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New Method for Generating New Families of Distributions

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Abstract: This article presents a new method for generating distributions. This method combines two techniques—the transformed—transformer and alpha power transformation approaches—allowing for tremendous flexibility in the resulting distributions. The new approach is applied to introduce the alpha power Weibull—exponential distribution. The density of this distribution can take asymmetric and near-symmetric shapes. Various asymmetric shapes, such as decreasing, increasing, L-shaped, near-symmetrical, and right-skewed shapes, are observed for the related failure rate function, making it more tractable for many modeling applications. Some significant mathematical features of the suggested distribution are determined. Estimates of the unknown parameters of the proposed distribution are obtained using the maximum likelihood method. Furthermore, some numerical studies were carried out, in order to evaluate the estimation performance. Three practical datasets are considered to analyze the usefulness and flexibility of the introduced distribution. The proposed alpha power Weibull–exponential distribution can outperform other well-known distributions, showing its great adaptability in the context of real data analysis.

Keywords: alpha power transformation; moment; order statistics; Rényi entropy; T-X family

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1. Introduction

Several statistical distributions have been extensively applied to describe and predict existing phenomena in several disciplines, such as economics, engineering, finance, insurance, demography, biology, and environmental and medical sciences. However, in many of these areas, the data usually demonstrate complicated behavior and varied shapes, associated with various degrees of skewness and kurtosis. Thus, many of the existing standard distributions have some limitations when fitting these data, such that applying these classical distributions may not provide an acceptable fit. Therefore, many researchers have attempted to extend these existing classical distributions, in order to obtain greater flexibility in modeling data from different fields of study. Some examples of these modified distributions in the literature include the exponentiated Weibull distribution [1], the exponentiated exponential [2], and the exponentiated Gumbel distribution [3], which are modifications of the well-known Weibull, exponential, and Gumbel distributions, respectively. Extensions of the existing standard models are usually obtained by developing techniques to generate new families of distributions; that is, new generators for families of distributions are defined, in order to improve the goodness-of-fit of the distributions.

Various methods for generating new families of distributions have been studied recently. These include the beta-generating (beta-G) family [4], as well as the Kumaraswamy-generating (Kw-G) family [5]. The beta-G and Kw-G families were developed using distributions defined on the support [0, 1] as generators. The auhtors of [6] developed a general method, called the transformed–transformer (T–X) family of distributions, which enable the use of any continuous distribution as the generator. This family can be described as follows:

If r(t) and R(t) are the probability density function (PDF) and cumulative distribution function (CDF) of a random variable (RV) T, respectively, where $T \in [a, b]$ for $-\infty \leq a < b \leq \infty$, then the CDF and PDF of the T–X family are expressed, respectively, as

$$F(x) = \int_a^{W(G(x))} r(t)dt, \tag{1}$$

$$f(x) = \left\{\frac{d}{dx} W(G(x))\right\} r\{W(G(x))\}, \tag{2}$$

where W(G(x)) is a function of the CDF G(x) of any RV X that meets the conditions described in [6].

Many new distributions have been defined and studied on the basis of the T–X technique, using different forms of W(G(x)). For example, the study in [7] used $W(G(x)) = \log[G(x)/(1 - G(x))]$ to introduce the Gumbel–Weibull distribution, the study in [8] used $W(G(x)) = -\log(1 - G^{\alpha}(x))$ to introduce the exponentiated Weibull–exponential distribution, and the studies in [9,10] used $W(G(x)) = -\log[1 - G(x)]$ to introduce the Weibull–Pareto and gamma–normal distributions, respectively.

A new innovative technique, named alpha power transformation (APT), has recently been developed [11]. The APT can be considered a useful method to incorporate skewness into any distribution. The CDF and PDF of an APT family can be expressed as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)} - 1}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1 \\ F(x) & \text{if } \alpha = 1, \end{cases} \tag{3}$$

$$f_{APT}(x) = \begin{cases} \frac{\log \alpha}{\alpha - 1} f(x) \alpha^{F(x)} & \text{if } \alpha > 0, \alpha \neq 1 \\ f(x) & \text{if } \alpha = 1, \end{cases} \tag{4}$$

where $F(x)$ and $f(x)$ represent the CDF and PDF of any continuous distribution, respectively.

Various studies have used this technique to introduce new distributions. These include the studies of [11], where the APT method was applied to the exponential distribution, [12], who presented the APT–Weibull distribution, [13] introduced the APT–Pareto distribution, [14] proposed the APT–inverse Lindley distribution, [15], introduced the APT–log logistic distribution, and [16] have recently proposed the alpha power exponentiated Weibull–exponential distribution.

In this paper, we combine the T–X family and APT techniques by replacing $F(x)$ in (3) by (1) and $f(x)$ in (4) by (2) to generate new families of distributions. This newly established approach can add great flexibility, in terms of fitting real-life applications.

If we choose $W(G(x)) = -\log[1 - G(x)]$ in (1), then (according to (2)) the PDF for the T–X family is written as

$$f(x) = \left\{\frac{g(x)}{1 - g(x)}\right\} r\{-\log\left(1 - (G(x)\right\}. \tag{5}$$

If an RV T follows the Weibull distribution, with a and γ as the shape and scale parameters, respectively, then $r(t) = \frac{a}{\gamma}\left(\frac{t}{\gamma}\right)^{a-1} e^{-\left(\frac{t}{\gamma}\right)^a}$, $t \geq 0$. From (5), the PDF of the Weibull–G family can be obtained as

$$f_{WG}(x) = \frac{a}{\gamma} \frac{g(x)}{1 - G(x)} \left(\frac{-\log\left(1 - G(x)\right)}{\gamma}\right)^{a-1} exp\left[-\left(\frac{-\log\left(1 - G(x)\right)}{\gamma}\right)^a\right], \tag{6}$$

with corresponding CDF

$$F_{WG}(x) = 1 - exp\left[-\left(\frac{-\log(1-G(x))}{\gamma}\right)^{a}\right]. \tag{7}$$

The Weibull–exponential distribution (WED) was derived as a member of the Weibull–G family [9], where G is an exponential RV with density function $g(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. Therefore, the PDF and CDF of the WED may be presented as

$$f_{WE}(x) = \frac{a\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{a-1} exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right], \tag{8}$$

$$F_{WE}(x) = 1 - exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right]. \tag{9}$$

This paper introduces a new distribution, named the alpha power Weibull–exponential distribution (APWED), based on a novel technique for generating new distributions with more flexibility in modeling real data in a variety of fields. The approach is based on a combination of the T–X and APT approaches.

The rest of this article is arranged as follows: In Section 2, the APWED is introduced, along with some special cases of the APWED. Section 3 discusses some fundamental properties of APWED. In Section 4, we derive the maximum likelihood estimates (MLEs) of APWED parameters. In Section 5, we carry out a simulation study. In Section 6, the APWED is applied to three real applications, in order to analyze its usefulness. Finally, we report our conclusions in Section 7.

2. The Alpha Power Weibull—Exponential Distribution

The CDF of an RV X that has a four-parameter APWED can be described as follows

$$F(x) = \begin{cases} \frac{\alpha^{1-exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right]}-1}{\alpha-1} & \text{if } \alpha > 0, \alpha \neq 1 \\ 1 - exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right] & \text{if } \alpha = 1, \end{cases} \tag{10}$$

where α, λ,γ, $a > 0$ and $x \geq 0$.

The corresponding PDF is given as

$$f(x) = \begin{cases} \frac{\log\alpha}{\alpha-1}\frac{a\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{a-1} exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right]\alpha^{1-exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right]} & \text{if } \alpha > 0, \alpha \neq 1 \\ \frac{a\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{a-1} exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right] & \text{if } \alpha = 1. \end{cases} \tag{11}$$

The survival function, $S(x)$, of the APWED can be obtained as

$$S(x) = \begin{cases} \frac{\alpha}{\alpha-1}\left(1 - \alpha^{-exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right]}\right) & \text{if } \alpha > 0, \alpha \neq 1 \\ exp\left[-\left(\frac{\lambda x}{\gamma}\right)^{a}\right] & \text{if } \alpha = 1. \end{cases} \tag{12}$$

The hazard rate function, $h(x)$, of the APWED is expressed as

$$h(x) = \begin{cases} \frac{a\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{a-1} exp\left[-(\frac{\lambda x}{\gamma})^{a}\right]\frac{\alpha^{-exp[-(\frac{\lambda x}{\gamma})^{a}]}}{1-\alpha^{-exp\left[-(\frac{\lambda x}{\gamma})^{a}\right]}} log\alpha & \text{if } \alpha > 0, \alpha \neq 1 \\ \frac{a\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{a-1} & \text{if } \alpha = 1. \end{cases} \tag{13}$$

2.1. *Special Cases of the APWED*

1. The APWED reduces to the WED at $\alpha = 1$.
2. The APWED reduces to the Weibull distribution at $\alpha = \lambda = 1$.
3. The APWED reduces to the exponential distribution when $\alpha = \gamma = a = 1$.

Figures 1 and 2 illustrate the various shapes of the PDF and $h(x)$ of the APWED for some particular parameters.

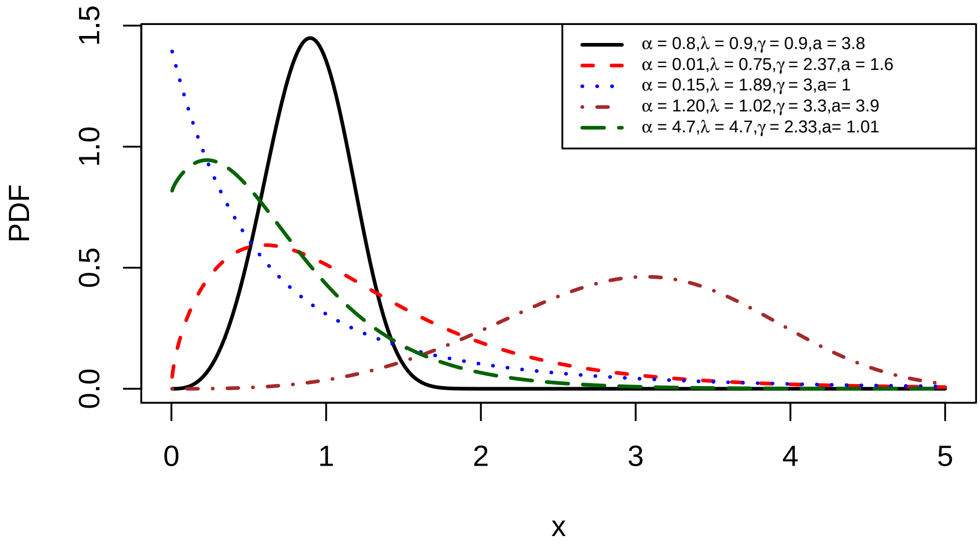

Figure 1. Plots of the APWED PDF, for some certain values.

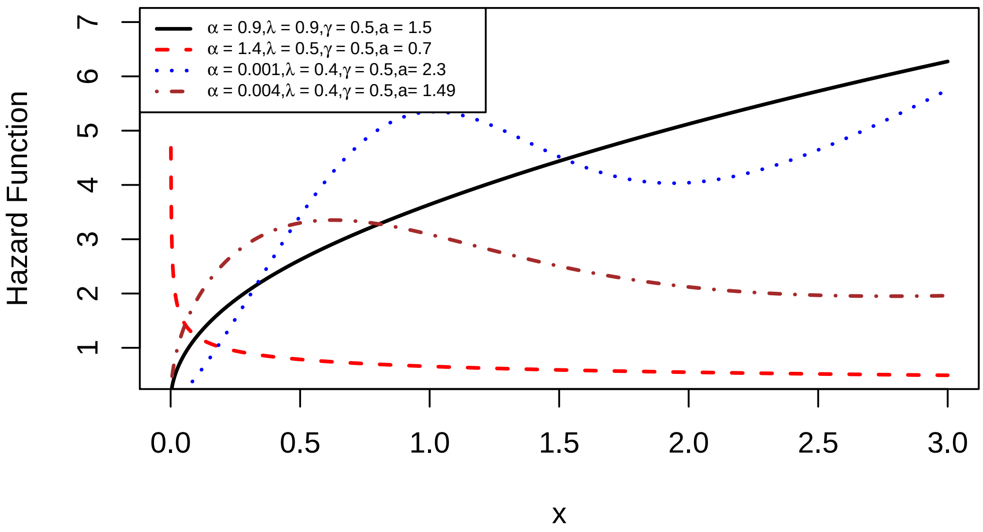

Figure 2. Plots of the APWED hazard function, for some certain values.

Figure 1 shows that the density of the APWED can take a number of forms, including symmetric, near symmetric, inverted J-shaped, right-skewed, and left-skewed shapes. Furthermore, Figure 2 shows that the hazard rate of the APWED features a wide variety of asymmetrical shapes. These results can be viewed as clear measures of the high degree of versatility of the APWED.

2.2. *Expansion for the PDF*

A simple expansion for the APWED PDF in (11) is provided, using the series representation, as follows

$$\alpha^{-z} = \sum_{q=0}^{\infty} (-1)^q \frac{(\log \alpha)^q}{q!} (z)^q. \tag{14}$$

Therefore, expanding $\alpha^{-exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right]}$ in (11) using (14), we have

$$f(x) = \frac{\alpha \log \alpha}{\alpha - 1} \frac{a\lambda}{\gamma} \left(\frac{\lambda x}{\gamma}\right)^{a-1} exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right] \sum_{q=0}^{\infty} (-1)^q \frac{(\log \alpha)^q}{q!} \left(exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right]\right)^q.$$

After some algebraic simplification, the PDF of APWED becomes

$$f(x) = \frac{\alpha}{\alpha - 1} \frac{a\lambda}{\gamma} \left(\frac{\lambda x}{\gamma}\right)^{a-1} \sum_{q=0}^{\infty} (-1)^q \frac{(\log \alpha)^{q+1}}{q!} exp\left[-(q+1)\left(\frac{\lambda x}{\gamma}\right)^a\right]. \tag{15}$$

3. Properties of the APWED

Some fundamental statistical properties of the APWED are presented in this section, as follows

3.1. *Quantile Function*

The pth quantile function $(0 < p < 1)$ of the APWED can be obtained as

$$x_p = \frac{\gamma}{\lambda} \left(-\log\left(1 - \frac{\log(p(\alpha - 1) + 1)}{\log \alpha}\right)\right)^{\frac{1}{a}}. \tag{16}$$

As a result, when setting $p = 0.5$, the median of the APWED can be obtained as

$$x_{0.50} = \frac{\gamma}{\lambda} \left(-\log\left(1 - \frac{\log(0.50(\alpha - 1) + 1)}{\log \alpha}\right)\right)^{\frac{1}{a}}. \tag{17}$$

For details, see [17].

3.2. *Moments*

If $X \sim APWED(\alpha, \lambda, \gamma, a)$, then the rth moment of X can be obtained as

$$\begin{aligned}\mu_r =& E(x^r) = \int_0^{\infty} x^r f(x) dx \\ =& \frac{\alpha}{\alpha - 1} \frac{a\lambda}{\gamma} \left(\frac{\lambda}{\gamma}\right)^{a-1} \sum_{q=0}^{\infty} (-1)^q \frac{(\log \alpha)^{q+1}}{q!} \int_0^{\infty} x^{r+a-1} exp\left[-(q+1)\left(\frac{\lambda x}{\gamma}\right)^a\right] dx.\end{aligned}$$

Substituting $y = (q+1)\left(\frac{\lambda x}{\gamma}\right)^a$, the rth moment of the APWED can be expressed as

$$\mu_r = E(x^r) = \frac{\alpha}{\alpha - 1} \sum_{q=0}^{\infty} (-1)^q \frac{(\log \alpha)^{q+1}}{q!} \left(\frac{\gamma}{\lambda}\right)^r \left(\frac{1}{q+1}\right)^{\frac{r}{a}+1} \Gamma\left(\frac{r}{a} + 1\right). \tag{18}$$

Therefore, the mean of the APWED is easily expressed as

$$\mu = E(x) = \frac{\alpha}{\alpha - 1} \sum_{q=0}^{\infty} (-1)^q \frac{(\log \alpha)^{q+1}}{q!} \left(\frac{\gamma}{\lambda}\right) \left(\frac{1}{q+1}\right)^{\frac{1}{a}+1} \Gamma\left(\frac{1}{a} + 1\right). \tag{19}$$

Additionally, from (18) and (19), the variance for the APWED can be given by

$$\begin{aligned}\sigma^2 =& E(x^2) - \mu^2\\ =& \left(\frac{\alpha}{\alpha-1}\sum_{q=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\left(\frac{\gamma}{\lambda}\right)^2\left(\frac{1}{q+1}\right)^{\frac{2}{a}+1}\Gamma\left(\frac{2}{a}+1\right)\right) - \mu^2.\end{aligned}$$

3.3. Moment Generating and Characteristic Functions

The moment generating function (MGF) of APWED is easily expressed as

$$M_x(t) = \frac{\alpha}{\alpha-1}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\frac{t^r}{r!}\left(\frac{\gamma}{\lambda}\right)^r\left(\frac{1}{q+1}\right)^{\frac{r}{a}+1}\Gamma\left(\frac{r}{a}+1\right). \tag{20}$$

Similarly, the characteristic function of APWED is as follows

$$\phi_x(t) = \frac{\alpha}{\alpha-1}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\frac{(it)^r}{r!}\left(\frac{\gamma}{\lambda}\right)^r\left(\frac{1}{q+1}\right)^{\frac{r}{a}+1}\Gamma\left(\frac{r}{a}+1\right). \tag{21}$$

3.4. Mean Residual Life and Mean Waiting Time

If X has the $S(x)$ in (12), then the mean residual life function of the APWED (say, $\mu(t)$) is obtained by

$$\mu(t) = \frac{1}{S(t)}\left(E(t) - \int_0^t xf(x)dx\right). \tag{22}$$

If we let $I = \int_0^t xf(x)dx$, from (15), we then have

$$I = \frac{\alpha}{\alpha-1}\sum_{q=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\left(\frac{\gamma}{\lambda}\right)\left(\frac{1}{q+1}\right)^{\frac{1}{a}+1}\gamma\left((q+1)\left(\frac{\lambda t}{\gamma}\right)^a, \frac{1}{a}+1\right), \tag{23}$$

where $\gamma(a,b) = \int_0^a x^{b-1}e^{-x}dx$ is the lower incomplete gamma function. Thus, by substituting (12), (19), and (23) into Equation (22), $\mu(t)$ can be derived as

$$\begin{aligned}\mu(t) =& \frac{1}{\left(1-\alpha^{-exp[-\left(\frac{\lambda t}{\gamma}\right)^a]}\right)}\left(\sum_{q=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\left(\frac{\gamma}{\lambda}\right)\left(\frac{1}{q+1}\right)^{\frac{1}{a}+1}\left[\Gamma\left(\frac{1}{a}+1\right)-\right.\right.\\ & \left.\left.\gamma\left((q+1)\left(\frac{\lambda t}{\gamma}\right)^a, \frac{1}{a}+1\right)\right]\right) - t.\end{aligned} \tag{24}$$

If X has the CDF in (10), then its mean waiting time, $(\bar{\mu}(t))$, can be obtained as follows

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t xf(x)dx, \tag{25}$$

where I is given by (23). Thus, by substituting (10) and (23) into Equation (25), $\bar{\mu}(t)$ can be derived as follows

$$\bar{\mu}(t) = t - \frac{\left(\alpha\sum_{q=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\left(\frac{\gamma}{\lambda}\right)\left(\frac{1}{q+1}\right)^{\frac{1}{a}+1}\gamma\left((q+1)\left(\frac{\lambda t}{\gamma}\right)^a, \frac{1}{a}+1\right)\right)}{\left(\alpha^{1-exp\left[-\left(\frac{\lambda t}{\gamma}\right)^a\right]}-1\right)}. \tag{26}$$

3.5. Rényi and Shannon Entropies

The measure of uncertainty of an RV X having the PDF in (11) is determined by its entropy. The Rényi entropy, $(RE_X(v))$, is defined as follows

$$RE_X(v) = \frac{1}{1-v}\log\left(\int_0^\infty f(x)^v dx\right); \quad v > 0, v \neq 1.$$

The Rényi entropy, $RE_X(v)$, of APWED is, therefore, given by

$$RE_x(v) = \frac{v}{1-v}\log\left(\frac{\alpha\log\alpha}{\alpha-1}\right) - \log\left(\frac{a\lambda}{\gamma}\right) + \frac{1}{1-v}\log\left(\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\frac{(\log\alpha)^q(v)^q}{(v+q)^{v-\frac{(v-1)}{a}}}\Gamma\left(v-\frac{(v-1)}{a}\right)\right).. \tag{27}$$

Furthermore, the Shannon entropy, (SE_X), of X is obtained as follows

$$SE_X = E[-\log f(x)] = -\int_0^\infty \log(f(x))f(x)dx.$$

Therefore, SE_X can be derived, after solving the integral, as

$$SE_X = \log\left(\frac{\alpha-1}{\alpha\log\alpha}\frac{\gamma}{a\lambda}\right) + \frac{\alpha}{\alpha-1}\sum_{q=0}^{\infty}(-1)^q\frac{(\log\alpha)^{q+1}}{q!}\left[\frac{a-1}{a(q+1)}(k+\log(q+1)) + \frac{1}{(q+1)^2} + \frac{(\log\alpha)}{(q+2)}\right], \tag{28}$$

where k is the Euler constant.

Table 1 presents the mean, variance, skewness, and Kurtosis of APWED for various values of α, λ, γ, and a. For fixed λ, γ, and a, the values of the mean and the variance of APWED increase with the increase inα. However, as the value of α increases, the skewness and Kurtosis values decrease. Furthermore, at fixed α and γ, the mean, variance, skewness, and Kurtosis decrease with increasing λ and a.

3.6. Order Statistics

Suppose $X_1, X_2, \ldots.., X_n$ are the observed values of a sample from the APWED and $X_{i:n}$ denotes the ith order statistic. The density of the order statistic, $X_{i:n}$, is defined in [18] as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}. \tag{29}$$

By substituting Equations (10) and (11) into (29), we have

$$f_{i:n}(x) = \frac{\alpha^{n-i}(-1)^{i-1}}{B(i,n-i+1)(\alpha-1)^{n-1}}f(x)\left(1-\alpha^{1-exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right]}\right)^{i-1} \left(1-\alpha^{-exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right]}\right)^{n-i}, \tag{30}$$

where $B(a,b)$ refers to the beta function. Applying the binomial series expansion, given as

$$(x-d)^n = \sum_{y=0}^{n}(-1)^y\binom{n}{y}x^{n-y}d^y, \tag{31}$$

then $f_{i:n}(x)$ can be expressed as

$$f_{i:n}(x) = \frac{a \log \alpha}{B(i, n-i+1)(\alpha-1)^n} \left(\frac{\lambda}{\gamma}\right)^a \sum_{y=0}^{i-1} \sum_{l=0}^{n-i} \binom{i-1}{y} \binom{n-i}{l} \frac{(-1)^{i+y+l-1}}{\alpha^{-(n+y-i+1)}} (x)^{a-1}$$
$$exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right] \alpha^{-(y+l+1)exp\left[-\left(\frac{\lambda x}{\gamma}\right)^a\right]}. \tag{32}$$

Table 1. The mean, variance, skewness, and kurtosis of the APWED for certain selected values of the parameters.

α	λ	γ	a	**Mean**	**Variance**	**Skewness**	**Kurtosis**
	0.5		1	3.3378	13.1798	5.3935	7.3632
	1		2	1.5962	0.7901	1.8985	2.1830
0.5	2	2	3	0.8298	0.1018	1.4105	1.5306
	3		5	0.5847	0.0199	1.1602	1.2061
	5.5		6	0.3246	0.0044	1.1148	1.1478
	0.5		1	3.8953	15.5766	4.6210	6.1819
	1		2	1.7452	0.8497	1.7866	2.0266
0.9	2	2	3	0.8833	0.1050	1.3672	1.4716
	3		5	0.6079	0.0197	1.1457	1.1867
	5.5		6	0.3354	0.0043	1.1049	1.1346
	0.5		1	4.4136	17.5837	4.0788	5.3732
	1		2	1.8783	0.8858	1.6952	1.9026
1.5	2	2	3	0.9303	0.1057	1.3291	1.4210
	3		5	0.6281	0.0192	1.1322	1.1689
	5.5		6	0.3448	0.0042	1.0954	1.1222
	0.5		1	4.7150	18.6460	3.8192	4.9917
	1		2	1.9534	0.8990	1.6473	1.8387
2	2	2	3	0.9566	0.1052	1.3082	1.3938
	3		5	0.6392	0.0188	1.1244	1.1588
	5.5		6	0.3499	0.0040	1.0899	1.1150
	0.5		1	4.9516	19.4253	3.6379	4.7276
	1		2	2.0114	0.9057	1.6121	1.7923
2.5	2	2	3	0.9767	0.1045	1.2925	1.3734
	3		5	0.6477	0.0184	1.1184	1.1511
	5.5		6	0.3538	0.0039	1.0856	1.1095
	0.5		1	5.8843	22.0241	3.0690	3.9099
	1		2	2.2316	0.9040	1.4921	1.6361
6	2	2	3	1.0519	0.0985	1.2367	1.3019
	3		5	0.6790	0.0165	1.0962	1.1227
	5.5		6	0.3683	0.0035	1.0696	1.0890

4. Maximum Likelihood Estimates

If $x_1, x_2, x_3, ..., x_n$ is a random sample of size n from the APWED, then the log-likelihood (ℓ) for the vector of parameters $\theta = (\alpha, \lambda, \gamma, a)$ can be written as

$$\ell(\alpha, \lambda, \gamma, a; x) = n \log\left(\frac{\log \alpha}{\alpha - 1}\right) + n \log\left(\frac{\alpha a \lambda}{\gamma}\right) + (a-1) \sum_{i=1}^{n} \log\left(\frac{\lambda x_i}{\gamma}\right) -$$
$$\sum_{i=1}^{n} \left(\frac{\lambda x_i}{\gamma}\right)^a - \log \alpha \sum_{i=1}^{n} e^{-\left(\frac{\lambda x_i}{\gamma}\right)^a}. \tag{33}$$

Therefore, the MLEs can be computed by differentiating (33) with respect to each parameter and solving the system of non-linear equations, as given in Appendix A, either numerically or by directly maximizing (33) using optimization techniques in any statistical program (i.e., R).

5. Simulation Study

Some simulations are detailed in this section, in order to evaluate the performance of the MLEs of the APWED parameters. The simulation was performed as follows

1. Different sample sizes (30, 50, 100, 150, 200, and 500) were drawn from the APWED under 1000 replicates;
2. Two different sets of parameters values were assigned
   Set 1 $(\alpha = 3, \lambda = 0.1, \gamma = 5, a = 9)$ and
   Set 2 $(\alpha = 0.5, \lambda = 2, \gamma = 1.5, a = 0.3)$.
3. For each sample size, the average estimates of the parameters and mean square error (MSE) were calculated using the "optim" function in R.

The simulation results are shown in Table 2. The MSE decreased as n increased and the MLEs of the parameters approached the true parameters.

Table 2. Simulation Study results for the APWED.

		Set 1		Set 2	
Sample Size	**Par.**	**MLE**	**MSE**	**MLE**	**MSE**
30	α	2.9422	2.1201	0.3806	4.1005
	λ	0.0755	0.0010	4.9901	23.3080
	γ	3.7871	2.4981	2.7951	11.2943
	a	9.5300	1.9036	0.3053	0.0034
50	α	2.9373	1.5402	1.2925	3.5021
	λ	0.0771	0.0010	4.4244	17.3286
	γ	3.8701	2.4263	2.6141	8.2774
	a	9.4225	1.1767	0.2989	0.0021
100	α	2.9961	0.8505	0.4001	0.0795
	λ	0.1036	0.0004	3.9395	12.6731
	γ	5.1892	0.9504	2.4014	5.3823
	a	9.0946	0.4844	0.2943	0.0014
150	α	3.0051	0.7086	0.3960	0.0687
	λ	0.1045	0.0004	3.6920	9.2768
	γ	5.2315	0.8841	2.3709	5.0723
	a	9.0983	0.3791	0.2943	0.0011
200	α	2.9866	0.5953	0.4138	0.0609
	λ	0.1037	0.0003	3.4956	8.3503
	γ	5.1882	0.7747	2.3642	4.2086
	a	9.0770	0.2935	0.2955	0.0009
500	α	2.9896	0.3320	0.4547	0.0524
	λ	0.1018	0.0002	2.9487	4.2672
	γ	5.0923	0.4310	2.1032	2.4497
	a	9.0334	0.1393	0.2969	0.0005

6. Applications

The APWED was applied to three sets of real data, and its fit was compared with those of some other well-known distributions.

6.1. Survival Time Data

The first dataset included the survival times of 55 patients with Head and Neck Cancer from [19], presented as

6.54, 10.42, 14.48, 16.10, 22.70, 3441.55, 4245.28 49.40 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

6.2. Failure Time Data

The second dataset was obtained from [20]. This data represented 84 failure times of aircraft windshields, described as

0.04, 1.87, 2.39, 3.44, 0.30, 1.88, 2.48, 3.47, 0.31, 1.81, 2.61, 3.48, 0.56, 1.91, 2.63, 3.58, 0.94, 1.91, 2.63, 3.51, 1.07, 1.91, 2.65, 3.61, 1.12, 1.98, 2.66, 3.78, 1.25, 2.01, 2.69, 3.92, 1.28, 2.038, 2.82,3, 4.04, 1.28, 2.09, 2.89, 4.12, 1.30, 2.09, 2.90, 4.17, 1.43, 2.01, 2.93, 4.24, 1.48, 2.14, 2.96, 4.26, 1.51, 2.15, 2.96, 4.28, 1.51, 2.19, 3.00, 4.31, 1.57, 2.19, 3.10, 4.38, 1.62, 2.22, 3.11, 4.45, 1.62, 2.22, 3.12, 4.49, 1.65, 2.23, 3.17, 4.57, 1.65, 2.30, 3.34, 4.60, 1.76, 2.32, 3.38, 4.66.

6.3. Strength Data

The third dataset was comprised of 63 values of the strengths of 1.5 cm glass fibers, obtained from the "UK National Physical Laboratory" and used in the work [21]. These values included

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

A comparison was made between the APWED and six other competing distributions with the following density functions:

1. Kumaraswamy–Weibull (Ku–W) by [22]

$$f(x)_{Ku-W} = \alpha\beta\frac{c}{\gamma}\left(\frac{x}{\gamma}\right)^{c-1} e^{-\left(\frac{x}{\gamma}\right)^c}\left[1-e^{-\left(\frac{x}{\beta}\right)^c}\right]^{\alpha-1}\left[1-\left(1-e^{-\left(\frac{x}{\beta}\right)^c}\right)^{\alpha}\right]^{\beta-1},$$

   where $x, \alpha, \beta, c, \gamma > 0$.
2. Exponentiated truncated inverse Weibull–inverse Weibull (ETIWIW) by [23]

$$f(x)_{ETIWIW} = ab\theta\mu^{\theta}x^{-\theta-1}e^{-\left(\frac{\mu}{x}\right)^{\theta}}\left[1-e^{-\left(\frac{\mu}{x}\right)^{\theta}}\right]^{-b-1} e^{1-\left[1-e^{-\left(\frac{\mu}{x}\right)^{\theta}}\right]^{-b}} \left[1-e^{1-\left[1-e^{-\left(\frac{\mu}{x}\right)^{\theta}}\right]^{-b}}\right]^{a-1},$$

   where $x, a, b, \theta, \mu > 0$.
3. Alpha power inverse Weibull (APIW) distribution by [24].

$$f_{APIW}(x) = \begin{cases} \frac{\log\alpha}{\alpha-1}\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{\beta}}\alpha^{e^{-\lambda x^{\beta}}} & \text{if } \alpha > 0, \alpha \neq 1 \\ \lambda\beta x^{-(\beta+1)}e^{-\lambda x^{\beta}} & \text{if } \alpha = 1, \end{cases}$$

   where $x, \lambda, \beta > 0$.
4. Alpha power exponential (APE) distribution by [11].

$$f_{APE}(x) = \begin{cases} \frac{\log\alpha}{\alpha-1}\lambda e^{-\lambda x}\alpha^{(1-e^{-\lambda x})} & \text{if } \alpha > 0, \alpha \neq 1 \\ \lambda e^{-\lambda x} & \text{if } \alpha = 1, \end{cases}$$

where $x, \lambda > 0$.

5. Weibull–Lomax (WL) distribution by [9].

$$f_{WL}(x) = \frac{a}{\gamma}\frac{\alpha}{\beta}\frac{\left[1+(\frac{x}{\beta})\right]^{-(\alpha+1)}}{\left[1+(\frac{x}{\beta})\right]^{-\alpha}}\left(\frac{-\log\left[1+(\frac{x}{\beta})\right]^{-\alpha}}{\gamma}\right)^{a-1}$$
$$exp\left[-\left(\frac{-\log\left[1+(\frac{x}{\beta})\right]^{-\alpha}}{\gamma}\right)^{a}\right],$$

where $x, \gamma, a, \alpha, \beta > 0$.

6. Exponential (E) distribution.

$$f(x)_E = \lambda e^{-\lambda x}, \quad x \geq 0,\ \lambda > 0.$$

For each dataset, the MLEs of the parameters, along with the standard errors (SEs) for the APWED and other competing distributions, are reported in Tables 3–5.

To examine the validity of the proposed model, in comparison with the other models, we considered the following goodness-of-fit (GOF) criteria: The negative $-\ell(\hat{\theta})$, Akaike's information Criterion (AIC), the Corrected Akaike Information Criterion (CAIC), the Hannan–Quinn Information Criterion (HQIC), Anderson–Darling (A), Cramer–von Mises (W), and Kolmogorov–Smirnov (K–S) statistics. The lower these values, the better the fit. Tables 6–8 present a comparison between the performance of the APWED and the other distributions for the three real data sets described above.

Tables 6–8 showed that the APWED have the lowest $-\ell(\hat{\theta})$, AIC, CAIC, HQIC, K-S, W, and A scores, indicating its superiority in fitting the three real data sets, as compared to the other distributions.

Table 3. MLEs (SEs in parentheses) for survival time data.

Distribution	Estimated Parameters			
APWED	0.0097	0.0021	2.2258	1.0213
($\hat{\alpha}$,$\hat{\lambda}$,$\hat{\gamma}$,$\hat{a}$)	(0.0134)	(0.0002)	(0.7427)	(0.0976)
Ku-W	0.9783	0.0848	0.3994	0.5239
($\hat{\alpha}$,$\hat{\beta}$,$\hat{c}$,$\hat{\gamma}$)	(0.4010)	(0.0119)	(0.0019)	(0.0109)
ETIWIW	28.0765	0.0612	3.5071	0.1089
($\hat{a}$,$\hat{b}$,$\hat{\theta}$,$\hat{\mu}$)	(7.3394)	(0.0025)	(0.0033)	(0.0033)
APIW	27.1117	19.1687	0.8835	-
($\hat{\alpha}$,$\hat{\lambda}$, $\hat{\beta}$)	(20.9187)	(6.9268)	(0.0772)	
APE	0.2756	0.0023	-	-
($\hat{\alpha}$,$\hat{\lambda}$)	(0.2311)	(0.0004)		
WL	5.1200	5.4303	0.7655	3.0234
($\hat{\gamma}$,$\hat{a}$,$\hat{\alpha}$,$\hat{\beta}$)	(6.5560)	(0.5031)	(0.8938)	(0.5809)
E	0.0027	-	-	-
($\hat{\lambda}$)	(0.0003)			

Table 4. MLEs (SEs in parentheses) for failure time data.

Distribution	Estimated Parameters			
APWED ($\hat{\alpha}$,$\hat{\lambda}$,$\hat{\gamma}$,$\hat{a}$)	25.6685 (40.1965)	2.6851 (199.6717)	4.9713 (369.7458)	1.5886 (0.3246)
Ku-W ($\hat{\alpha}$,$\hat{\beta}$,$\hat{c}$,$\hat{\gamma}$)	1.3649 (0.0095)	0.0812 (0.0089)	1.9011 (0.0040)	0.7197 (0.0045)
ETIWIW ($\hat{a}$,$\hat{b}$,$\hat{\theta}$,$\hat{\mu}$)	2.3249 (0.8317)	3.5489 (0.9676)	0.4390 (0.0619)	5.7867 (2.7310)
APIW ($\hat{\alpha}$,$\hat{\lambda}$, $\hat{\beta}$)	3.9152 (1.2192)	0.1948 (0.0235)	1.2339 (0.0695)	-
APE ($\hat{\alpha}$,$\hat{\lambda}$)	34.1428 (13.3994)	0.7591 (0.0592)	-	-
WL ($\hat{\gamma}$,$\hat{a}$,$\hat{\alpha}$,$\hat{\beta}$)	9.8170 (10.0656)	4.5320 (0.3975)	5.6555 (4.8035)	1.7000 (0.0898)
E ($\hat{\lambda}$)	0.3902 (0.0423)	-	-	-

Table 5. MLEs (SEs in parentheses) for strength data.

Distribution	Estimated Parameters			
APWED ($\hat{\alpha}$,$\hat{\lambda}$,$\hat{\gamma}$,$\hat{a}$)	10.8494 (12.8014)	10.4039 (1625.4758)	14.9854 (2341.4107)	4.4840 (0.7681)
Ku-W ($\hat{\alpha}$,$\hat{\beta}$,$\hat{c}$,$\hat{\gamma}$)	8.3440 (0.6533)	0.0923 (0.0117)	2.8288 (0.0024)	0.6285 (0.0024)
ETIWIW ($\hat{a}$,$\hat{b}$,$\hat{\theta}$,$\hat{\mu}$)	1.0768 (0.4988)	23.9679 (18.3488)	0.9791 (0.2967)	5.8196 (3.0173)
APIW ($\hat{\alpha}$,$\hat{\lambda}$, $\hat{\beta}$)	193.0604 (267.2453)	0.6365 (0.1822)	3.8769 (0.3096)	-
APE ($\hat{\alpha}$,$\hat{\lambda}$)	33.0483 (11.9269)	1.2993 (0.1080)	-	-
WL ($\hat{\gamma}$,$\hat{a}$,$\hat{\alpha}$,$\hat{\beta}$)	2.7539 (1.7144)	8.7108 (0.8782)	2.8070 (1.0768)	1.0327 (0.0211)
E ($\hat{\lambda}$)	0.6636 (0.0836)	-	-	-

Table 6. GOF criteria for survival time data.

Distribution	AIC	CAIC	HQIC	K-S	W	A	$-\ell(\hat{\theta})$
APWED	751.5332	752.3332	754.6382	0.1596	0.3468	1.8001	371.7666
Ku-W	795.5555	796.3555	798.6605	0.3165	0.3808	1.9230	393.7778
ETIWIW	753.0122	753.8122	756.1172	0.1940	0.3507	1.9772	372.5061
APIW	752.8410	753.3116	755.1698	0.1927	0.4975	2.5184	373.4205
APE	757.1062	757.3369	758.6587	0.1730	0.3904	2.1997	376.5531
WL	752.0366	752.8366	755.1416	0.1620	0.3766	1.8987	372.0183
E	764.0156	764.0911	764.7919	0.2763	0.9638	4.5921	381.0078

Table 7. GOF criteria for failure time data.

Distribution	AIC	CAIC	HQIC	K-S	W	A	$-\ell(\hat{\theta})$
APWED	267.4350	267.9350	271.3650	0.0652	0.0478	0.4565	129.7175
Ku-W	278.3159	278.8159	282.2459	0.1288	0.0987	0.9209	135.158
ETIWIW	303.1870	303.6870	307.1170	0.1626	0.5833	3.4447	147.5935
APIW	354.8962	355.1925	357.8437	0.2619	1.2682	7.6828	174.4481
APE	281.9912	282.1375	283.9562	0.1603	0.6024	3.4923	138.9956
WL	285.1424	285.6424	289.0724	0.0916	0.1492	1.2825	138.5712
E	331.9754	332.0236	332.9579	0.3032	2.3607	11.81	164.9877

Table 8. GOF criteria for strength data.

Distribution	AIC	CAIC	HQIC	K-S	W	A	$-\ell(\hat{\theta})$
APWE	34.9483	35.6379	38.3199	0.1225	0.1314	0.8554	13.4741
Ku-W	66.1026	66.7922	69.4742	0.2845	0.6943	3.7613	29.0513
ETIWIW	42.8534	43.5431	46.2251	0.1920	0.3909	1.9878	17.4267
APIW	81.7724	82.1791	84.3011	0.2163	0.8679	4.9894	37.8862
APE	124.7411	124.9411	126.4269	0.3388	2.4692	12.379	60.3706
WL	40.5321	41.2217	43.9037	0.1728	0.2859	1.5654	16.2660
E	179.6606	179.7262	180.5035	0.4180	3.8618	18.424	88.8303

Figures 3–5 display the observed density (histogram) for the three real datasets with the estimated PDF of the APWED, along with those of the competing distributions. These figures show that a closer fit to the observed density was provided by the APWED for all datasets.

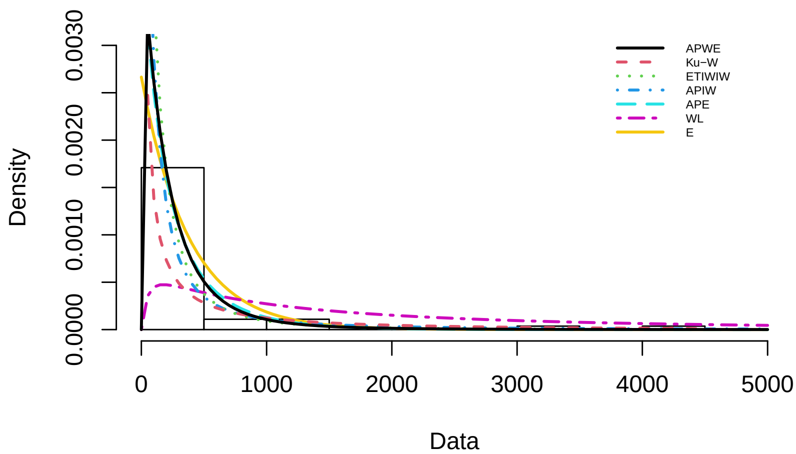

Figure 3. Fitted distributions for survival time data.

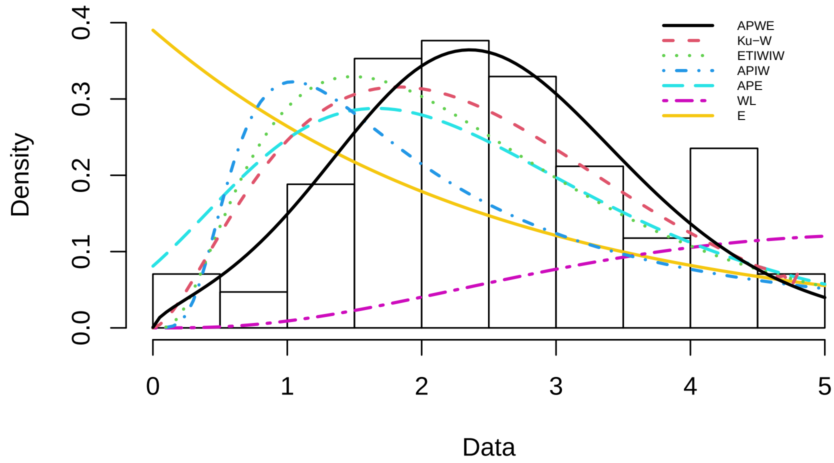

Figure 4. Fitted distributions for failure time data.

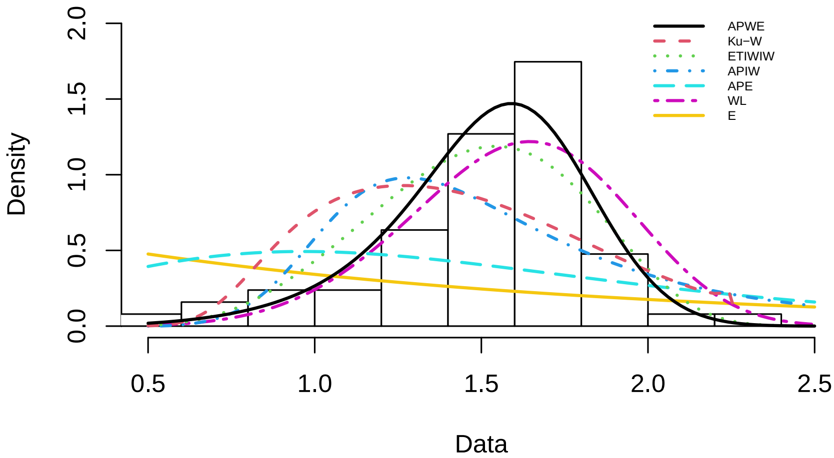

Figure 5. Fitted distributions for strength data.

7. Conclusions

In this article, we introduced a novel approach for generating distributions that provide great flexibility for modeling real data in a variety of fields. The method combines two well-known techniques: T–X and APT. The new distribution, APWED, is introduced as an application of this approach. The density and hazard rate functions of the proposed distribution have appealing shapes for implementing various data behaviors. For example, the APWED can be used to analyze both symmetrical and asymmetrical data shapes. Different fundamental statistical properties of the APWE were provided, such as the moments, quantile, median, mean residual life, order statistics, and entropy. MLEs were obtained for the unknown parameters of the APWED, and the conducted simulation studies showed the consistency and efficacy of the estimators. The application of the new distribution was also demonstrated by fitting three real datasets. The APWED generally provided the best fit, compared to the results of well-known competitive models. Thus, APWED is a promising distribution, which can be used to fit a variety of real-world data.

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Appendix A. The Partial Derivatives of (33), with Respect to Each Parameter

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \frac{n(\alpha - 1 - \alpha \log \alpha)}{\alpha(\alpha - 1)\log \alpha} - \frac{1}{\alpha}\sum_{i=1}^{n} e^{-\left(\frac{\lambda x_i}{\gamma}\right)^a},$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{na}{\lambda} - \frac{a}{\gamma}\sum_{i=1}^{n} x_i \left(\frac{\lambda x_i}{\gamma}\right)^{a-1}\left[1 - \log \alpha \, e^{-\left(\frac{\lambda x_i}{\gamma}\right)^a}\right],$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{-na}{\gamma} + \frac{a}{\gamma}\sum_{i=1}^{n} \left(\frac{\lambda x_i}{\gamma}\right)^{a}\left[1 - \log \alpha \, e^{-\left(\frac{\lambda x_i}{\gamma}\right)^a}\right],$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log\left(\frac{\lambda x_i}{\gamma}\right)\left[1 - \left(\frac{\lambda x_i}{\gamma}\right)^{a} + \log \alpha \left(\frac{\lambda x_i}{\gamma}\right)^{a} e^{-\left(\frac{\lambda x_i}{\gamma}\right)^a}\right] .$$

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