



Donghai Liu^{1,*}, An Huang¹, Yuanyuan Liu¹ and Zaiming Liu²

- ¹ Department of Applied Statistics, Hunan University of Science and Technology, Xiangtan 411201, China; huangan@mail.hnust.edu.cn (A.H.); 16011502003@mail.hnust.edu.cn (Y.L.)
- ² Department of Mathematics and Statistics, Central South University, Changsha 410083, China; zmliu@csu.edu.cn
- * Correspondence: dhliu@hnust.edu.cn

Abstract: The paper studies an extension TOPSIS method with the adjusted probabilistic linguistic fuzzy set in which the decision maker's behavior tendency is considered. Firstly, we propose a concept of probabilistic linguistic q-rung orthopair set (PLQROS) based on the probability linguistic fuzzy set (PLFS) and linguistic q-rung orthopair set (LQROS). The operational laws are introduced based on the transformed probabilistic linguistic q-rung orthopair sets (PLQROSs) which have the same probability. Through this adjustment method, the irrationality of the existing methods in the aggregated operators. The distance measure of PLQROSs is also defined, which can deal with the symmetric information in multi-attribute decision making problems. Considering that the decision maker's behavior has a very important impact on decision-making results, we propose a behavioral TOPSIS decision making method for PLQROS. Finally, we apply the practical problem of investment decision to demonstrate the validity of the extension TOPSIS method, and the merits of the behavior decision method is testified by comparing with the classic TOPSIS method. The sensitivity analysis results of decision-maker's behavior are also given.

Keywords: behavioral decision making; risk attitude; adjusted probabilistic fuzzy set

1. Introduction

There is much uncertainty in decision making problems. It is not easy to describe the evaluation information with accurate numerical value, and they can only be described with linguistic values. Zadeh [1–3] defined the linguistic term set (LTS) and applied it to express the qualitative evaluation. For example, the commonly used seven valued LTS is in the form of $S = \{s_0 : great \ distaste, s_1 : distaste, s_2 : a \ bit \ distaste, s_3 : generally, s_4 :$ a bit favorite, s_5 : favorite, s_6 : great favorite}, it can be used to describe how much the decision maker likes the object. However, when the decision maker hesitates about the preference of the evaluation object, the single linguistic term is no longer describe such information. So Rodríguez et al. [4] presented the hesitant fuzzy linguistic term set (HFLTS), each of elements is a collection of linguistic terms. For example, the decision maker thinks that the audience's opinion of the program is "favorite" or "great favorite", the evaluation about the opinion of the program can be only expressed in the form of $\{s_5, s_6\}$. Since the HFLTS was put forward, some extensions of HFLTS were developed and applied in many fields [5–11]. Beg and Rashid [5] proposed the TOPSIS method of HFLTS and applied it to sort the alternative, Liao et al. [6] defined the preference relation of HFLTS, Liu et al. [7] applied the HFLTSs to the generalized TOPSIS method and presented a new similarity measure of HFLTS. In these studies however, all the linguistic terms in HFLTS have the same weights, it rarely happens in reality. In fact, the decision makers may have different degrees to the possible linguistic evaluations. Therefore, Pang et al. [12] developed the HFLTS to the PLFS through adding the probabilities to each element. For



Citation: Liu, D.; Huang, A.; Liu, Y.; Liu, Z. An Extension TOPSIS Method Based on the Decision Maker's Risk Attitude and the Adjusted Probabilistic Fuzzy Set. *Symmetry* 2021, *13*, 891. https://doi.org/ 10.3390/sym13050891

Academic Editor: Juan Luis García Guirao

Received: 19 April 2021 Accepted: 13 May 2021 Published: 17 May 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). example, the decision maker believes that the possibility of favorite to the program is 0.4 and the possibility of great favorite to the program is 0.6, then the above evaluation can be represented as $\{s_5(0.4), s_6(0.6)\}$.

In the aboved LTS, they only describe the membership degree of elements. To improve the range of their application, Chen [13] defined the linguistic intuitionistic fuzzy set (LIFS) $LI = \{ \langle x_i, s_{\theta_1}(x_i), s_{\phi_1}(x_i) \rangle | x_i \in X \}, \text{ where } s_{\theta_1}(x_i), s_{\phi_1}(x_i) \in S(S = \{ s_i | s_0 \le s_i \le s_{2\tau} \}), \}$ the membership $s_{\theta_1}(x_i)$ and the non-membership $s_{\phi_1}(x_i)$ satisfy the following condition: $0 \le \theta_1 + \phi_1 \le 2\tau$. Furthermore, Garg [14] defined the linguistic Pythagorean fuzzy set (LPFS) $LP = \{ \langle x_i, s_{\theta_2}(x_i), s_{\phi_2}(x_i) \rangle | x_i \in X \}$, where $s_{\theta_2}(x_i), s_{\phi_2}(x_i) \in S$, the following condition of the membership $s_{\theta_2}(x_i)$ and the non-membership $s_{\phi_2}(x_i)$ must be satisfied: $0 \le (\theta_2)^2 + (\phi_2)^2 \le (2\tau)^2$, its advantage is it has a wider range of uncertainty than LIFS. Furthermore, in order to better describe the uncertainty in decision making problems, Liu et al. [7] proposed the LQROS $LQ = \{\langle x_i, s_{\theta}(x_i), s_{\phi}(x_i) \rangle | x_i \in X\}$ based on the q-rung orthopair fuzzy set (QROFS) [15], where the following condition of the membership $s_{\theta}(x_i)$ and the non-membership $s_{\phi}(x_i)$ should be met: $0 \leq (\theta)^q + (\phi)^q \leq (2\tau)^q (q \geq 1)$. Obviously, when q = 1 or 2, the LQROS is reduced to LIFS or LPFS, respectively. Although the LQROS extends the scope of information representation, it cannot describe the following evaluation information. For the given LTS $S = \{s_0 : extreme \ slowly, s_1 : slowly, s_2 : slightly \ slowly, s_3 :$ generally, s_4 : slightly high, s_5 : high, s_6 : extreme high}, one expert believes that 30% possibility of profit from the investment in the project is high, and 70% possibility of profit from the investment in the project is extreme high. While the other expert may believe that 10% possibility of not making a profit is extreme slowly and 90% possibility of not making a profit is slightly slowly. Up to now, we cannot apply the existing LTS to describe the above evaluation information. Motivated by this, we introduce the PLQROS by integrating the LQROS and the probabilistic fuzzy set. Then the above information can be represented by $Q_s(p) = \{\{s_5(0.3), s_6(0.7)\}, \{s_0(0.1), s_2(0.9)\}\}$, the detailed definition is given in Section 3.1.

On the other side, the TOPSIS is a classical method to handle the multiple criteria decision making problems. Since it was introduced by Hwang and Yoon [16], there were many literatures on TOPSIS method, we can refer to [17–22]. The TOPSIS method is a useful technique for choosing an alternative that is closet to the best alternative and farthest from the worst alternative simultaneously. Furthermore, Yoon and Kim [23] proposed a behavioral TOPSIS method that incorporates the gain and loss in behavioral economics, which makes the decision results more reasonable. As all we know, there is no related research study the behavioral TOPSIS method in uncertain decision environments problems. Inspired by this, we study the TOPSIS method which consider the decision maker's risk attitude and the adjusted probabilistic fuzzy set, the main contributions of the paper are given as follows:

- (1) The operational laws of PLQROS are given based on the adjusted PLQROS with the same probability, which can avoid the unreasonable calculation and improve the adaptability of PLQROS in reality.
- (2) The new aggregation operators and distance measures between PLQROSs are presented, which can represent the differences between PLQROSs and deal with the symmetry information.
- (3) The behavioral TOPSIS method is introduced into the uncertain multi-attribute decision making process, which changes the behavioral TOPSIS method only used in the deterministic environment.

The remainder of paper is organized as follows: in the second section, some related concepts are reviewed. In the third section, we introduce the operational laws of PLQROSs, the aggregated operators and distance measures of PLQROSs, and we also give their corresponding properties. In Section 4, the process steps of the behavioral decision algorithm are given. In Section 5, a practical example is utilized to prove the availability of the extension TOPSIS method. Furthermore, the sensitivity analysis of the behavioral factors of decision

maker's risk attitude is provided. Finally, we made a summary of the paper and expanded the future studies.

2. Preliminaries

In order to define the PLQROS, we introduce some concepts of LTS, PLFS, QROFS and LQROS. Throughout the paper, assume $X = \{x_1, x_2, \dots, x_m\}$ to be a non-empty and finite set.

In the uncertain decision making environment, the experts applied the LTS to make a qualitative description, it is defined as follows:

Definition 1 ([1]). Assume $S = \{s_{\alpha} | \alpha = 0, 1, \dots, 2g\}$ is a finite set, where s_{α} is a linguistic term and g is a natural number, the LTS S should satisfy two properties:

- (1) *if* $\alpha \leq \beta$ *, then* $s_{\alpha} \leq s_{\beta}$ *;*
- (2) $s_{\alpha} = neg(s_{\beta})$, where $\alpha + \beta = 2g$.

In order to describe the decision information more objective, Xu [24] extended the LTS S to a continuous LTS $\overline{S} = \{s_{\alpha} | \alpha \in [0, \rho]\}$, where $\rho(\rho > 2g)$ is a natural number.

The PLFS is regarded as an extension of HFLTS, which considers the elements of LTSs with different weights, the PLFS can be denoted as:

Definition 2 ([12]). Assume $X = \{x_1, x_2, \dots, x_m\}$ and $S = \{s_0, s_1, \dots, s_{2g}\}$ is a LTS, the PLFS $Z_s(r)$ in X is defined as:

$$Z_s(r) = \{ \langle x_i, z_s(r)(x_i) \rangle | x_i \in X \},\$$

where $z_s(r)(x_j) = \{s^{j(u)}(r^{(u)}) | s^{j(u)} \in S, r^{(u)} \ge 0, u = 1, 2, \cdots, U; \sum_{u=1}^{U} r^{(u)} \le 1\}, s^{j(u)}$ is the linguistic term and $r^{(u)}$ is the corresponding probability of $s^{j(u)}$, U is the number of linguistic terms $s^{j(u)}$.

Next, we introduce the concept of QROFS as follows:

Definition 3 ([15]). Assume $X = \{x_1, x_2, \dots, x_m\}$, the QROFS Q is represented as:

$$Q = \{ \langle x_j, \mu_Q(x_j), \nu_Q(x_j) \rangle | x_j \in X \}, \ q \ge 1,$$

where the membership $\mu_Q(x_j)(0 \le \mu_Q(x_j) \le 1)$ and the non-membership $\nu_Q(x_j)(0 \le \nu_Q(x_j) \le 1)$ satisfy $0 \le (\mu_Q(x_j))^q + (\nu_Q(x_j))^q \le 1$, $\pi_Q(x_j) = \sqrt[q]{1 - (\mu_Q(x_j))^q - (\nu_Q(x_j))^q}$ is the indeterminacy degree of QROFS Q.

If *X* = {*x*}, the QROFS *Q* is reduced to a q-rung orthopair fuzzy number (QROFN) $Q \triangleq \langle \mu_Q, \nu_Q \rangle$.

Remark 1. If q = 1 or 2, the QROFS Q is degenerated to an intuitionistic fuzzy set (IFS) or a Pythagorean fuzzy set (PFS).

In some realistic decision making problems, the set should be described qualitatively. So we review the concept of LQROS as follows:

Definition 4 ([25]). Assume $X = \{x_1, x_2, \dots, x_m\}$, $\overline{S} = \{s_{\alpha} | \alpha \in [0, \rho]\}$ ($\rho > 2g$) is a LTS, the LQROS Y is represented as:

$$Y = \{ \langle x_j, s_{\theta}(x_j), s_{\phi}(x_j) \rangle | x_j \in X \},\$$

where the membership $s_{\theta}(x_j)$ and the non-membership $s_{\phi}(x_j)$ satisfy $0 \le \theta_j^q + \phi_j^q \le \rho^q \ (q \ge 1)$, the indeterminacy degree $\pi_Y(x_j) = s_{q/\rho^q - \theta_i^q - \phi_i^q}$.

If $X = \{x\}$, the LQROS *Y* is degenerated to a linguistic q-rung orthopair number (LQRON) $Y \triangleq \langle s_{\theta}, s_{\phi} \rangle$.

Definition 5 ([25]). Let $y_1 = \langle s_{\theta_1}, s_{\phi_1} \rangle$ and $y_2 = \langle s_{\theta_2}, s_{\phi_2} \rangle$ be two LQRONs, $s_{\theta_i}, s_{\phi_i} \in \overline{S}_{[0,\rho]}$ ($\iota = 1, 2$), $\epsilon > 0$, the algorithms of the LQRONs can be expressed as follows:

- $(a) \quad y_1 \oplus y_2 = \langle s_{((\theta_1)^q + (\theta_2)^q (\frac{\theta_1 \theta_2}{\rho})^q)^{\frac{1}{q}}}, s_{\frac{\phi_1 \phi_2}{\rho}} \rangle;$
- $(b) \quad y_1 \otimes y_2 = \langle s_{\frac{\theta_1\theta_2}{\rho}}, s_{((\phi_1)^q + (\phi_2)^q (\frac{\phi_1\phi_2}{\rho})^q)^{\frac{1}{q}}} \rangle;$

$$\begin{array}{ll} (c) \quad \epsilon y_1 = \langle s_{(\rho^q - \rho^q (1 - \frac{(\theta_1)^q}{\rho^q})^\epsilon)^{\frac{1}{q}}, s_{\rho(\frac{\theta_1}{\rho})^\epsilon} \rangle; \\ (d) \quad y_1^\epsilon = \langle s_{\rho(\frac{\theta_1}{\rho})^\epsilon}, s_{(\rho^q - \rho^q (1 - \frac{(\theta_1)^q}{\sigma^q})^\epsilon)^{\frac{1}{q}} \rangle. \end{array}$$

3. The Proposed Probabilistic Fuzzy Set

Now we propose a new probabilistic fuzzy set—PLQROS, which not only allows the experts to express evaluation information with multiple linguistic terms but contains the possibility of each linguistic terms. The difficulty is how to define the operational laws of PLQROS reasonably when the corresponding probability distributions are different.

3.1. The Basic Definition of PLQROS

Definition 6. Let $X = \{x_1, x_2, \dots, x_m\}$ and $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a continuous LTS, the PLQROS $PL_s(r)$ in X can be defined as:

$$PL_s(r) = \{ \langle x_j, H_s(\hat{r})(x_j), G_s(\tilde{r})(x_j) \rangle | x_j \in X \},\$$

where $H_{s}(\hat{r})(x_{j}) = \{s_{\theta^{j(u)}}(\hat{r}^{(u)})|s_{\theta^{j(u)}} \in \bar{S}_{[0,\rho]}, \hat{r}^{(u)} \ge 0, \sum_{u=1}^{U} \hat{r}^{(u)} \le 1\}$ is the membership and $G_{s}(\tilde{r})(x_{j}) = \{s_{\phi^{j(v)}}(\tilde{r}^{(v)})|s_{\phi^{j(v)}} \in \bar{S}, \tilde{r}^{(v)} \ge 0, \sum_{v=1}^{V} \tilde{r}^{(v)} \le 1\}$ is the non-membership, respectively. For any $x_{j} \in X$, they satisfy with: $0 \le (\max_{u=1}^{U} \{\theta^{j(u)}\})^{q} + (\max_{v=1}^{V} \{\phi^{j(v)}\})^{q} \le \rho^{q} (q \ge 1).$

If $X = \{x\}$, the PLQROS $PL_s(r)$ is degenerated to a PLQRON $pl_s(r) \triangleq \langle \{s_{\theta^{(u)}}(\hat{r}^{(u)})\}\rangle$, where $s_{\theta^{(u)}}, s_{\phi^{(v)}} \in \bar{S}_{[0,\rho]}, \sum_{u=1}^{U} \hat{r}^{(u)} \leq 1$ and $\sum_{v=1}^{V} \tilde{r}^{(v)} \leq 1$.

Example 1. Let $S = \{s_0 : very slowly, s_1 : slowly, s_2 : slightly slowly, s_3 : generally, s_4 : slightly fast, s_5 : fast, s_6 : very fast \}. Two groups of experts inspected the development of the company, one group may think that "the speed of company development is slightly slowly with 100% possibility, with 40% probability that it is not slightly slowly, with 40% probability that the speed of company development is not generally and with 20% probability that the speed of company development is not slightly fast". The other group think that "with 20% probability that the speed of company development is slowly, with 60% probability that the speed of company development is slowly fast". The other group think that "with 20% probability that the speed of company development is slowly, with 60% probability that the speed of company development is slowly and with 20% probability that the speed of company development is slowly and with 20% probability that the speed of company development is slowly and with 20% probability that the speed of company development is slowly and with 20% probability that it is not very fast". Then the above evaluation information can be denoted as <math>pl_s^1(r) = \langle \{s_2(1)\}, \{s_2(0.4), s_3(0.4), s_4(0.2)\} \rangle$ and $pl_s^2(r) = \langle \{s_1(0.2), s_2(0.6), s_3(0.2)\}, \{s_5(0.5), s_6(0.5)\} \rangle$.

According to Example 1, we can see that the probabilities and numbers of elements in $pl_s^1(r)$ and $pl_s^2(r)$ are not same, the general operations on PLFSs multiply the probabilities of corresponding linguistic terms directly, which may cause the unreasonable result. There-

fore, Wu et al. [26] presented a method to modify the probabilities of linguistic terms to be same, which is given as follows:

Let $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a continuous LTS, $pl_s^1(r) = \langle \{s_{\theta^{1}(u)}(\hat{r}^{(u)})\}, \{s_{\phi^{1}(v)}(\tilde{r}^{(v)})\}|u = 1, 2, \cdots, V_1 \rangle$ and $pl_s^2(r) = \langle \{s_{\theta^{2}(u)}(\hat{r}^{(u)})\}, \{s_{\phi^{2}(v)}(\tilde{r}^{(v)})\}|u = 1, 2, \cdots, U_2;$ $v = 1, 2, \cdots, V_2 \rangle$ are two PLQRONs. We adjust the probability distributions of $pl_s^1(r)$ and $pl_s^2(r)$ to be same, respectively. That is to say, $pl_s^{*1}(r) = \langle \{s_{\theta^{1}(k)}(\hat{r}^{*(k)})\}, \{s_{\phi^{1}(b)}(\tilde{r}^{*(b)})\}|k = 1, 2, \cdots, K; b = 1, 2, \cdots, B \rangle$ and $pl_s^{*2}(r) = \langle \{s_{\theta^{2}(k)}(\hat{r}^{*(k)})\}, \{s_{\phi^{2}(b)}(\tilde{r}^{*(b)})\}|k = 1, 2, \cdots, K; b = 1, 2, \cdots, B \rangle$. Applying the method of Wu et al. [26] to adjust the PLQRONs, the linguistic terms and the sum of probabilities of each linguistic term set are not changed, which means that the adjustment method does not result in the loss of evaluation information.

Example 2. Let $S = \{s_{\alpha} | \alpha = 0, 1, 2, \dots, 6\}$, $pl_{s}^{1}(r) = \langle \{s_{2}(1)\}, \{s_{3}(0.4), s_{4}(0.4), s_{6}(0.2)\} \rangle$ and $pl_{s}^{2}(r) = \langle \{s_{1}(0.2), s_{2}(0.6), s_{3}(0.2)\}, \{s_{5}(0.5), s_{6}(0.5)\} \rangle$ be two PLQRONs, the adjusted PLQRONs are $pl_{s}^{*1}(r) = \langle \{s_{2}(0.2), s_{2}(0.6), s_{2}(0.2)\}, \{s_{3}(0.4), s_{4}(0.1), s_{4}(0.3), s_{6}(0.2)\} \rangle$ and $pl_{s}^{*2}(r) = \langle \{s_{1}(0.2), s_{2}(0.6), s_{3}(0.2)\}, \{s_{5}(0.4), s_{5}(0.1), s_{6}(0.3), s_{6}(0.2)\} \rangle$, respectively. The adjustment process is shown in Figure 1.

$\mathbf{pl}_{s}^{1}(r)$		S₂(1)		S ₃ (0.4)	S ₄ (0.1)	S ₄ (0.3)	S ₆ (0.2)
	×			V	Ļ	Ļ	Ļ
$\mathbf{pl}_{s}^{*1}(r)$	S ₂ (0.2)	S ₂ (0.6)	S ₂ (0.2)	S₃(0.4)	S ₄ (0.1)	S ₄ (0.3)	S ₆ (0.2)
$\mathrm{pl}_{s}^{*2}(r)$	S₁(0.2) ↑	S₂(0.6)	S₃(0.2) ↑	S₅(0.4)	S₅(0.1)	S ₆ (0.3) ↑	S ₆ (0.2) ↑
$\mathrm{pl}_s^2(r)$	S ₁ (0.2)	S ₂ (0.6)	S₃(0.2)	S ₅ (0.4)	S ₅ (0.1)	S ₆ (0.3)	S ₆ (0.2)

Figure 1. The adjust process of PLQRONs $pl_s^1(r)$ and $pl_s^2(r)$.

3.2. Some Properties for PLQRONs

Firstly, we apply the adjustment method to adjust the probabilistic linguistic terms with same probability, which can overcome the defects that may occur in process of aggregation. Then, we propose the operation rules of the adjusted PLQRONs, and their properties.

Definition 7. Let $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a LTS, $pl_s^{*1}(r) = \langle \{s_{\theta^{1}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{1}(v)}(\tilde{r}^{*(v)})\} \rangle$ and $pl_s^{*2}(r) = \langle \{s_{\theta^{2}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{2}(v)}(\tilde{r}^{*(v)})\} \rangle$ ($u = 1, 2, \cdots, U$; $v = 1, 2, \cdots, V$) are two adjusted PLQRONs, where $\theta^{\iota(u)}, \phi^{\iota(v)}$ ($\iota = 1, 2$) are the subscript of $s_{\theta^{\iota(u)}}, s_{\phi^{\iota(v)}}$ ($\iota = 1, 2$), $\eta > 0$, the operational laws of the PLQRONs can be expressed as follows:

(a) $neg(pl_s^{*1}(r)) = \langle \{s_{\phi^{1(v)}}(\tilde{r}^{*(v)})\}, \{s_{\theta^{1(u)}}(\hat{r}^{*(u)})\} \rangle;$

$$(b) \quad pl_{s}^{*1}(r) \oplus pl_{s}^{*2}(r) = \langle \{s_{((\theta^{1(u)})^{q} + (\theta^{2(u)})^{q} - (\frac{(\theta^{1(u)})(\theta^{2(u)})}{\rho})^{q}}, \{s_{\frac{\theta^{1(v)}\theta^{2(v)}}{\rho}}(\tilde{r}^{*(v)})\} \rangle;$$

$$(c) \quad pl_{s}^{*1}(r) \otimes pl_{s}^{*2}(r) = \langle \{s_{\frac{\theta^{1}(u)}{\rho}2^{(u)}}(\hat{r}^{*(u)})\}, \{s_{((\phi^{1(v)})^{q} + (\phi^{2(v)})^{q} - (\frac{(\phi^{1(v)})(\phi^{2(v)})}{\rho})^{q}}, \frac{\theta^{*(v)}}{\rho})\}\rangle;$$

$$\begin{array}{ll} (d) & \eta p l_{s}^{*1}(r) = \langle \{s_{(\rho^{q} - \rho^{q}(1 - \frac{(\rho^{1}(u))^{q}}{\rho^{q}})^{\eta}}, \frac{1}{q}(\tilde{r}^{*(u)})\}, \{s_{\rho(\frac{\phi^{1}(v)}{\rho})^{\eta}}(\tilde{r}^{*(v)})\}\rangle; \\ (e) & (nl^{*1}(r))^{\eta} - \langle ls_{s-1}(r), (\hat{r}^{*(u)})\}, \{s_{s-1}(r), (\hat{r}^{*(v)})\}\rangle \\ \end{array}$$

$$(e) \quad (pl_s^{*1}(r))^{\eta} = \langle \{s_{\rho(\frac{\theta^{1(u)}}{\rho})^{\eta}}(\hat{r}^{*(u)})\}, \{s_{(\rho^q - \rho^q(1 - \frac{(\phi^{1(v)})^q}{\rho^q})^{\eta}})^{\frac{1}{q}}(\tilde{r}^{*(v)})\}\rangle.$$

Example 3. Let $S = \{s_{\alpha} | \alpha = 0, 1, 2, \dots, 6\}$, $pl_{s}^{1}(r) = \langle \{s_{4}(0.4), s_{5}(0.6)\}, \{s_{1}(0.7), s_{2}(0.3)\} \rangle$ and $pl_{s}^{2}(r) = \langle \{s_{3}(0.2), s_{4}(0.5), s_{5}(0.3)\}, \{s_{3}(0.5), s_{4}(0.5)\} \rangle$ be two PLQRONs, the modified PLQRONs are $pl_{s}^{*1}(r) = \langle \{s_{4}(0.2), s_{4}(0.2), s_{5}(0.3)\}, \{s_{5}(0.3)\}, \{s_{1}(0.5), s_{1}(0.2), s_{2}(0.3)\} \rangle$ and $pl_s^{*2}(r) = \langle \{s_3(0.2), s_4(0.2), s_4(0.3), s_5(0.3)\}, \{s_3(0.5), s_4(0.2), s_4(0.3)\} \rangle$, The adjustment process is shown in Figure 2. Let $\eta = 0.5$ and q = 3, then we have

 $neg(pl_s^{*1}(r)) = \langle \{s_1(0.5), s_1(0.2), s_2(0.3)\}, \{s_4(0.2), s_4(0.2), s_5(0.3), s_5(0.3)\} \rangle;$ $pl_s^{*1}(r) \oplus pl_s^{*2}(r) = \langle \{s_{4.3621}(0.2), s_{4.7774}(0.2), s_{5.3364}(0.3), s_{5.6217}(0.3)\}, \{s_{0.5}(0.5), s_{0.6667}(0.2), s_{1.3333}(0.3)\} \rangle;$ $pl_s^{*1}(r) \otimes pl_s^{*2}(r) = \langle \{s_2(0.2), s_{2.6667}(0.2), s_{3.333}(0.3), s_{4.1667}(0.3)\}, \{s_{3.0321}(0.5), s_{4.0146}(0.2), s_{4.114}(0.3)\} \rangle;$ $0.5pl_s^{*1}(r) = \langle \{s_{3.2649}(0.2), s_{3.2649}(0.2), s_{4.2321}(0.3), s_{4.2321}(0.3)\}, \{s_{2.4495}(0.5), s_{2.4495}(0.2), s_{3.4641}(0.3)\} \rangle;$ $(pl_s^{*1}(r))^{0.5} = \langle \{s_{4.899}(0.2), s_{4.899}(0.2), s_{5.4772}(0.3), s_{5.4772}(0.3)\}, \{s_{0.794}(0.5), s_{0.794}(0.2), s_{1.5924}(0.3)\} \rangle.$



Figure 2. The adjust process of PLQRONs $pl_s^1(r)$ and $pl_s^2(r)$.

Theorem 1. Let $pl_s^{*1}(r) = \langle \{s_{\theta^{1}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{1}(v)}(\tilde{r}^{*(v)})\} \rangle$ and $pl_s^{*2}(r) = \langle \{s_{\theta^{2}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{2}(v)}(\tilde{r}^{*(v)})\} \rangle$ $(u = 1, 2, \dots, U; v = 1, 2, \dots, V)$ be any two adjusted PLQRONs, $\eta, \eta_1, \eta_2 > 0$, then

- (1) $pl_s^{*1}(r) \oplus pl_s^{*2}(r) = pl_s^{*2}(r) \oplus pl_s^{*1}(r);$
- (2) $pl_s^{*1}(r) \otimes pl_s^{*2}(r) = pl_s^{*2}(r) \otimes pl_s^{*1}(r);$
- (3) $\eta(pl_s^{*1}(r) \oplus pl_s^{*2}(r)) = \eta pl_s^{*1}(r) \oplus \eta pl_s^{*2}(r);$
- (4) $\eta_1 p l_s^{*1}(r) \oplus \eta_2 p l_s^{*1}(r) = (\eta_1 + \eta_2) p l_s^{*1}(r);$
- (5) $(pl_s^{*1}(r))^{\eta_1} \otimes (pl_s^{*1}(r))^{\eta_2} = (pl_s^{*1}(r))^{\eta_1+\eta_2};$
- (6) $(pl_s^{*1}(r))^{\eta} \otimes (pl_s^{*2}(r))^{\eta} = (pl_s^{*1}(r) \otimes pl_s^{*2}(r))^{\eta}.$

Here we prove the property (1) and (3), other properties proof process are similar, we omit them.

(1) By Definition 7, we have

$$\begin{split} pl_{s}^{*1}(r) \oplus pl_{s}^{*2}(r) &= \langle \{s_{((\theta^{1(u)})^{q} + (\theta^{2(u)})^{q} - (\frac{(\theta^{1(u)})(\theta^{2(u)})}{\rho})^{\frac{1}{q}}}(\hat{r}^{*(u)})\}, \{s_{\frac{\phi^{1(v)}\phi^{2(v)}}{\rho}}(\tilde{r}^{*(v)})\}\rangle \\ &= \langle \{s_{((\theta^{2(u)})^{q} + (\theta^{1(u)})^{q} - (\frac{(\theta^{2(u)})(\theta^{1(u)})}{\rho})^{\frac{1}{q}}}(\hat{r}^{*(u)})\}, \{s_{\frac{\phi^{2(v)}\phi^{1(v)}}{\rho}}(\tilde{r}^{*(v)})\}\rangle \\ &= pl_{s}^{*2}(r) \oplus pl_{s}^{*1}(r). \end{split}$$

Therefore $pl_s^{*1}(r) \oplus pl_s^{*2}(r) = pl_s^{*2}(r) \oplus pl_s^{*1}(r)$ is obtained.

(3) By Definition 7, we can get

$$\begin{split} \eta(pl_{s}^{*1}(r) \oplus pl_{s}^{*2}(r)) &= \eta \langle \{s_{((\theta^{1(u)})^{q} + (\theta^{2(u)})^{q} - (\frac{\theta^{1(u)}\theta^{2(u)}}{\rho})^{q}} | \frac{1}{q} (\hat{r}^{*(u)}) \}, \{s_{\frac{\theta^{1(v)}\theta^{2(v)}}{\rho}} (\tilde{r}^{*(v)}) \} \rangle \\ &= \langle \{s_{(\rho^{q} - \rho^{q}(1 - \frac{(\theta^{1(u)})^{q} + (\theta^{2(u)})^{q} - (\frac{(\theta^{1(u)})(\theta^{2(u)})}{\rho})^{q}}{\rho^{q}})^{\eta}} | \frac{1}{q} (\hat{r}^{*(u)}) \}, \{s_{\rho(\frac{\theta^{1(v)}\theta^{2(v)}}{\rho^{2}})^{\eta}} (\tilde{r}^{*(v)}) \} \rangle. \end{split}$$

Moreover, since

$$\begin{split} \eta(pl_{s}^{*1}(r)) &= \langle \{s_{(\rho^{q}-\rho^{q}(1-\frac{(\theta^{1}(u))^{q}}{\rho^{q}})^{\eta}} | \stackrel{1}{\stackrel{q}{\stackrel{}}}(\hat{r}^{*(u)}) \}, \{s_{\rho(\frac{\phi^{1}(v)}{\rho})^{\eta}}(\tilde{r}^{*(v)}) \} \rangle, \\ \eta(pl_{s}^{*2}(r)) &= \langle \{s_{(\rho^{q}-\rho^{q}(1-\frac{(\theta^{2}(u))^{q}}{\rho^{q}})^{\eta}} | \stackrel{1}{\stackrel{q}{\stackrel{}}}(\hat{r}^{*(u)}) \}, \{s_{\rho(\frac{\phi^{2}(v)}{\rho})^{\eta}}(\tilde{r}^{*(v)}) \} \rangle, \end{split}$$

let $\varpi = (\rho^q - \rho^q (1 - \frac{(\theta^{1(u)})^q}{\rho^q})^\eta)^{\frac{1}{q}}$ and $\chi = (\rho^q - \rho^q (1 - \frac{(\theta^{2(u)})^q}{\rho^q})^\eta)^{\frac{1}{q}}$, the above formulas can be denoted as:

$$\begin{split} \eta p l_s^{*1}(r) \oplus \eta p l_s^{*2}(r) &= \langle \{s_{(\omega^q + \chi^q - (\frac{\omega \cdot \chi}{\rho})^q)^{1/q}}(\hat{r}^{*(u)})\}, \{s_{\rho(\frac{\phi^{1(v)}\phi^{2(v)}}{\rho^2})^\eta}(\tilde{r}^{*(v)})\} \rangle \\ &= \langle \{s_{\{\rho^q - \rho^q \cdot [(1 - (\frac{\theta^{1(u)})^q}{\rho^q})^\eta + (1 - (\frac{\theta^{2(u)})^q}{\rho^q})^\eta] - (1 - (1 - (\frac{\theta^{1(u)})^q}{\rho^q})^\eta) \cdot [\rho^q - \rho^q (1 - (\frac{\theta^{2(u)})^q}{\rho^q})^\eta]\}^{\frac{1}{q}}(\hat{r}^{*(u)})\}, \{s_{\rho(\frac{\phi^{1(v)}\phi^{2(v)}}{\rho^q})^\eta}(\tilde{r}^{*(v)})\} \rangle \\ &= \langle \{s_{\{\rho^q - \rho^q \cdot [(1 - (\frac{\theta^{1(u)})^q}{\rho^q})^\eta + (1 - (\frac{\theta^{2(u)})^q}{\rho^q})^\eta] + \rho^q[(1 - (\frac{\theta^{1(u)})^q}{\rho^q})^\eta + (1 - (\frac{\theta^{2(u)})^q}{\rho^q})^\eta - (1 - (\frac{\theta^{1(u)})^q}{\rho^q})^\eta \cdot (1 - (\frac{\theta^{2(u)})^q}{\rho^q})^\eta]\}^{\frac{1}{q}}(\hat{r}^{*(u)})\}, \\ &= \langle \{s_{\{\rho^q - \rho^q \cdot [(1 - (\frac{\theta^{1(u)})^q}{\rho^q})^\eta \cdot (1 - (\frac{\theta^{2(u)})^q}{\rho^q})^\eta]\}^{\frac{1}{q}}(\hat{r}^{*(u)})\}, \{s_{\rho(\frac{\phi^{1(v)}\phi^{2(v)}}{\rho^2})^\eta}(\tilde{r}^{*(v)})\} \rangle \\ &= \langle \{s_{(\rho^q - \rho^q (1 - (\frac{\theta^{1(u)})^q + (\theta^{2(u)})^q - ((\frac{\theta^{1(u)})(\theta^{2(u)}}{\rho^q})^\eta))^{\frac{1}{q}}}(\hat{r}^{*(u)})\}, \{s_{\rho(\frac{\phi^{1(v)}\phi^{2(v)}}{\rho^2})^\eta}(\tilde{r}^{*(v)})\} \rangle \\ &= \eta(p l_s^{*1}(r) \oplus p l_s^{*2}(r)). \end{split}$$

Therefore $\eta(pl_s^{*1}(r) \oplus pl_s^{*2}(r)) = \eta pl_s^{*1}(r) \oplus \eta pl_s^{*2}(r)$ is proved.

In order to compare the order relation of PLQROSs, we present the comparison rules as follows:

Definition 8. Assume $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a LTS, for any adjusted PLQRON $pl_s^*(r) = \langle \{s_{\theta^{(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{(v)}}(\tilde{r}^{*(v)})\}\rangle$, where $s_{\theta^{(u)}}, s_{\phi^{(v)}} \in \tilde{S}_{[0,\rho]}$, $(u = 1, 2, \cdots, U; v = 1, 2, \cdots, V)$, the score function of $pl_s^*(r)$ is

$$A(pl_{s}^{*}(r)) = \sum_{u=1}^{\#U_{\theta}} (\frac{\theta^{(u)} \cdot \hat{r}^{*(u)}}{\rho})^{q} - \sum_{v=1}^{\#V_{\phi}} (\frac{\phi^{(v)} \cdot \tilde{r}^{*(v)}}{\rho})^{q},$$
(1)

where $\theta^{(u)}, \phi^{(v)} \in [0, \rho]$, # U_{θ} and # V_{ϕ} represent the number of elements in the corresponding set, respectively.

The accuracy function of $pl_s^*(r)$ is

$$H(pl_{s}^{*}(r)) = \sum_{u=1}^{\#U_{\theta}} \left(\frac{\theta^{(u)} \cdot \tilde{r}^{*(u)}}{\rho}\right)^{q} + \sum_{v=1}^{\#V_{\phi}} \left(\frac{\phi^{(v)} \cdot \tilde{r}^{*(v)}}{\rho}\right)^{q},$$
(2)

where $\theta^{(u)}$, $\phi^{(v)} \in [0, \rho]$, # U_{θ} and # V_{ϕ} represent the number of elements in the corresponding set, respectively.

Theorem 2. Let $pl_s^{*1}(r) = \langle \{s_{\theta^{1}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{1}(v)}(\tilde{r}^{*(v)})\} \rangle$ and $pl_s^{*2}(r) = \langle \{s_{\theta^{2}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{2}(v)}(\tilde{r}^{*(v)})\} \rangle$ be two adjusted PLQRONs. $A(pl_s^{*1}(r))$ and $A(pl_s^{*2}(r))$ are the score function of $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$, the accuracy function of $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$ are $H(pl_s^{*1}(r))$ and $H(pl_s^{*2}(r))$, respectively, then the order relation of $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$ are given as follows: (1) If $A(pl_s^{*1}(r)) > A(pl_s^{*2}(r))$, then $pl_s^{*1}(r) > pl_s^{*2}(r)$;

(2) If $A(pl_s^{*1}(r)) = A(pl_s^{*2}(r))$, then

- (a) If $H(pl_s^{*1}(r)) = H(pl_s^{*2}(r))$, then $pl_s^{*1}(r) \simeq pl_s^{*2}(r)$;
- (b) If $H(pl_s^{*1}(r)) < H(pl_s^{*2}(r))$, then $pl_s^{*1}(r) \prec pl_s^{*2}(r)$;
- (c) If $H(pl_s^{*1}(r)) > H(pl_s^{*2}(r))$, then $pl_s^{*1}(r) \succ pl_s^{*2}(r)$;
- (3) If $A(pl_s^{*1}(r)) < A(pl_s^{*2}(r))$, then $pl_s^{*1}(r) \prec pl_s^{*2}(r)$.

3.3. The Aggregation Operators of PLQROSs

In order to aggregate the multi-attribute information well, we introduce the aggregation operators of PLQRONs as follows.

Definition 9. Let $\bar{S}_{[0,\rho]}$ $(\rho > 2g)$ be a LTS, $pl_s^{*1}(r) = \langle \{s_{\rho^{l}(u)}(\hat{r}^{*(u)})\}, \{s_{\sigma^{l}(v)}(\hat{r}^{*(v)})\}\rangle (\iota = 1)$ 1, 2, · · · , n; u = 1, 2, . · · , U; v = 1, 2, . · · , V) are n adjusted PLQRONs, where $s_{\theta^{\iota(u)}}, s_{\phi^{\iota(v)}} \in \bar{S}_{[0,\rho]}$, the probabilistic linguistic q-rung orthopair weighted averaging (PLQROWA) operator can be *expressed as:*

$$PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) = \omega_{1}pl_{s}^{*1}(r) \oplus \omega_{2}pl_{s}^{*2}(r) \oplus \cdots \oplus \omega_{n}pl_{s}^{*n}(r)$$

$$= \langle \{s_{(\rho^{q} - \rho^{q}\prod_{l=1}^{n}(1 - \frac{(\theta^{l}(u))^{q}}{\rho^{q}})^{\omega_{l}})^{\frac{1}{q}}(\hat{r}^{*(u)})\}, \{s_{\prod_{l=1}^{n}\rho(\frac{\phi^{l}(v)}{\rho})^{\omega_{l}}}(\tilde{r}^{*(v)})\}\rangle,$$
(3)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector, and it satisfies $\sum_{l=1}^n \omega_l = 1 (0 \le \omega_l \le 1)$.

Theorem 3. Let $pl_s^{*\iota}(r) = \langle \{s_{\theta^{\iota(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{\iota(v)}}(\tilde{r}^{*(v)})\}\rangle (\iota = 1, 2, \cdots, n; u = 1, 2, \cdots, U;$ $v = 1, 2, \cdots, V$ be ι adjusted PLQRONs, the weight ω_{ι} ($\iota = 1, 2, \cdots, n$) satisfies with $0 \le \omega_{\iota} \le 1$ and $\sum_{\iota=1}^{n} \omega_{\iota} = 1$, then the properties of PLQROWA are shown as follows:

(1) Idempotency: if $pl_s^{*\iota}(r)$ ($\iota = 1, 2, \dots, n$) are equal, i.e., $pl_s^{*\iota}(r) = pl_s^*(r) \triangleq \langle \{s_{\theta^{(u)}}(\hat{r}^{*(u)})\},$ $\{s_{\phi^{(v)}}(\tilde{r}^{*(v)})\}\rangle$, then

$$PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) = pl_{s}^{*}(r) = \langle \{s_{\theta^{(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{(v)}}(\tilde{r}^{*(v)})\} \rangle.$$

Monotonicity: let $pl_{s}^{*1}(r)$, $pl_{s}^{*2}(r)$, \cdots , $pl_{s}^{*n}(r)$ and $pl_{s}^{'*1}(r)$, $pl_{s}^{'*2}(r)$, \cdots , $pl_{s}^{'*n}(r)$ be two (2) collections of adjusted PLQRONs, for all ι , $s_{\theta^{\mu(u)}} < s'_{\theta^{\mu(u)}}$ and $s_{\theta^{\mu(v)}} > s'_{\theta^{\mu(v)}}$, then

$$PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) < PLQROWA(pl_{s}^{'*1}(r), pl_{s}^{'*2}(r), \cdots, pl_{s}^{'*n}(r)).$$

(3) Boundedness: let $s_{\theta^{l}(+)} = \max_{u=1}^{U} s_{\theta^{l}(u)}, s_{\theta^{l}(-)} = \min_{u=1}^{U} s_{\theta^{l}(u)}, s_{\phi^{l}(+)} = \max_{v=1}^{V} s_{\phi^{l}(v)}$ and $s_{\phi^{l}(-)} = \sum_{v=1}^{V} s_{\theta^{l}(v)} s_{\theta^{l}(v)} s_{\theta^{l}(v)} = \sum_{v=1}^{V} s_{\theta^{l}(v)} s_{\theta^{l}(v)} s_{\theta^{l}(v)} = \sum_{v=1}^{V} s_{\theta^{l}(v)} s_{\theta^{l}($ $\min_{v=1} s_{\phi^{\iota(v)}}$, then

$$\langle \{s_{\theta^{l(-)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{l(+)}}(\tilde{r}^{*(v)})\} \rangle \leq PLQROWA(pl_s^{*1}(r), pl_s^{*2}(r), \cdots, pl_s^{*n}(r)) \leq \langle \{s_{\theta^{l(+)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{l(-)}}(\tilde{r}^{*(v)})\} \rangle.$$

(1)

$$(1) \text{ For all } \iota, \text{ since } pl_{s}^{*\iota}(r) = pl_{s}^{*}(p) = \langle \{s_{\theta^{(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{(v)}}(\tilde{r}^{*(v)})\} \rangle, \text{ then}$$

$$PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) = \omega_{1}pl_{s}^{*1}(r) \oplus \omega_{2}pl_{s}^{*2}(r) \oplus \cdots \oplus \omega_{n}pl_{s}^{*n}(r)$$

$$= \langle \{s_{(\rho^{q} - \rho^{q}\prod_{l=1}^{n}(1 - \frac{(\theta^{l}(u))^{q}}{\rho^{q}})\omega_{l})\frac{1}{q}}(\hat{r}^{*(u)})\}, \{s_{\prod_{l=1}^{n}\rho(\frac{\theta^{l}(v)}{\rho})\omega_{l}}(\tilde{r}^{*(v)})\} \rangle$$

$$= \langle \{s_{(\rho^{q} - \rho^{q}(1 - \frac{(\theta^{l}(u))^{q}}{\rho^{q}})\sum_{l=1}^{n}\omega_{l})\frac{1}{q}}(\hat{r}^{*(u)})\}, \{s_{\rho(\frac{\theta^{l}(v)}{\rho})}\sum_{l=1}^{n}\omega_{l}}(\tilde{r}^{*(v)})\} \rangle$$

$$= \langle \{s_{\theta^{(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{(v)}}(\tilde{r}^{*(v)})\} \rangle.$$

Therefore $PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) = \langle \{s_{\theta^{(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{(v)}}(\tilde{r}^{*(v)})\} \rangle$ is proved.

(2) For all ι , $s_{\theta^{\iota(u)}} < s'_{\theta^{\iota(u)}}$ and $s_{\phi^{\iota(v)}} > s'_{\phi^{\iota(v)}}$, then we have

$$s'_{1-\frac{\theta^{l}(u)}{\rho}} < s_{1-\frac{\theta^{l}(u)}{\rho}} \Rightarrow s_{(\rho-\rho\prod_{l=1}^{n}(1-(\frac{\theta^{l}(u)}{\rho})^{q})^{\omega_{l}})^{\frac{1}{q}}} < s'_{(\rho-\rho\prod_{l=1}^{n}(1-(\frac{\theta^{l}(u)}{\rho})^{q})^{\omega_{l}})^{\frac{1}{q}}}, s_{\prod_{l=1}^{n}(\phi^{l}(v))^{\omega_{l}}} > s'_{\prod_{l=1}^{n}(\phi^{l}(v))^{\omega_{l}}}.$$
Assume *PLQROWA*($pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)$) = $pl_{s}(r)$ and *PLQROWA*($pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)$) = $pl_{s}(r)$ and *PLQROWA*($pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)$) = $pl_{s}(r)$ and *PLQROWA*($pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)$) = $pl_{s}(r)$, by (1), we can get

$$A(pl_{s}^{*}(r)) = \sum_{u=1}^{\#U_{\theta}} \left(\frac{(\rho - \rho \cdot \prod_{i=1}^{n} (1 - (\frac{\theta^{(u)}}{\rho})^{q})^{\omega_{i}})^{\frac{1}{q}}}{\rho} \cdot (\hat{r}^{*(u)}))^{q} - \sum_{v=1}^{\#V_{\phi}} (\frac{(\phi^{(v)})^{\omega_{i}}}{\rho} \cdot \tilde{r}^{*(v)})^{q};$$

$$A(pl_{s}^{'*}(r)) = \sum_{u=1}^{\#U_{\theta^{'}}} (\frac{(\rho - \rho \cdot \prod_{\iota=1}^{n} (1 - (\frac{\theta^{\prime(u)}}{\rho})^{q})^{\omega_{\iota}})^{\frac{1}{q}}}{\rho} \cdot (\hat{r'}^{*(u)}))^{q} - \sum_{v=1}^{\#V_{\phi^{'}}} (\frac{(\phi^{\prime(v)})^{\omega_{\iota}}}{\rho} \cdot \tilde{r'}^{*(v)})^{q}.$$

Then we have $A(pl_{s}^{*}(r)) < A(pl_{s}^{**}(r))$, that is $pl_{s}^{*}(r) < pl_{s}^{**}(r)$. Therefore $PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) < PLQROWA(pl_{s}^{'*1}(r), pl_{s}^{'*2}(r), \cdots, pl_{s}^{'*n}(r))$ is proved.

(3) For all $\iota, s_{\theta^{l(-)}} \leq s_{\theta^{l(u)}} \leq s_{\theta^{l(+)}}, s_{\phi^{l(-)}} \leq s_{\phi^{l(v)}} \leq s_{\phi^{l(+)}}$, according to the properties (1) and (2), we can easily have

$$\langle \{s_{\theta^{l}(-)}(\hat{r}^{*(u)})\}, \{s_{\phi^{l}(+)}(\tilde{r}^{*(u)})\} \rangle \leq PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) \leq \langle \{s_{\theta^{l}(+)}(\hat{r}^{*(u)})\}, \{s_{\phi^{l}(-)}(\tilde{r}^{*(v)})\} \rangle$$

Remark 2. Especially, when $\omega_{\iota} = \frac{1}{n}(\iota = 1, 2, \dots, n)$, the PLQROWA operator is reduced to a probabilistic linguistic q-rung orthopair averaging (PLQROA) operator:

$$PLQROA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) = \frac{1}{n}pl_{s}^{*1}(r) \oplus \frac{1}{n}pl_{s}^{*2}(r) \oplus \cdots \oplus \frac{1}{n}pl_{s}^{*n}(r)$$
$$= \langle \{s_{(\rho^{q} - \rho^{q}\prod_{i=1}^{n}(1 - \frac{(\rho^{i}(u))^{q}}{\rho^{q}})^{\frac{1}{n}})^{\frac{1}{q}}(\hat{r}^{*(u)})\}, \{s_{\prod_{i=1}^{n}\rho(\frac{\phi^{i}(v)}{\rho})^{\frac{1}{n}}}(\hat{r}^{*(v)})\}\rangle$$

Definition 10. Let $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a LTS, $pl_s^{*\iota}(r) = \langle \{s_{\theta^{\iota(u)}}(\hat{r}^{*(u)})\}, \{s_{\phi^{\iota(v)}}(\tilde{r}^{*(v)})\}\rangle(\iota = 1, 2, \cdots, n; u = 1, 2, \cdots, U; v = 1, 2, \cdots, V)$ is a collection of adjusted PLQRONs, where $s_{\theta^{\iota(u)}}, s_{\phi^{\iota(v)}} \in \bar{S}_{[0,\rho]}$, the probabilistic linguistic q-rung orthopair weighted geometric (PLQROWG) operator is given as follows:

$$PLQROWG(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) = (pl_{s}^{*1}(r))^{\omega_{1}} \oplus (pl_{s}^{*2}(r))^{\omega_{2}} \oplus \cdots \oplus (pl_{s}^{*n}(r))^{\omega_{n}} \\ = \langle \{s_{\prod_{l=1}^{n} \rho(\frac{\rho^{l}(\omega)}{\rho})^{\omega_{l}}}(\tilde{r}^{*(\omega)})\}, \{s_{(\rho^{q}-\rho^{q}\prod_{l=1}^{n}(1-\frac{(\phi^{l}(\upsilon))^{q}}{\rho^{q}})^{\omega_{l}}})^{\frac{1}{q}}(\tilde{r}^{*(\upsilon)})\}\rangle,$$
(4)

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector and satisfies with $\sum_{l=1}^n \omega_l = 1 (0 \le \omega_l \le 1)$.

Theorem 4. The PLQROWG operator satisfies the properties in Theorem 3.

Proof. Because the proof is similar to Theorem 3, we omit it here. \Box

Remark 3. Especially, when $\omega_{\iota} = \frac{1}{n}(\iota = 1, 2, \dots, n)$, the PLQROWG operator is degenerated into the probabilistic linguistic q-rung orthopair geometric (PLQROG) operator

$$\begin{aligned} PLQROG(pl_{s}^{*1}(r), pl_{s}^{*2}(r), \cdots, pl_{s}^{*n}(r)) &= (pl_{s}^{*1}(r))^{\frac{1}{n}} \otimes (pl_{s}^{*2}(r))^{\frac{1}{n}} \otimes \cdots \otimes (pl_{s}^{*n}(r))^{\frac{1}{n}} \\ &= \langle \{s_{\prod_{l=1}^{n} \rho(\frac{\theta^{l}(u)}{\rho})^{\frac{1}{n}}}(\hat{r}^{*(u)})\}, \{s_{(\rho^{q} - \rho^{q}\prod_{l=1}^{n}(1 - \frac{(\phi^{l}(v))^{q}}{\rho^{q}})^{\frac{1}{n}}})^{\frac{1}{q}}}(\tilde{r}^{*(v)})\} \rangle. \end{aligned}$$

Example 4. Let $\bar{S} = \{s_{\alpha} | 0 \le \alpha \le 6\}$ be a LTS, $pl_s^1(r) = \langle \{s_2(1)\}, \{s_1(0.9), s_2(0.1)\} \rangle$, $pl_s^2(r) = \langle \{s_1(0.2), s_2(0.6), s_3(0.2)\}, \{s_1(0.9), s_2(0.1)\} \rangle$ and $pl_s^3(r) = \langle \{s_2(0.8), s_3(0.2)\}, \{s_0(0.3), s_1(0.5), s_2(0.2)\} \rangle$ be three PLQRONs, $\omega = (0.3, 0.5, 0.2)^T$ is the corresponding weight vector, then the calculation results of the PLQROWA and the PLQROWG are given as follows.

Firstly, we adjust the corresponding probability distributions of $pl_s^1(r)$, $pl_s^2(r)$ and $pl_s^3(r)$, the adjusted PLQRONs obtained as follows:

$$pl_{s}^{*1}(r) = \langle \{s_{2}(0.2), s_{2}(0.6), s_{2}(0.2)\}, \{s_{1}(0.3), s_{1}(0.5), s_{1}(0.1), s_{2}(0.1)\} \rangle;$$

$$pl_{s}^{*2}(r) = \langle \{s_{1}(0.2), s_{2}(0.6), s_{3}(0.2)\}, \{s_{1}(0.3), s_{1}(0.5), s_{1}(0.1), s_{2}(0.1)\} \rangle;$$

$$pl_{s}^{*3}(r) = \langle \{s_{2}(0.2), s_{2}(0.6), s_{3}(0.2)\}, \{s_{0}(0.3), s_{1}(0.5), s_{2}(0.1), s_{2}(0.1)\} \rangle.$$

If q = 3, according to the Formula (3), we can get

$$\begin{split} PLQROWA(pl_{s}^{*1}(r), pl_{s}^{*2}(r), pl_{s}^{*3}(r)) &= \langle \{s_{(6^{3}-6^{3}\prod_{l=1}^{3}(1-\frac{(\theta^{l}(u))^{3}}{6^{3}})^{\omega_{l}})^{\frac{1}{3}}}(\hat{r}^{*(u)})\}, \{s_{\prod_{l=1}^{3}6(\frac{\phi^{l}(v)}{6})^{\omega_{l}}}(\tilde{r}^{*(v)})\}\rangle \\ &= \langle \{s_{(6^{3}-6^{3}(1-\frac{2^{3}}{6^{3}})^{0.5}(1-\frac{1^{3}}{6^{3}})^{0.5})^{\frac{1}{3}}}(0.2), s_{(6^{3}-6^{3}(1-\frac{2^{3}}{6^{3}})^{1})^{\frac{1}{3}}}(0.6), s_{(6^{3}-6^{3}(1-\frac{2^{3}}{6^{3}})^{0.3}(1-\frac{3^{3}}{6^{3}})^{0.7})^{\frac{1}{3}}}(0.2)\}, \\ &\{s_{6(\frac{1}{6})^{0.8}(\frac{0}{6})^{0.2}}(0.3), s_{6(\frac{1}{6})^{1}}(0.5), s_{6(\frac{1}{6})^{0.8}(\frac{2}{6})^{0.2}}(0.1), s_{6(\frac{2}{6})^{1}}(0.1)\}\rangle \\ &= \langle \{s_{1.65}(0.2), s_{2}(0.6), s_{2.98}(0.2)\}, \{s_{0}(0.3), s_{1}(0.5), s_{1.15}(0.1), s_{2}(0.1)\}\rangle. \end{split}$$

If q = 3, according to the Formula (4), we can get

$$PLQROWG(pl_{s}^{*1}(r), pl_{s}^{*2}(r), pl_{s}^{*3}(r)) = \langle \{s_{\prod_{i=1}^{3} 6(\frac{\theta^{i}(u)}{6})^{\omega_{i}}}(\hat{r}^{*(u)})\}, \{s_{(6^{3}-6^{3}\prod_{i=1}^{3}(1-\frac{(\theta^{i}(v))^{3}}{6^{3}})^{\omega_{i}})^{\frac{1}{3}}}(\tilde{r}^{*(v)})\}\rangle$$

$$= \langle \{s_{6(\frac{1}{6})^{0.5}(\frac{2}{6})^{0.5}}(0.2), s_{6(\frac{2}{6})^{1}}(0.6), s_{6(\frac{2}{6})^{0.2}(\frac{3}{6})^{0.7}}(0.2)\}, \{s_{(6^{3}-6^{3}(1-\frac{13}{6^{3}})^{0.8}(1-\frac{(0)^{3}}{6^{3}})^{0.2}})^{\frac{1}{3}}(0.3), s_{(6^{3}-6^{3}(1-\frac{13}{6^{3}})^{1})^{\frac{1}{3}}}(0.5), s_{(6^{3}-6^{3}(1-\frac{23}{6^{3}})^{1})^{\frac{1}{3}}}(0.1)\}\rangle$$

$$= \langle \{s_{1.41}(0.2), s_{2}(0.6), s_{2.66}(0.2)\}, \{s_{0}(0.3), s_{1}(0.5), s_{1.34}(0.1), s_{2}(0.1)\}\rangle.$$

3.4. Distance Measures between PLQRONs

In order to compare the differences between different alternatives, we introduced the distance measure between PLQRONs, which is an important tool to process multiattribute decision problems. In this subsection, we first propose the distance measures between PLQRONs.

Definition 11. Let $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a LTS, $pl_s^{*1}(r) = \langle H_s^{*1}(\hat{r}), G_s^{*1}(\hat{r}) \rangle = \langle \{s_{\theta^{1}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{1}(v)}(\hat{r}^{*(v)})\} \rangle$ and $pl_s^{*2}(r) = \langle H_s^{*2}(\hat{r}), G_s^{*2}(\hat{r}) \rangle = \langle \{s_{\theta^{2}(u)}(\hat{r}^{*(u)})\}, \{s_{\phi^{2}(v)}(\hat{r}^{*(v)})\} \rangle$ ($u = 1, 2, \cdots, U$; $v = 1, 2, \cdots, V$) are two adjusted PLQRONs, where $s_{\theta^{i}(u)}, s_{\phi^{i}(v)} \in \bar{S}_{[0,\rho]}$ (i = 1, 2), the Hamming distance measure D_{dhd} between $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$ can be defined as:

$$D_{dhd}(pl_s^{*1}(r), pl_s^{*2}(r)) = \sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^q - (\theta^{2(u)})^q|}{\rho^q}) + \sum_{v=1}^{\#V_{\phi}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^q - (\phi^{2(v)})^q|}{\rho^q}),$$
(5)

where $q \ge 1$, $\theta^{\iota(u)}$ and $\phi^{\iota(v)}$ ($\iota = 1, 2$) are the subscripts of $s_{\theta^{\iota(u)}}$ and $s_{\phi^{\iota(v)}}$ ($\iota = 1, 2$), # U_{θ} and # V_{ϕ} represent the number of elements in $H_s^{*\iota}(\hat{r})$ and $G_s^{*\iota}(\tilde{r})$ ($\iota = 1, 2$), respectively.

The Euclidean distance measure D_{ded} between $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$ can be defined as follows:

$$D_{ded}(pl_s^{*1}(r), pl_s^{*2}(r)) = \sqrt{\sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^q - (\theta^{2(u)})^q|}{\rho^q})^2 + \sum_{v=1}^{\#V_{\theta}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^q - (\phi^{2(v)})^q|}{\rho^q})^2},$$
(6)

where $q \ge 1$, $\theta^{\iota(u)}$ and $\phi^{\iota(v)}$ ($\iota = 1, 2$) are the subscripts of $s_{\theta^{\iota(u)}}$ and $s_{\phi^{\iota(v)}}$ ($\iota = 1, 2$), # U_{θ} and $#V_{\phi}$ represent the number of elements in $H_s^{*\iota}(\hat{r})$ and $G_s^{*\iota}(\tilde{r})$ ($\iota = 1, 2$), respectively.

The generalized distance measure D_{dgd} between $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$ can be defined as:

$$D_{dgd}(pl_s^{*1}(r), pl_s^{*2}(r)) = \sqrt[\lambda]{\sum_{u=1}^{\lambda} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^q - (\theta^{2(u)})^q|}{\rho^q})^{\lambda}} + \sum_{v=1}^{\#V_{\phi}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^q - (\phi^{2(v)})^q|}{\rho^q})^{\lambda},$$
(7)

where $\lambda > 0$, $q \ge 1$, $\theta^{\iota(u)}$ and $\phi^{\iota(v)}$ ($\iota = 1, 2$) are the subscripts of $s_{\theta^{\iota(u)}}$ and $s_{\phi^{\iota(v)}}$ ($\iota = 1, 2$), $#U_{\theta}$ and $#V_{\phi}$ represent the number of elements in $H_s^{*\iota}(\hat{r})$ and $G_s^{*\iota}(\tilde{r})$ ($\iota = 1, 2$), respectively.

Remark 4. In particular, if $\lambda = 1$ or $\lambda = 2$, D_{dgd} is degenerated into D_{dhd} or D_{ded} , respectively.

Theorem 5. Assume $pl_s^{*1}(r)$ and $pl_s^{*2}(r)$ are two adjusted PLQRONs, the distance measure D_{dgd} satisfies the following properties:

- (1) Non-negativity: $0 \le D_{dgd}(pl_s^{*1}(r), pl_s^{*2}(r)) \le 1, D_{dgd}(pl_s^{*1}(r), pl_s^{*1}(r)) = 0;$
- (2) Symmetry: $D_{dgd}(pl_s^{*1}(r), pl_s^{*2}(r)) = D_{dgd}(pl_s^{*1}(r), pl_s^{*1}(r));$ (3) Triangle inequality: $D_{dgd}(pl_s^{*1}(r), pl_s^{*2}(r)) + D_{dgd}(pl_s^{*2}(r), pl_s^{*3}(p)) \ge D_{dgd}(pl_s^{*1}(r), pl_s^{*1}(r))$ $pl_{s}^{*3}(p)).$

Obviously, D_{dgd} satisfies the property (1) and (2). The symmetry information can be expressed by the distance measure D_{dgd} .

The proof of property (3) is given as follows:

$$D_{dgd}(pl_{s}^{*1}(r), pl_{s}^{*3}(r)) = \sqrt[\lambda]{\sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^{q} - (\theta^{3(u)})^{q}|}{\rho^{q}})^{\lambda} + \sum_{v=1}^{\#V_{\theta}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^{q} - (\phi^{3(v)})^{q}|}{\rho^{q}})^{\lambda}}$$

$$= \sqrt[\lambda]{\sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^{q} - (\theta^{2(u)})^{q} + (\theta^{2(u)})^{q} - (\theta^{3(u)})^{q}|}{\rho^{q}})^{\lambda} + \sum_{v=1}^{\#V_{\theta}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^{q} - (\phi^{2(v)})^{q} + (\phi^{2(v)})^{q} - (\phi^{3(v)})^{q}|}{\rho^{q}})^{\lambda}}}$$

$$\leq \sqrt[\lambda]{\sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^{q} - (\theta^{2(u)})^{q}| + |(\theta^{2(u)})^{q} - (\theta^{3(u)})^{q}|}{\rho^{q}})^{\lambda} + \sum_{v=1}^{\#V_{\theta}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^{q} - (\phi^{2(v)})^{q}| + |(\phi^{2(v)})^{q} - (\phi^{3(v)})^{q}|}{\rho^{q}})^{\lambda}}$$

$$\leq \sqrt[\lambda]{\sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{1(u)})^{q} - (\theta^{2(u)})^{q}|}{\rho^{q}})^{\lambda} + \sum_{v=1}^{\#L_{\theta}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{1(v)})^{q} - (\phi^{2(v)})^{q}|}{\rho^{q}})^{\lambda}} + \sqrt[\lambda]{\sum_{u=1}^{\#U_{\theta}} (\hat{r}^{*(u)} \cdot \frac{|(\theta^{2(u)})^{q} - (\phi^{3(v)})^{q}|}{\rho^{q}})^{\lambda} + \sum_{v=1}^{\#V_{\theta}} (\tilde{r}^{*(v)} \cdot \frac{|(\phi^{2(v)})^{q} - (\phi^{3(v)})^{q}|}{\rho^{q}})^{\lambda}}$$

Therefore $D_{dgd}(pl_s^{*1}(r), pl_s^{*3}(r)) \leq D_{dgd}(pl_s^{*1}(r), pl_s^{*2}(r)) + D_{dgd}(pl_s^{*2}(r), pl_s^{*3}(r))$ is proved.

Example 5. Assume $\bar{S} = \{s_{\alpha} | 0 \le \alpha \le 6\}$ is a LTS, $pl_s^{*1}(r) = \{\{s_2(0.2), s_2(0.6), s_2(0.2)\}, \}$ $\{s_3(0.4), s_4(0.1), s_4(0.3), s_5(0.2)\}\$ and $pl_s^{*2}(r) = \langle\{s_1(0.2), s_2(0.6), s_3(0.2)\}, \{s_5(0.4), s_5(0.1), s_5(0.2)\}\$ $s_6(0.3), s_6(0.2)$ are two adjusted PLQRONs. If q = 2, the calculation result of $D_{dgd}(pl_s^{*1}(r), r)$ $pl_s^{*2}(r)$) is given as follows:

$$\begin{split} D_{dgd}(pl_s^{*1}(r), pl_s^{*2}(r)) &= ((\frac{0.2 * |(2^2 - 1^2)|}{6^2})^{\lambda} + (\frac{0.6 * |(2^2 - 2^2)|}{6^2})^{\lambda} + (\frac{0.2 * |(2^2 - 3^2)|2}{6^2})^{\lambda} + (\frac{0.4 * |(3^2 - 5^2)|}{6^2})^{\lambda} + (\frac{0.1 * |(4^2 - 5^2)|}{6^2})^{\lambda} + (\frac{0.3 * |(4^2 - 6^2)|}{6^2})^{\lambda} + (\frac{0.2 * |(5^2 - 6^2)|}{6^2})^{\lambda})^{\frac{1}{\lambda}}. \end{split}$$

If $\lambda = 1$, we can get $D_{dhd} = 0.475$. If $\lambda = 2$, we can get $D_{ded} = 0.2545$.

4. The Behavioral Decision Method

Since Hwang and Yoon [16] proposed the TOPSIS method, it has been widely applied in solving multiple criteria group decision making (MCGDM) problems. The traditional TOPSIS [17,18] method is an effective method in ranking the alternative. However, in practical decision making problems, the conditions in traditional TOPSIS method does not consider the behavior factors of decision makers. Thus, Yoon and Kim [23] introduced a behavioral TOPSIS method, which consider the behavioral tendency of decision makers and incorporate it into traditional TOPSIS method. However, in the uncertain decision making environment, how to represent the decision maker's behavior factors is a difficult problem. In order to solve this problem, we deal with it as follows. The gain can be viewed as the earns from taking the alternative instead of the anti-ideal solution, and the loss can be considered as the decision maker's pays from taking the alternative instead of the ideal solution, they can be expressed by the distance measure of related uncertain sets. So the behavioral TOPSIS method contains the loss aversion of decision maker in behavioral economics, and the decision maker can select the appropriate loss aversion ratio to express his/her choice preference. The method is proved to give a better choice than other methods (including traditional TOPSIS method) particularly in many fields, such as emergency decision making, selection for oil pipeline routes, etc., because the behavioral TOPSIS method precisely reflects the behavior tendency of decision maker.

Assume a group of experts $e = \{e_1, e_2, \dots, e_W\}$ evaluate a series of alternatives $Q = \{Q_1, Q_2, \dots, Q_m\}$ under the criteria $C = \{C_1, C_2, \dots, C_n\}$, let $\bar{S}_{[0,\rho]}$ ($\rho > 2g$) be a continuous LTS, the evaluation of experts are represented in the form of PLQRONs $l_s^{o_l}(r) = \langle H_s^{o_l}(\hat{r}), G_s^{o_l}(\tilde{r}) \rangle$, where $H_s^{o_l}(\hat{r}) = \{s_{\phi^{o_l}(v)}(\hat{r}^{(u)})|s_{\phi^{o_l}(v)} \in \bar{S}_{[0,\rho]}, \hat{r}^{(u)} \ge 0, u = 1, 2, \cdots, U;$ $\sum_{u=1}^{U} \hat{r}^{(u)} \le 1\}$ and $G_s^{o_l}(\tilde{r}) = \{s_{\phi^{o_l}(v)}(\tilde{r}^{(v)})|s_{\phi^{o_l}(v)} \in \bar{S}_{[0,\rho]}, \tilde{r}^{(v)} \ge 0, v = 1, 2, \cdots, V; \sum_{v=1}^{V} \tilde{r}^{(v)} \le 1\}, o = 1, 2, \dots, m; \iota = 1, 2, \dots, n$. The criteria's weight vector are $\omega_{eu} = (\omega_{e1}, \omega_{e2}, \dots, \omega_{em})^T$, where $\sum_{l=1}^{n} \omega_{cl} = 1$ ($0 \le \omega_{eu} \le 1$), the experts' weight vector are $\omega_{ew} = (\omega_{e1}, \omega_{e2}, \dots, \omega_{em})^T$ ($\sum_{w=1}^{W} \omega_{ew} = 1, 0 \le \omega_{ew} \le 1$), then the *w*th expert's decision matrix $F^{(w)}$ can be given as follows:

$$F^{(w)} = \begin{pmatrix} P_{11}^{w} & P_{12}^{w} & \dots & P_{1n}^{w} \\ P_{21}^{w} & P_{22}^{w} & \dots & P_{2n}^{w} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m1}^{w} & P_{m2}^{w} & \dots & P_{mn}^{w} \end{pmatrix},$$

where $P_{ol}^w = (P_s^{ol}(r))^w (o = 1, 2, ..., m; i = 1, 2, ..., n; w = 1, 2, ..., W)$ are PLQRONs. The steps of decision making are given as follows:

Step 1. Apply the adjustment method to adjust the probability distribution of PLQRONs, the adjusted decision matrix of the *w*th expert can be denoted as $F^{(*w)} = (P_s^{*ol}(r))_{m \times n}^w$.

Step 2. Apply the PLQROWA operator or PLQROWG operator to obtain the aggregated decision matrix $F^{(*)} = (Ps^{*ot}(r))_{m \times n}$. Furthermore, normalize the aggregated decision matrix $F^{(*)}$ based on the type of criteria. If it is a benefit-type criterion, there is no need to adjust; if it is a cost-type criterion, we need utilize the negation operator to normalize the decision matrix.

Step 3. Determine the ideal solution $Q^+ = \{Q_1^+, Q_2^+, \dots, Q_i^+\}$ and the anti-ideal solution $Q^- = \{Q_1^-, Q_2^-, \dots, Q_i^-\}$, respectively, where

$$Q_{\iota}^{+} = \{ \max_{o=1}^{m} \{ pl_{s}^{*o\iota}(p) \} \}, \ Q_{\iota}^{-} = \{ \min_{o=1}^{m} \{ pl_{s}^{*o\iota}(p) \} \}.$$

For the criterion C_i , we apply the Formulas (1) and (2) to calculate Q^+ and Q^- .

Step 4. Utilize D_{dgd} to calculate the distance between each alternative and Q^+ , Q^- , respectively. That is D_o^+ and D_o^- , where $D_o^+ = \sum_{i=1}^n \omega_{ci} D_{dgd}(Q_o, Q^+)$, and $D_o^- = \sum_{i=1}^n \omega_{ci} D_{dgd}(Q_o, Q^-)$. Step 5. Calculate the value function V_o for alternative $Q_o(o = 1, 2, ..., m)$.

$$V_o = (D_o^-)^{\alpha} + [-\gamma (D_o^+)^{\beta}], (0 \le \alpha \le 1, 0 \le \beta \le 1)$$

where γ is the decision maker's loss aversion ratio, if $\gamma > 1$, it implies the decision maker's behavior is more sensitive to losses than gains; if $\gamma = 1$, it implies the decision maker have neutral attitude towards losses or gains; if $\gamma < 1$, it means the decision maker's behavior is more sensitive to gains than losses; α and β reflects the decision maker's risk aversion attitudes and the risk seeking attitudes in decision process, respectively.

Step 6. The greater value of V_o , the better alternatives Q_o will be, then we can obtain the rank of the alternatives.

5. Numerical Example

Here we present a practical multiple criteria group decision making example about investment decision (Beg et al. [27]), and the behavioral TOPSIS method is utilized to deal with this problem. The advantages of the behavioral TOPSIS method with PLQROSs are highlighted by the comparison analysis with the traditional TOPSIS method. Furthermore, we analyzed the stability and sensitivity of decision makers' behavior.

5.1. Background

There are three investors e_1 , e_2 and e_3 , who want to invest the following three types of projects: real estate (Q_1) , the stock market (Q_2) and treasury bills (Q_3) . In order to decide which project to invest, they consider from the following attributes: the risk factor (C_1) , the growth factor (C_2) , the return rate (C_3) and the complexity of the document requirements (C_4) . The weight vector of investors is $(0.3, 0.5, 0.2)^T$ and the criteria's weight vector is $(0.4, 0.2, 0.3, 0.1)^T$. The evaluation for the criterion C_1 is LTS $S_1 = \{s_0 : extreme \ low, s_1 : low, s_2 : slightly \ low, s_3 : generally, s_4 : slightly \ high, s_5 : high, s_6 : extreme \ high\}$, for the criteria C_i (i = 2, 3) is $S_i = \{s_0 : extreme \ slowly, s_1 : slowly, s_2 : slightly \ slowly, s_3 : generally, s_4 : slightly \ fast, s_5 : fast, s_6 : extreme \ fast\}$ (i = 2, 3); and the evaluation for criterion C_4 is LTS $S_4 = \{s_0 : extreme \ easy, s_1 : easy, s_2 : slightly \ easy, s_3 : generally, s_4 : slightly \ complexity, s_5 : complexity, s_6 : extreme \ complexity\}$. Then, the decision matrices of each experts are expressed in Tables 1–3.

Table 1. The decision matrix $F^{(1)}$.

	C_1	<i>C</i> ₂
Q_1	$\langle \{s_6(1)\}, \{s_0(1)\} angle$	$\langle \{s_4(0.4), s_5(0.6)\}, \{s_0(0.7), s_1(0.3)\} \rangle$
Q_2	$\langle \{s_4(0.7), s_5(0.3)\}, \{s_1(0.6), s_2(0.4)\} \rangle$	$\langle \{s_3(0.3), s_4(0.7)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
Q_3	$\langle \{s_5(0.6), s_6(0.4)\}, \{s_0(0.5), s_1(0.5)\} \rangle$	$\langle \{s_1(0.5), s_2(0.5)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
	<i>C</i> ₃	C_4
	$\langle \{s_0(0.3), s_1(0.7)\}, \{s_2(0.4), s_3(0.6)\} \rangle$	$\langle \{s_1(0.7), s_2(0.3)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
	$\langle \{s_3(0.3), s_4(0.7)\}, \{s_0(0.7), s_1(0.3)\} \rangle$	$\langle \{s_4(0.3), s_5(0.7)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
	$\{\{s_1(0.5), s_2(0.5)\}, \{s_3(0.7), s_4(0.3)\}\}$	$\{s_4(0.3), s_5(0.7)\}, \{s_1(0.5), s_2(0.5)\}$

Table 2. The decision matrix $F^{(2)}$.

	C_1	C_2
Q_1	$\langle \{s_0(0.3), s_1(0.7)\}, \{s_2(0.4), s_3(0.6)\} \rangle$	$\langle \{s_4(0.7), s_5(0.3)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
Q_2	$\langle \{s_3(0.3), s_4(0.7)\}, \{s_0(0.5), s_1(0.5)\} \rangle$	$\langle \{s_1(0.5), s_2(0.5)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
Q_3	$\langle \{s_5(0.6), s_6(0.4)\}, \{s_0(1)\} \rangle$	$\langle \{s_3(0.5), s_4(0.5)\}, \{s_1(0.1), s_2(0.3), s_3(0.6)\} \rangle$
	<i>C</i> ₃	C_4
	$\langle \{s_4(0.5), s_5(0.5)\}, \{s_0(0.7), s_1(0.3)\} \rangle$	$\langle \{s_5(0.7), s_6(0.3)\}, \{s_0(1)\} \rangle$
	$\langle \{s_4(0.3), s_5(0.7)\}, \{s_1(0.7), s_2(0.3)\} \rangle$	$\langle \{s_5(0.5), s_6(0.5)\}, \{s_0(1)\} \rangle$
	$\langle \{s_1(0.5), s_2(0.5)\}, \{s_2(0.2), s_3(0.6), s_4(0.2)\} \rangle$	$\langle \{s_4(0.5), s_5(0.5)\}, \{s_0(1)\} \rangle$

	C_1	<i>C</i> ₂
Q_1	$\langle \{s_4(0.5), s_5(0.5)\}, \{s_0(0.4), s_1(0.6)\} \rangle$	$\langle \{s_5(0.7), s_6(0.3)\}, \{s_0(1)\} \rangle$
Q_2	$\langle \{s_1(0.5), s_2(0.5)\}, \{s_2(0.5), s_3(0.3), s_4(0.2)\} \rangle$	$\langle \{s_5(0.7), s_6(0.3)\}, \{s_0(1)\} \rangle$
Q_3	$\langle \{s_4(0.2), s_5(0.8)\}, \{s_1(0.7), s_2(0.3)\} \rangle$	$\langle \{s_4(0.6), s_5(0.4)\}, \{s_0(0.5), s_1(0.5)\} \rangle$
	C_3	C_4
	$\langle \{s_2(0.5), s_3(0.5), \}, \{s_3(0.5), s_4(0.5)\} \rangle$	$\langle \{s_0(0.3), s_1(0.7)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
	$\langle \{s_4(0.4), s_5(0.6)\}, \{s_1(0.7), s_2(0.3)\} \rangle$	$\langle \{s_3(0.3), s_4(0.7)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
	$\langle \{s_0(0.2), s_1(0.4), s_2(0.4)\}, \{s_2(0.4), s_3(0.6)\} \rangle$	$\langle \{s_6(1), \}, \{s_0(1)\} \rangle$

Table 3. The decision matrix $F^{(3)}$.

Where $F^{(w)}$ represents the *w*th investor's evaluation information.

5.2. The Behavioral TOPSIS Method

Step 1. According to the adjustment method, we adjust the probability distribution of decision matrices $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$, and the corresponding adjusted matrices $F^{(*1)}$, $F^{(*2)}$ and $F^{(*3)}$ are given in Tables 4–6.

Table 4. The adjusted decision matrix $F^{(*1)}$.

	C_1	C_2
Q_1	$\langle \{s_6(0.3), s_6(0.2), s_6(0.5)\}, \{s_0(0.4), s_0(0.6)\} \rangle$	$\langle \{s_4(0.4), s_5(0.3), s_5(0.3)\}, \{s_0(0.5), s_0(0.2), s_1(0.3)\} \rangle$
Q_2	$\langle \{s_4(0.3), s_4(0.2), s_4(0.2), s_5(0.3)\}, \{s_1(0.5), s_1(0.1), s_2(0.2), s_2(0.2)\} \rangle$	$\langle \{s_3(0.3), s_4(0.2), s_4(0.2), s_4(0.3)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
Q_3	$\langle \{s_5(0.2), s_5(0.4), s_6(0.4)\}, \{s_0(0.5), s_1(0.2), s_1(0.3)\} \rangle$	$\langle \{s_1(0.5), s_2(0.1), s_2(0.4)\}, \{s_3(0.1), s_3(0.3), s_3(0.1), s_4(0.5)\} \rangle$
	<i>C</i> ₃	C_4
	$\langle \{s_0(0.3), s_1(0.2), s_1(0.5)\}, \{s_2(0.4), s_3(0.1), s_3(0.2), s_3(0.3)\} \rangle$	$\langle \{s_1(0.3), s_1(0.4), s_2(0.3)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
	$\langle \{s_3(0.3), s_4(0.1), s_4(0.6)\}, \{s_0(0.7), s_0(0.3)\} \rangle$	$\langle \{s_4(0.3), s_5(0.2), s_5(0.5)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
	$\langle \{s_1(0.2), s_1(0.3), s_2(0.1), s_2(0.4)\}, \{s_3(0.2), s_3(0.2), s_3(0.3), s_4(0.1), s_4(0.2)\} \rangle$	$\langle \{s_4(0.3), s_5(0.2), s_5(0.5)\}, \{s_1(0.5), s_2(0.5)\} \rangle$

Table 5. The adjusted decision matrix $F^{(*2)}$.

	0	2
	\mathcal{L}_1	\mathcal{L}_2
Q_1	$\langle \{s_0(0.3), s_1(0.2), s_1(0.5)\}, \{s_2(0.4), s_3(0.6)\} \rangle$	$\langle \{s_4(0.4), s_4(0.3), s_5(0.3)\}, \{s_1(0.5), s_2(0.2), s_2(0.3)\} \rangle$
Q_2	$\langle \{s_3(0.3), s_4(0.2), s_4(0.2), s_4(0.3)\}, \{s_0(0.5), s_1(0.1), s_1(0.2), s_1(0.2)\} \rangle$	$\langle \{s_1(0.3), s_1(0.2), s_2(0.2), s_2(0.3)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
Q_3	$\langle \{s_5(0.2), s_5(0.4), s_6(0.4)\}, \{s_0(0.5), s_0(0.2), s_0(0.3)\} \rangle$	$\langle \{s_3(0.5), s_4(0.1), s_4(0.4)\}, \{s_1(0.1), s_2(0.3), s_3(0.1), s_3(0.5)\} \rangle$
	C_3	C_4
	$\langle \{s_4(0.3), s_4(0.2), s_5(0.5)\}, \{s_0(0.4), s_0(0.1), s_0(0.2), s_1(0.3)\} \rangle$	$\langle \{s_5(0.3), s_5(0.4), s_6(0.3)\}, \{s_0(0.5), s_0(0.5)\} \rangle$
	$\langle \{s_4(0.3), s_5(0.1), s_5(0.6)\}, \{s_1(0.7), s_2(0.3)\} \rangle$	$\langle \{s_5(0.3), s_5(0.2), s_6(0.5)\}, \{s_0(0.5), s_0(0.5)\} \rangle$
	$\langle \{s_1(0.2), s_1(0.3), s_2(0.1), s_2(0.4)\}, \{s_2(0.4), s_3(0.2), s_3(0.3), s_3(0.1), s_4(0.2)\} \rangle$	$\langle \{s_4(0.3), s_4(0.2), s_5(0.5)\}, \{s_0(0.5), s_0(0.5)\} \rangle$

Table 6. The adjusted decision matrix $F^{(*3)}$.

	C_1	C_2
Q_1	$\langle \{s_2(0.3), s_4(0.2), s_5(0.5)\}, \{s_0(0.4), s_1(0.6)\} \rangle$	$\langle \{s_5(0.4), s_5(0.3), s_6(0.3)\}, \{s_0(0.5), s_0(0.2), s_1(0.3)\} \rangle$
Q_2	$\langle \{s_1(0.3), s_1(0.2), s_2(0.2), s_2(0.3)\}, \{s_2(0.5), s_3(0.1), s_3(0.2), s_4(0.2)\} \rangle$	$\langle \{s_5(0.3), s_5(0.2), s_5(0.2), s_6(0.3)\}, \{s_0(0.5), s_0(0.5)\} \rangle$
Q_3	$\langle \{s_4(0.2), s_5(0.4), s_5(0.4)\}, \{s_1(0.5), s_1(0.2), s_2(0.3)\} \rangle$	$\langle \{s_4(0.5), s_4(0.1), s_5(0.4)\}, \{s_0(0.1), s_0(0.3), s_0(0.1), s_1(0.5)\} \rangle$
	<i>C</i> ₃	C_4
	$\langle \{s_2(0.3), s_2(0.2), s_3(0.5)\}, \{s_3(0.4), s_3(0.1), s_4(0.2), s_4(0.3)\} \rangle$	$\langle \{s_0(0.3), s_1(0.4), s_1(0.3)\}, \{s_3(0.5), s_4(0.5)\} \rangle$
	$\langle \{s_4(0.3), s_4(0.1), s_5(0.6)\}, \{s_1(0.7), s_2(0.3)\} \rangle$	$\langle \{s_3(0.3), s_4(0.2), s_4(0.5)\}, \{s_1(0.5), s_2(0.5)\} \rangle$
	$\langle \{s_0(0.2), s_1(0.3), s_1(0.1), s_2(0.4)\}, \{s_2(0.4), s_2(0.2), s_3(0.3), s_3(0.1), s_3(0.2)\} \rangle$	$\langle \{s_6(0.3), s_6(0.2), s_6(0.5)\}, \{s_0(0.5), s_0(0.5)\} \rangle$

Step 2. Firstly, we aggregate the adjusted decision matrices based on the PLQROWA operator. Then we normalize the aggregated matrix according to the type of criteria (criteria C_2 and C_3 belong to the benefit-type criteria, criterion C_1 , C_4 belongs to the cost-type criteria). If q = 2, we can get the normalized decision matrix $F^{(*)}$ in Table 7.

	C_1
Q_1	$\langle \{s_0(0.4), s_0(0.6)\}, \{s_6(0.3), s_6(0.2), s_6(0.5)\} \rangle$
Q_2	$\langle \{s_0(0.5), s_{1.25}(0.1), s_{1.53}(0.2), s_{1.62}(0.2) \}, \{s_{3.16}(0.3), s_{3.69}(0.2), s_{3.75}(0.2), s_{4.2}(0.3) \} \rangle$
Q_3	$\langle \{s_0(0.5), s_0(0.2), s_0(0.3)\}, \{s_{4.86}(0.2), s_5(0.4), s_6(0.4)\} \rangle$
	<i>C</i> ₂
	$\langle \{s_{4.27}(0.4), s_{4.6}(0.3), s_6(0.3)\}, \{s_0(0.5), s_0(0.2), s_{1.41}(0.3)\} \rangle$
	$\langle \{s_{3,21}(0.3), s_{3.54}(0.2), s_{3.68}(0.2), s_6(0.3)\}, \{s_0(0.5), s_0(0.5)\} \rangle$
	$\langle \{s_{2.92}(0.5), s_{3.6}(0.1), s_{3.95}(0.4)\}, \{s_0(0.1), s_0(0.3), s_0(0.1), s_{2.63}(0.5)\} \rangle$
	0
	C3
	$\frac{c_{3}}{\langle \{s_{3.13}(0.3), s_{3.16}(0.2), s_{4.17}(0.5)\}, \{s_{0}(0.4), s_{0}(0.1), s_{0}(0.2), s_{1.83}(0.3)\} \rangle}$
	$ \begin{array}{c} C_3 \\ \hline \\ \{s_{3.13}(0.3), s_{3.16}(0.2), s_{4.17}(0.5)\}, \{s_0(0.4), s_0(0.1), s_0(0.2), s_{1.83}(0.3)\} \\ \\ \\ \{s_{3.76}(0.3), s_{4.6}(0.1), s_{4.78}(0.6)\}, \{s_0(0.7), s_{1.62}(0.3)\} \\ \end{array} $
	$ \begin{array}{c} & & \\ \hline & & \\ & $
	$ \begin{array}{c} C_3 \\ \\ & \langle \{s_{3.13}(0.3), s_{3.16}(0.2), s_{4.17}(0.5) \}, \{s_0(0.4), s_0(0.1), s_0(0.2), s_{1.83}(0.3) \} \rangle \\ & \langle \{s_{3.76}(0.3), s_{4.6}(0.1), s_{4.78}(0.6) \}, \{s_0(0.7), s_{1.62}(0.3) \} \rangle \\ & \langle \{s_{0.9}(0.2), s_1(0.3), s_{1.85}(0.1), s_2(0.4) \}, \{s_{2.26}(0.2), s_{2.77}(0.2), s_3(0.3), s_{3.27}(0.1), s_{3.78}(0.2) \} \rangle \\ \hline & \mathbf{C_4} \end{array} $
	$ \begin{array}{c} C_3 \\ \\ \langle \{s_{3.13}(0.3), s_{3.16}(0.2), s_{4.17}(0.5)\}, \{s_0(0.4), s_0(0.1), s_0(0.2), s_{1.83}(0.3)\}\rangle \\ \langle \{s_{3.76}(0.3), s_{4.6}(0.1), s_{4.78}(0.6)\}, \{s_0(0.7), s_{1.62}(0.3)\}\rangle \\ \langle \{s_{0.9}(0.2), s_1(0.3), s_{1.85}(0.1), s_2(0.4)\}, \{s_{2.26}(0.2), s_{2.77}(0.2), s_3(0.3), s_{3.27}(0.1), s_{3.78}(0.2)\}\rangle \\ \hline \\ \hline \\ \hline \\ C_4 \\ \langle \{s_0(0.5), s_0(0.5)\}, \{s_{4.03}(0.3), s_{4.05}(0.4), s_6(0.3)\}\rangle \\ \end{array}$
	$\begin{array}{c} C_3 \\ \\ \langle \{s_{3.13}(0.3), s_{3.16}(0.2), s_{4.17}(0.5)\}, \{s_0(0.4), s_0(0.1), s_0(0.2), s_{1.83}(0.3)\}\rangle \\ \langle \{s_{3.76}(0.3), s_{4.6}(0.1), s_{4.78}(0.6)\}, \{s_0(0.7), s_{1.62}(0.3)\}\rangle \\ \langle \{s_{0.9}(0.2), s_1(0.3), s_{1.85}(0.1), s_2(0.4)\}, \{s_{2.26}(0.2), s_{2.77}(0.2), s_3(0.3), s_{3.27}(0.1), s_{3.78}(0.2)\}\rangle \\ \hline \\ \hline \\ \hline \\ \hline \\ C_4 \\ \langle \{s_0(0.5), s_0(0.5)\}, \{s_{4.03}(0.3), s_{4.05}(0.4), s_6(0.3)\}\rangle \\ \langle \{s_0(0.5), s_0(0.5)\}, \{s_{4.5}(0.3), s_{4.86}(0.2), s_6(0.5)\}\rangle \\ \end{array}$

Table 7. The aggregate decision matrix $F^{(*)}$.

Step 3. According to Definition 8, the score function matrix can be obtained as follows:

	-0.38	0.219	0.1478	-0.2035	
$\mathbf{A} =$	-0.0939	0.1448	0.263	-0.3269	
	-0.2973	0.0841	-0.0334	-0.38	

Furthermore, we can obtain the ideal solution as follows:

$$\begin{aligned} Q^+ &= \{ \langle \{s_0(0.5), s_{1.25}(0.1), s_{1.53}(0.2), s_{1.62}(0.2) \}, \{s_{3.16}(0.3), s_{3.69}(0.2), s_{3.75}(0.2), s_{4.2}(0.3) \} \rangle, \langle \{s_{4.27}(0.4), s_{4.6}(0.3), s_{6}(0.3) \}, \{s_0(0.5), s_0(0.2), s_{1.41}(0.3) \} \rangle, \langle \{s_{3.76}(0.3), s_{4.6}(0.1), s_{4.78}(0.6) \}, \{s_0(0.7), s_{1.62}(0.3) \} \rangle, \langle \{s_0(0.5), s_0(0.5) \}, \{s_{4.03}(0.3), s_{4.05}(0.4), s_{6}(0.3) \} \rangle \}, \end{aligned}$$

The anti-ideal solution is given as follows:

$$\begin{aligned} Q^{-} &= \{ \langle \{s_{0}(0.4), s_{0}(0.6)\}, \{s_{6}(0.3), s_{6}(0.2), s_{6}(0.5)\} \rangle, \langle \{s_{2.92}(0.5), s_{3.6}(0.1), s_{3.95}(0.4)\}, \{s_{0}(0.1), s_{0}(0.3), s_{0}(0.1), s_{2.63}(0.5)\} \rangle, \langle \{s_{0.9}(0.2), s_{1}(0.3), s_{1.85}(0.1), s_{2}(0.4)\}, \{s_{2.26}(0.2), s_{2.77}(0.2), s_{3}(0.3), s_{3.27}(0.1), s_{3.78}(0.2)\} \rangle, \langle \{s_{0}(0.5), s_{0}(0.5)\}, s_{6}(0.3), s_{6}(0.2), s_{6}(0.5)\} \rangle \}. \end{aligned}$$

Step 4. Calculate $D(Q_o, Q^+)$ an $D(Q_o, Q^-)(o = 1, 2, 3)$, respectively. If q = 2, here we apply that the Euclidean distance measure D_{ded} , then $D_o^+ = \sum_{l=1}^4 \omega_{cl} D_{ded}(Q_o, Q^+)$ and $D_o^- = \sum_{l=1}^4 \omega_{cl} D_{ded}(Q_o, Q^-)$. So the separation measures between the alternative and the ideal/anti-ideal solution are obtained in Table 8.

Table 8. The separation measures for each alternative.

	Q1	Q2	Q3
D_o^+	0.1561	0.0549	0.2055
D_o^-	0.1236	0.2626	0.0561

Step 5. Calculate the value function V_o about alternatives $Q_o(o = 1, 2, 3)$. Here the parameters α , β and γ are used to describe the decision maker's behavior tendency. Here we assume $\gamma = 2.25$, $\alpha = \beta = 0.88$ [28], then we have $V_1 = -0.2801$, $V_2 = 0.1334$, $V_3 = -0.4797$. **Step 6.** According to the values of V_o , we have $Q_2 \succ Q_1 \succ Q_3$, so Q_2 is the best alternative. Next, we consider the relationship between the decision conclusion and the change of parameter λ . We still take the PLQROWA operator as an example. Assume q = 2, $\alpha = \beta = 0.88$, $\gamma = 2.25$, and $\lambda = 2, 3, 5, 8, 10, 12$, respectively. The Figure 3 shows the corresponding ranking results (Table 9 shows the detailed calculation results). Obviously, the varies of value function V_0 is not sensitive to the parameter λ , which indicates that the parameter λ has little effect on the decision results.



Figure 3. The results of change λ in behavioral TOPSIS method.

Table 9. The detailed results of parameter λ .

	$\lambda = 2$	$\lambda = 3$	$\lambda = 5$	$\lambda = 8$	$\lambda = 10$	$\lambda = 12$
V_1	-0.2801	-0.2335	-0.2056	-0.197	-0.1957	-0.1953
V_2	0.1334	0.1078	0.0959	0.0929	0.0925	0.0923
V_3	-0.4797	-0.4052	-0.3806	-0.3768	-0.3763	-0.3761

5.3. Comparison Analysis with Existed Method

Here, the traditional TOPSIS method is used to compare with the behavioral TOPSIS method, the algorithm steps [29] are given as follow.

Step 1. Adjust the probability distribution of PLQRONs, the corresponding matrices $F^{(*1)}$, $F^{(*2)}$ and $F^{(*3)}$ are obtained.

Step 2. Apply the PLQROWA operator to aggregate the evaluation information, then we normalize the aggregated decision matrix; the result is same as Section 5.2.

Step 3. Similarly, we can obtain the positive ideal solution Q^+ as follows:

$$\begin{aligned} Q^+ &= \{ \langle \{s_0(0.5), s_{1.25}(0.1), s_{1.53}(0.2), s_{1.62}(0.2) \}, \{s_{3.16}(0.3), s_{3.69}(0.2), s_{3.75}(0.2), s_{4.2}(0.3) \} \rangle, \langle \{s_{4.27}(0.4), s_{4.6}(0.3), s_6(0.3) \}, \{s_0(0.5), s_0(0.2), s_{1.41}(0.3) \} \rangle, \langle \{s_{3.76}(0.3), s_{4.6}(0.1), s_{4.78}(0.6) \}, \{s_0(0.7), s_{1.62}(0.3) \} \rangle, \langle \{s_0(0.5), s_0(0.5) \}, \{s_{4.03}(0.3), s_{4.05}(0.4), s_6(0.3) \} \rangle \}. \end{aligned}$$

The anti-ideal solution Q^- is also obtained as follows:

$$\begin{aligned} Q^{-} &= \{ \langle \{s_{0}(0.4), s_{0}(0.6)\}, \{s_{6}(0.3), s_{6}(0.2), s_{6}(0.5)\} \rangle, \langle \{s_{2.92}(0.5), s_{3.6}(0.1), s_{3.95}(0.4)\}, \{s_{0}(0.1), s_{0}(0.3), s_{0}(0.1), s_{2.63}(0.5)\} \rangle, \langle \{s_{0.9}(0.2), s_{1}(0.3), s_{1.85}(0.1), s_{2}(0.4)\}, \{s_{2.26}(0.2), s_{2.77}(0.2), s_{3}(0.3), s_{3.27}(0.1), s_{3.78}(0.2)\} \rangle, \langle \{s_{0}(0.5), s_{0}(0.5)\}, s_{6}(0.3), s_{6}(0.2), s_{6}(0.5)\} \rangle \}. \end{aligned}$$

Step 4. If q = 2, we apply D_{ded} to calculate the distance of each alternative between Q^+ and Q^- , the results are obtained in Table 10.

	<i>Q</i> ₁	Q_2	Q3
D_o^+	0.1561	0.0549	0.2055
D_o^-	0.1236	0.2626	0.0561

Table 10. The separation measures for each alternative.

Step 5. Calculate the closeness coefficient $R_o(o = 1, 2, 3)$,

$$R_o = \frac{D_o^-}{D_o^- + D_o^+}.$$

By calculation, we get $R_1 = 0.4419$, $R_2 = 0.8271$ and $R_3 = 0.2145$. So the ranking order of the alternatives is $Q_2 \succ Q_1 \succ Q_3$. The decision result is same as behavioral TOPSIS method, which shows the proposed method is effective.

Similarly, we consider the relationship between the decision result and the change of λ based on the traditional TOPSIS method. Here q = 2, $\alpha = \beta = 0.88$ and $\gamma = 2.25$, the parameter $\lambda = 2, 3, 5, 8, 10, 12$, we apply the PLQROWA operator to calculate the closeness coefficient R_o of each alternative, Figure 4 shows the ranking results (Table 11 shows the detailed calculation results). As can be seen from Figure 4, the closeness coefficient R_o remains unchanged and the decision result is also tend to stable.



Figure 4. The results of change the parameter λ in traditional TOPSIS method.

Table 11. The detailed results with the parameter λ .

	$\lambda = 2$	$\lambda = 3$	$\lambda = 5$	$\lambda = 8$	$\lambda = 10$	$\lambda = 12$
R_1	0.4419	0.4472	0.4582	0.4635	0.4644	0.4647
R_2	0.8271	0.8202	0.8156	0.8142	0.814	0.814
R_3	0.2145	0.2306	0.2347	0.2349	0.235	0.2351

5.4. The Sensitivity of Decision Maker's Behavior

Here, we make the analysis of the influence of loss aversion parameter γ , the risk preference parameter α and β in the proposed behavioral TOPSIS method.

Firstly, the impact of the loss aversion parameter γ in the value function is considered. We take the PLQROWA operator as an example, if q = 2, $\alpha = \beta = 0.88$ and $\lambda = 2$, let $\gamma = 0.5, 0.8, 1, 2.25, 5$, the ranking results of the value function V_o are shown in Figure 5 (Table 12 shows the detailed calculation results). As can be seen from Figure 5, when $\gamma \leq 2.25$, the values of V_1 , V_2 and V_3 are less sensitive to the change of the loss aversion parameter γ ; while $\gamma > 2.25$, the values of $V_o(o = 1, 2, 3)$ is changing obviously. In comparison, the loss aversion parameter γ has a significant influence on V_2 and V_3 . When $\gamma > 2.25$, the values of $V_o(o = 1, 2, 3)$ decrease sharply at the same time, which means if the parameter γ becomes larger, the loss aversion has a greater impact on the value function V_o .



Figure 5. The results of changed loss aversion parameter γ .

Tab	le 12	Pre	terence	ranking	under	various	loss	aversion	paramet	ter γ .
-----	-------	-----	---------	---------	-------	---------	------	----------	---------	----------------

	Distance	e Measure					В	ehaviora	al TOPSIS					
			$\gamma =$	$\gamma = 0.5$ $\gamma = 0.8$		$\gamma = 1$		$\gamma = 1.5$		$\gamma =$ 2.25		$\gamma = 5$		
	D_i^+	D_i^-	V_i	Rank	V_i	Rank	V_i	Rank	V_i	Rank	V_i	Rank	V_i	Rank
Q_1	0.1561	0.1236	0.0613	2	0.0028	2	-0.0362	2	-0.1338	2	-0.2801	2	-0.8167	2
Q_2	0.0549	0.2626	0.2695	1	0.2461	1	0.2306	1	0.1917	1	0.1334	1	-0.0805	1
Q_3	0.2055	0.0561	-0.0449	3	-0.1195	3	-0.1691	3	-0.2934	3	-0.4797	3	-1.1629	3

Next, we consider the influence of the risk preference parameters α and β in the value function, respectively. We take the PLQROWA operator as an example, suppose that q = 2, $\beta = 0.88$ and $\gamma = 2.25$, $\lambda = 2$, let $\alpha = 0.1, 0.3, 0.5, 0.8, 1$, the results of value functions change with the parameter α are shown in Figure 6. It is easy to know that the values of the function V_o (o = 1, 2, 3) descend with the parameter α . We know Q_2 is always the best alternative from the Figure 6. If $\alpha > 0.88$, the values of v_o (o = 1, 2, 3) also tend to stable.

Furthermore, assume that q = 2, $\alpha = 0.88$, $\gamma = 2.25$ and $\lambda = 2$, let $\beta = 0.1$, 0.3, 0.5, 0.8, 1, the results of value functions change with the parameter β are shown in Figure 7. Similarly, we know that the values of V_o (o = 1, 2, 3) increase with the parameter β , and the best alternative remains unchanged. If $\beta > 0.88$, the values of V_o (o = 1, 2, 3) tend to stable. In conclusion, the change of value function V_o is consistent with expert's risk preference, if the expert is risk averse, the parameter α increases, he/she is more sensitive to the loss, and the overall value functions are decreasing. If the decision maker is risk appetite, when the parameter β increases, he/she becomes more sensitive to gains, the overall value functions are increasing.



Figure 6. The results of change the risk preference parameter α .



Figure 7. The results of change the risk preference parameter β .

According to the above comparison analyses, we can find that the proposed method has the following advantages. First, the behavioral TOPSIS method implements the decision maker's choice by adopting the gain and loss. Second, it has been demonstrated that the traditional TOPSIS method is a special case of the proposed behavioral TOPSIS method [23], while the behavioral TOPSIS method involves a wider range of situations. In addition, there are three parameters (α , β and γ) in the value function of V_o , the decision maker can choose the appropriate numerical value according to his/her risk preference and loss aversion, which makes the proposed behavioral TOPSIS method more flexible in practical application.

6. Conclusions

The main conclusions of the paper are given as follows:

- (1) The operations of PLQROS are proposed based on the adjusted PLQROS with the same probability. Then we present the PLQROWA operator, PLQROWG operator and the distance measures between the PLQROSs based on the proposed operational laws.
- (2) We develop the fuzzy behavior TOPSIS method to PLQROS, which consider the behavioral tendency in decision making process.
- (3) We utilize a numerical example to demonstrate the validity and feasibility of the fuzzy behavior TOPSIS method, and we prove the superiority of the method by comparison with the traditional TOPSIS method.

Next, we will apply the proposed method to deal with the multi-attribute decision making problems, such as the emergency decision, supplier selection and investment decision, etc.

Author Contributions: Conceptualization, D.L. and A.H.; methodology, D.L. and Y.L.; writing, D.L. and A.H.; supervision, Z.L. All authors have read and agreed to the published version of the manuscript.

Funding: The research was funded by National Natural Science Foundation of China (11501191) and Hunan Postgraduate Scientific Research and Innovation project (CX20190812).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of this study are included within the article.

Acknowledgments: We thank the editor and anonymous reviewers for their helpful comments on an earlier draft of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

- 1. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning—I. *Inf. Sci.* **1975**, *8*, 199–249. [CrossRef]
- 2. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning—II. *Inf. Sci.* **1975**, *8*, 301–357. [CrossRef]
- 3. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning—III. *Inf. Sci.* **1975**, *9*, 43–80. [CrossRef]
- 4. Rodriguez, R.M.; Luis, M.; Francisco, H. Hesitant fuzzy linguistic term sets for decision making. *IEEE Trans. Fuzzy Syst.* 2012, 20, 109–119. [CrossRef]
- 5. Beg, I.; Rashid, T. TOPSIS for hesitant fuzzy linguistic term sets. Int. J. Intell. Syst. 2013, 28, 1162–1171. [CrossRef]
- 6. Liao, H.; Xu, Z.; Herrera-Viedma, E.; Herrera, F. Hesitant fuzzy linguistic term set and its application in decision making: A state of the art survey. *Int. J. Fuzzy Syst.* **2018**, *20*, 2084–2110. [CrossRef]
- Liu, D.H.; Liu, Y.Y.; Chen, X.H. The new similarity measure and distance measure of a hesitant fuzzy linguistic term set based on a linguistic scale function. *Symmetry* 2018, 10, 367. [CrossRef]
- 8. Hai, W.; Xu, Z.; Zeng, X.J. Hesitant fuzzy linguistic term sets for linguistic decision making: Current developments, issues and challenges. *Inf. Fusion* **2018**, *43*, 1–12. [CrossRef]
- 9. Wu, Z.; Xu, J.; Jiang, X.; Zhong, L. Two MAGDM models based on hesitant fuzzy linguistic term sets with possibility distributions: VIKOR and TOPSIS. *Inf. Sci.* **2019**, *473*, 101–120. [CrossRef]
- 10. Kong, M.; Pei, Z.; Ren, F.; Hao, F. New operations on generalized hesitant fuzzy linguistic term Sets for linguistic decision making. *Int. J. Fuzzy Syst.* 2019, *21*, 243–262. [CrossRef]
- 11. Liu, D.H.; Chen, X.H.; Peng, D. Distance measures for hesitant fuzzy linguistic sets and their applications in multiple criteria decision making. *Int. J. Fuzzy Syst.* **2018**, *20*, 2111–2121. [CrossRef]
- 12. Pang, Q.; Wang, H.; Xu, Z.S. Probabilistic linguistic term sets in multi-attribute group decision making. *Inf. Sci.* **2016**, *369*, 128–143. [CrossRef]
- 13. Chen, Z.C. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *Int. J. Comput. Intell. Syst.* 2015, *8*, 747–760. [CrossRef]
- 14. Garg, H. Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process. *Int. J. Intell. Syst.* **2018**, 33, 1234–1263. [CrossRef]
- 15. Yager, R.R. Generalized orthopair fuzzy sets. IEEE Trans. Fuzzy Syst. 2016, 25, 1222–1230.
- 16. Hwang, C.L.; Yoon, K. Multiple Attribute Decision Making: Methods and Applications a State-of-the-Art Survey; Springer: Berlin/Heidelberg, Germany, 2012. [CrossRef]
- 17. Lai, Y.J.; Liu, T.Y.; Hwang, C.L. TOPSIS for MODM. Eur. J. Oper. Res. 1994, 76, 486–500. [CrossRef]
- Chen, C.T. Extensions of the TOPSIS for group decision-making under fuzzy environment. *Fuzzy Sets Syst.* 2000, 114, 1–9. [CrossRef]
- 19. Zyoud, S.H.; Fuchs-Hanusch, D. A bibliometric-based survey on AHP and TOPSIS techniques. *Expert Syst. Appl.* 2017, 78, 158–181. [CrossRef]
- 20. Liu, D.H.; Chen, X.H.; Peng, D. Cosine distance measure between neutrosophic hesitant fuzzy linguistic sets and its application in multiple criteria decision making. *Symmetry* **2018**, *10*, 602. [CrossRef]
- 21. Liu, D.H.; Chen, X.H.; Peng, D. Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets. *Int. J. Intell. Syst.* **2019**, *34*, 1572–1587. [CrossRef]

- 22. Liu, H.C.; Wang, L.E.; Li, Z.; Hu, Y.P. Improving risk evaluation in FMEA with cloud model and hierarchical TOPSIS method. *IEEE Trans. Fuzzy Syst.* **2019**, *27*, 84–95. [CrossRef]
- 23. Yoon, K.P.; Kim, W.K. The behavioral TOPSIS. Expert Syst. Appl. 2017, 89, 266–272. [CrossRef]
- 24. Xu, Z.S. A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Inf. Sci.* 2004, *166*, 19–30. [CrossRef]
- 25. Liu, P.D.; Liu, W.Q. Multiple-attribute group decision-making based on power Bonferroni operators of linguistic q-rung orthopair fuzzy numbers. *Int. J. Intell. Syst.* 2019, 34, 652–689. [CrossRef]
- 26. Wu, X.; Liao, H.; Xu, Z.; Hafezalkotob, A.; Herrera, F. Probabilistic linguistic MULTIMOORA: A multi-attributes decision making method based on the probabilistic linguistic expectation function and the improved Borda rule. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 3688–3702.
- 27. Beg, I.; Rashid, T. Hesitant intuitionistic fuzzy linguistic term sets. Notes Intuit. Fuzzy Sets 2014, 20, 53-64. [CrossRef]
- 28. Tversky, A.; Kahneman, D. Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertain*. **1992**, *5*, 297–323. [CrossRef]
- 29. Herrera, F.; Herrera-Viedma, E. Linguistic decision analysis: Steps for solving decision problems under linguistic information. *Fuzzy Sets Syst.* **2000**, *115*, 67–82.