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The Properties of Eigenvalues and Eigenfunctions for Nonlocal Sturm–Liouville Problems

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Abstract: The present paper is concerned with the spectral theory of nonlocal Sturm–Liouville eigenvalue problems on a finite interval. The continuity, differentiability and comparison results of eigenvalues with respect to the nonlocal potentials are studied, and the oscillation properties of eigenfunctions are investigated. The comparison result of eigenvalues and the oscillation properties of eigenfunctions indicate that the spectral properties of nonlocal problems are very different from those of classical Sturm–Liouville problems. Some examples are given to explain this essential difference.

Keywords: nonlocal potential; Sturm–Liouville problem; eigenvalue

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1. Introduction

This paper is concerned with the spectral problems of the nonlocal Sturm–Liouville differential equation

$$-y''(x)+q(x)y(x)+a(x)y(1)=\lambda y(x),\ x\in(0,1) \tag{1}$$

associated to boundary value conditions

$$y(0)=0,\ y'(1)+\int_0^1 a(x)y(x)\mathrm{d}x=0, \tag{2}$$

where $q\in L^1([0,1],\mathbb{R})$ is the "local" potential and $a\in L^1([0,1],\mathbb{R})$ is called the "nonlocal" potential. The authors in [1] considered the inverse eigenvalue problems (1) where $q(x)\equiv 0$ with the boundary conditions (2).

Models similar to the nonlocal differential equation (1) have been used in the study of voltage-driven electrical systems, population dynamics, processes with conserved first integral and nonlocal problems with convective terms. Such nonlocal operators appear not only in quantum mechanics [2] but also in the theory of diffusion processes [1].

Other nonlocal problems result from the linear ordinary differential equation

$$-y''(x)+q(x)y(x)=\lambda w(x)y(x),\ x\in[0,1] \tag{3}$$

associated to nonlocal boundary conditions such as multi-point boundary value conditions and integral boundary conditions which involve values of the unknown function inside the interval (see [3–6] and the references cited therein).

The spectrum of the nonlocal problem associating (1) with boundary conditions has been studied by many authors. The authors in [7,8] investigate the behaviors of eigenvalues for the similar case with (1) with different nonlocal potential functions and Dirichlet boundary conditions, whereas, unlike the boundary conditions (2), the authors in [9,10] studied the inverse eigenvalue problems (1) where $q(x)\equiv 0$ with the boundary conditions

$$y(0)=y(1),\ y'(1)-y'(0)+\int_0^1 a(t)y(t)\mathrm{d}t=0 \text{ and} \tag{4}$$

$$y(0) = y(1) = 0,\ y'(x_0 + 0) - y'(x_0 - 0) - \int_0^1 a(t)y(t)\mathrm{d}t = 0, \tag{5}$$

respectively.

For the purpose of the clear statement of our methods, we only consider the case of $q = 0$, i.e., the nonlocal Sturm–Liouville differential equation

$$-y''(x) + a(x)y(1) = \lambda y(x),\ x \in (0,1) \tag{6}$$

with the boundary condition (2). For the case of $q \neq 0$ and other self-adjoint nonlocal boundary conditions, the corresponding results can be achieved in the similar way. In the present paper, we mainly focus on the continuity, differentiability and comparison properties of eigenvalues with respect to the nonlocal potentials and the oscillation of eigenfunctions of the nonlocal boundary value problems (6) and (2).

Following this section, some preliminary knowledge is listed in Section 2. In Section 3, we study the continuity and differentiability of eigenvalues with respect to the nonlocal potentials in Theorems 8 and 9, respectively. Theorem 10 of Section 4 gives the comparison result of eigenvalues. The oscillation properties of eigenfunctions are studied in Theorem 11 of Section 5, and we present some examples to explain the difference between the classical cases and the nonlocal cases.

2. Some Known Results of the Problem

In this section, some preliminary knowledge on the eigenvalues, eigenfunctions and the characteristic function of the nonlocal boundary value problems (6) and (2) is given (see [1]).

Lemma 1. *Assume that the real-valued function $a \in L^1[0,1]$. Then $\lambda = \rho^2 \in (-\infty, \infty)$ is an eigenvalue of (6) and (2) if and only if*

$$\chi(\lambda, a) := \cos\rho + \frac{2}{\rho}\int_0^1 a(t)\sin(\rho t)dt - \int_0^1\int_0^1 a(x)G(x,t;\rho)a(t)dxdt = 0, \tag{7}$$

where $\rho = \sqrt{\lambda}$ and the kernel $G(x,t;\rho)$ is defined as

$$G(x,t;\rho) = \frac{1}{\rho^2}\begin{cases}\sin\rho x\sin\rho(1-t),\ 0 \le x \le t \le 1,\\ \sin\rho t\sin\rho(1-x),\ 0 \le t \le x \le 1,\end{cases} \tag{8}$$

and $\chi(\lambda, a)$ is called the characteristic function of the nonlocal problems (6) and (2).

(i) (6) and (2) has a discrete spectrum consisting of real eigenvalues, say $\{\lambda_n(a)\}$, such that

$$\lambda_1(a) \in (-\infty, \pi^2],\ \lambda_n(a) \in [(n-1)^2\pi^2, n^2\pi^2],\ n \ge 2. \tag{9}$$

(ii) It holds that $(-1)^n\chi(n^2\pi^2, a) \ge 0$ for $n \ge 1$.

(iii) For an integer m, $m^2\pi^2$ is an eigenvalue of (6) and (2) if and only if

$$(-1)^m m\pi + \int_0^1 a(x)\sin(m\pi x)dx = 0. \tag{10}$$

Remark 1. *(i) For the case $\rho = 0$, we write $\frac{\sin\rho x}{\rho} = x$, and then the expression of G is given by*

$$G(x,t;0) = \begin{cases}x(1-t),\ 0 \le x \le t \le 1,\\ t(1-x),\ 0 \le t \le x \le 1.\end{cases} \tag{11}$$

(ii) *For the case* $\rho = i\tau$ *with* $\tau > 0$ *the functions* $\sin\rho$ *and* $\cos\rho$ *are defined, respectively, by*

$$\sin(i\tau) = \frac{e^{-\tau} - e^{\tau}}{2i},\ \cos(i\tau) = \frac{e^{-\tau} + e^{\tau}}{2}$$

due to the Euler's formula. Therefore, for all cases, the characteristic function $\chi(\lambda, a)$ *is real-valued for* $\lambda \in \mathbb{R}$.

For the multiplicity of eigenvalues, the following conclusions are proven in Theorem 2.2 of [1].

Lemma 2. *Let* $\lambda_n(a)$ *be the nth eigenvalues of* (6) *and* (2). *Then*

(i) *the multiplicity of* $\lambda_n(a)$ *does not exceed* 2.

(ii) *If* $\lambda \neq m^2\pi^2$ *is an eigenvalue, then it is simple, i.e.,* $\frac{\partial\chi}{\partial\lambda}(\lambda; a) \neq 0$. *The corresponding eigenfunction is given by*

$$\phi(x) = \sin\rho x/\rho - \int_0^1 G(x, t; \rho)a(t)dt.$$

(iii) *If* $n^2\pi^2$ *is a simple eigenvalue, then the corresponding eigenfunction is given by* $\phi(x) = \sin(n\pi x)$.

(iv) $n^2\pi^2$ *is a double eigenvalue if and only if*

$$\chi(n^2\pi^2; a) = \frac{\partial\chi}{\partial\lambda}(n^2\pi^2; a) = 0$$

and the corresponding linearly independent eigenfunctions are given by

$$\phi(x) = \sin(n\pi x),\ \psi(x) = \int_0^x \sin n\pi(x - t)a(t)dt.$$

Lemma 3. *Let* $\chi(\lambda, a)$ *be defined as in* (7). *If* $\chi(m^2\pi^2, a) = 0$ *or* (10) *holds, then*

$$\frac{\partial\chi}{\partial\lambda}(m^2\pi^2; a) = \int_0^1\int_0^1 a(x)\widehat{G}(x, t; m\pi)a(t)dxdt, \tag{12}$$

where $\rho = \sqrt{\lambda}$ *and*

$$\widehat{G}(x, t, \rho) = \frac{(-1)^{n-1}}{2\rho^3}\begin{cases}\sin\rho x\cos\rho t,\ x \leq t,\\ \sin\rho t\cos\rho x,\ x \geq t.\end{cases} \tag{13}$$

It is easy to see that $\chi(\lambda; a) \to \infty$ as $\lambda \to -\infty$ by the expression of $\chi(\lambda; a)$ in (7). As a result the following corollary is immediately a consequence of the above lemmas.

Lemma 4. *Let* $\chi(\lambda, a)$ *be defined as in* (7). *If* $\chi(\pi^2, a) = 0$, *then* $\frac{\partial\chi}{\partial\lambda}(\pi^2, a) < 0$ *(respectively* $\frac{\partial\chi}{\partial\lambda}(\pi^2, a) > 0$*) means that* π^2 *is the first (respectively second) and simple eigenvalue. If* $\chi(4\pi^2, a) = 0$ *and* $\frac{\partial\chi}{\partial\lambda}(4\pi^2, a) > 0$, *then* $4\pi^2$ *is the second and simple eigenvalue.*

3. Continuity and Differentiability of Eigenvalues and Eigenfunctions

In this section, we prove the continuity of eigenvalues on nonlocal potentials of (6) and (2). This kind of result for classical Sturm–Liouville problems has been given in [11]. Here, we will use different methods to prove such results for nonlocal Sturm–Liouville problems. we need the following lemma as a preparation of our main results in Section 3.

Lemma 5. *Let $F_n(\lambda)$, $F(\lambda)$, $n = 1, 2, \cdots$, be analytic functions and $F_n(\lambda) \to F(\lambda)$ as $n \to \infty$ uniform convergence on any bounded domain of $\mathbb{C}$. $F \neq$ constant. Let Σ_n and Σ be the zero sets of F_n and F, respectively. Set*

$$\Sigma_\infty = \{\lambda : \exists\, \lambda_n \in \Sigma_n, \text{ such that } \lambda_n \to \lambda,\ n \to +\infty\}.$$

Then $\Sigma = \Sigma_\infty$. Moreover, if there exists $\lambda_{n1} \neq \lambda_{n2} \in \Sigma_n$ such that $\lambda_{nj} \to \lambda_0$ as $n \to \infty$ for $j = 1, 2$, then $F'(\lambda_0) = 0$.

Proof. By the definition, for $\lambda_0 \in \Sigma_\infty$, there exists $\{\lambda_n,\ n = 1, 2, \cdots\}$ such that $\lambda_n \to \lambda_0$, $n \to \infty$. The analyticity implies that $F_n' \to F'$ uniform convergence on any bounded domain of $\mathbb{C}$, $\{F_n'\}$ is bounded on any bounded subset of $\mathbb{C}$, and hence

$$|F_n(\lambda_n) - F_n(\lambda_0)| = \left| \int_{\lambda_0}^{\lambda_n} F_n'(z)\mathrm{d}z \right| \leq M|\lambda_n - \lambda_0| \to 0.$$

As a result, $F_n(\lambda_n) = 0$ yields that $F_n(\lambda_0) \to 0$ or $F(\lambda_0) = 0$, i.e., $\lambda_0 \in \Sigma$.
Conversely, for $\lambda_0 \in \Sigma$, if $\lambda_0 \notin \Sigma_\infty$, then there exists $\varepsilon_0 > 0$ such that

$$B(\lambda_0, \varepsilon_0) \cap \Sigma = \{\lambda_0\},\ d(\lambda_0, \Sigma_n) \geq 2\varepsilon_0 \tag{14}$$

as $n \geq N$ for some $N > 0$, where $B(\lambda_0, \varepsilon_0) := \{\lambda : |\lambda - \lambda_0| \leq \varepsilon_0\}$. From (14), we have

$$F_n(\lambda) \neq 0,\ \forall\, \lambda \in B(\lambda_0, \varepsilon_0),\ n \geq N.$$

Therefore, $F_n^{-1}(\lambda)$ is analytic on $B(\lambda_0, \varepsilon_0)$ for $n > N$. By the Cauchy integral formula,

$$F_n^{-1}(\lambda_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(\lambda_0, \varepsilon_0)} \frac{F_n^{-1}(z)}{z - \lambda_0} \mathrm{d}z. \tag{15}$$

Note that $F(z) \neq 0$ on $\partial B(\lambda_0, \varepsilon_0)$ implies

$$|F^{-1}(z)| \leq \frac{1}{A},\ A := \min_{\partial B(\lambda_0, \varepsilon_0)} |F(z)| > 0.$$

Since $F_n \to F$ as $n \to \infty$ uniformly on $\partial B(\lambda_0, \varepsilon_0)$, we know there exists a sufficiently large number $N_1 (> N)$ such that

$$|F_n^{-1}(z)| \leq \frac{2}{A},\ z \in \partial B(\lambda_0, \varepsilon_0)$$

for $n \geq N_1$. This together with (15) gives $|F_n^{-1}(\lambda_0)| \leq 2/A$ for $n \geq N_1$. This clearly contradicts $F_n(\lambda_0) \to F(\lambda_0) = 0$ as $n \to \infty$.

For the proof of the second part, we note that for every $\lambda_0 \in \Sigma$, there exists $\varepsilon > 0$ such that $B(\lambda_0, \varepsilon) \cap \Sigma = \{\lambda_0\}$. Every set Σ_n is countable, and hence $\bigcup_1^\infty \Sigma_n$ is countable. Therefore, there exists $\varepsilon_0 \in (0, \varepsilon]$ such that $\partial B(\lambda_0, \varepsilon_0) \cap \Sigma_n = \emptyset$ for $n \geq 1$. As a result, there exists $\varepsilon_0 > 0$ such that

$$B(\lambda_0, \varepsilon_0) \cap \Sigma = \{\lambda_0\},\ \partial B(\lambda_0, \varepsilon_0) \cap \Sigma_n = \emptyset,\ n \geq 1.$$

Moreover there exists $N > 0$ such that for any $n \geq N$, $\{\lambda_{n1}, \lambda_{n2}\} \subset B(\lambda_0, \varepsilon_0)$. From the argument principle (see Chapter 4, Section 5 of [12]), we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B(\lambda_0, \varepsilon_0)} \frac{F_n'(z)}{F_n(z)} \mathrm{d}z \geq 2.$$

Let $n \to \infty$; we get

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B(\lambda_0, \varepsilon_0)} \frac{F'(z)}{F(z)} \mathrm{d}z \geq 2.$$

This implies that $F'(\lambda_0) = 0$. □

In order to prove the main result in this section, we also need to estimate the lower bound of the eigenvalues.

Lemma 6. *Let $\lambda_1(a)$ be the first eigenvalue of (6) and (2). Then*

$$\lambda_1(a) \geq \begin{cases} 2(1 - 2\|a\|), & \|a\| \leq 1/2, \\ -4\|a\|^2, & \|a\| > 1/2, \end{cases} \tag{16}$$

where $\|a\| = \int_0^1 |a(x)| dx$.

Proof. Let λ be an eigenvalue of (6) and (2) and ϕ be a real eigenfunction of λ. Since $\phi(0) = 0$,

$$\int_0^1 |\phi(x)|^2 \mathrm{d}x = \int_0^1 \left| \int_0^x \phi'(t) \mathrm{d}t \right|^2 \mathrm{d}x \leq \int_0^1 x \int_0^x |\phi'(t)|^2 \mathrm{d}t \mathrm{d}x \leq \frac{1}{2} \int_0^1 |\phi'(t)|^2 \mathrm{d}t,$$

and hence

$$\int_0^1 |\phi'(x)|^2 \mathrm{d}x \geq 2 \int_0^1 |\phi(x)|^2 \mathrm{d}x. \tag{17}$$

On the other hand, since $\phi(x) = \int_0^x \phi'(t) \mathrm{d}t$,

$$\begin{aligned} 2\left|\phi(1) \int_0^1 a(x)\phi(x) \mathrm{d}x\right| &\leq 2 \max\{|\phi(x)|^2 : \ x \in [0,1]\} \|a\| \\ \leq 2\left(\int_0^1 |\phi'(x)| \mathrm{d}x\right)^2 \|a\| &\leq 2 \int_0^1 |\phi'(x)|^2 \mathrm{d}x \|a\|. \end{aligned}$$

This inequality, together with

$$\lambda \int_0^1 |\phi|^2 = \int_0^1 \left(|\phi'|^2 + 2\phi(1) a \phi \right)$$

gives that

$$\lambda \int_0^1 |\phi|^2 \geq \int_0^1 |\phi'|^2 (1 - 2\|a\|) \geq 2(1 - 2\|a\|) \int_0^1 |\phi|^2$$

by (17) if $\|a\| \leq 1/2$, and hence the first part of (16) holds.

For the case $\|a\| > 1/2$, we set $\max\{|\phi(x)|^2 : \ x \in [0,1]\} = |\phi(x_0)|^2$ for some $x_0 \in (0,1]$. Since

$$|\phi(x_0)|^2 = 2\left| \int_0^{x_0} \phi'(x)\phi(x) \mathrm{d}x \right| \leq 2\|\phi'\| \|\phi\|, \ \|\phi\| = \left(\int_0^1 |\phi|^2 \right)^{1/2},$$

we get

$$\lambda \int_0^1 |\phi|^2 \geq \|\phi'\|^2 - 4\|a\| \|\phi\| \|\phi'\| = (\|\phi'\| - 2\|a\| \|\phi\|)^2 - 4\|a\|^2 \|\phi\|^2,$$

and hence the second part of (16) holds. □

Clearly, we have from (7) that

Lemma 7. *For fixed $a \in L^1[0,1]$, $\chi(\lambda, a)$ is an entire function of λ and for fixed λ, $\chi(\lambda, a)$ is continuous with respect to a in $L^1[0,1]$.*

Now we prove the continuity of eigenvalues on the nonlocal potentials in $L^1[0,1]$.

Theorem 8. *Let $a_n, a \in L^1[0,1]$ and $\lambda_n(a)$, $\phi_n(x;a)$ be the nth eigenpair of (6) and (2). If $a_n \to a$ as $n \to \infty$, then for any $k \geq 1$, $\lambda_k(a_n) \to \lambda_k(a)$ as $n \to \infty$.*

Proof. Set $F_n(\lambda) = \chi(\lambda, a_n)$ and $F(\lambda) = \chi(\lambda, a)$. Then $F_n, F \not\equiv 0$ and $F_n(\lambda) \to F(\lambda)$ uniformly on any bounded domain of $\mathbb{C}$ as $n \to \infty$ by Lemma 7.

Since $\{\|a_n\|\}$ is bounded, $\{\lambda_1(a_n)\}$ is bounded below by Lemma 6. This, together with $\lambda_1(a_n) \leq \pi^2$, yields that $\{\lambda_1(a_n) : n \geq 1\}$ is bounded. Applying Lemma 5, one can verify that $\lambda_1(a_n) \to \lambda_1 \in \Sigma_\infty = \Sigma$ as $n \to \infty$. Hence $\lambda_1 = \inf \Sigma_\infty = \inf \Sigma = \lambda_1(a)$.

For the case $k = 2$, the same method as above proves that $\lambda_2(a_n) \to \lambda_2 \in [\pi^2, 2\pi^2]$ as $n \to \infty$ by Lemma 5. $\lambda_2 = \lambda_1(a)$ only takes place as $\lambda_1(a) = \pi^2$ since $\lambda_1(a) \leq \pi^2$, or equivalently

$$\int_0^1 a(t) \sin \pi t \mathrm{d}t = \pi$$

by (iii) of Lemma 1 and $\chi'(\pi^2, a) = 0$ by Lemma 5. Hence π^2 is a double eigenvalue of the problem, and hence $\lambda_2(a) = \lambda_2$ by Lemma 2. If $\lambda_2 > \lambda_1(a)$, then

$$\lambda_2 = \inf \Sigma_\infty \setminus \{\lambda_1(a)\} = \inf \Sigma \setminus \{\lambda_1(a)\} = \lambda_2(a).$$

By mathematical deduction, the conclusion of Theorem 8 is true. □

The following theorem gives the differentiability of eigenvalues and eigenfunctions.

Theorem 9. *Let $\lambda_n(t)$ and $\phi_n(x;t)$ be the nth eigenpair of*

$$-y''(x) + a(x;t)y(1) = \lambda y(x),\ y(0) = 0,\ y'(1) + \int_0^1 a(s;t)y(s)ds = 0, \tag{18}$$

where $a(x;t) = a_0(x) + th(x)$ with $a_0, h \in L^1([0,1], \mathbb{R})$, $t \in \mathbb{C}$. Then

(i) $\lambda_n(t)$ is analytic and $\lambda_n'(t_0) = 0$ if $\lambda_n(t_0) = m^2\pi^2$, $m \geq 1$.
(ii) For every $t_0 \in \mathbb{C}$, there exists a neighborhood of t_0 and an eigenfunction of $\phi_n(\cdot;t)$ defined on the neighborhood such that $\phi_n(\cdot;t)$ is analytic at $t = t_0$.

Proof. (i) We only need to prove the conclusion holds at $t_0 = 0$ since we can replace a_0 by $a_0 + t_0 h$. If $\lambda_n(0)$ is simple, then

$$\frac{\partial \chi}{\partial \lambda}(\lambda_n(0), 0) \neq 0$$

by Lemma 2. Clearly, $\chi(\lambda, t)$ is analytic on (λ, t). Then by the existence of the implicit function for an analytic function, we know that there exists single-valued analytic function $\lambda(t)$ such that $\chi(\lambda(t), a_0 + th) \equiv 0$ on a neighborhood of $t = 0$, and hence $\lambda(t) = \lambda_n(t)$.

Assume that $\lambda_n(0)$ is not simple. Note that this takes place only for the case $\lambda_n(0) = m^2\pi^2$ by Lemma 2, and it holds that

$$(-1)^m m\pi + \int_0^1 a_0(x) \sin(m\pi x) \mathrm{d}x = 0 \tag{19}$$

by Lemma 1. If $\int_0^1 h(x) \sin(m\pi x) \mathrm{d}x = 0$. Then, for arbitrary $t \in \mathbb{C}$,

$$(-1)^m m\pi + \int_0^1 (a_0(x) + th(x)) \sin(m\pi x) \mathrm{d}x = 0,$$

which means that $\lambda_n(t) \equiv m^2\pi^2$, and hence the conclusion is clearly true.

Now suppose that

$$\int_0^1 h(x)\sin(m\pi x)\mathrm{d}x \neq 0. \tag{20}$$

Then, for any $0 < |t| \leq \delta$, $(-1)^m m\pi + \int_0^1 (a_0(x) + th(x))\sin(m\pi x)\mathrm{d}x \neq 0$. Thus, the above argument proves that $\lambda_n(t)$ is analytic on $0 < |t| \leq \delta$. This, together with the continuity of $\lambda_n(t)$ and Morera's theorem (see [13]), yields that $\lambda_n(t)$ is analytic on $|t| \leq \delta$.

If $\lambda_n(t_0) = m^2\pi^2$ with $m = n-1$ or $m = n$, then $t_0 \in \mathbb{R}$ and $\lambda_n(t) \in [(n-1)^2\pi^2, n^2\pi^2]$ on $\mathbb{R}$ means that $\lambda_n(t_0)$ is the minimum of $\lambda_n(t)$ on $\mathbb{R}$ for $m = n-1$ or is the maximum for $m = n$, and hence $\lambda_n'(t_0) = 0$.

(ii) Set $\rho_n(t) = \sqrt{\lambda_n(t)}$,

$$C_n(t) = \rho_n(t) - \int_0^1 (a_0(x) + th(x))\sin\rho_n(t)(1-x)\mathrm{d}x,$$
$$\phi(x,\rho,t) = \sin(\rho x)/\rho - \int_0^1 G(x,s,\rho)(a_0(s) + th(s))\mathrm{d}s.$$

Since $\phi(x;\rho_n(t),t)$ is an eigenfunction for those t such that $\rho_n(t) \neq m\pi$ by Lemma 2 and $\rho_n(t)$ is analytic, one sees that the conclusion of (ii) is true for those t such that $\rho_n(t) \neq m\pi$.

Suppose that $\rho_n(t_0) = m\pi$ $(m \geq 1)$ and take $t_0 = 0$ for the sake of simplicity. Then it follows from (19) that $C_n(0) = 0$. If $\int_0^1 h(x)\sin(m\pi x)\mathrm{d}x = 0$, then

$$(-1)^m m\pi + \int_0^1 (a_0(x) + th(x))\sin(m\pi x)\mathrm{d}x = 0$$

for all $t \in \mathbb{C}$, which means that $\lambda_n(t) \equiv m^2\pi^2$ and $\phi(x;m\pi,t) = \sin(m\pi x)/(m\pi)$, and hence the conclusion is clearly true.

Now suppose that $\int_0^1 h(x)\sin(m\pi x)\mathrm{d}x \neq 0$. Since

$$C_n'(t) = \rho_n'(t)\left[1 - \int_0^1 (a_0(x) + th(x))(1-x)\cos[\rho_n(t)(1-x)]\mathrm{d}x\right]$$
$$- \int_0^1 h(x)\sin\rho_n(t)(1-x)\mathrm{d}x,$$

$\rho_n(0) = m\pi$ $(m \geq 1)$ and $\rho_n'(0) = 0$, one sees that

$$C_n'(0) = (-1)^m \int_0^1 h(x)\sin(m\pi x)\mathrm{d}x \neq 0.$$

This, together with $C_n(0) = 0$, implies that there exists $\delta > 0$ such that $C_n(t) \neq 0$ for $0 < |t| < \delta$. Now define

$$\phi_n(x;t) = \begin{cases} \dfrac{\rho_n(t)}{C_n(t)}\phi(x,\rho_n;t), & 0 < |t| < \delta, \\ \dfrac{\sin(m\pi x)}{m\pi}, & t = 0. \end{cases} \tag{21}$$

From the above discussion, we know that $\phi_n(x;t)$ is analytic for $t \neq 0$, and calculation gives that

$$\phi(x;\rho_n(0),0) = 0, \ \frac{\partial(\rho_n\phi)}{\partial t}(x;\rho_n(0),0) = (-1)^m \int_0^1 h(s)\sin(m\pi s)\mathrm{d}s\frac{\sin(m\pi x)}{m\pi}.$$

This, combined with

$$C_n(0) = 0,\ C_n'(0) = (-1)^m \int_0^1 h(x)\sin(m\pi x)\mathrm{d}x$$

yields the continuity of $\phi_n(x;t)$ at $t = 0$:

$$\lim_{t\to 0}\phi_n(x;t) = \frac{\frac{\partial(\rho_n\phi)}{\partial t}(x;\rho_n(0),0)}{C_n'(0)} = \phi_n(x;0),$$

and hence $\phi_n(x;t)$ is analytic at $t = 0$ by Morera's theorem (see [13]). The proof of Theorem 9 is finished. □

Remark 2. *Note that $C_n(t) \neq 0$ implies that $\rho_n(t) \neq m\pi$. Then the above proof has proven that $\phi_n(x;t)$ is always analytic at $t \in \mathbb{C}$ and $C_n(t) \neq 0$.*

4. Comparison of Eigenvalues with Respect to Nonlocal Potentials

In this section, we derive the comparison result for eigenvalues with respect to the nonlocal potentials.

Theorem 10. *Let $a_1, a_2 \in L^1[0,1]$ and $\lambda_n(a_j)$ be the nth eigenvalue of (6) and (2) with a replaced by a_j, for $j = 1,2$. Let $G(x,t,\rho)$ be defined as in (8). Define*

$$\phi(x;\rho,a_1) = \sin\rho x/\rho - \int_0^1 G(x,t;\rho)a_1(t)dt.$$

Then

$$\lambda_n(a_2) - \lambda_n(a_1)\begin{cases} \geq 0\ (respectively \leq 0),\ n = odd, \\ \leq 0\ (respectively \geq 0),\ n = even \end{cases} \tag{22}$$

if and only if

$$2\int_0^1 h\phi_n - \int_D hG_nh \geq 0\ (respectively \leq 0), \tag{23}$$

where $h = a_2 - a_1$, $\phi_n(x) = \phi(x;\rho_n,a_1)$, $G_n(x,t) = G(x,t,\rho_n)$, $\rho_n = \sqrt{\lambda_n(a_1)}$ and

$$\int_D fG_ng = \int_0^1\int_0^1 f(x)G_n(x,t)g(t)dxdt,\ D = [0,1]\times[0,1].$$

Proof. Let $\chi(\lambda,a_1)$ and $\chi(\lambda,a_2)$ be defined as in (7) with a replaced by a_1 and a_2, respectively. Set $h = a_2 - a_1$. Define

$$F(\lambda,s) = \chi(\lambda,a_1 + sh),\ s \in [0,1].$$

It follows from the expression (7) of χ that F is continuously differentiable with respect to s. Clearly $F(\lambda,0) = \chi(\lambda,a_1)$, $F(\lambda,1) = \chi(\lambda,a_2)$ and

$$\frac{\partial F}{\partial s}(\lambda,s) = \frac{2}{\rho}\int_0^1 \sin\rho x h(x)\mathrm{d}x - \int_D hG(a_1 + sh) - \int_D (a_1 + sh)Gh.$$

Since the kernel G is symmetric, we have that

$$\begin{aligned}\frac{\partial F}{\partial s}(\lambda,s) &= \frac{2}{\rho}\int_0^1 \sin\rho x h(x)\mathrm{d}x - 2\int_D hG(a_1 + sh) \\ &= \frac{2}{\rho}\int_0^1 h(x)\left(\sin\rho x - \rho\int_0^1 G(x,t;\rho)a_1(t)\mathrm{d}t\right)\mathrm{d}x - 2s\int_D hGh \\ &= 2\int_0^1 h(x)\phi(x;\rho,a_1)\mathrm{d}x - 2s\int_D hGh.\end{aligned}$$

Since $F(\lambda,1)-F(\lambda,0)=\int_0^1 \frac{\partial F}{\partial s}(\lambda,s)\mathrm{d}s$, one has

$$\chi(\lambda,a_2)-\chi(\lambda,a_1)=2\int_0^1 h(x)\phi(x;\rho)\mathrm{d}x-\int_D hGh. \tag{24}$$

Take $\lambda=\lambda_n(a_1)$ in (24) and note that $\chi(\lambda_n(a_1),a_1)=0$, we get

$$\chi(\lambda_n(a_1),a_2)=2\int_0^1 h\phi_n-\int_D hG_nh.$$

There are three cases to be considered:
Case 1 $\lambda_n(a_1)\in\big((n-1)^2\pi^2,n^2\pi^2\big)$;
Case 2 $\lambda_n(a_1)=n^2\pi^2$;
Case 3 $\lambda_n(a_1)=(n-1)^2\pi^2$.
For **Case 1** now suppose that

$$2\int_0^1 h\phi_n-\int_D hG_nh\geq 0$$

in (23) of Theorem 10 holds. Clearly, $2\int_0^1 h\phi_n-\int_D hG_nh=0$ if and only if $\chi(\lambda_n(a_1),a_2)=0$, or equivalently $\lambda_n(a_2)=\lambda_n(a_1)$ by Lemma 1. Now suppose that $\chi(\lambda_n(a_1),a_2)>0$. We first suppose that n is an odd number. It follows from (ii) of Lemma 1 that

$$\chi((n-1)^2\pi^2,a_2)\geq 0,\ \chi(n^2\pi^2,a_2)\leq 0.$$

If $\chi(n^2\pi^2,a_2)<0$, then there exists one zero of $\chi(\lambda,a_2)$ on $(\lambda_n(a_1),n^2\pi^2)$, which must be the nth eigenvalue, say $\lambda_n(a_2)$, associated to a_2 by Lemma 1, and hence $\lambda_n(a_2)>\lambda_n(a_1)$. If $\chi(n^2\pi^2,a_2)=0$, then we can choose $\varepsilon_k\in L^1[0,1]$ such that

$$\chi(n^2\pi^2,a_2+\varepsilon_k)\neq 0\ (i.e.,<0),\ \varepsilon_k\to 0$$

as $k\to\infty$. Since $\chi(\lambda_n(a_1),a_2)>0$ and $\chi(\lambda,a)$ is continuous on $a\in L^1[0,1]$, we have $\chi(\lambda_n(a_1),a_2+\varepsilon_k)>0$ for sufficiently large k, say $k\geq 1$ for simplicity. Therefore, there exists a zero, say $\lambda_n(a_2+\varepsilon_k)$, of $\chi(\lambda,a_2+\varepsilon_k)$ on $(\lambda_n(a_1),n^2\pi^2)$. By Theorem 8, $\lambda_n(a_2+\varepsilon_k)\to\lambda_n(a_2)$ as $k\to\infty$, and hence $\lambda_n(a_2)\in[\lambda_n(a_1),n^2\pi^2]$. This, together with $\chi(\lambda_n(a_1),a_2)>0$, gives that $\lambda_n(a_2)>\lambda_n(a_1)$.

If n is even, then we have from (ii) of Lemma 1 that

$$\chi((n-1)^2\pi^2,a_2)\leq 0,\ \chi(n^2\pi^2,a_2)\geq 0.$$

Then, in the similar way as above, one can prove that $\chi(\lambda,a_2)$ has a zero on the interval $\big((n-1)^2\pi^2,\lambda_n(a_1)\big)$, and hence $\lambda_n(a_2)<\lambda_n(a_1)$.

Conversely, assume that $\lambda_n(a_2)-\lambda_n(a_1)\geq 0$. We claim that $\chi(\lambda_n(a_1),a_2)\geq 0$ if n is odd. Suppose on the contrary that $\chi(\lambda_n(a_1),a_2)<0$. Since $\chi((n-1)^2\pi^2,a_2)\geq 0$, then the similar argument as above proves that the nth eigenvalue associated to a_2 belongs to the interval $[(n-1)^2\pi^2,\lambda_n(a_1))$, which contradicts $\lambda_n(a_2)-\lambda_n(a_1)\geq 0$. Applying $\chi(\lambda_n(a_1),a_2)\geq 0$ and the fact $\chi(\lambda_n(a_1),a_1)=0$, one sees from (24) that the inequality in (23) is true. The same argument as above also yields that $\chi(\lambda_n(a_1),a_2)\leq 0$ if n is even. The proof for Case 1 is finished.

For **Case 2**, we note that if $\lambda_n(a_1)=n^2\pi^2$, then

$$G_n(x,t)=(-1)^{n-1}\frac{1}{n^2\pi^2}\sin(n\pi x)\sin(n\pi t)$$

and by (10)

$$(-1)^n n\pi+\int_0^1 a_1(t)\sin(n\pi t)\mathrm{d}t=0.$$

As a result

$$\phi_n(x) = \frac{1}{n\pi}\Big[\sin(n\pi x) - (-1)^{2n}\sin(n\pi x)\Big] = 0,$$

and hence

$$2\int_0^1 h\phi_n - \int_D hG_n h = (-1)^n\left(\int_0^1 h(x)\frac{\sin(n\pi x)}{n\pi}\mathrm{d}x\right)^2 \begin{cases} \le 0, & n = odd, \\ \ge 0, & n = even. \end{cases}$$

Since $\lambda_n(a_2) \in [(n-1)^2\pi^2, n^2\pi^2]$, we have $\lambda_n(a_2) \le \lambda_n(a_1) = n^2\pi^2$. Therefore, the conclusion is valid for this case. In a similar way, one can prove the conclusion is true for **Case 3**. This completes the proof of Theorem 10. □

Example 1.

(i) *If $a \in L^1[0,1]$ and $a(t) \le 0$, a.e. on $[0,1]$, then $\lambda_1(a) \le \pi^2/4$.*

(ii) *If $a(x) \equiv a > 0$, then $\lambda_1(a) \le \pi^2/4$ if $a \ge a_0$ and $\lambda_1(a) \ge \pi^2/4$ if $a \le a_0$, where $a_0 = \frac{\pi^2}{4-\pi}$.*

Proof. Take $a_1 \equiv 0$ and $a_2(x) = a(x)$. Then $\lambda_1(0) = \pi^2/4$,

$$\phi_1(x) = \frac{2}{\pi}\sin\Big(\frac{\pi x}{2}\Big),\ G\Big(x,t,\frac{\pi}{2}\Big) = \frac{4}{\pi^2}\begin{cases} \sin\dfrac{\pi x}{2}\cos\dfrac{\pi t}{2}, & x \le t, \\ \sin\dfrac{\pi t}{2}\cos\dfrac{\pi x}{2}, & x \ge t. \end{cases}$$

Clearly, ϕ_1 and G are both non-negative on $[0,1]$.

(i) As a result, (23) gives

$$2\int_0^1 a\phi_1 - \int_D aGa \le 0$$

if $a(x) \le 0$ on $[0,1]$. Then $\lambda_1(a) \le \pi^2/4$ by Theorem 10.

(ii) If $a(x) \equiv a > 0$, then it follows from

$$\int_0^1 \sin\frac{\pi x}{2}\mathrm{d}x = \frac{2}{\pi},\ \int_D G(x,t;\pi/2)\mathrm{d}x\mathrm{d}t = \frac{8(4-\pi)}{\pi^4}$$

that (23) is reduced to

$$\chi(\frac{\pi^2}{4},a) = \frac{8a}{\pi^2} - \frac{8(4-\pi)}{\pi^4}a^2 = \frac{8a}{\pi^2}\left(1 - \frac{4-\pi}{\pi^2}a\right)\begin{cases} \le 0, & a \ge \dfrac{\pi^2}{4-\pi}, \\ \ge 0, & a \le \dfrac{\pi^2}{4-\pi}, \end{cases}$$

and hence the conclusion of (ii) is true by Theorem 10.

□

Remark 3. *For the classical Sturm–Liouville problem $-y'' + qy = \lambda y$, $y(0) = 0 = y(1)$ with $q \in (L^1[0,1], \mathbb{R})$, it is well known that $\lambda_n(q_1) \ge \lambda_n(q_2)$ for all $n \ge 1$ if $q_1(x) \ge q_2(x)$ on $[0,1]$, where λ_n is the nth eigenvalue. However, from Theorem 10, we find that eigenvalues of nonlocal problem do not possess the monotonicity with respect to nonlocal potentials. This is an essential difference from the classical Sturm–Liouville problems.*

5. Oscillation of Eigenfunctions

In this section, we study the oscillation properties of eigenfunctions to the nonlocal problem. Let $\lambda_n(a)$ and $\phi_n(x;a)$ be the nth eigenpair of (6) and (2). We prove that

Theorem 11. *For given $a \in L^1[0,1]$, if $n \geq \int_0^1 |a|/\pi + 1$, then ϕ_n has exactly $n-1$ zeros on $(0,1)$.*

Proof. For $a(x) = 0$, *a.e.* $x \in [0,1]$, the problem (6) and (2) is reduced to the Laplace equation

$$-y''(x) = \lambda y(x),\ x \in (0,1),$$

with the boundary condition $y(0) = 0,\ y'(1) = 0$; hence, the conclusion clearly holds.

Let $a \in L^1[0,1]$ and

$$\text{mes}\{x \in [0,1] :\ a(x) \neq 0\} > 0. \tag{25}$$

If $\phi_n(1) = 0$, then $\lambda_n(a) = m^2\pi^2$ with $m = n-1$ or n by Lemma 2 and it holds that

$$(-1)^m m\pi + \int_0^1 a(t)\sin(m\pi x)\mathrm{d}x = 0$$

by Lemma 1. This together with (25) gives

$$\int_0^1 |a| > \int_0^1 |\sin(m\pi t)a(t)|\mathrm{d}t \geq \left|\int_0^1 \sin(m\pi t)a(t)\mathrm{d}t\right| = m\pi \geq (n-1)\pi$$

which means that $n < \int_0^1 |a|/\pi + 1$, a contradiction.

Now suppose that $\phi_n(1) \neq 0$. We claim that the zeros of ϕ_n are isolated and simple if $n \geq \int_0^1 |a|/\pi + 1$. Otherwise, we must have $\phi_n(x_0) = \phi_n'(x_0) = 0$ for some $x_0 \in [0,1)$. Therefore, by the constant variation formula,

$$\phi_n(x) = \frac{1}{\rho}\int_{x_0}^x \sin\rho(x-t)a(t)\mathrm{d}t\phi_n(1), \tag{26}$$

where $\rho = \sqrt{\lambda_n(a)} \in [(n-1)\pi, n\pi]$. Since $\phi_n(1) \neq 0$, we have from (26) that

$$\rho - \int_{x_0}^1 \sin\rho(1-t)a(t)\mathrm{d}t = 0,$$

and combining it with (25), we have

$$\int_0^1 |a| > \int_{x_0}^1 |\sin\rho(1-t)a(t)|\mathrm{d}t \geq \left|\int_{x_0}^1 \sin\rho(1-t)a(t)\mathrm{d}t\right| = \rho \geq (n-1)\pi$$

which means $n < \int_0^1 |a|/\pi + 1$, a contradiction as well.

Let $\lambda(t)$ and $\phi(x;t)$ be the nth eigenpair of the nonlocal problem

$$-y''(x) + ta(x)y(1) = \lambda y(x),\ y(0) = 0,\ y'(1) + t\int_0^1 a(s)y(s)\mathrm{d}s = 0 \tag{27}$$

and $\rho(t) = \sqrt{\lambda(t)},\ t \in [0,1]$. Then $\rho(t) \geq (n-1)\pi$. Note that if for some $t \in [0,1]$,

$$\rho(t) - t\int_0^1 \sin\rho(t)(1-x)a(x)\mathrm{d}x = 0,$$

and the same argument as above gives a contradiction. Therefore, we assume that

$$\rho(t) - t\int_0^1 \sin\rho(t)(1-x)a(x)\mathrm{d}x \neq 0$$

for all $t \in [0,1]$, and hence by Remark 2 in Section 3, we know that both $\lambda(t)$ and $\phi_n(\cdot,t)$ are continuously differentiable on $[0,1]$.

Set $N(t) = \sharp\phi(x;t)$ as the number of zeros of ϕ on $(0,1)$. Clearly, $N(0) = n-1$ since $\phi(x;0) = \sin[(n-1/2)\pi x]$. Since for all $t \in [0,1]$, the zeros of $\phi(x;t)$ are simple, it follows from the deformation lemma (see page 41 of [14]) that $N(0) = N(1)$. □

Note that the conclusion of Theorem 11 is similar to that of classical Sturm–Liouville problems for sufficient large n. However, generally speaking, the number of zeros of eigenfunctions to the nonlocal problem is very different from that of the classical one. The following examples explain the difference mentioned above.

Example 2. *In these examples, three cases will occur for the first eigenfunction.*

Case 1 *$\phi_1(x;a)$ has no zero. Take $a(x) \equiv 0$. Then, $\phi_1(x) = \sin(\pi x/2)$ has no zero on $(0,1)$.*

Caes 2 *$\phi_1(x;a)$ has exactly 1 zero. Take $a(x) = -90x^2 + 60x + 6$. Then $\phi_1(x) = -x^2 + \frac{2}{3}x$ has one zero on $(0,1)$.*

Caes 3 *$\phi_1(x;a)$ has two zeros. Take $a(x) = -\frac{889}{8}(8x^3 - 10x^2 + 3x) + 48x - 20$. Then $\phi_1(x) = x(x-\frac{1}{2})(x-\frac{3}{4})$ has two zeros on $(0,1)$.*

Example 3. *In these examples, three cases occur for the second eigenfunction.*

Case 1 *$\phi_2(x;a)$ has exactly 1 zero. Take $a(x) \equiv 0$. Then $\phi_2(x) = \sin(3\pi x/2)$ has one zero on $(0,1)$.*

Caes 2 *$\phi_2(x;a)$ has no zero. Take $a(x) = [\frac{10597}{1051}(x^2 - x_0 x) + 2]/(1-x_0)$, where $x_0 = \frac{101}{100}$. Then $\phi_2(x) = x^2 - x_0 x$ has no zero on $(0,1)$.*

Caes 3 *$\phi_2(x;a)$ has two zeros. Take $a(x) = \frac{15x(x-0.1)(x-b)+6x-0.2-2b}{0.9-0.9b}$. Then $\phi_2(x) = x(x-0.1)(x-b)$ has two zeros on $(0,1)$, where $0 < b < 1$ and such that $-\frac{1373}{300}b^2 + \frac{5869}{600}b - \frac{10919}{2100} = 0$.*

6. Conclusions

In this work, we obtain continuity, differentiability and comparison results of eigenvalues for nonlocal Sturm–Liouville problems on a finite interval, and the oscillation properties of eigenfunctions are researched. The above properties will play a key role in future research, and we will discuss extremal problems of L^1-norm for "local" potentials by using the above results.

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References

1. Albeverio, S.; Hryniv, R.; Nizhnik, L. Inverse spectral problems for nonlocal Sturm–Liouville operators. *Inverse Probl.* **2007**, *23*, 523–535. [CrossRef]
2. Albeverio, S.; Nizhnik, L.P. Schrödinger operators with nonlocal point interactions. *J. Math. Anal. Appl.* **2007**, *332*, 884–895. [CrossRef]
3. Pečiulyte, S.; Štikoniene, O.; Štikonas, A. Sturm–Liouville problem for stationary differential operator with nonlocal integral boundary condition. *Math. Model. Anal.* **2005**, *10*, 377–392. [CrossRef]
4. Štikonas, A.; Štikoniene, O. Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions. *Math. Model. Anal.* **2009**, *14*, 229–246. [CrossRef]
5. Bingelė, K.; Bankauskienė, A.; Štikonas, A. Spectrum curves for a discrete Sturm–Liouville problem with one integral boundary condition. *Nonlinear Anal. Model. Control* **2019**, *24*, 755–774. [CrossRef]

6. Bingelė, K.; Bankauskienė, A.; Štikonas, A. Investigation of spectrum curves for a Sturm–Liouville problem with two-point nonlocal boundary conditions. *Math. Model. Anal.* **2020**, *25*, 53–70. [CrossRef]
7. Freitas, P. A nonlocal Sturm–Liouville eigenvalue problem. *Proc. R. Soc. Edinb. A* **1994**, *124*, 169–188. [CrossRef]
8. Wu, X.; Niu, P.; Wei, G. An inverse eigenvalue problem for a nonlocal Sturm–Liouville operator. *J. Math. Anal. Appl.* **2021**, *494*, 124661. [CrossRef]
9. Nizhnik, L.P. Inverse eigenvalue problems for nonlocal Sturm–Liouville operators. *Methods Funct. Anal. Topol.* **2009**, *15*, 41–47.
10. Nizhnik, L.P. Inverse nonlocal Sturm–Liouville problem. *Inverse Probl.* **2010**, *26*, 125006. [CrossRef]
11. Zettl, A. *Sturm–Liouville Theory*; American Mathematical Society: Providence, RI, USA, 2005; Volume 121.
12. Ahlfors, L. *Complex Analysis*, 3rd ed.; McGraw Hill: New York, NY, USA, 1979.
13. Springer, G. On Morera's theorem. *Am. Math. Mon.* **1957**, *64*, 323–331. [CrossRef]
14. Jöschel, P.; Trubowitz, E. *Inverse Spectral Theory*; Academic Press: Boston, MA, USA, 1987.