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An Oscillation Criterion of Nonlinear Differential Equations with Advanced Term

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Abstract: The aim of the present paper is to provide oscillation conditions for fourth-order damped differential equations with advanced term. By using the Riccati technique, some new oscillation criteria, which ensure that every solution oscillates, are established. In fact, the obtained results extend, unify and correlate many of the existing results in the literature. Furthermore, two examples with specific parameter values are provided to confirm our results.

Keywords: oscillation; fourth-order; damped differential equations

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1. Introduction

Fourth-order advanced differential equations have an enormous potential for applications in engineering, medicine, aviation and physics, etc. The oscillation of differential equations contributes to many applications in science and technology and self-excited oscillation phenomena which occur in bridges and in the oscillatory muscle movement model; see [1,2].

In this article, we study some oscillation properties of the solutions to fourth-order advanced differential equations

$$\begin{cases} \left(j(z)\Phi_p[\zeta'''(z)]\right)' + a(z)f(\zeta'''(z)) + q(z)g(\zeta(c(z))) = 0, \\ j(z) > 0,\ j'(z) + a(z) \geq 0,\ z \geq z_0 > 0, \end{cases} \tag{1}$$

where $1 < p < \infty$ and p is an even number. Throughout this work, we assume that

L1: $\Phi_p[s] = |s|^{p-2}s$,

L2: $j,\ a,\ c,\ q \in C([z_0,\infty),[0,\infty))$, $q > 0$, $c(z) \geq z$, $\lim_{z\to\infty} c(z) = \infty$ and under condition

$$\int_{z_0}^{\infty} \left[\frac{1}{j(s)} \exp\left(-\int_{z_0}^{s} \frac{a(y)}{j(y)} dy\right)\right]^{1/p-1} ds < \infty. \tag{2}$$

L3: $f,\ g \in C(\mathbb{R},\mathbb{R})$ such that $f(w)/|w|^{p-2}w \geq k_f > 0$, $g(w)/|w|^{p-2}w \geq k_g > 0$, for $w \neq 0$, $k_f \geq 1, k_g$ are constants.

Definition 1. *When a solution of (1) has arbitrarily large zeros on $[z_\zeta,\infty)$, then it is termed oscillatory; otherwise, it is termed as non-oscillatory.*

Definition 2. *When all the solutions of the equation in (1) are oscillatory, the equation is called oscillatory.*

Definition 3. *If condition* $c(z) \geq z$ *hold, then the Equation* (1) *is called an advanced differential equation.*

Asymptotic behavior of solutions of differential equations have been the objective of many authors. Oscillation and asymptotic theory, however, has gained particular attention due to its widespread applications in clinical applications, earthquake structures, which involve symmetrical properties; see [3–8]. Nowadays, there has been an increasing interest in studying the asymptotic behavior of differential equations, see [9–21].

Park et al. [22] studied some oscillation properties of the solutions of differential equations with advanced term, by employing the comparison technique. Agarwal et al. [23,24] established the properties of oscillation for advanced equations using integral averaging technique.

Bazighifan et al. [25,26] considered fourth-order differential equations with advanced term

$$\begin{cases} \left(j(z)\left|\zeta^{(m-1)}(z)\right|^{p-2}\zeta^{(m-1)}(z)\right)' + \sum_{i=1}^{j} q_i(z) g(\zeta(\eta_i(z))) = 0, \\ j \geq 1,\ z \geq z_0 > 0, \end{cases}$$

where m is even and $p > 1$.

The authors in [4], obtained some oscillation conditions for equation

$$\begin{cases} \left(j(z)\Phi_p\left(\zeta^{(m-1)}(z)\right)\right)' + a(z)\Phi_p\left(\zeta^{(m-1)}(z)\right) + q(z)\Phi_p(\zeta(g(z))) = 0, \\ \Phi_p = |s|^{p-2}s,\ z \geq z_0 > 0, \end{cases}$$

where m is even and $p > 1$. Moreover, the authors used the comparison method to obtain oscillation conditions for this equation.

Other work has been done on similar equations with advanced term. Li et al. [3] investigated some oscillation criteria of equation

$$\begin{cases} \left(j(z)|\zeta'''(z)|^{p-2}\zeta'''(z)\right)' + \sum_{i=1}^{j} q_i(z)|\zeta(G_i(z))|^{p-2}\zeta(G_i(z)) = 0, \\ 1 < p < \infty,\ ,\ z \geq z_0 > 0. \end{cases}$$

The purpose of this paper is to establish new oscillation criteria for (1). The methods used in this paper simplify and extend some of the known results that are reported in the literature [4,26]. The authors in [4,26] used a comparison technique that differs from the one we used in this article. Moreover, the authors in [4,26] also studied the equation under the condition $\int_{z_0}^{\infty}\left[\frac{1}{j(s)}\exp\left(-\int_{z_0}^{s}\frac{a(y)}{j(y)}dy\right)\right]^{1/\alpha}ds = \infty$ which is different from our condition $\int_{z_0}^{\infty}\left[\frac{1}{j(s)}\exp\left(-\int_{z_0}^{s}\frac{a(y)}{j(y)}dy\right)\right]^{1/\alpha}ds < \infty.$

The organization of this article is as follows. After this introduction, in Section 2, we propose some preliminary lemmas that are used in the proof of our main theorems. In Section 3, we establish some oscillation criteria for (1) by Riccati technique; our results extend and correlate many of the existing results in the literature. Then, some examples are considered to check the efficiency of our main results.

2. Some Lemmas

These are some of the important Lemmas

Lemma 1 ([27])**.** *Let* $\alpha \geq 1$. *Then*

$$Dy - Cy^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{D^{\alpha+1}}{C^{\alpha}},$$

for all positive y, $C > 0$ and D be positive constant.

Lemma 2 ([28])**.** *Let $\zeta \in C^m([z_0,\infty),(0,\infty))$ and $\zeta^{(m-1)}(z)\zeta^{(m)}(z) \le 0$ such that m a positive integer, then*

$$\zeta(\theta z) \ge M t^{m-1} \zeta^{(m-1)}(z),$$

for all $\theta \in (0,1)$ there exists a constant $M > 0$.

Lemma 3 ([29])**.** *Let $\zeta \in C^m([z_0,\infty),(0,\infty))$ and*

$$\zeta^{(m-1)}(z)\zeta^{(m)}(z) \le 0,$$

then

$$\zeta(z) \ge \frac{\lambda}{(m-1)!} z^{m-1} \left| \zeta^{(m-1)}(z) \right|,$$

Lemma 4 ([30])**.** *Let ζ is a positive solution of (1). Then, there exist two possible cases*

$$(\mathbf{D}_1)\ \zeta(z) > 0,\ \zeta'(z) > 0,\ \zeta'''(z) > 0,\ \zeta^{(4)}(z) < 0;$$
$$(\mathbf{D}_2)\ \zeta(z) > 0,\ \zeta''(z) > 0,\ \zeta'''(z) < 0.$$

for $z \ge z_1$ where $z_1 \ge z_0$ is sufficiently large.

3. Oscillation Criteria

The motivation for this section is to create new oscillation criteria, established for (1) by the Riccati technique.

For ease of use, here are some notations.

$$\begin{aligned}
G(z_0,z) &= \exp\left(\int_{z_0}^{z} \frac{a(z)}{j(z)} du\right), \\
\xi(z) &= \int_z^\infty \frac{ds}{(j(s)G(z_0,s))^{\frac{1}{p-1}}}, \\
\phi(z) &= \frac{\delta'(z)}{\delta(z)} - \frac{k_f a(z)}{j(z)}, \\
\varphi(z) &= \frac{1}{G^{\frac{1}{p-1}}(z_0,z)} - \frac{\xi(z)a(z)j^{(2-p)/(p-1)}(z)}{p-1}
\end{aligned}$$

and

$$\tilde{\varphi}(z) = \frac{a(z)}{j(z)} + \frac{(p-1)^p \delta(z) \varphi^p(z) G(z_0,z)}{\xi(z) j^{\frac{1}{p-1}}(z)}.$$

Theorem 1. *Let (2) holds. Suppose that $\delta, \vartheta \in C^1([z_0,\infty),(0,\infty))$ and $M > 0$ and $k_g > 1$ are constants such that*

$$\limsup_{z\to\infty} \int_{z_0}^{z} \left(k_g \delta(s) q(s) - \left(\frac{2}{Ms^2}\right)^{p-1} \frac{j(s)\delta(s)(\phi(s))^p}{p^p} \right) ds = \infty. \tag{3}$$

If

$$\frac{\vartheta(z)}{\xi(z)(j(z)G(z_0,z))^{1/(p-1)}} + \vartheta'(z) \le 0 \tag{4}$$

and, for some $\mu \in (0,1)$,

$$\limsup_{z\to\infty} \int_{z_0}^{z} \left(k_g q(s) \left(\frac{\mu c^2(s)}{2} \frac{\vartheta(c(s))}{\vartheta(s)} \xi(s) \right)^{p-1} G(z_0,s) - \tilde{\varphi}(s) \right) ds = \infty, \tag{5}$$

then Equation (1) is oscillatory.

Proof. Let ζ be a nonoscillatory solution of Equation (1), then $\zeta \geq 0$. From Lemma 4, let case $(\mathbf{D}_1)$ hold. By Lemma 2, we obtain

$$\zeta'(z/2) \geq Mt^2\zeta'''(z). \tag{6}$$

Define

$$\psi(z) := \delta(z)\frac{j(z)(\zeta''')^{p-1}(z)}{\zeta^{p-1}(z/2)} \tag{7}$$

and

$$\begin{aligned}\psi'(z) &= \delta'(z)\frac{j(z)(\zeta''')^{p-1}(z)}{\zeta^{p-1}(z/2)} + \delta(z)\frac{\left(j(\zeta''')^{p-1}\right)'(z)}{\zeta^{p-1}(z/2)} \\ &\quad -(p-1)\delta(z)\frac{\zeta'(z/2)j(z)(\zeta''')^{p-1}(z)}{2\zeta^{p}(z/2)}.\end{aligned}$$

Using (7) and (6), we find

$$\begin{aligned}\psi'(z) &\leq \frac{\delta'(z)}{\delta(z)}\psi(z) + \delta(z)\frac{\left(j(\zeta''')^{p-1}\right)'(z)}{\zeta^{p-1}(z/2)} \\ &\quad -(p-1)Mt^2\delta(z)\frac{j(z)(\zeta''')^{p}(z)}{2\zeta^{p}(z/2)}.\end{aligned}$$

From (1), we obtain

$$\begin{aligned}\psi'(z) &\leq \frac{\delta'(z)}{\delta(z)}\psi(z) - k_f a(z)\frac{\psi(z)}{j(z)} \\ &\quad -k_g\delta(z)q(z)\frac{\zeta^{p-1}(c(z))}{\zeta^{p-1}(z/2)} - (p-1)Mt^2\frac{\psi^{\frac{p}{p-1}}(z)}{2(\delta(z)j(z))^{1/(p-1)}} \\ &\leq -k_g\delta(z)q(z) + \left(\frac{\delta'(z)}{\delta(z)} - k_f\frac{a(z)}{j(z)}\right)\psi(z) - (p-1)Mt^2\frac{\psi^{\frac{p}{p-1}}(z)}{2(\delta(z)j(z))^{1/(p-1)}}.\end{aligned}$$

So, we find

$$\psi'(z) \leq -k_g\delta(z)q(z) + \phi(z)\psi(z) - \frac{(p-1)Mt^2}{2(j(z)\delta(z))^{1/(p-1)}}\psi^{\frac{p}{p-1}}(z). \tag{8}$$

Using Lemma 1, we set

$$D = \phi(z), \quad C = (p-1)Mt^2/\left(2(j(z)\delta(z))^{1/(p-1)}\right) \quad \text{and} \quad y = \psi,$$

we have

$$\psi'(z) \leq -k_g\delta(z)q(z) + \left(\frac{2}{Mt^2}\right)^{p-1}\frac{j(z)\delta(z)(\phi(z))^p}{p^p}. \tag{9}$$

Integrating from z_1 to z, we obtain

$$\int_{z_1}^{z}\left(k_g\delta(s)q(s) - \left(\frac{2}{Ms^2}\right)^{p-1}\frac{j(s)\delta(s)(\phi(s))^p}{p^p}\right)ds \leq \psi(z_1),$$

which contradicts (3).

For case $(\mathbf{D}_2)$. Since

$$\begin{aligned}
\left(-j(z)(-\zeta'''(z))^{p-1}G(z_0,z)\right)' &= \left(-j(z)(-\zeta'''(z))^{p-1}\right)'G(z_0,z) \\
&\quad +\left(-j(z)(-\zeta'''(z))^{p-1}\right)G(z_0,z)\frac{a(z)}{j(z)} \\
&= (-1)^p(-a(z)f(\zeta'''(z)) - q(z)g(\zeta(c(z))))G(z_0,z) \\
&\quad -a(z)(-\zeta'''(z))^{p-1}G(z_0,z) \\
&\leq (-1)^p\left(-k_f a(z)(\zeta'''(z))^{p-1} - k_g q(z)\zeta^{p-1}(c(z))\right)G(z_0,z) \\
&\quad -a(z)(-\zeta'''(z))^{p-1}G(z_0,z) \\
&= \left(-a(z)(-\zeta'''(z))^{p-1}(1-k_f) + k_g q(z)\left(-\zeta^{p-1}(c(z))\right)\right)G(z_0,z) \\
&= (-1)^{p-1}\left(-a(z)(\zeta'''(z))^{p-1}(1-k_f) + k_g q(z)\left(\zeta^{p-1}(c(z))\right)\right)G(z_0,z) \\
&\leq -k_g q(z)\zeta^{p-1}(c(z))G(z_0,z) < 0,
\end{aligned}$$

we deduce that $-j(z)(-\zeta'''(z))^{p-1}G(z_0,z)$ is decreasing. Thus, for $s \geq z \geq z_1$

$$(j(s)G(z_0,s))^{1/(p-1)}\zeta'''(s) \leq (j(z)G(z_0,z))^{1/(p-1)}\zeta'''(z). \tag{10}$$

Dividing both sides of (10) by $(j(s)G(z_0,s))^{1/(p-1)}$ and integrating from z to h, we get

$$\int_z^h \zeta'''(z)dt \leq (j(z)G(z_0,z))^{1/(p-1)}\zeta'''(z)\int_z^h \frac{ds}{(j(s)G(z_0,s))^{1/(p-1)}}.$$

Easily, we find that

$$\zeta''(h) - \zeta''(z) \leq (j(z)G(z_0,z))^{1/(p-1)}\zeta'''(z)\int_z^h \frac{ds}{(j(s)G(z_0,s))^{1/(p-1)}}.$$

Letting $h \to \infty$, we find that

$$-\zeta''(z) \leq (j(z)G(z_0,z))^{1/(p-1)}\zeta'''(z)\int_z^\infty \frac{ds}{(j(s)G(z_0,s))^{1/(p-1)}}.$$

Therefore, we see that

$$-\zeta''(z) \leq (j(z)G(z_0,z))^{1/(p-1)}\zeta'''(z)\xi(z),$$

which yields

$$-\frac{\zeta'''(z)}{\zeta''(z)}\xi(z)(j(z)G(z_0,z))^{1/(p-1)} \leq 1.$$

Hence,

$$\frac{j(z)(\zeta'''(z))^{p-1}}{(\zeta''(z))^{p-1}} \geq \frac{-1}{\xi^{p-1}(z)G(z_0,z)}. \tag{11}$$

Define

$$B(z) := -\frac{j(z)(-\zeta''')^{p-1}(z)}{(\zeta''(z))^{p-1}}, \tag{12}$$

we obtain $B(z) < 0$ also, from (11) and (12), we have

$$B(z) \geq \frac{-1}{\xi^{p-1}(z)G(z_0,z)}. \tag{13}$$

From (12), we find

$$B'(z) = \frac{\left(-j(z)(-\zeta'''(z))^{p-1}\right)'}{(\zeta''(z))^{p-1}} - (p-1)\frac{-j(z)(-\zeta'''(z))^{p}}{(\zeta''(z))^{p}}.$$

Using (1), we obtain

$$B'(z) \leq \frac{-a(z)k_f(-\zeta'''(z))^{p-1}}{(\zeta''(z))^{p-1}} - \frac{q(z)k_g\zeta^{p-1}(c(z))}{(\zeta''(z))^{p-1}} - (p-1)\frac{-j(z)(-\zeta'''(z))^{p}}{(\zeta''(z))^{p}}.$$

From (12), we see

$$\begin{aligned} B'(z) &= -k_f\frac{a(z)}{j(z)}B(z) - k_g q(z)\frac{\zeta^{p-1}(c(z))}{(\zeta''(z))^{p-1}} - (p-1)\frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{1}{\alpha}}(z)}. \\ &= -k_f\frac{a(z)}{j(z)}B(z) - k_g q(z)\frac{\zeta^{p-1}(c(z))}{(\zeta''(c(z)))^{p-1}}\frac{(\zeta''(c(z)))^{p-1}}{(\zeta''(z))^{p-1}} - (p-1)\frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{1}{p-1}}(z)}. \end{aligned} \tag{14}$$

Using Lemma 3, we find

$$\zeta(z) \geq \frac{\mu}{2}z^2\zeta''(z). \tag{15}$$

From (13)–(15), we obtain

$$B'(z) \leq \frac{k_f a(z)}{j(z)\xi^{p-1}(z)G(z_0,z)} - k_g q(z)\left(\frac{\mu}{2}c^2(z)\right)^{p-1} - (p-1)\frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{1}{p-1}}(z)}. \tag{16}$$

From (11), we see

$$\frac{\zeta'''(z)}{\zeta''(z)} \geq \frac{-1}{\xi(z)(j(z)G(z_0,z))^{1/(p-1)}}.$$

Using the latter inequality and (4), we see

$$\begin{aligned} \left(\frac{\zeta''(z)}{\vartheta(z)}\right)' &= \frac{\zeta'''(z)\vartheta(z) - \zeta''(z)\vartheta\prime(z)}{\vartheta^2(z)} \\ &\geq \frac{-\zeta''(z)}{\vartheta^2(z)}\left(\frac{\vartheta(z)}{\xi(z)(j(z)G(z_0,z))^{1/(p-1)}} + \vartheta'(z)\right) \geq 0, \end{aligned}$$

which implies that $\zeta''(z)/\vartheta(z)$ is nondecreasing. Thus, it follows from $c(z) \geq z$ that

$$\frac{\zeta''(c(z))}{\zeta''(z)} \geq \frac{\vartheta(c(z))}{\vartheta(z)}.$$

So, by (14) and (15), we see

$$B'(z) \leq \frac{k_f a(z)}{j(z)\xi^{p-1}(z)G(z_0,z)} - k_g q(z)\left(\frac{\mu}{2}c^2(z)\right)^{p-1}\left(\frac{\vartheta(c(z))}{\vartheta(z)}\right)^{p-1} - (p-1)\frac{B^{\frac{p}{p-1}}(z)}{j^{\frac{1}{p-1}}(z)}. \tag{17}$$

Multiplying (17) by $\xi^{p-1}(z)G(z_0,z)$ and integrating from z_1 to z, we get

$$\begin{aligned}
&\xi^{p-1}(z)G(z_0,z)B(z) - \xi^{p-1}(z_1)G(z_0,z_1)B(z_1) - k_f\int_{z_1}^{z}\frac{a(s)}{j(s)}ds \\
&-(p-1)\int_{z_1}^{z} j^{\frac{-1}{p-1}}(s)\xi^{p-2}(s)G(s_0,s)\varphi(s)B(s)ds \\
&+\int_{z_1}^{z} k_g q(s)\left(\frac{\mu}{2}c^2(s)\right)^{p-1}\left(\frac{\vartheta(c(s))}{\vartheta(s)}\right)^{p-1}\xi^{p-1}(s)G(s_0,s)ds \\
&+(p-1)\int_{z_1}^{z}\frac{B^{\frac{p}{p-1}}(s)}{j^{\frac{1}{p-1}}(s)}\xi^{p-1}(s)G(s_0,s)ds \\
\leq\ & 0.
\end{aligned}$$

By Lemma 1, we set

$$C = \xi^{p-1}(s)G(s_0,s)/j^{\frac{1}{p-1}}(s),\ D = \int_{z_1}^{z} j^{\frac{-1}{p-1}}(s)\xi^{p-2}(s)G(s_0,s)\varphi(s)ds,\ y = B(s).$$

Thus, we see

$$\begin{aligned}
&\xi^{p-1}(z)G(z_0,z)B(z) - \xi^{p-1}(z_1)G(z_0,z_1)B(z_1) - k_f\int_{z_1}^{z}\frac{a(s)}{j(s)}ds \\
&+\int_{z_1}^{z} k_g q(s)\left(\frac{\mu}{2}c^2(s)\right)^{p-1}\left(\frac{\vartheta(c(s))}{\vartheta(s)}\right)^{p-1}\xi^{p-1}(s)G(z_0,s)ds \\
&-\int_{z_1}^{z}\frac{(p-1)^p\delta(s)\varphi^p(s)G(s_0,s)}{\xi(s)j^{\frac{1}{p-1}}(s)}ds \\
\leq\ & 0.
\end{aligned}$$

Hence, by (13), we obtain

$$\int_{z_1}^{z}\left(k_g q(s)\left(\frac{\mu c^2(s)}{2}\frac{\vartheta(c(s))}{\vartheta(s)}\xi(s)\right)^{p-1}G(s_0,s) - \tilde{\varphi}(s)\right)ds \leq \xi^{p-1}(z_1)G(z_0,z_1)B(z_1) + 1,$$

which contradicts (5).

Theorem 1 is proved. □

Remark 1. *As a special case of (1), we can apply the same method used in this work to second-order, advanced differential equations to obtain new results. We set*

$$B(z) := \vartheta(z)\frac{\zeta'(z)}{\zeta(z)},$$

in second-order, advanced differential equation with middle term. We find the following result

Corollary 1. *Let (2) holds. If $\delta, \vartheta \in C^1([z_0,\infty),(0,\infty))$ such that*

$$\limsup_{z\to\infty}\int_{z_0}^{z}\left(k_g\delta(s)q(s) - \frac{j(s)\delta(s)(\phi(s))^p}{p^p}\right)ds = \infty, \tag{18}$$

additionally, (4) is satisfied and

$$\limsup_{z\to\infty}\int_{z_0}^{z}\left(k_g q(s)\left(\frac{\vartheta(c(s))}{\vartheta(s)}\xi(s)\right)^{p-1}G(z_0,s) - \tilde{\varphi}(s)\right)ds = \infty, \tag{19}$$

then equation

$$\left(j(z)\Phi_p[\zeta'(z)]\right)' + a(z)f\left(\zeta'(z)\right) + q(z)g(\zeta(c(z))) = 0, \tag{20}$$

is oscillatory.

Now, we present two examples that illustrate the applicability of the obtained results. It is worth noting that these examples represent many physical phenomena, such as their application in earthquake structures, mechanical oscillations and clinical applications.

Example 1. *Consider the differential equation*

$$\left(z^2\left(\zeta'''(z)\right)\right)' + \frac{z}{2}\zeta'''(z) + \frac{q_0}{z^2}\zeta(2z) = 0. \tag{21}$$

where $q_0 > 0$, $p = 2$, $z_0 = 1$, $j(z) = z^2$, $a(z) = z/2$, $q(z) = z$, $c(z) = 2z$, *we now set* $\delta(z) = z$, $k_f = k_g = 1$, *then*

$$\begin{aligned} G(z_0, z) &= \exp\left(\int_{z_0}^{z} \frac{a(z)}{j(z)} du\right) = z^{1/2},\ \xi(z) = \int_z^\infty \frac{ds}{(j(s)G(z_0,s))^{\frac{1}{\kappa}}} = \frac{2z^{-3/2}}{3}, \\ \vartheta(z) &= \frac{2z^{-3/2}}{3},\ \varphi(z) = \frac{1}{G^{\frac{1}{p-1}}(z_0,z)} - \frac{\xi(z)a(z)j^{(2-p)/(p-1)}(z)}{p-1} = \frac{2z^{-1/2}}{3},\ \phi(z) = \frac{-1}{2z} \end{aligned}$$

and

$$\tilde{\varphi}(z) = \frac{a(z)}{j(z)} + \frac{(p-1)^p\delta(z)\varphi^p(z)G(z_0,z)}{\xi(z)j^{\frac{1}{p-1}}(z)} = \frac{2z^{-1/3}}{3}.$$

Thus, we obtain

$$\limsup_{z\to\infty} \int_{z_0}^{z} \left(k_g\delta(s)q(s) - \left(\frac{2}{Ms^2}\right)^{p-1} \frac{j(s)\delta(s)(\phi(s))^p}{p^p}\right) ds = \infty$$

and

$$\frac{\vartheta(z)}{\xi(z)(j(z)G(z_0,z))^{1/(p-1)}} + \vartheta'(z) = 0$$

also, for some $\mu \in (0,1)$,

$$\limsup_{z\to\infty} \int_{z_0}^{z} \left(k_g q(s)\left(\frac{\mu c^2(s)}{2}\frac{\vartheta(c(s))}{\vartheta(s)}\xi(s)\right)^{p-1} G(z_0,s) - \tilde{\varphi}(s)\right) ds = \infty.$$

Using Theorem 1, the Equation (21) is oscillatory if $q_0 > 0$.

Example 2. *Consider the differential equation*

$$\left(z^2\left(\zeta'(z)\right)\right)' + \frac{z}{2}\zeta'(z) + q_0\zeta(2z) = 0. \tag{22}$$

where $q_0 > 0$, $p = 2$, $z_0 = 1$, $j(z) = z^2$, $a(z) = z/2$, $q(z) = q_0$, $c(z) = 2z$, *we now set* $\delta(z) = k_f = k_g = 1$, *then* $\xi(z) = \vartheta(z) = 2z^{-3/2}/3$.

Using Corollary 1, the Equation (22) is oscillatory if $q_0 > 2\sqrt{2}$.

4. Conclusions

In this article, we establish oscillation conditions of advanced nonlinear differential equations of fourth-order with middle term of the form (1). Our approach is different and obtained by using Riccati technique to reduce the main equation into a first-order equation. The new proposed criteria complement several results in the literature. We provide two examples with specific parameters to illustrate the applicability of our theorems. In future work, we will discuss the oscillatory behavior of these equations by using the integral averaging technique and under the condition

$$\int_{z_0}^{\infty} \left[\frac{1}{j(s)} \exp\left(- \int_{z_0}^{s} \frac{a(y)}{j(y)} dy \right) \right]^{1/\alpha} ds = \infty. \tag{23}$$

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References

1. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
2. Aronsson, G.; Janfalk, U. On Hele-Shaw flow of power-law fluids. *Eur. J. Appl. Math.* **1992**, *3*, 343–366. [CrossRef]
3. Li, T.; Baculikova, B.; Dzurina, J.; Zhang, C. Oscillation of fourth order neutral differential equations with p-Laplacian like operators. *Bound. Value Probl.* **2014**, *56*, 41–58. [CrossRef]
4. Liu, S.; Zhang, Q.; Yu, Y. Oscillation of even-order half-linear functional differential equations with damping. *Comput. Math. Appl.* **2011**, *61*, 2191–2196. [CrossRef]
5. Zhang, C.; Agarwal, R.; Li, T. Oscillation and asymptotic behavior of higher-order delay differential equations with p-Laplacian like operators. *J. Math. Anal. Appl.* **2014**, *409*, 1093–1106. [CrossRef]
6. Nofal, T.A.; Bazighifan, O.; Khedher, K.M.; Postolache, M. More Effective Conditions for Oscillatory Properties of Differential Equations. *Symmetry* **2021**, *13*, 278. [CrossRef]
7. Bazighifan, O.; Dassios, I. Riccati Technique and Asymptotic Behavior of Fourth-Order Advanced Differential Equations. *Mathematics* **2020**, *8*, 590. [CrossRef]
8. Sidorov, N.A. Explicit and implicit parametrizations in the construction of branching solutions by iterative methods. *Sb. Math.* **1995**, *186*, 297–310. [CrossRef]
9. Dzurina, J.; Jadlovska, I. Oscillation theorems for fourth order delay differential equations with a negative middle term. *Math. Meth. Appl. Sci.* **2017**, *4563*, 1–13. [CrossRef]
10. Lozi, R.; Abdelouahab, M.-S.; Chen, G. Mixed-Mode Oscillations Based on Complex Canard Explosion in a Fractional-Order Fitzhugh-Nagumo Model. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 239–256. [CrossRef]
11. Selvi, M.; Salai, M.; Rajendran, L. Application of modified wavelet and homotopy perturbation methods to nonlinear oscillation problems. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 351–364. [CrossRef]
12. Sanz-Serna, J.M.; Zhu, B. Word series high-order averaging of highly oscillatory differential equations with delay. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 445–454. [CrossRef]
13. Grace, S.; Agarwal, R.; Graef, J. Oscillation theorems for fourth order functional differential equations. *J. Appl. Math. Comput.* **2009**, *30*, 75–88. [CrossRef]
14. Bhatti, M.M.; Marin, M.; Zeeshan, A.; Ellahi, R.; Abdelsalam, S.I. Swimming of Motile Gyrotactic Microorganisms and Nanoparticles in Blood Flow Through Anisotropically Tapered Arteries. *Front. Phys.* **2020**, *8*, 95. [CrossRef]
15. Marin, M.; Vlase, S.; Paun, M. Considerations on double porosity structure for micropolar bodies. *AIP Adv.* **2015**, *5*, 3. [CrossRef]
16. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
17. Bazighifan, O.; Alotaibi, H.; Mousa, A.A.A. Neutral Delay Differential Equations: Oscillation Conditions for the Solutions. *Symmetry* **2021**, *13*, 101. [CrossRef]
18. Treanta, S. On a Class of Differential Variational Inequalities in Infinite-Dimensional Spaces. *Mathematics* **2021**, *9*, 266. [CrossRef]
19. Doroftei, M.; Treanta, S. Higher order hyperbolic equations involving a finite set of derivations. *Balk. J. Geom. Appl.* **2012**, *17*, 22–33.
20. Treanta, S. Gradient Structures Associated with a Polynomial Differential Equation. *Mathematics* **2020**, *8*, 535. [CrossRef]
21. Treanta, S. On the Kernel of a Polynomial of Scalar Derivations. *Mathematics* **2020**, *8*, 515. [CrossRef]
22. Park, C.; Moaaz, O.; Bazighifan, O. Oscillation Results for Higher Order Differential Equations. *Axioms* **2020**, *9*, 14. [CrossRef]

23. Agarwal, R.; Grace, S.R. Oscillation theorems for certain functional differential equations of higher order. *Math. Comput. Model.* **2004**, *39*, 1185–1194. [CrossRef]
24. Agarwal, R.; Grace, S.R.; O'Regan, D. Oscillation criteria for certain *m*th order differential equations with deviating arguments. *J. Math. Anal. Appl.* **2001**, *262*, 601–622. [CrossRef]
25. Bazighifan, O.; Abdeljawad, T.; Al-Mdallal, Q.M. Differential equations of even-order with p-Laplacian like operators: Qualitative properties of the solutions. *Adv. Differ. Equ.* **2021**, *2021*, 96. [CrossRef]
26. Bazighifan, O.; Abdeljawad, T. Improved Approach for Studying Oscillatory Properties of Fourth-Order Advanced Differential Equations with p-Laplacian Like Operator. *Mathematics* **2020**, *8*, 656. [CrossRef]
27. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *336*, 1–9.
28. Moaaz, O.; Elabbasy, E.M.; Muhib, A. Oscillation criteria for even-order neutral differential equations with distributed deviating arguments. *Adv. Differ. Equ.* **2019**, *297*, 2019. [CrossRef]
29. Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear Neutral Delay Differential Equations of Fourth-Order: Oscillation of Solutions. *Entropy* **2021**, *23*, 129. [CrossRef] [PubMed]
30. Zhang, C.; Li, T.; Sun, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* **2011**, *24*, 1618–1621. [CrossRef]