbb{R}$ is characterized by a mapping $\zeta : \mathbb{R} \to [0,1]$ called the membership function, for each fuzzy set and $\gamma \in (0,\ 1]$, then γ-level sets of ζ is denoted and defined as follows $\zeta_\gamma = \{u \in \mathbb{R} |\ \zeta(u) \geq \gamma\}$. If $\gamma = 0$, then $supp(\zeta) = \{\ddagger \in \mathbb{R} |\ \zeta(\ddagger)\rangle 0\}$ is called support of ζ. By $[\zeta]^0$ we define the closure of $supp(\psi)$.

Let $\mathbb{F}(\mathbb{R})$ be the family of all fuzzy sets and $\zeta \in \mathbb{F}(\mathbb{R})$ denote the family of all nonempty sets. $\zeta \in \mathbb{F}(\mathbb{R})$ is a fuzzy set. Then, we define the following:

(1) ζ is said to be normal if there exists $\ddagger \in \mathbb{R}$ and $\zeta(\ddagger) = 1$;
(2) ζ is said to be upper semicontinuous on $\mathbb{R}$ if for given $\ddagger \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\zeta(\ddagger) - \zeta(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\ddagger - y| < \delta$;
(3) ζ is said to be fuzzy convex if ζ_γ is convex for every $\gamma \in [0,\ 1]$;
(4) ζ is compactly supported if $supp(\zeta)$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it has properties (1)–(4). We denote by $\mathbb{F}_0$ the family of all intervals.

Let $\zeta \in \mathbb{F}_0$ be a fuzzy interval if and only if γ-levels $[\zeta]^\gamma$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\zeta]^\gamma = [\zeta_*(\gamma),\ \zeta^*(\gamma)],$$

where

$$\zeta_*(\gamma) = inf\{\ddagger \in \mathbb{R} |\ \zeta(\ddagger) \geq \gamma\},\ \zeta^*(\gamma) = sup\{\ddagger \in \mathbb{R} |\ \zeta(\ddagger) \geq \gamma\}.$$

Proposition 1. [14] *If $\zeta, \eta \in \mathbb{F}_0$, then relation "$\preccurlyeq$" defined on $\mathbb{F}_0$ by*

$$\zeta \preccurlyeq \eta \text{ if and only if, } [\zeta]^{\gamma} \leq_I [\eta]^{\gamma}, \text{ for all } \gamma \in [0,\ 1], \tag{18}$$

this relation is known as partial order relation.

For $\zeta, \eta \in \mathbb{F}_0$ and $\varrho \in \mathbb{R}$, the sum $\zeta \tilde{+} \eta$, product $\zeta \tilde{\times} \eta$, scalar product $\varrho.\zeta$ and sum with scalar are defined by:

Then, for all $\gamma \in [0,\ 1]$, we have

$$[\zeta \tilde{+} \eta]^{\gamma} = [\zeta]^{\gamma} + [\eta]^{\gamma}, \tag{19}$$

$$[\zeta \tilde{\times} \eta]^{\gamma} = [\zeta]^{\gamma} \times [\ \eta]^{\gamma}, \tag{20}$$

$$[\varrho.\zeta]^{\gamma} = \varrho.[\zeta]^{\gamma}. \tag{21}$$

$$[\varrho \tilde{+}\ \zeta]^{\gamma} = \varrho + [\zeta]^{\gamma}. \tag{22}$$

For $\psi \in \mathbb{F}_0$ such that $\zeta = \eta \tilde{+} \psi$, then by this result we have existence of Hukuhara difference of ζ and η, and we say that ψ is the H-difference of ζ and η, and denoted by $\zeta \tilde{-} \eta$. If H-difference exists, then

$$(\psi)^{*}(\gamma) = (\zeta \tilde{-} \eta)^{*}(\gamma) = \zeta^{*}(\gamma) - \eta^{*}(\gamma), (\psi)_{*}(\gamma) = (\zeta \tilde{-} \eta)_{*}(\gamma) = \zeta_{*}(\gamma) - \eta_{*}(\gamma). \tag{23}$$

A partition of $[u, \nu]$ is any finite ordered subset P having the form

$$P = \{u = \ddagger_1 <\ \ddagger_2 <\ \ddagger_3 < \ddagger_4 <\ \ddagger_5, \ldots < \ddagger_k = \nu\}.$$

The mesh of a partition P is the maximum length of the subintervals containing P that is,

$$\text{mash}(P) = max\{\ddagger_j -\ \ddagger_{j-1} :\ j = 1, 2, 3 \ldots k\}.$$

Let $\mathcal{P}(\delta,\ [u,\ \nu])$ be the set of all $P \in \mathcal{P}(\delta,\ [u,\ \nu])$ such that mesh $(P) < \delta$. For each interval $[\ddagger_{j-1}, \ddagger_j]$, where $1 \leq j \leq k$, choose an arbitrary point η_j and taking the sum

$$S(\mathcal{Q}, P,\ \delta) = \sum_{j=1}^{k} \mathcal{Q}(\eta_j)(\ddagger_j - \ddagger_{j-1}),$$

where $\mathcal{Q} : [u, \nu] \to \mathbb{R}_I$. We call $S(\mathcal{Q}, P,\ \delta)$ a Riemann sum of $\mathcal{Q}$ corresponding to $P \in \mathcal{P}(\delta,\ [u,\ \nu])$.

Definition 1. *[10] A function $\mathcal{Q} : [u, \nu] \to \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[u, \nu]$ if there exists $B \in \mathbb{R}_I$ such that, for each ϵ, there exists $\delta > 0$ such that*

$$d(S(\mathcal{Q}, P,\ \delta),\ B) < \epsilon,$$

for every Riemann sum of $\mathcal{Q}$ corresponding to $P \in \mathcal{P}(\delta,\ [u,\ \nu])$ and for arbitrary choice of $\eta_j \in [\ddagger_{j-1},\ \ \ddagger_j]$ for $1 \leq j \leq k$. Then, we say that B is the IR-integral of $\mathcal{Q}$ on $[u, \nu]$ and is denoted by $B = (IR)\int_u^{\nu} \mathcal{Q}(\ddagger)d\ddagger$.

Moore [9] first proposed the concept of Riemann integral for IVF, and it is defined as follows:

Theorem 1. *[9] If $\mathcal{Q} : [u, \nu] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF on such that $\mathcal{Q}(\ddagger) = [\mathcal{Q}_*,\ \mathcal{Q}^*]$. Then, $\mathcal{Q}$* is Riemann integrable over *$[u, \nu]$ if and only if $\mathcal{Q}_*$ and $\mathcal{Q}^*$ are both Riemann integrable over $[u, \nu]$ such that*

$$(IR)\int_u^{\nu} \mathcal{Q}(\ddagger)d\ddagger = \left[(R)\int_u^{\nu} \mathcal{Q}_*(u)d\ddagger,\ (R)\int_u^{\nu} \mathcal{Q}^*(u)d\ddagger\right]. \tag{24}$$

The collection of all Riemann integrable real valued functions and Riemann integrable IVF is denoted by $\mathcal{R}_{[c,\ d]}$ *and* $\mathcal{IR}_{[c,\ d]}$*, respectively.*

Definition 2. *[30] A fuzzy map* $\mathcal{Q} : K \subset \mathbb{R} \to \mathbb{F}_0$ *is called fuzzy-IVF. For each* $\gamma \in [0,\ 1]$, whose γ*-levels define the family of IVFs* $\mathcal{Q}_\gamma : K \subset \mathbb{R} \to \mathcal{K}_C$ *are given by* $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ *for all* $\ddagger \in K$*. Here, for each* $\gamma \in [0,\ 1]$*, the left and right real valued functions* $\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma) : K \to \mathbb{R}$ *are also called lower and upper functions of* $\mathcal{Q}$*.*

Remark 2. *If* $\mathcal{Q} : K \subset \mathbb{R} \to \mathbb{F}_0$ *is a fuzzy-IVF, then* $\mathcal{Q}(\ddagger)$ *is called continuous function at* $\ddagger \in K$*, if for each* $\gamma \in [0,\ 1]$*, both left and right real valued functions* $\mathcal{Q}_*(\ddagger, \gamma)$ *and* $\mathcal{Q}^*(\ddagger, \gamma)$ *are continuous at* $\ddagger \in K$*.*

The following conclusion can be drawn from the above literature review, see [10,14,30]:

Definition 3. *Let* $\mathcal{Q} : [c,\ d] \subset \mathbb{R} \to \mathbb{F}_0$ *is called fuzzy-IVF. The fuzzy Riemann integral of* $\mathcal{Q}$ *over* $[c,\ d]$*, denoted by* $(FR)\int_c^d \mathcal{Q}(\ddagger)d\ddagger$*, it is defined level by level*

$$\left[(FR)\int_c^d \mathcal{Q}(\ddagger)d\ddagger\right]^\gamma = (IR)\int_c^d \mathcal{Q}_\gamma(\ddagger)d\ddagger = \left\{\int_c^d \mathcal{Q}(\ddagger, \gamma)d\ddagger : \mathcal{Q}(\ddagger, \gamma) \in \mathcal{R}_{[c,\ d]}\right\}, \quad (25)$$

for all $\gamma \in [0,\ 1]$*, where* $\mathcal{R}_{[c,\ d]}$ *contains the family of left and right functions of IVFs.* $\mathcal{Q}$ *is* (FR)*-integrable over* $[c,\ d]$ *if* $(FR)\int_c^d \mathcal{Q}(\ddagger)d\ddagger \in \mathbb{F}_0$*. Note that, if left and right real valued functions are Lebesgue-integrable, then* $\mathcal{Q}$ *is fuzzy Aumann-integrable over* $[c,\ d]$*, denoted by* $(FA)\int_c^d \mathcal{Q}(\ddagger)d\ddagger$*, see [30].*

Theorem 2. *Let* $\mathcal{Q} : [c,\ d] \subset \mathbb{R} \to \mathbb{F}_0$ *be a fuzzy-IVF, whose* γ*-levels obtain the collection of IVFs* $\mathcal{Q}_\gamma : [c,\ d] \subset \mathbb{R} \to \mathcal{K}_C$ *are defined by* $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ *for all* $\ddagger \in [c,\ d]$ *and for all* $\gamma \in [0,\ 1]$*. Then,* $\mathcal{Q}$ *is* (FR)*-integrable over* $[c,\ d]$ *if and only if* $\mathcal{Q}_*(\ddagger, \gamma)$ *and* $\mathcal{Q}^*(\ddagger, \gamma)$ *both are R-integrable over* $[c,\ d]$*. Moreover, if* $\mathcal{Q}$ is (FR)*-integrable over* $[c,\ d]$*, then*

$$\left[(FR)\int_c^d \mathcal{Q}(\ddagger)d\ddagger\right]^\gamma = \left[(R)\int_c^d \mathcal{Q}_*(\ddagger, \gamma)d\ddagger,\ (R)\int_c^d \mathcal{Q}^*(\ddagger, \gamma)d\ddagger\right] = (IR)\int_c^d \mathcal{Q}_\gamma(\ddagger)d\ddagger, \quad (26)$$

for all $\gamma \in [0,\ 1]$*. For each* $\gamma \in [0,\ 1]$*,* $\mathcal{FR}_{([c,\ d],\ \gamma)}$ *and* $\mathcal{R}_{([c,\ d],\ \gamma)}$ *denote the collection of all* (FR)*-integrable fuzzy-IVFs and, R-integrable left and right functions over* $[c,\ d]$*.*

Definition 4. *[4] A real valued function* $\mathcal{Q} : [u,\ \nu] \to \mathbb{R}^+$ *is called P-convex function if*

$$\mathcal{Q}(\varrho x + (1-\varrho)y) \leq \mathcal{Q}(x) + \mathcal{Q}(y), \quad (27)$$

for all $x,\ y \in [u,\ \nu]$*,* $\varrho \in [0,\ 1]$*. If (27) is reversed, then* $\mathcal{Q}$ *is called P-concave.*

Definition 5. *[6] The fuzzy-IVF* $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ *is called convex fuzzy-IVF on* $[u,\ \nu]$ *if*

$$\mathcal{Q}(\varrho x + (1-\varrho)y\ ) \preccurlyeq \varrho\mathcal{Q}(x)\widetilde{+}(1-\varrho)\mathcal{Q}(y), \quad (28)$$

for all $x,\ y \in [u,\ \nu]$*,* $\varrho \in [0,\ 1]$*. where* $\mathcal{Q}(\ddagger) \succcurlyeq \widetilde{0}$ *for all* $\ddagger \in [u,\ \nu]$*. If (28) is reversed, then* $\mathcal{Q}$ *is called concave fuzzy-IVF on* $[u,\ \nu]$*.* $\mathcal{Q}$ *is affine if and only if it is both convex and concave fuzzy-IVF.*

Remark 3. *If* $\mathcal{Q}_*(\ddagger, \gamma) = \mathcal{Q}^*(\ddagger, \gamma)$ *and* $\gamma = 1$*, then we obtain the inequality (1).*

3. Fuzzy-Interval Jensen and Schur Inequalities

In this section, the discrete Jensen and Schur inequalities for convex fuzzy-IVF are proposed. Firstly, we give the following results connected with Jensen inequality for convex fuzzy-IVF by means of fuzzy order relation.

Theorem 3. *(Jensen inequality for convex fuzzy-IVF) Let* $\sqsupseteq_j \in \mathbb{R}^+$, $u_j \in [u,\ \nu]$,$(j = 1,\ 2,\ 3,\ \ldots,\ k,\ k \geq 2)$ *and* $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ *be a convex fuzzy-IVF, whose* γ*-levels define the family of IVFs* $\mathcal{Q}_\gamma : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are given by* $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ *for all* $\ddagger \in [u,\ \nu]$ *and for all* $\gamma \in (0,\ 1]$. *Then,*

$$\mathcal{Q}\left(\frac{1}{W_k}\sum_{j=1}^{k} \sqsupseteq_j u_j\right) \preccurlyeq \sum_{j}^{k} \frac{\sqsupseteq_j}{W_k}\mathcal{Q}(u_j), \tag{29}$$

where $W_k = \sum_{j=1}^{k} \sqsupseteq_j$. *If* $\mathcal{Q}$ *is concave, then inequality (29) is reversed.*

Proof. When $k = 2$ then inequality (29) is true. Consider inequality (28) is true for $k = n - 1$, then

$$\mathcal{Q}\left(\frac{1}{W_{n-1}}\sum_{j=1}^{n-1} \sqsupseteq_j u_j\right) \preccurlyeq \sum_{j=1}^{n-1} \frac{\sqsupseteq_j}{W_{n-1}}\mathcal{Q}(u_j).$$

Now, let us prove that inequality (29) holds for $k = n$.

$$\mathcal{Q}\left(\tfrac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j\right) = \mathcal{Q}\left(\tfrac{1}{W_n}\sum_{j=1}^{n-2} \sqsupseteq_j u_j + \tfrac{\sqsupseteq_{n-1}+\sqsupseteq_n}{W_n}\left(\tfrac{\sqsupseteq_{n-1}}{\sqsupseteq_{n-1}+\sqsupseteq_n}u_{n-1} + \tfrac{\sqsupseteq_n}{\sqsupseteq_{n-1}+\sqsupseteq_n}u_n\right)\right)$$

Therefore, for each $\gamma \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j, \gamma\right)\\
&= \mathcal{Q}_*\left(\tfrac{1}{W_n}\sum_{j=1}^{n-2} \sqsupseteq_j u_j + \tfrac{\sqsupseteq_{n-1}+\sqsupseteq_n}{W_n}\left(\tfrac{\sqsupseteq_{n-1}}{\sqsupseteq_{n-1}+\sqsupseteq_n}u_{n-1} + \tfrac{\sqsupseteq_n}{\sqsupseteq_{n-1}+\sqsupseteq_n}u_n\right), \gamma\right)\\
&\leq \sum_{j=1}^{n-2} \tfrac{\sqsupseteq_j}{W_n}\mathcal{Q}_*(u_j, \gamma) + \tfrac{\sqsupseteq_{n-1}+\sqsupseteq_n}{W_n}\mathcal{Q}_*\left(\left(\tfrac{\sqsupseteq_{n-1}}{\sqsupseteq_{n-1}+\sqsupseteq_n}u_{n-1} + \tfrac{\sqsupseteq_n}{\sqsupseteq_{n-1}+\sqsupseteq_n}u_n\right), \gamma\right)\\
&\leq \sum_{j=1}^{n-2} \tfrac{\sqsupseteq_j}{W_n}\mathcal{Q}_*(u_j, \gamma) + \tfrac{\sqsupseteq_{n-1}+\sqsupseteq_n}{W_n}\left[\tfrac{\sqsupseteq_{n-1}}{\sqsupseteq_{n-1}+\sqsupseteq_n}\mathcal{Q}_*(u_{n-1}, \gamma) + \tfrac{\sqsupseteq_n}{\sqsupseteq_{n-1}+\sqsupseteq_n}\mathcal{Q}_*(u_n, \gamma)\right]\\
&\leq \sum_{j=1}^{n-2} \tfrac{\sqsupseteq_j}{W_n}\mathcal{Q}_*(u_j, \gamma) + \left[\tfrac{\sqsupseteq_{n-1}}{W_n}\mathcal{Q}_*(u_{n-1}, \gamma) + \tfrac{\sqsupseteq_n}{W_n}\mathcal{Q}_*(u_n, \gamma)\right]\\
&= \sum_{j=1}^{n} \tfrac{\sqsupseteq_j}{W_n}\mathcal{Q}_*(u_j, \gamma).
\end{aligned} \tag{30}$$

Similarly, for $\mathcal{Q}^*(\sqsupseteq, \gamma)$, we have

$$\mathcal{Q}^*\left(\frac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j, \gamma\right) \leq \sum_{j=1}^{n} \frac{\sqsupseteq_j}{W_n}\mathcal{Q}^*(u_j, \gamma). \tag{31}$$

From (30) and (31), we have

$$\left[\mathcal{Q}_*\left(\frac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j, \gamma\right),\ \mathcal{Q}^*\left(\frac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j, \gamma\right)\right] \leq_I \left[\sum_{j=1}^{n} \frac{\sqsupseteq_j}{W_n}\mathcal{Q}_*(u_j, \gamma),\ \sum_{j=1}^{n} \frac{\sqsupseteq_j}{W_n}\mathcal{Q}^*(u_j, \gamma)\right],$$

that is,

$$\mathcal{Q}\left(\frac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j\right) \preccurlyeq \sum_{j=1}^{n} \frac{\sqsupseteq_j}{W_n}\mathcal{Q}(u_j),$$

and the theorem has been proved. □

If $\sqsupseteq_1 = \sqsupseteq_2 = \sqsupseteq_3 = \ \ldots = \sqsupseteq_k = 1$, then Theorem 3 reduces to the following result:

Corollary 1. *Let $u_j \in [u,\ \nu]$, $(j = 1,\ 2,\ 3,\ \ldots, k,\ k \geq 2)$ and $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ be a convex fuzzy-IVF, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ for all $\ddagger \in [u,\ \nu]$ and for all $\gamma \in (0,\ 1]$. Then,*

$$\mathcal{Q}\left(\frac{1}{W_n}\sum_{j=1}^{n} \sqsupseteq_j u_j\right) \preccurlyeq \sum_{J=1}^{k} \frac{1}{k}\mathcal{Q}(u_j). \tag{32}$$

If $\mathcal{Q}$ is a concave fuzzy-IVF, then inequality (32) is reversed.

The next Theorem 4 gives the Schur inequality for convex fuzzy-IVFs.

Theorem 4. *(Schur inequality for convex fuzzy-IVF)Let $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$* be a convex fuzzy-IVF, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ for all $\ddagger \in [u,\ \nu]$ and for all $\gamma \in [0,\ 1]$. If, for $u_1,\ u_2,\ u_3 \in [u,\ \nu]$, such that $u_1 <\ u_2 < u_3$ and $u_3 - u_1, u_3 - u_2, u_2 - u_1 \in [0,\ 1]$, we have

$$(u_3 - u_1)\mathcal{Q}(u_2) \preccurlyeq (u_3 - u_2)\mathcal{Q}(u_1) + (u_2 - u_1)\mathcal{Q}(u_3). \tag{33}$$

If $\mathcal{Q}$ is concave, then inequality (33) is reversed.

Proof. Let $u_1,\ u_2,\ u_3 \in [u,\ \nu]$ and $u_3 - u_1 > 0$. Consider $\varrho = \frac{u_3 - u_2}{u_3 - u_1}$, then $u_2 = \varrho u_1 + (1 - \varrho)u_3$. Since $\mathcal{Q}$ is a convex fuzzy-IVF, then, by hypothesis, we have

$$\mathcal{Q}(u_2) \preccurlyeq \frac{u_3 - u_2}{u_3 - u_1}\mathcal{Q}(u_1) + \frac{u_2 - u_1}{u_3 - u_1}\mathcal{Q}(u_3).$$

Therefore, for each $\gamma \in [0,\ 1]$, we have

$$\mathcal{Q}_*(u_2, \gamma) \leq \tfrac{u_3 - u_2}{u_3 - u_1}\mathcal{Q}_*(u_1, \gamma) + \tfrac{u_2 - u_1}{u_3 - u_1}\mathcal{Q}_*(u_3, \gamma). \tag{34}$$

From (34), we have

$$(u_3 - u_1)\mathcal{Q}_*(u_2, \gamma) \leq (u_3 - u_2)\mathcal{Q}_*(u_1, \gamma) + (u_2 - u_1)\mathcal{Q}_*(u_3, \gamma). \tag{35}$$

Similarly, for $\mathcal{Q}^*(\sqsupseteq, \gamma)$, we have

$$(u_3 - u_1)\mathcal{Q}^*(u_2, \gamma) \leq (u_3 - u_2)\mathcal{Q}^*(u_1, \gamma) + (u_2 - u_1)\mathcal{Q}^*(u_3, \gamma). \tag{36}$$

From (35) and (36), we have

$$\begin{aligned} &[(u_3 - u_1)\mathcal{Q}_*(u_2, \gamma),\ (u_3 - u_1)\mathcal{Q}^*(u_2, \gamma)] \\ \leq_I\ &[(u_3 - u_2)\mathcal{Q}_*(u_1, \gamma) + (u_2 - u_1)\mathcal{Q}_*(u_3, \gamma),\ \ (u_3 - u_2)\mathcal{Q}^*(u_1, \gamma) + (u_2 - u_1)\mathcal{Q}^*(u_3, \gamma)]. \end{aligned}$$

That is

$$(u_3 - u_1)\mathcal{Q}(u_2) \preccurlyeq (u_3 - u_2)\mathcal{Q}(u_1) + (u_2 - u_1)\mathcal{Q}(u_3).$$

Hence, the result has been proved. □

Now we obtain a refinement of Schur inequality for convex fuzzy-IVF is given in the following result.

Theorem 5. *Let $\sqsupseteq_j \in \mathbb{R}^+$, $u_j \in [u,\ \nu]$, $(j = 1,\ 2,\ 3,\ \ldots, k,\ k \geq 2)$ and $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ be a convex fuzzy-IVF, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ for all $\ddagger \in [u,\ \nu]$ and for all $\gamma \in [0,\ 1]$. If $(L,\ U) \subseteq [u, \nu]$, then*

$$\sum_{j=1}^{k} \frac{\sqsupseteq_j}{W_k}\mathcal{Q}(u_j) \preccurlyeq \sum_{j=1}^{k}\left(\left(\frac{U - u_j}{U - L}\right)\left(\frac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}(L, \gamma) + \left(\frac{u_j - L}{U - L}\right)\left(\frac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}(U, \gamma)\right), \tag{37}$$

where $W_k = \sum_{j=1}^{k} \sqsupseteq_j$. *If* $\mathcal{Q}$ *is concave, then inequality (37) is reversed.*

Proof. Consider $= u_1,\ u_j = u_2, (j = 1,\ 2,\ 3,\ \ldots, k), U = u_3$. Then, by hypothesis, we have

$$\mathcal{Q}(u_j) \leq \frac{U-u_j}{U-L}\mathcal{Q}(L,\gamma) + \frac{u_j - L}{U-L}\mathcal{Q}(U,\gamma).$$

Therefore, for each $\gamma \in [0,\ 1]$, we have

$$\mathcal{Q}_*(u_j,\gamma) \leq \tfrac{U-u_j}{U-L}\mathcal{Q}_*(L,\gamma) + \tfrac{u_j-L}{U-L}\mathcal{Q}_*(U,\gamma).$$

Above inequality can be written as,

$$\tfrac{\sqsupseteq_j}{W_k}\mathcal{Q}_*(u_j,\gamma) \leq \left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}_*(L,\gamma) + \left(\tfrac{u_j-L}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}_*(U,\gamma). \tag{38}$$

Taking sum of all inequalities (38) for $j = 1,\ 2,\ 3,\ \ldots k$, we have

$$\textstyle\sum_{j=1}^{k}\tfrac{\sqsupseteq_j}{W_k}\mathcal{Q}_*(u_j,\gamma) \leq \sum_{j=1}^{k}\left(\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}_*(L,\gamma) + \left(\tfrac{u_j-L}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}_*(U,\gamma)\right). \tag{39}$$

Similarly, for $\mathcal{Q}^*(\sqsupseteq,\gamma)$, we have

$$\sum_{j=1}^{k}\frac{\sqsupseteq_j}{W_k}\mathcal{Q}^*(u_j,\gamma) \leq \sum_{j=1}^{k}\left(\left(\frac{U-u_j}{U-L}\right)\left(\frac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}^*(L,\gamma) + \left(\frac{u_j-L}{U-L}\right)\left(\frac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}^*(U,\gamma)\right). \tag{40}$$

From (39) and (40), we have

$$\begin{aligned}
\textstyle\sum_{j=1}^{k}\tfrac{\sqsupseteq_j}{W_k}\left[\mathcal{Q}(u_j)\right]^{\gamma} &= \left[\textstyle\sum_{j=1}^{k}\tfrac{\sqsupseteq_j}{W_k}\mathcal{Q}_*(u_j,\gamma),\ \sum_{j=1}^{k}\tfrac{\sqsupseteq_j}{W_k}\mathcal{Q}^*(u_j,\gamma)\right] \\
&\leq_I \left[\textstyle\sum_{j=1}^{k}\begin{pmatrix}\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}_*(L,\gamma) \\ +\left(\tfrac{u_j-L}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}_*(U,\gamma)\end{pmatrix},\ \sum_{j=1}^{k}\begin{pmatrix}\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}^*(L,\gamma) \\ +\left(\tfrac{u_j-L}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}^*(U,\gamma)\end{pmatrix}\right], \\
&\leq_I \textstyle\sum_{j=1}^{k}\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\left[\mathcal{Q}_*(L,\gamma),\ \mathcal{Q}^*(L,\gamma)\right] + \sum_{j=1}^{k}\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\left[\mathcal{Q}_*(U,\gamma),\ \mathcal{Q}^*(U,\gamma)\right], \\
&= \textstyle\sum_{j=1}^{k}\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}(L,\gamma) + \sum_{j=1}^{k}\left(\tfrac{U-u_j}{U-L}\right)\left(\tfrac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}(U,\gamma).
\end{aligned}$$

Thus,

$$\sum_{j=1}^{k}\frac{\sqsupseteq_j}{W_k}\mathcal{Q}(u_j) \preccurlyeq \sum_{j=1}^{k}\left(\left(\frac{U-u_j}{U-L}\right)\left(\frac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}(L) + \left(\frac{u_j-L}{U-L}\right)\left(\frac{\sqsupseteq_j}{W_k}\right)\mathcal{Q}(U)\right),$$

this completes the proof. □

We now consider some special cases of Theorems 3 and 5.

If $\mathcal{Q}_*(\sqsupseteq,\gamma) = \mathcal{Q}_*(\sqsupseteq,\gamma)$ with $\gamma = 1$, then Theorems 3 and 5 reduce to the following results:

Corollary 2. *(Jensen inequality for convex function) Let* $\sqsupseteq_j \in \mathbb{R}^+$, $u_j \in [u,\ v]$, $(j = 1,\ 2,\ 3,\ \ldots,$ $k,\ k \geq 2)$ *and let* $\mathcal{Q} : [u,\ v] \to \mathbb{R}^+$ *be a non-negative real-valued function. If* $\mathcal{Q}$ *is a convex function, then*

$$\mathcal{Q}\left(\frac{1}{W_k}\sum_{j=1}^{k}\sqsupseteq_j u_j\right) \leq \sum_{j=1}^{k}\frac{\sqsupseteq_j}{W_k}\mathcal{Q}(u_j), \tag{41}$$

where $W_k = \sum_{j=1}^{k}\sqsupseteq_j$. If $\mathcal{Q}$ *is concave function, then inequality (41) is reversed.*

Corollary 3. *Let* $\sqsupseteq_j \in \mathbb{R}^+$, $u_j \in [u,\ \nu]$, $(j = 1,\ 2,\ 3,\ \ldots, k,\ k \geq 2)$, *and* $\mathcal{Q} : [u,\ \nu] \to \mathbb{R}^+$ *be a non-negative real-valued function. If* $\mathcal{Q}$ *is a convex function and* $u_1,\ u_2, \ldots, u_j \in (L,\ U) \subseteq [u, \nu]$, *then*

$$\sum_{j=1}^{k} \frac{\sqsupseteq_j}{W_k} \mathcal{Q}(u_j) \leq \sum_{j=1}^{k} \left(\left(\frac{U - u_j}{U - L} \right) \left(\frac{\sqsupseteq_j}{W_k} \right) \mathcal{Q}(L) + \left(\frac{u_j - L}{U - L} \right) \left(\frac{\sqsupseteq_j}{W_k} \right) \mathcal{Q}(U) \right), \tag{42}$$

where $W_k = \sum_{j=1}^{k} \sqsupseteq_j$. *If* $\mathcal{Q}$ *is a concave function, then inequality (42) is reversed.*

4. Fuzzy-Interval Fractional Hermite–Hadamard Inequalities

In this section, we shall continue with the following fractional HH-inequality for convex fuzzy-IVFs, and we also give fractional HH–Fejér inequality for convex fuzzy-IVF through fuzzy order relation. In what follows, we denote by $L([u,\ \nu], \mathbb{F}_0)$ the family of Lebesgue measurable fuzzy-IVFs.

Theorem 6. *Let* $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ *be a convex fuzzy-IVF on* $[u,\ \nu]$, *whose* γ*-levels define the family of IVFs* $\mathcal{Q}_\gamma : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are given by* $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ *for all* $\ddagger \in [u,\ \nu]$ *and for all* $\gamma \in [0,\ 1]$. *If* $\mathcal{Q} \in L([u,\ \nu], \mathbb{F}_0)$, *then*

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{\Gamma(\alpha+1)}{2(\nu-u)^\alpha} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}(\nu) \widetilde{+} \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}(u)\right] \preccurlyeq \frac{\mathcal{Q}(u) \widetilde{+} \mathcal{Q}(\nu)}{2}. \tag{43}$$

If $\mathcal{Q}(\sqsupseteq)$ *is concave fuzzy-IVF, then*

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \succcurlyeq \frac{\Gamma(\alpha+1)}{2(\nu-u)^\alpha} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}(\nu) \widetilde{+} \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}(u)\right] \succcurlyeq \frac{\mathcal{Q}(u) \widetilde{+} \mathcal{Q}(\nu)}{2}. \tag{44}$$

Proof. Let $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ be a convex fuzzy-IVF. Then, by hypothesis, we have

$$2\mathcal{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \mathcal{Q}(\varrho u + (1-\varrho)\nu) \widetilde{+} \mathcal{Q}((1-\varrho)u + \varrho\nu).$$

Therefore, for every $\gamma \in [0,\ 1]$, we have

$$2\mathcal{Q}_*\left(\frac{u+\nu}{2},\ \gamma\right) \leq \mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma) + \mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma).$$

Multiplying both sides by $\varrho^{\alpha-1}$ and integrating the obtained result with respect to ϱ over $(0,1)$, we have

$$\begin{aligned} &2\int_0^1 \varrho^{\alpha-1} \mathcal{Q}_*\left(\tfrac{u+\nu}{2},\ \gamma\right) d\varrho \\ &\leq \int_0^1 \varrho^{\alpha-1} \mathcal{Q}_*(\varrho u + (1-\varrho)\nu, \gamma) d\varrho + \int_0^1 \varrho^{\alpha-1} \mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma) d\varrho. \end{aligned}$$

Let $\sqsupseteq = \varrho u + (1-\varrho)\nu$ and $y = (1-\varrho)u + \varrho\nu$. Then, we have

$$\begin{aligned} \tfrac{2}{\alpha}\mathcal{Q}_*\left(\tfrac{u+\nu}{2},\ \gamma\right) &\leq \tfrac{1}{(\nu-u)^\alpha} \int_u^\nu (\nu-y)^{\alpha-1} \mathcal{Q}_*(y, \gamma) dy + \tfrac{1}{(\nu-u)^\alpha} \int_u^\nu (\sqsupseteq - u)^{\alpha-1} \mathcal{Q}_*(\sqsupseteq, \gamma) dz \\ &\leq \tfrac{\Gamma(\alpha)}{(\nu-u)^\alpha} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}_*(\nu,\ \gamma) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}_*(u,\ \gamma)\right]. \end{aligned} \tag{45}$$

Similarly, for $\mathcal{Q}^*(\sqsupseteq, \gamma)$, we have

$$\frac{2}{\alpha}\mathcal{Q}^*\left(\frac{u+\nu}{2},\ \gamma\right) \leq \frac{\Gamma(\alpha)}{(\nu-u)^\alpha} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}^*(\nu,\ \gamma) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}^*(u,\ \gamma)\right]. \tag{46}$$

From (45) and (46), we have

$$\begin{gathered}\frac{2}{\alpha}\left[\mathcal{Q}_*\left(\frac{u+\nu}{2},\,\gamma\right),\,\mathcal{Q}^*\left(\frac{u+\nu}{2},\,\gamma\right)\right]\\ \leq_I \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}_*(\nu,\,\gamma)+\mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}_*(u,\,\gamma)\right],\,\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}^*(\nu,\,\gamma)+\mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}^*(u,\,\gamma)\right]\right].\end{gathered}$$

That is

$$\frac{2}{\alpha}\,\mathcal{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}(\nu)\widetilde{+}\mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}(u)\right]. \tag{47}$$

In a similar way as above, we have

$$\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}(\nu)\widetilde{+}\mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}(u)\right] \preccurlyeq \frac{\mathcal{Q}(u)\widetilde{+}\mathcal{Q}(\nu)}{\alpha}. \tag{48}$$

Combining (47) and (48), we have

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{\Gamma(\alpha+1)}{2(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}(\nu)\widetilde{+}\mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}(u)\right] \preccurlyeq \frac{\mathcal{Q}(u)\widetilde{+}\mathcal{Q}(\nu)}{2}.$$

Hence, the required result. □

Remark 4. *From Theorem 6 we clearly see that*

Let $\alpha = 1$. Then, Theorem 6 reduces to the result for convex-IVF given in [31]:

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\,(FR)\int_u^\nu \mathcal{Q}(\ddagger)d\ddagger \preccurlyeq \frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}.$$

If $\mathcal{Q}_*(\ddagger,\,\gamma) = \mathcal{Q}^*(\ddagger,\,\gamma)$ and $\gamma = 1$, then from Theorem 6 we get inequality (3).

Let $\alpha = 1 = \gamma$ and $\mathcal{Q}_*(\ddagger,\,\gamma) = \mathcal{Q}^*(\ddagger,\,\gamma)$. Then, from Theorem 6 we obtain inequality (2).

Example 1. *Let $\alpha = \frac{1}{2}$, $\ddagger \in [2,3]$, and the fuzzy-IVF $\mathcal{Q} : [u,\,\nu] = [2,\,3] \to \mathbb{F}_0$, defined by*

$$\mathcal{Q}(\ddagger)(\theta) = \begin{cases} \frac{\theta}{2-\ddagger^{\frac{1}{2}}} & \theta \in \left[0,\,2-\ddagger^{\frac{1}{2}}\right] \\ \frac{2\left(2-\ddagger^{\frac{1}{2}}\right)-\theta}{2-\ddagger^{\frac{1}{2}}} & \theta \in \left(2-\ddagger^{\frac{1}{2}},\,2\left(2-\ddagger^{\frac{1}{2}}\right)\right] \\ 0 & \text{otherwise,} \end{cases}$$

Then, for each $\gamma \in [0,\,1]$, we have $\mathcal{Q}_\gamma(\ddagger) = \left[\gamma\left(2-\ddagger^{\frac{1}{2}}\right),(2-\gamma)\left(2-\ddagger^{\frac{1}{2}}\right)\right]$. Since left and right end point functions $\mathcal{Q}_*(\ddagger,\gamma) = \gamma\left(2-\ddagger^{\frac{1}{2}}\right)$, $\mathcal{Q}^*(\ddagger,\,\gamma) = (2-\gamma)\left(2-\ddagger^{\frac{1}{2}}\right)$ are convex functions for each $\gamma \in [0,\,1]$, then $\mathcal{Q}(\ddagger)$ is convex fuzzy-IVF. We clearly see that $\mathcal{Q} \in L([u,\,\nu],\mathbb{F}_0)$ and

$$\begin{gathered}\mathcal{Q}_*\left(\frac{u+\nu}{2},\,\gamma\right) = \mathcal{Q}_*\left(\frac{5}{2},\,\gamma\right) = \gamma\frac{4-\sqrt{10}}{2}\\ \mathcal{Q}^*\left(\frac{u+\nu}{2},\,\gamma\right) = \mathcal{Q}^*\left(\frac{5}{2},\,\gamma\right) = (2-\gamma)\frac{4-\sqrt{10}}{2}\\ \frac{\mathcal{Q}_*(u,\gamma)+\mathcal{Q}_*(\nu,\,\gamma)}{2} = \gamma\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)\\ \frac{\mathcal{Q}^*(u,\gamma)+\mathcal{Q}^*(\nu,\,\gamma)}{2} = (2-\gamma)\left(\frac{4-\sqrt{2}+\sqrt{3}}{2}\right).\end{gathered}$$

Note that

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}_*(\nu,\ \gamma)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}_*(u,\gamma)\right]\\&=\frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(3-\ddagger)^{\frac{-1}{2}}\ .\ \gamma\left(2-\ddagger^{\frac{1}{2}}\right)d\ddagger\\&+\frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(\ddagger-2)^{\frac{-1}{2}}\ .\ \gamma\left(2-\ddagger^{\frac{1}{2}}\right)d\ddagger\\&=\frac{1}{4}\gamma\left[\frac{7393}{10{,}000}+\frac{9501}{10{,}000}\right]\\&=\gamma\frac{8447}{20{,}000}.\end{aligned}$$

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}^*(\nu,\ \gamma)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}^*(u,\gamma)\right]\\&=\frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(3-\ddagger)^{\frac{-1}{2}}\ .\ (2-\gamma)\left(2-\ddagger^{\frac{1}{2}}\right)d\ddagger\\&+\frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_2^3(\ddagger-2)^{\frac{-1}{2}}\ .\ (2-\gamma)\left(2-\ddagger^{\frac{1}{2}}\right)d\ddagger\\&=\frac{1}{4}(2-\gamma)\left[\frac{7393}{10{,}000}+\frac{9501}{10{,}000}\right]\\&=(2-\gamma)\frac{8447}{20{,}000}\end{aligned}$$

Therefore

$$\left[\gamma\frac{4-\sqrt{10}}{2},(2-\gamma)\frac{4-\sqrt{10}}{2}\right]\leq_I\left[\gamma\frac{8447}{20,000},(2-\gamma)\frac{8447}{20,000}\right]\leq_I\left[\gamma\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right),(2-\gamma)\left(\frac{4-\sqrt{2}+\sqrt{3}}{2}\right)\right],$$

and Theorem 6 is verified.

Now, we derive fractional HH–Fejér inequality for convex fuzzy-IVF, which generalize the classical fractional HH and HH–Fejér inequality. Firstly, we give the following result connected with the right part of the classical HH–Fejér inequality for convex fuzzy-IVF through fuzzy order relation, which is known as second fuzzy fractional HH–Fejér inequality.

Theorem 7. *Second fuzzy fractional HH–Fejér inequality) Let $\mathcal{Q}:[u,\ \nu]\to\mathbb{F}_0$ be a convex fuzzy-IVF with $u<\ \nu$, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma:[u,\ \nu]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ are given by $\mathcal{Q}_\gamma(\ddagger)=[\mathcal{Q}_*(\ddagger,\gamma),\ \mathcal{Q}^*(\ddagger,\gamma)]$ for all $\ddagger\in[u,\ \nu]$ and for all $\gamma\in[0,\ 1]$. If $\mathcal{Q}\in L([u,\ \nu],\mathbb{F}_0)$ and $\Omega:[u,\ \nu]\to\mathbb{R}$, $\Omega(\ddagger)\geq0$, symmetric with respect to $\frac{u+\nu}{2}$, then*

$$\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}\Omega(\nu)\widetilde{+}\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}\Omega(u)\right]\preccurlyeq\frac{\mathcal{Q}(u)\widetilde{+}\mathcal{Q}(\nu)}{2}\left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu)+\mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right].\tag{49}$$

If $\mathcal{Q}$ is concave fuzzy-IVF, then inequality (49) is reversed.

Proof. Let $\mathcal{Q}$ be a convex fuzzy-IVF and $\varrho^{\alpha-1}\Omega(\varrho u+(1-\varrho)\nu)\ \geq\ 0$. Then, for each $\gamma\in[0,\ 1]$, we have

$$\begin{aligned}&\varrho^{\alpha-1}\mathcal{Q}_*(\varrho u+(1-\varrho)\nu,\ \gamma)\Omega(\varrho u+(1-\varrho)\nu)\\&\leq\varrho^{\alpha-1}(\varrho\mathcal{Q}_*(u,\ \gamma)+(1-\varrho)\mathcal{Q}_*(\nu,\ \gamma))\Omega(\varrho u+(1-\varrho)\nu),\end{aligned}\tag{50}$$

and

$$\begin{aligned}&\varrho^{\alpha-1}\mathcal{Q}_*((1-\varrho)u+\varrho\nu,\ \gamma)\Omega((1-\varrho)u+\varrho\nu)\\&\leq\varrho^{\alpha-1}((1-\varrho)\mathcal{Q}_*(u,\ \gamma)+\varrho\mathcal{Q}_*(\nu,\ \gamma))\Omega((1-\varrho)u+\varrho\nu).\end{aligned}\tag{51}$$

After adding (50) and (51), and integrating over $[0,\ 1]$, we get

$$\begin{aligned}&\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*(\varrho u+(1-\varrho)\nu,\ \gamma)\Omega(\varrho u+(1-\varrho)\nu)d\varrho\\&+\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*((1-\varrho)u+\varrho\nu,\ \gamma)\Omega((1-\varrho)u+\varrho\nu)d\varrho\\&\leq\int_0^1\left[\begin{array}{c}\varrho^{\alpha-1}\mathcal{Q}_*(u,\ \gamma)\{\varrho\Omega(\varrho u+(1-\varrho)\nu)+(1-\varrho)\Omega((1-\varrho)u+\varrho\nu)\}\\+\varrho^{\alpha-1}\mathcal{Q}_*(\nu,\ \gamma)\{(1-\varrho)\Omega(\varrho u+(1-\varrho)\nu)+\varrho\Omega((1-\varrho)u+\varrho\nu)\}\end{array}\right]d\varrho,\\&=\mathcal{Q}_*(u,\ \gamma)\int_0^1\ \varrho^{\alpha-1}\Omega(\varrho u+(1-\varrho)\nu)\ d\varrho+\mathcal{Q}_*(\nu,\ \gamma)\int_0^1\ \varrho^{\alpha-1}\Omega((1-\varrho)u+\varrho\nu)\ d\varrho.\end{aligned}$$

Since Ω is symmetric, then

$$= [\mathcal{Q}_*(u,\ \gamma) + \mathcal{Q}_*(\nu,\ \gamma)] \int_0^1 \ \varrho^{\alpha-1}\Omega((1-\varrho)u + \varrho\nu)\ d\varrho.$$

$$= \frac{\mathcal{Q}_*(u,\ \gamma)+\mathcal{Q}_*(\nu,\ \gamma)}{2} \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right]. \tag{52}$$

Since

$$\begin{aligned}
&\int_0^1 \varrho^{\alpha-1}\mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma)\Omega((1-\varrho)u + \varrho\nu)d\varrho \\
&+ \int_0^1 \varrho^{\alpha-1}\mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma)\Omega((1-\varrho)u + \varrho\nu)d\varrho \\
&= \frac{1}{(\nu-u)^{\alpha}} \int_u^{\nu} (\ddagger - u)^{\alpha-1}\mathcal{Q}_*(u-\nu-\ddagger,\gamma)\Omega(\ddagger)d\ddagger \\
&\quad + \frac{1}{(\nu-u)^{\alpha}} \int_u^{\nu} (\ddagger - u)^{\alpha-1}\mathcal{Q}_*(\ddagger,\gamma)\Omega(\ddagger)d\ddagger \\
&= \frac{1}{(\nu-u)^{\alpha}} \int_u^{\nu} (\nu - u)^{\alpha-1}\mathcal{Q}_*(\ddagger,\gamma)\Omega(u-\nu-\ddagger)d\ddagger \\
&\quad + \frac{1}{(\nu-u)^{\alpha}} \int_u^{\nu} (\ddagger - u)^{\alpha-1}\mathcal{Q}_*(\ddagger,\gamma)\Omega(\ddagger)d\ddagger \\
&= \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}_*\Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}_*\Omega(u)\right].
\end{aligned} \tag{53}$$

Then, from (53), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}_*\Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}_*\Omega(u)\right] \\
&\le \frac{\mathcal{Q}_*(u,\ \gamma)+\mathcal{Q}_*(\nu,\ \gamma)}{2} \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right].
\end{aligned} \tag{54}$$

Similarly, for $\mathcal{Q}^*(\sqsupseteq,\gamma)$, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}^*\Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}^*\Omega(u)\right] \\
&\le \frac{\mathcal{Q}^*(u,\ \gamma)+\mathcal{Q}^*(\nu,\ \gamma)}{2} \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right].
\end{aligned} \tag{55}$$

From (54) and (55), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}_*\Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}_*\Omega(u),\ \mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}^*\Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}^*\Omega(u)\right] \\
&\le_I \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}} \left[\frac{\mathcal{Q}_*(u,\ \gamma)+\mathcal{Q}_*(\nu,\ \gamma)}{2},\ \frac{\mathcal{Q}^*(u,\ \gamma)+\mathcal{Q}^*(\nu,\ \gamma)}{2}\right] \left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right].
\end{aligned}$$

That is

$$\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}\Omega(\nu) \widetilde{+} \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}\Omega(u)\right] \preccurlyeq \frac{\mathcal{Q}(u) \widetilde{+} \mathcal{Q}(\nu)}{2} \left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right].$$

Now, we obtain the following result connected with left part of classical HH–Fejér inequality for convex fuzzy-IVF through fuzzy order relation, which is known as first fuzzy fractional HH–Fejér inequality. □

Theorem 8. *(First fuzzy fractional HH–Fejér inequality) Let* $\mathcal{Q} : [u,\ \nu] \to \mathbb{F}_0$ *be a convex fuzzy-IVF with* $u < \nu$*, whose* γ*-levels define the family of IVFs* $\mathcal{Q}_{\gamma} : [u,\ \nu] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ *are given by* $\mathcal{Q}_{\gamma}(\ddagger) = [\mathcal{Q}_*(\ddagger,\gamma),\ \mathcal{Q}^*(\ddagger,\gamma)]$ *for all* $\ddagger \in [u,\ \nu]$ *and for all* $\gamma \in [0,\ 1]$*. If* $\mathcal{Q} \in\ L([u,\ \nu],\mathbb{F}_0)$ *and* $\Omega : [u,\ \nu] \to \mathbb{R},\ \Omega(\ddagger) \ge 0,$ *symmetric with respect to* $\frac{u+\nu}{2}$*, then*

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right)\left[\mathcal{I}_{u^+}^{\alpha}\ \Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \Omega(u)\right] \preccurlyeq \left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}\Omega(\nu) + \mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}\Omega(u)\right]. \tag{56}$$

If $\mathcal{Q}$ *is concave fuzzy-IVF, then inequality (56) is reversed.*

Proof. Since $\mathcal{Q}$ is a convex fuzzy-IVF, then for $\gamma \in [0,\ 1]$, we have

$$\mathcal{Q}_*\left(\frac{u+\nu}{2},\ \gamma\right) \le \frac{1}{2}(\mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma) + \mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma)). \tag{57}$$

Since $\Omega(\varrho u+(1-\varrho)\nu)=\Omega((1-\varrho)u+\varrho\nu)$, then, by multiplying (57) by $\varrho^{\alpha-1}\Omega((1-\varrho)u+\varrho\nu)$ and integrating it with respect to ϱ over $[0,\ 1]$, we obtain

$$\begin{array}{c} \mathcal{Q}_*\left(\frac{u+\nu}{2},\ \gamma\right)\int_0^1\varrho^{\alpha-1}\Omega((1-\varrho)u+\varrho\nu)d\varrho \\ \leq\frac{1}{2}\left(\begin{array}{c}\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*(\varrho u+(1-\varrho)\nu,\ \gamma)\Omega((1-\varrho)u+\varrho\nu)d\varrho \\ +\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*((1-\varrho)u+\varrho\nu,\ \gamma)\Omega((1-\varrho)u+\varrho\nu)d\varrho\end{array}\right).\end{array} \tag{58}$$

Let $\ddagger=(1-\varrho)u+\varrho\nu$. Then, we have

$$\begin{array}{c} \int_0^1\varrho^{\alpha-1}\mathcal{Q}_*(\varrho u+(1-\varrho)\nu,\ \gamma)\Omega((1-\varrho)u+\varrho\nu)d\varrho \\ +\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*((1-\varrho)u+\varrho\nu,\ \gamma)\Omega((1-\varrho)u+\varrho\nu)d\varrho \\ =\frac{1}{(\nu-u)^\alpha}\ \int_u^\nu(\sqsupseteq-u)^{\alpha-1}\mathcal{Q}_*(u-\nu-\sqsupseteq,\gamma)\Omega(\sqsupseteq)d\sqsupseteq \\ +\frac{1}{(\nu-u)^\alpha}\ \int_u^\nu(\sqsupseteq-u)^{\alpha-1}\mathcal{Q}_*(\sqsupseteq,\gamma)\Omega(\sqsupseteq)d\sqsupseteq \\ =\frac{1}{(\nu-u)^\alpha}\ \int_u^\nu(\nu-u)^{\alpha-1}\mathcal{Q}_*(\ddagger,\gamma)\Omega(u-\nu-\ddagger)d\ddagger \\ +\frac{1}{(\nu-u)^\alpha}\ \int_u^\nu(\ddagger-u)^{\alpha-1}\mathcal{Q}_*(\sqsupseteq,\gamma)\Omega(\ddagger)d\ddagger \\ =\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}_*\Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}_*\Omega(u)\right].\end{array} \tag{59}$$

Then, from (59), we have

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\mathcal{Q}_*\left(\frac{u+\nu}{2},\ \gamma\right)\left[\mathcal{I}_{u^+}^\alpha\ \Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \Omega(u)\right] \\ \leq\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\ \left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}_*\Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}_*\Omega(u)\right].\end{array} \tag{60}$$

Similarly, for $\mathcal{Q}^*(\ddagger,\gamma)$, we have

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\mathcal{Q}^*\left(\frac{u+\nu}{2},\ \gamma\right)\left[\mathcal{I}_{u^+}^\alpha\ \Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \Omega(u)\right] \\ \leq\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\ \left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}^*\Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}^*\Omega(u)\right].\end{array} \tag{61}$$

From (60) and (61), we have

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{Q}_*\left(\frac{u+\nu}{2},\ \gamma\right),\ \ \mathcal{Q}^*\left(\frac{u+\nu}{2},\ \gamma\right)\right]\left[\mathcal{I}_{u^+}^\alpha\ \Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \Omega(u)\right] \\ \leq_I\ \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\ \mathcal{I}_{u^+}^\alpha\ \mathcal{Q}_*\Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}_*\Omega(u),\ \ \mathcal{I}_{u^+}^\alpha\ \mathcal{Q}^*\Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}^*\Omega(u)\right],\end{array}$$

that is

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right)\left[\mathcal{I}_{u^+}^\alpha\ \Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \Omega(u)\right]\preccurlyeq\ \left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}\Omega(\nu)+\mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}\Omega(u)\right].$$

This completes the proof. □

Remark 5. *If* $\Omega(\ddagger)\alpha=1$ *then from Theorem 7 and Theorem 8, we get Theorem 3.*

Let $\alpha=1$. Then, we obtain the following HH–Fejér inequality for convex fuzzy-IVF, which is also new one.

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{1}{\int_u^\nu\mathcal{W}(\ddagger)d\ddagger}\ (FR)\int_u^\nu\mathcal{Q}(\ddagger)\mathcal{W}(\ddagger)d\ddagger\preccurlyeq\frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}.$$

If $\mathcal{Q}_*(\ddagger,\ \gamma)=\mathcal{Q}^*(\ddagger,\ \gamma)$ and $\Omega(\ddagger)=\alpha=1=\gamma$, then from Theorems 7 and 8, we get the classical HH-inequality (2).

If $\mathcal{Q}_*(\ddagger,\ \gamma)=\mathcal{Q}^*(\ddagger,\ \gamma)$ and $\alpha=1$, then from Theorems 7 and 8, we obtain the classical HH–Fejér inequality [50].

Example 2. *We consider the fuzzy-IVF* $\mathcal{Q} : [0,\ 2] \to \mathbb{F}_0$ *defined by,*

$$\mathcal{Q}(\ddagger)(\sigma) = \begin{cases} \frac{\sigma}{2-\sqrt{\ddagger}}, & \sigma \in [0,\ 2-\sqrt{\ddagger}], \\ \frac{2(2-\sqrt{\ddagger})-\sigma}{2-\sqrt{\ddagger}}, & \sigma \in (2-\sqrt{\ddagger},\ 2(2-\sqrt{\ddagger})], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\gamma \in [0,\ 1]$, we have $\mathcal{Q}_\gamma(\ddagger) = [\gamma(2-\sqrt{\ddagger}), (2-\gamma)(2-\sqrt{\ddagger})]$. Since end point functions $\mathcal{Q}_*(\ddagger, \gamma)$, $\mathcal{Q}^*(\ddagger, \gamma)$ are convex functions for each $\gamma \in [0,\ 1]$, then $\mathcal{Q}(\ddagger)$ is convex fuzzy-IVF. If

$$\Omega(\ddagger) = \begin{cases} \sqrt{\ddagger}, & \sigma \in [0, 1], \\ \sqrt{2-\ddagger}, & \sigma \in (1,\ 2], \end{cases}$$

then $\Omega(2-\ddagger) = \Omega(\ddagger) \geq 0$, for all $\ddagger \in [0,\ 2]$. Since $\mathcal{Q}_*(\ddagger, \gamma) = \gamma(2-\sqrt{\ddagger})$ and $\mathcal{Q}^*(\ddagger, \gamma) = (2-\gamma)(2-\sqrt{\ddagger})$. If $\alpha = \frac{1}{2}$, then we compute the following:

$$\begin{aligned} \frac{\mathcal{Q}_*(u,\ \gamma)+\mathcal{Q}_*(\nu,\ \gamma)}{2}\left[\mathcal{I}^\alpha_{u^+}\ \Omega(\nu) + \mathcal{I}^\alpha_{\nu^-}\ \Omega(u)\right] &= \sqrt{\pi}\ .\gamma\left(\frac{4-\sqrt{2}}{2}\right), \\ \frac{\mathcal{Q}^*(u,\ \gamma)+\mathcal{Q}^*(\nu,\ \gamma)}{2}\left[\mathcal{I}^\alpha_{u^+}\ \Omega(\nu) + \mathcal{I}^\alpha_{\nu^-}\ \Omega(u)\right] &= \sqrt{\pi}.(2-\gamma)\left(\frac{4-\sqrt{2}}{2}\right), \end{aligned} \tag{62}$$

and

$$\begin{aligned} \left[\mathcal{I}^\alpha_{u^+}\ \mathcal{Q}_*\Omega(\nu) \widetilde{+} \mathcal{I}^\alpha_{\nu^-}\ \mathcal{Q}_*\Omega(u)\right] &= \frac{1}{\sqrt{\pi}}\gamma\left(2\pi + \frac{4-8\sqrt{2}}{3}\right), \\ \left[\mathcal{I}^\alpha_{u^+}\ \mathcal{Q}^*\Omega(\nu) \widetilde{+} \mathcal{I}^\alpha_{\nu^-}\ \mathcal{Q}^*\Omega(u)\right] &= \frac{1}{\sqrt{\pi}}(2-\gamma)\left(2\pi + \frac{4-8\sqrt{2}}{3}\right). \end{aligned} \tag{63}$$

From (62) and (63), we have

$$\frac{1}{\sqrt{\pi}}\left[\gamma\left(2\pi + \frac{4-8\sqrt{2}}{3}\right),\ (2-\gamma)\left(2\pi + \frac{4-8\sqrt{2}}{3}\right)\right] \quad \leq_I \frac{\pi}{\sqrt{2}}\left[\gamma\left(\frac{4-\sqrt{2}}{2}\right),\ (2-\gamma)\left(\frac{4-\sqrt{2}}{2}\right)\right],$$

for each $\gamma \in [0,\ 1]$. Hence, Theorem 8 is verified.

For Theorem 8, we have

$$\begin{aligned} \mathcal{Q}_*\left(\tfrac{u+\nu}{2},\ \gamma\right)\left[\mathcal{I}^\alpha_{u^+}\ \Omega(\nu) + \mathcal{I}^\alpha_{\nu^-}\ \Omega(u)\right] &= \gamma\sqrt{\pi}, \\ \mathcal{Q}^*\left(\tfrac{u+\nu}{2},\ \gamma\right)\left[\mathcal{I}^\alpha_{u^+}\ \Omega(\nu) + \mathcal{I}^\alpha_{\nu^-}\ \Omega(u)\right] &= (2-\gamma)\sqrt{\pi}. \end{aligned} \tag{64}$$

From (63) and (64), we have

$$\sqrt{\pi}[\gamma,\ (2-\gamma)] \quad \leq_I \frac{1}{\sqrt{\pi}}\left[\gamma\left(2\pi + \frac{4-8\sqrt{2}}{3}\right),\ (2-\gamma)\left(2\pi + \frac{4-8\sqrt{2}}{3}\right)\right], \text{ for each } \gamma \in [0,\ 1].$$

From Theorems 9 and 10, we obtain some fuzzy-interval fractional integral inequalities related to fuzzy-interval fractional HH-inequalities.

Theorem 9. *Let* $\mathcal{Q}, \mathcal{P} : [u,\ \nu] \to \mathbb{F}_0$ *be two convex fuzzy-IVFs on* $[u,\ \nu]$, *whose* γ*-levels* $\mathcal{Q}_\gamma$, $\mathcal{P}_\gamma : [u,\ \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are defined by* $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma),\ \mathcal{Q}^*(\ddagger, \gamma)]$ *and* $\mathcal{P}_\gamma(\ddagger) = [\mathcal{P}_*(\ddagger, \gamma), \mathcal{P}^*(\ddagger, \gamma]$ *for all* $\ddagger \in [u,\ \nu]$ *and for all* $\gamma \in [0,\ 1]$. *If* $\mathcal{Q}$, $\mathcal{P}$ *and* $\mathcal{Q}\widetilde{\times}\mathcal{P} \in L([u,\ \nu], \mathbb{F}_0)$, *then*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\ \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(\nu) + \mathcal{I}^\alpha_{\nu^-}\mathcal{Q}(u)\widetilde{\times}\mathcal{P}(u)\right] \\ &\preccurlyeq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta(u,\nu)\widetilde{+}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla(u,\nu). \end{aligned}$$

where $\Delta(u,\nu) = \mathcal{Q}(u)\widetilde{\times}\mathcal{P}(u)\ \widetilde{+}\ \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(\nu)$, $\nabla(u,\nu) = \mathcal{Q}(u)\widetilde{\times}\mathcal{P}(\nu)\ \widetilde{+}\ \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(u)$, *and* $\Delta_\gamma(u,\nu) = [\Delta_*((u,\nu),\ \gamma),\ \Delta^*((u,\nu),\ \gamma)]$ *and* $\nabla_\gamma(u,\nu) = [\nabla_*((u,\nu),\ \gamma),\ \nabla^*((u,\nu),\ \gamma)]$.

Proof. Since $\mathcal{Q}$, $\mathcal{P}$ both are convex fuzzy-IVFs, then for each $\gamma \in [0,\ 1]$ we have

$$\mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \leq \varrho \mathcal{Q}_*(u,\gamma) + (1-\varrho)\mathcal{Q}_*(\nu,\ \gamma).$$

and

$$\mathcal{P}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \leq \varrho \mathcal{P}_*(u,\gamma) + (1-\varrho)\mathcal{P}_*(\nu,\ \gamma).$$

From the definition of convex fuzzy-IVFs it follows that $\widetilde{0} \preccurlyeq \mathcal{Q}(\ddagger)$ and $\widetilde{0} \preccurlyeq \mathcal{P}(\ddagger)$, so

$$\begin{array}{c} \mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \times \mathcal{P}(\varrho u + (1-\varrho)\nu,\ \gamma) \\ \leq \left(\ \varrho \mathcal{Q}_*(u,\gamma) + (1-\varrho)\mathcal{Q}_*(\nu,\ \gamma)\ \right)\left(\ \varrho \mathcal{P}_*(u,\gamma) + (1-\varrho)\mathcal{P}_*(\nu,\ \gamma)\ \right) \\ = \varrho^2 \mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(u,\gamma) + (1-\varrho)^2 \mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(\nu,\ \gamma) \\ +\varrho(1-\varrho)\mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(\nu,\ \gamma) + \varrho(1-\varrho)\mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(u,\gamma). \end{array} \tag{65}$$

Analogously, we have

$$\begin{array}{c} \mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma)\mathcal{P}((1-\varrho)u + \varrho\nu,\ \gamma) \\ \leq (1-\varrho)^2 \mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(u,\gamma) + \varrho^2 \mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(\nu,\ \gamma) \\ +\varrho(1-\varrho)\mathcal{Q}_*(u,\gamma) \times \mathcal{P}(\nu,\ \gamma) + \varrho(1-\varrho)\mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(u,\gamma)\ . \end{array} \tag{66}$$

Adding (65) and (66), we have

$$\begin{array}{c} \mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \times \mathcal{P}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \\ +\mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma) \times \mathcal{P}_*((1-\varrho)u + \varrho\nu,\ \gamma) \\ \leq \left[\varrho^2 + (1-\varrho)^2\right]\left[\mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(u,\gamma) + \mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(\nu,\ \gamma)\right] \\ +2\varrho(1-\varrho)\left[\mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(u,\gamma) + \mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(\nu,\ \gamma)\right]. \end{array} \tag{67}$$

Taking multiplication of (67) by $\varrho^{\alpha-1}$ and integrating the obtained result with respect to ϱ over (0,1), we have

$$\begin{array}{c} \int_0^1 \varrho^{\alpha-1}\mathcal{Q}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \times \mathcal{P}_*(\varrho u + (1-\varrho)\nu,\ \gamma) \\ +\varrho^{\alpha-1}\mathcal{Q}_*((1-\varrho)u + \varrho\nu,\ \gamma) \times \mathcal{P}_*((1-\varrho)u + \varrho\nu,\ \gamma)d\varrho \\ \leq \Delta_*((u,\nu),\ \gamma)\int_0^1 \varrho^{\alpha-1}\left[\varrho^2 + (1-\varrho)^2\right]d\varrho + 2\nabla_*((u,\nu),\ \gamma)\int_0^1 \varrho^{\alpha-1}\varrho(1-\varrho)d\varrho. \end{array}$$

It follows that,

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(\nu,\ \gamma) + \mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(u,\gamma)\right] \\ \leq \frac{2}{\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta_*((u,\nu),\ \gamma) + \frac{2}{\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla_*((u,\nu),\ \gamma). \end{array} \tag{68}$$

Similarly, for $\mathcal{Q}^*(\ddagger,\gamma)$, we have

$$\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}^*(\nu,\ \gamma) \times \mathcal{P}^*(\nu,\ \gamma) + \mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}^*(u,\gamma) \times \mathcal{P}^*(u,\gamma)\right]. \tag{69}$$

$$\leq\ \frac{2}{\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta^*((u,\nu),\ \gamma) + \frac{2}{\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla^*((u,\nu),\ \gamma).$$

From (68) and (69), we have

$$\begin{array}{c} \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}_*(\nu,\ \gamma) \times \mathcal{P}_*(\nu,\ \gamma) + \mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}_*(u,\gamma) \times \mathcal{P}_*(u,\gamma),\right. \\ \left.\mathcal{I}_{u^+}^\alpha\ \mathcal{Q}^*(\nu,\ \gamma) \times \mathcal{P}^*(\nu,\ \gamma) + \mathcal{I}_{\nu^-}^\alpha\ \mathcal{Q}^*(u,\gamma) \times \mathcal{P}^*(u,\ \gamma)\right] \end{array}$$

$$\leq_I \frac{2}{\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[\Delta_*((u,\nu),\ \gamma),\ \Delta^*((u,\nu),\ \gamma)]+\frac{2}{\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[\nabla_*((u,\nu),\ \gamma),\ \nabla^*((u,\nu),\ \gamma)].$$

That is

$$\frac{\Gamma(\alpha+1)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(\nu)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}(u)\widetilde{\times}\mathcal{P}(u)\right]\preccurlyeq\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta(u,\nu)\widetilde{+}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla(u,\nu).$$

and the theorem has been established. □

Example 3. *Let* $[u,\nu]=[0,2]$, $\alpha=\frac{1}{2}$, $\mathcal{Q}(\ddagger)=[\ddagger,2\ddagger]$, *and* $\mathcal{P}(\ddagger)=[\ddagger,3\ddagger]$.

$$\mathcal{Q}(\ddagger)(\theta)=\begin{cases}\frac{\theta}{\ddagger} & \theta\in[0,\ \ddagger]\\ \frac{2\ddagger-\theta}{\ddagger} & \theta\in(z,\ 2z]\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathcal{P}(\ddagger)(\theta)=\begin{cases}\frac{\theta}{2\ddagger} & \theta\in[0,\ 2\ddagger]\\ \frac{4\ddagger-\theta}{2\ddagger} & \theta\in(2z,\ 4z]\\ 0 & \text{otherwise.}\end{cases}$$

Then, for each $\gamma\in[0,\ 1]$, *we have* $\mathcal{Q}_\gamma(\ddagger)=[\gamma\ddagger,(2-\gamma)\ddagger]$ *and* $\mathcal{P}_\gamma(\ddagger)=[2\gamma\ddagger,2(2-\gamma)\ddagger]$. *Since left and right end point functions* $\mathcal{Q}_*(\ddagger,\gamma)=\gamma\ddagger$, $\mathcal{Q}^*(\ddagger,\ \gamma)=(2-\gamma)\ddagger$, $\mathcal{P}_*(\ddagger,\gamma)=2\gamma\ddagger$ *and* $\mathcal{P}^*(\ddagger,\ \gamma)=2(2-\gamma)\ddagger$ *are convex functions for each* $\gamma\in[0,\ 1]$, *then* $\mathcal{Q}(\ddagger)$ *and* $\mathcal{P}(\ddagger)$ *both are convex fuzzy-IVF. We clearly see that* $\mathcal{Q}(\ddagger)\widetilde{\times}\mathcal{P}(\ddagger)\in L([u,\ \nu],\mathbb{F}_0)$ *and*

$$\begin{aligned}&\tfrac{\Gamma(1+\alpha)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}_*(\nu)\times\mathcal{P}_*(\nu)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}_*(u)\times\mathcal{P}_*(u)\right]\\ &=\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(2-\ddagger)^{\frac{-1}{2}}\ \left(2\gamma^2\ddagger^2\right)d\ddagger+\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(\ddagger)^{\frac{-1}{2}}\ \left(2\gamma^2\ddagger^2\right)d\ddagger\approx 2.9332\gamma^2,\end{aligned}$$

$$\begin{aligned}&\tfrac{\Gamma(1+\alpha)}{2(\nu-u)^{\alpha}}\left[\mathcal{I}_{u^+}^{\alpha}\ \mathcal{Q}^*(\nu)\times\mathcal{P}^*(\nu)+\mathcal{I}_{\nu^-}^{\alpha}\ \mathcal{Q}^*(u)\times\mathcal{P}^*(u)\right]\\ &=\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(2-\ddagger)^{\frac{-1}{2}}\ .2(2-\gamma)^2\ddagger^2d\ddagger+\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int_0^2(\ddagger)^{\frac{-1}{2}}\ .2(2-\gamma)^2\ddagger^2d\ddagger\\ &\approx 2.9332(2-\gamma)^2.\end{aligned}$$

Note that

$$\begin{aligned}\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta_*(u,\nu)&=[\mathcal{Q}_*(u)\times\mathcal{P}_*(u)+\mathcal{Q}_*(\nu)\times\mathcal{P}_*(\nu)]=\tfrac{11}{30}.8\gamma^2,\\ \left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta^*(u,\nu)&=[\mathcal{Q}^*(u)\times\mathcal{P}^*(u)+\mathcal{Q}^*(\nu)\times\mathcal{P}^*(\nu)]=\tfrac{11}{30}.8(2-\gamma)^2,\\ \left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla_*(u,\nu)&=[\mathcal{Q}_*(u)\times\mathcal{P}_*(\nu)+\mathcal{Q}_*(\nu)\times\mathcal{P}_*(u)]=\tfrac{2}{15}(0),\\ \left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla_*(u,\nu)&=[\mathcal{Q}^*(u)\times\mathcal{P}^*(\nu)+\mathcal{Q}^*(\nu)\times\mathcal{P}^*(u)]=\tfrac{2}{15}(0).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta_\gamma((u,\nu),\ \gamma)+\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla_\gamma((u,\nu),\ \gamma)\\ &=\tfrac{11}{30}\left[8\gamma^2,8(2-\gamma)^2\right]+\tfrac{2}{15}[0,0]\approx\left[2.9332\gamma^2,2.9332(2-\gamma)^2\right].\end{aligned}$$

It follows that

$$\left[2.9332\gamma^2,2.9332(2-\gamma)^2\right]\leq_I\left[2.9332\gamma^2,2.9332(2-\gamma)^2\right],$$

and Theorem 9 has been demonstrated.

Theorem 10. *Let* $\mathcal{Q}, \mathcal{P} : [u, \nu] \to \mathbb{F}_0$ *be two convex fuzzy-IVFs, whose* γ*-levels define the family of IVFs* $\mathcal{Q}_\gamma$, $\mathcal{P} : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are given by* $\mathcal{Q}_\gamma(\ddagger) = [\mathcal{Q}_*(\ddagger, \gamma), \mathcal{Q}^*(\ddagger, \gamma)]$ *and* $\mathcal{P}_\gamma(\ddagger) = [\mathcal{P}_*(\ddagger, \gamma), \mathcal{P}^*(\ddagger, \gamma)]$ *for all* $\ddagger \in [u, \nu]$ *and for all* $\gamma \in [0, 1]$. *If* $\mathcal{Q}\widetilde{\times}\mathcal{P} \in L([u, \nu], \mathbb{F}_0)$, *then*

$$\begin{aligned}\frac{1}{\alpha}\mathcal{Q}\left(\frac{u+\nu}{2}\right)\widetilde{\times}\mathcal{P}\left(\frac{u+\nu}{2}\right) &\preccurlyeq \frac{\Gamma(\alpha+1)}{4(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(\nu) + \mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}(u)\widetilde{\times}\mathcal{P}(u)\right]\\ &+\frac{1}{2\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla(u,\nu) + \frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta(u,\nu).\end{aligned}$$

where $\Delta(u,\nu) = \mathcal{Q}(u)\widetilde{\times}\mathcal{P}(u) \mathbin{\widetilde{+}} \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(\nu)$, $\nabla(u,\nu) = \mathcal{Q}(u)\widetilde{\times}\mathcal{P}(\nu) \mathbin{\widetilde{+}} \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(u)$, *and* $\Delta_\gamma(u,\nu) = [\Delta_*((u,\nu), \gamma), \Delta^*((u,\nu), \gamma)]$ *and* $\nabla_\gamma(u,\nu) = [\nabla_*((u,\nu), \gamma), \nabla^*((u,\nu), \gamma)]$.

Proof. Consider $\mathcal{Q}, \mathcal{P} : [u, \nu] \to \mathbb{F}_0$ are convex fuzzy-IVFs. Then, by hypothesis, for each $\gamma \in [0, 1]$, we have

$$\begin{aligned}&\mathcal{Q}_*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}_*\left(\frac{u+\nu}{2}, \gamma\right)\\
&\le \frac{1}{4}\begin{bmatrix}\mathcal{Q}_*(\varrho u + (1-\varrho)\nu, \gamma) \times \mathcal{P}_*(\varrho u + (1-\varrho)\nu, \gamma)\\ +\mathcal{Q}_*(\varrho u + (1-\varrho)\nu, \gamma) \times \mathcal{P}_*((1-\varrho)u + \varrho\nu, \gamma)\end{bmatrix}\\
&+\frac{1}{4}\begin{bmatrix}\mathcal{Q}_*((1-\varrho)u + \varrho\nu, \gamma) \times \mathcal{P}_*(\varrho u + (1-\varrho)\nu, \gamma)\\ +\mathcal{Q}_*((1-\varrho)u + \varrho\nu, \gamma) \times \mathcal{P}_*((1-\varrho)u + \varrho\nu, \gamma)\end{bmatrix}\\
&\le \frac{1}{4}\begin{bmatrix}\mathcal{Q}_*(\varrho u + (1-\varrho)\nu, \gamma) \times \mathcal{P}_*(\varrho u + (1-\varrho)\nu, \gamma)\\ +\mathcal{Q}_*((1-\varrho)u + \varrho\nu, \gamma) \times \mathcal{P}_*((1-\varrho)u + \varrho\nu, \gamma)\end{bmatrix}\\
&+\frac{1}{4}\begin{bmatrix}(\varrho\mathcal{Q}_*(u, \gamma) + (1-\varrho)\mathcal{Q}_*(\nu, \gamma))\\ \times((1-\varrho)\mathcal{P}_*(u, \gamma) + \varrho\mathcal{P}_*(\nu, \gamma))\\ +((1-\varrho)\mathcal{Q}_*(u, \gamma) + \varrho\mathcal{Q}_*(\nu, \gamma))\\ \times(\varrho\mathcal{P}_*(u, \gamma) + (1-\varrho)\mathcal{P}_*(\nu, \gamma))\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}\mathcal{Q}_*(\varrho u + (1-\varrho)\nu, \gamma) \times \mathcal{P}_*(\varrho u + (1-\varrho)\nu, \gamma)\\ +\mathcal{Q}_*((1-\varrho)u + \varrho\nu, \gamma) \times \mathcal{P}_*((1-\varrho)u + \varrho\nu, \gamma)\end{bmatrix}\\
&+\frac{1}{4}\begin{bmatrix}\left\{\varrho^2 + (1-\varrho)^2\right\}\nabla_*((u,\nu), \gamma)\\ +\{\varrho(1-\varrho) + (1-\varrho)\varrho\}\Delta_*((u,\nu), \gamma)\end{bmatrix}.\end{aligned} \tag{70}$$

Taking multiplication of (67) with $\varrho^{\alpha-1}$ and integrating over $(0, 1)$, we get

$$\begin{aligned}&\frac{1}{\alpha}\mathcal{Q}_*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}_*\left(\frac{u+\nu}{2}, \gamma\right)\\
&\le \frac{1}{4(\nu-u)^\alpha}\left[\int_u^\nu(\nu-\ddagger)^{\alpha-1}\mathcal{Q}_*(\ddagger,\gamma)\times\mathcal{P}_*(\ddagger,\gamma)d\ddagger + \int_u^\nu(y-u)^{\alpha-1}\mathcal{Q}_*(y,\gamma)\times\mathcal{P}_*(y,\gamma)dy\right]\\
&+\frac{1}{2\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla_*((u,\nu), \gamma) + \frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta_*((u,\nu), \gamma)\\
&= \frac{\Gamma(\alpha+1)}{4(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}_*(\nu) \times \mathcal{P}_*(\nu) + \mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}_*(u) \times \mathcal{P}_*(u)\right]\\
&+\frac{1}{2\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla_*((u,\nu), \gamma) + \frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta_*((u,\nu), \gamma).\end{aligned} \tag{71}$$

Similarly, for $\mathcal{Q}^*(\ddagger, \gamma)$, we have

$$\begin{aligned}&\frac{1}{\alpha}\mathcal{Q}^*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}^*\left(\frac{u+\nu}{2}, \gamma\right)\\
&\le \frac{\Gamma(\alpha+1)}{4(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}^*(\nu) \times \mathcal{P}^*(\nu) + \mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}^*(u) \times \mathcal{P}^*(u)\right]\\
&+\frac{1}{2\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla^*((u,\nu), \gamma) + \frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta^*((u,\nu), \gamma).\end{aligned} \tag{72}$$

From (71) and (72), we have

$$\begin{aligned}&\frac{1}{\alpha}\left[\mathcal{Q}_*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}_*\left(\frac{u+\nu}{2}, \gamma\right), \mathcal{Q}^*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}^*\left(\frac{u+\nu}{2}, \gamma\right)\right]\\
&\le_I \frac{\Gamma(\alpha)}{4(\nu-u)^\alpha}\left[\mathcal{I}^\alpha_{u^+}\,\mathcal{Q}_*(\nu) \times \mathcal{P}_*(\nu) + \mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}_*(u) \times \mathcal{P}_*(u),\ \mathcal{I}^\alpha_{u^+}\,\mathcal{Q}^*(\nu) \times \mathcal{P}^*(\nu) + \mathcal{I}^\alpha_{\nu^-}\,\mathcal{Q}^*(u) \times \mathcal{P}(u)\right].\\
&+\frac{1}{2\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[\nabla_*((u,\nu), \gamma), \nabla^*((u,\nu), \gamma)] + \frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)[\nabla_*((u,\nu), \gamma), \nabla^*((u,\nu), \gamma)].\end{aligned}$$

That is

$$\frac{1}{\alpha}\,\mathcal{Q}\left(\frac{u+\nu}{2}\right)\widetilde{\times}\mathcal{P}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{\Gamma(\alpha+1)}{4(\nu-u)^{\alpha}}\left[\mathcal{I}^{\alpha}_{u^+}\ \mathcal{Q}(\nu)\widetilde{\times}\mathcal{P}(\nu) + \mathcal{I}^{\alpha}_{\nu^-}\ \mathcal{Q}(u)\widetilde{\times}\mathcal{P}(u)\right]$$
$$+\frac{1}{2\alpha}\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\nabla(u,\nu) + \frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\Delta(u,\nu).$$

Hence, the required result. □

5. Conclusions

In this study, we established the Jensen, Schur, HH-inequalities, and
–Fejér inequalities for convex fuzzy-IVF by means of fuzzy order relation and fuzzy-interval Riemann–Liouville fractional integrals. Our results generalize some well-known inequalities. In the future, we intend to use different classes of convex and nonconvex to study the interval-valued and fuzzy-IVFs and some applications in direction interval optimization and fuzzy interval optimization theory as future research directions.

In our final view, we believe that our work can be generalized to other models of fractional calculus such as Atangana–Baleanue and Prabhakar fractional operators with Mittag–Liffler functions in their kernels. We have left this consideration as an open problem for the researchers who are interested in this field. The interested researchers can proceed as done in references [34,35].

Author Contributions: Conceptualization, M.B.K. and M.A.N.; Data curation, P.O.M., Y.S.H.; Formal analysis, M.B.K. and M.A.N.; Funding acquisition, P.O.M.; Investigation, M.B.K., P.O.M., M.A.N. and Y.S.H.; Methodology, P.O.M. and Y.S.H.; Project administration, M.A.N. and Y.S.H.; Resources, M.A.N. and M.B.K.; Supervision, P.O.M. and M.A.N.; Validation, Y.S.H.; Visualization, Y.S.H.; Writing—original draft, M.B.K.; Writing—review & editing, P.O.M., M.B.K. and Y.S.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This work was supported by the Taif University Researchers Supporting Project Number (TURSP-2020/155), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sawano, Y.; Wadade, H. On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space. *J. Fourier Anal. Appl.* **2013**, *19*, 20–47. [CrossRef]
2. Gavrea, B.; Gavrea, I. On some Ostrowski type inequalities. *Gen. Math.* **2010**, *18*, 33–44.
3. Gunawan, H.; Eridani, E. Fractional integrals generalized Olsen inequalities. *Kyungpook Math. J.* **2009**, *49*, 31–39. [CrossRef]
4. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs, Victoria University: Melbourne, Australia, 2000.
5. Ciatti, P.; Cowling, M.G.; Ricci, F. Hardy and uncertainty inequalities on stratified Lie groups. *Adv. Math.* **2015**, *277*, 365–387. [CrossRef]
6. Mohammed, P.O. On new trapezoid type inequalities for h-convex functions via generalized fractional integral. *Turk. J. Anal. Number Theory* **2018**, *6*, 125–128. [CrossRef]
7. Hadamard, J. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *7*, 171–215.
8. Hermite, C. Sur deux limites d'une intégrale définie. *Mathesis* **1883**, *3*, 82–97.
9. Moore, R.E. *Interval Analysis*; Prentice Hall: Englewood Cliffs, NJ, USA, 1966.
10. Stefanini, L.; Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal.* **2010**, *71*, 1311–1328. [CrossRef]
11. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]

12. Zhao, D.F.; An, T.Q.; Ye, G.J.; Liu, W. New Jensen and Hermite-Hadamard type inequalities for h-convex interval-valued functions. *J. Inequal. Appl.* **2018**, *2018*, 1–14. [CrossRef]
13. Costa, T.M. Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst.* **2017**, *327*, 31–47. [CrossRef]
14. Costa, T.M.; Roman-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inform. Sci.* **2017**, *420*, 110–125. [CrossRef]
15. Román-Flores, H.; Chalco-Cano, Y.; Lodwick, W.A. Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* **2018**, *37*, 1306–1318. [CrossRef]
16. Roman-Flores, H.; Chalco-Cano, Y.; Silva, G.N. A note on Gronwall type inequality for interval-valued functions. In Proceedings of the IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013; Volume 35, pp. 1455–1458.
17. Chalco-Cano, Y.; Flores-Franulič, A.; Román-Flores, H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* **2012**, *31*, 457–472.
18. Chalco-Cano, Y.; Lodwick, W.A.; Condori-Equice, W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput.* **2015**, *19*, 3293–3300. [CrossRef]
19. Nikodem, K.; Snchez, J.L.; Snchez, L. Jensen and Hermite–Hadamard inequalities for strongly convex set-valued maps. *Math. Aterna* **2014**, *4*, 979–987.
20. Matkowski, J.; Nikodem, K. An integral Jensen inequality for convex multifunctions. *Results Math.* **1994**, *26*, 348–353. [CrossRef]
21. Zhang, D.; Guo, C.; Chen, D.; Wang, G. Jensen's inequalities for set-valued and fuzzy set-valued functions. *Fuzzy Sets Syst.* **2020**, *2020*, 1–27.
22. Abdeljawad, T.; Baleanu, D. Monotonicity results for fractional difference operators with discrete exponential kernels. *Adv. Differ. Equ.* **2017**, *2017*, 1–9. [CrossRef]
23. Agarwal, R.; Purohit, S.D.; Kritika. A mathematical fractional model with nonsingular kernel for thrombin receptor activation in calcium signaling. *Math. Methods Appl. Sci.* **2019**, *42*, 7160–7171. [CrossRef]
24. Agarwal, R.; Yadav, M.P.; Baleanu, D.; Purohit, S.D. Existence and uniqueness of miscible flow equation through porous media with a non-singular fractional derivative. *AIMS Math.* **2019**, *5*, 1062–1073. [CrossRef]
25. Kumar, D.; Singh, J.; Purohit, S.D.; Swroop, R. A hybrid analytical algorithm for nonlinear fractional wave-like equations. *Math. Model. Nat. Phenom.* **2019**, *14*, 1–13. [CrossRef]
26. Budak, H.; Tunç, T.; Sarikaya, M.Z. Fractional Hermite–Hadamard type inequalities for interval-valued functions. *Proc. Am. Math. Soc.* **2019**, *148*, 705–718. [CrossRef]
27. Adil Khan, M.; Begum, S.; Khurshid, Y.; Chu, Y.-M. Ostrowski type inequalities involving conformable fractional integrals. *J. Inequal. Appl.* **2018**, *2018*, 70. [CrossRef]
28. Adil Khan, M.; Chu, Y.-M.; Kashuri, A.; Liko, R.; Ali, G. Conformable fractional integrals versions of Hermite–Hadamard inequalities and their generalizations. *J. Funct. Spaces* **2018**, *2018*, 6928130. [CrossRef]
29. Adil Khan, M.; Khurshid, Y.; Du, T.-S.; Chu, Y.-M. Generalization of Hermite–Hadamard type inequalities via conformable fractional integrals. *J. Funct. Spaces* **2018**, *2018*, 5357463. [CrossRef]
30. Kaleva, O. Fuzzy differential equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317. [CrossRef]
31. Khan, M.B.; Noor, M.A.; Noor, K.I.; Ab Ghani, A.T.; Abdullah, L. Extended Perturbed Mixed Variational-Like Inequalities for Fuzzy Mappings. *J. Math.* **2021**, *2021*, 1–16.
32. Khan, M.B.; Noor, M.A.; Noor, K.I.; Chu, Y.-M. New Hermite-Hadamard Type Inequalities for (h1, h2)-Convex Fuzzy-Interval-Valued Functions. *Adv. Differ. Equ.* **2021**, *2021*, 6–20. [CrossRef]
33. Chu, Y.-M.; Adil Khan, M.; Ail, T.; Dragomir, S.S. Inequalities for α-fractional differentiable functions. *J. Inequal. Appl.* **2017**, *2017*, 93. [CrossRef]
34. Fernandez, A.; Mohammed, P. Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. *Math. Meth. Appl. Sci.* **2020**, *2020*, 1–18. [CrossRef]
35. Mohammed, P.O.; Abdeljawad, T. Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. *Adv. Differ. Equ.* **2020**, *2020*, 363. [CrossRef]
36. Mohammed, P.O.; Aydi, H.; Kashuri, A.; Hamed, Y.S.; Abualnaja, K.M. Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels. *Symmetry* **2021**, *13*, 550. [CrossRef]
37. Mohammed, P.O.; Abdeljawad, T.; Alqudah, M.A.; Jarad, F. New discrete inequalities of Hermite-Hadamard type for convex functions. *Adv. Differ. Equ.* **2021**, *2021*, 122. [CrossRef]
38. Kashuri, A.; Meftah, B.; Mohammed, P.O. Some weighted Simpson type inequalities for differentiable s-convex functions and their applications. *J. Fract. Calc. Nonlinear Syst.* **2021**, *1*, 75–94. [CrossRef]
39. Agarwal, R.P.; Baleanu, D.; Nieto, J.J.; Torres, D.F.M.; Zhou, Y. A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *J. Comput. Appl. Math.* **2018**, *339*, 3–29. [CrossRef]
40. Ahmad, M.Z.; Hasan, M.K.; De Baets, B. Analytical and numerical solutions of fuzzy differential equations. *Inf. Sci.* **2013**, *236*, 156–167. [CrossRef]
41. Bede, B.; Stefanini, L. Generalized differentiability of fuzzy-valued functions. *Fuzzy Sets Syst.* **2013**, *230*, 119–141. [CrossRef]
42. Diamond, P.; Kloeden, P.E. *Metric Spaces of Fuzzy Sets: Theory and Applications*; World Scientific: Singapore, 1994.
43. Goetschel, R., Jr.; Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43. [CrossRef]
44. Nanda, N.; Kar, K. Convex fuzzy mappings. *Fuzzy Sets Syst.* **1992**, *48*, 129–132. [CrossRef]

45. Noor, M.A. Fuzzy preinvex functions. *Fuzzy Sets Syst.* **1994**, *64*, 95–104. [CrossRef]
46. Stefanini, L. A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets Syst.* **2010**, *161*, 1564–1584. [CrossRef]
47. Kulish, U.; Miranker, W. *Computer Arithmetic in Theory and Practice*; Academic Press: New York, NY, USA, 2014.
48. Allahviranloo, T.; Salahshour, S.; Abbasbandy, S. Explicit solutions of fractional differential equations with uncertainty. *Soft Comput.* **2012**, *16*, 297–302. [CrossRef]
49. Bede, B.; Gal, S.G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets Syst.* **2005**, *151*, 581–599. [CrossRef]
50. Fejér, L. Uberdie Fourierreihen II. *Math. Nat. Anz. Ungar Akad Wiss.* **1906**, *24*, 369–390.