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Integrability of Riemann-Type Hydrodynamical Systems and Dubrovin's Integrability Classification of Perturbed KdV-Type Equations

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Abstract: Dubrovin's work on the classification of perturbed KdV-type equations is reanalyzed in detail via the gradient-holonomic integrability scheme, which was devised and developed jointly with Maxim Pavlov and collaborators some time ago. As a consequence of the reanalysis, one can show that Dubrovin's criterion inherits important parts of the gradient-holonomic scheme properties, especially the necessary condition of suitably ordered reduction expansions with certain types of polynomial coefficients. In addition, we also analyze a special case of a new infinite hierarchy of Riemann-type hydrodynamical systems using a gradient-holonomic approach that was suggested jointly with M. Pavlov and collaborators. An infinite hierarchy of conservation laws, bi-Hamiltonian structure and the corresponding Lax-type representation are constructed for these systems.

Keywords: bi-hamiltonian systems; lie-poisson structure; conservation laws; dubrovin integrability scheme; gradient-holonomic integrability scheme; symmetry analysis; riemann type hydrodynamical systems



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1. Introduction: Dubrovin's Integrability Scheme

We begin by recalling some very interesting works by B. Dubrovin and collaborators [1–3], in which the following classification problem was posed:

Consider a general evolution equation:

$$u_t = f(u)u_x + \varepsilon[f_{21}(u)u_{xx} + f_{22}(u)u_x^2] + \varepsilon^2[f_{31}(u)u_{xxx} + f_{32}(u)u_x u_{xx} + f_{33}(u)u_x^3] + \dots + \varepsilon^{N-1}[f_{N,1}(u)u_{Nx} + \dots], \quad (1)$$

with graded homogeneous polynomials of the jet-variables $\{u_x, u_{xx}, \dots, u_{kx}\dots\}$ with $\deg u_{kx} := k \in \mathbb{N}$, where $f'(u) \neq 0$ for arbitrary $u \in C^\infty(\mathbb{R}; \mathbb{R})$. Introduce now the set \mathcal{F} of smooth functions $f_{jk}(u)$, $k = \overline{1, j}$, $j = \overline{1, N}$, with fixed $N \in \mathbb{N}$, such that Equation (1) reduces by means of the following transformation:

$$u \rightarrow v + \sum_{k \in \mathbb{N}} \varepsilon^k F_k(v, v_x, v_{xx}, \dots, v^{(m_k)}), \quad (2)$$

to the Riemann-type equation

$$v_t = f(v)v_x, \quad (3)$$

where numbers $m_k \in \mathbb{Z}_+$, $k \in \mathbb{N}$, are finite and ε is a formal parameter. The transformation (2) is often called a quasi-Miura transformation and naturally acts as an automorphism of the ring $\mathcal{A}_\varepsilon := C^\infty(u)[u_1, u_2, \dots, u_k, \dots][[\varepsilon]]$ of formal functional series with respect to the parameter ε . It is worth mentioning that this ring is a topological ring \mathcal{A}_ε with respect to the natural metric, within which adding and multiplication of series is continuous. More-

over, the related group of Miura-type automorphisms, which is the semidirect product of the local diffeomorphism group $\text{Diff}_{\text{loc}}(\mathbb{R})$ of the real axis \mathbb{R} and the quasi-identical automorphism subgroup of self-mappings,

$$u \rightarrow u + \sum_{k \in \mathbb{N}} \varepsilon^k F_k(u, u_x, u_{xx}, \dots, u^{(m_k)}), \quad (4)$$

with finite $m_k \in \mathbb{N}, k \in \mathbb{N}$, naturally generates the Lie subalgebra $\mathcal{D}(\mathcal{A}_\varepsilon)$ of the natural derivations of the ring \mathcal{A}_ε , whose representatives coincide with the above Equation (1).

Dubrovin formulated the following *integrability criterion*:

Definition 1. The evolution Equation (1) is defined to be formally integrable, iff the corresponding inverse to (2) transformation

$$v \rightarrow u + \sum_{k \in \mathbb{N}} \varepsilon^k G_k(u, u_x, u_{xx}, \dots, u^{(m_k)}) \in \mathcal{A}_\varepsilon, \quad (5)$$

with finite orders $m_k \in \mathbb{N}, k \in \mathbb{N}$, upon application to an arbitrary Riemann type symmetry flow

$$v_s = h(v)v_x, \quad (6)$$

with respect to an evolution parameter $s \in \mathbb{R}$, reduces to the form

$$u_s = h(u)u_x + \sum_{k \in \mathbb{N}} \varepsilon^k W_k(u, x, u_{xx}, \dots, u^{(k)}) \in \mathcal{A}_\varepsilon, \quad (7)$$

with uniform homogeneous differential polynomials $W_k(u, x, u_{xx}, \dots, u^{(k)})$ of order $k \in \mathbb{N}$.

In their works, B. Dubrovin and his collaborators applied this scheme to the equation

$$u_t = uu_x + \varepsilon^2 [f_{31}(u)u_{xxx} + f_{32}(u)u_x u_{xx} + f_{33}(u)u_x^3] \quad (8)$$

and presented a list of 9 (!) equations [3]:

$$\begin{aligned} (1) \quad & u_t = uu_x + \varepsilon u_{xxx} \quad (\text{KdV}) \\ (2) \quad & w_t = w^2 w_x + \varepsilon^2 w_{xxx} \quad (w^2 := u) \\ (3) \quad & u_t = uu_x + \varepsilon^2 \left(u_{xxx} - \frac{3u_x u_{xx}}{u} + \frac{15u_x^2}{8u^2} \right) \\ (4) \quad & w_t = w^3 w_x + \varepsilon^2 w^3 w_{xxx} \quad (w^3 := u) \\ (5) \quad & w_t = w^3 w_x + \varepsilon^2 (3w^2 w_x w_{xx} + w^3 w_{xxx}), \quad (w^3 := u), \\ (6) \quad & w_t = w^2 w_x + \varepsilon^2 (3w^2 w_x w_{xx} + w^3 w_{xxx}), \quad (w^2 := u) \\ (7) \quad & w_t = w^2 w_x + \varepsilon^2 \left(\frac{3}{2} w^2 w_x w_{xx} + w^3 w_{xxx} \right), \quad (w^2 := u) \\ (8) \quad & u_t = uu_x + \varepsilon^2 \left(u^2 u_{xxx} + \frac{1}{9} u_x^3 \right) \\ (9) \quad & w_t = \frac{w_x}{w} + \varepsilon^2 w^3 w_{xxx}, \quad \left(\frac{1}{w} := u \right). \end{aligned} \quad (9)$$

The first two equations are the KdV and mKdV, the third equation is equivalent via the Miura transformation $u \rightarrow u + \varepsilon^2 [3u_{xx}/(2u) - 15u_x^2/(8u^2)]$ to the KdV Equation (1). The last ones (4)–(9) are reduced by means of suitable reciprocity transformations

$$dy = \alpha(u)dx + \rho dt, \quad ds = dt,$$

parametrized by a smooth function $\alpha(u)$, to the Equations (1)–(3) from this listing.

Keeping in mind this result, we have decided to reanalyze the integrability of the evolution Equation (3), having rewritten it in the following generalized form:

$$u_t = uu_x + \varepsilon^2 \left(u_{xxx} + c_1 \frac{u_x u_{xx}}{u} + c_2 \frac{u_x^3}{u^2} \right), \quad (10)$$

where $c_1, c_2 \in \mathbb{R}$ are constants. As the right-hand side of the flow (10) defines a vector field $u_t = K[u]$ on a suitably chosen smooth functional manifold $M \subset C^\infty(\mathbb{R}; \mathbb{R})$ (being here locally diffeomorphic to the jet-manifold $J^\infty(\mathbb{R}; \mathbb{R})$), we checked the existence of a suitable infinite hierarchy of conservation laws for the flow (10) and related Hamiltonian structures on M via the gradient-holonomic integrability scheme [4,5].

In particular, it means that this hierarchy is suitably ordered and satisfies the well known Noether-Lax equation $d\varphi/dt + K'^*[u] \cdot \varphi = 0$ on M , where $\varphi := \varphi[u; \lambda] = \text{grad } \zeta[u; \lambda] \in T^*(M)$ —the functional gradient of a functional $\zeta(\lambda)$ on M , depending on the constant parameter $\lambda \in \mathbb{C}$ as $|\lambda| \rightarrow \infty$, and chosen to be a generating function of conservation laws to the vector field $K : M \rightarrow T(M)$.

As a result of calculations, we obtained the following two cases:

$$(i) \quad c_1 = -3, c_2 = \frac{15}{8}; \quad (ii) \quad c_1 = -\frac{3}{2}, c_2 = \frac{3}{4}. \quad (11)$$

The first case gives rise to Equation (3) from the (9), and the second one gives rise to the new evolution equation

$$u_t = uu_x + \varepsilon^2 \left[u_{xxx} - \frac{3}{4} \left(\frac{u_x^2}{u} \right)_x \right], \quad (12)$$

which possesses infinite hierarchy of suitably ordered conservation laws:

$$\begin{aligned} H_1 &= \int dx u, \quad H_2 = \int dx \left(\frac{3}{2} \frac{u_x^2}{u} - u^2 \right), \\ H_3 &= \int dx \left(\frac{9u_x u_{xx}}{2u} + \frac{15u_{xx}^2}{4u} - \frac{21u_x^2 u_{xx}}{2u^2} - \right. \\ &\quad \left. - 3uu_{xx} + \frac{69u_x^4}{16u^3} - \frac{7u_x^2}{4} - \frac{u^3}{6} \right), \quad \dots \text{ and so on,} \end{aligned} \quad (13)$$

where we put, for brevity, $\varepsilon^2 = 1$.

The new evolution Equation (12) can be represented in the following Hamiltonian form

$$u_t = -\theta \text{grad} H_3[u] = -\eta \text{grad} H_2[u], \quad (14)$$

where the Poisson operators $\theta, \eta : T^*(M) \rightarrow T(M)$ are given by the expressions

$$\theta^{-1} = D_x^{-1} + \frac{3}{4} \left(\frac{1}{u} D_x + D_x \frac{1}{u} \right), \quad D_x := \frac{d}{dx}, \quad (15)$$

and

$$\eta = \sqrt{u} D_x \sqrt{u}, \quad (16)$$

and are compatible on the functional manifold M , that is for any $\lambda \in \mathbb{R}$ the operator $(\theta + \lambda \eta)^{-1} : T(M) \rightarrow T^*(M)$ is Hamiltonian.

Proposition 1. *The above result simply means that the dynamical system*

$$u_t = uu_x + \varepsilon^2 \left[u_{xxx} - \frac{3}{4} \left(\frac{u_x^2}{u} \right)_x \right] \quad (17)$$

is a new completely integrable bi-Hamiltonian system on the functional manifold M .

Remark 1. Concerning the Dubrovin-Zhang Equation (3) from the listing (9), we have stated, as a by-product, that it is also a bi-Hamiltonian system and representable in the form

$$u_t = -\vartheta \operatorname{grad} H_1[u] = -\eta \operatorname{grad} H_2[u], \quad (18)$$

where the compatible Poisson operators $\vartheta, \eta : T^*(M) \rightarrow T(M)$ are given, respectively, by the expressions

$$\vartheta = D_x \sqrt{u} D_x^{-1} \sqrt{u} D_x \quad (19)$$

and

$$\eta^{-1} = \frac{3}{u} D_x \frac{1}{u} - \frac{1}{\sqrt{u}} D_x^{-1} \frac{1}{\sqrt{u}}, \quad (20)$$

jointly with the Hamiltonian operators

$$\begin{aligned} H_1 &= \int dx \left(\frac{u_x^2}{2u^2} + \frac{4u}{3} \right), \\ H_2 &= \int dx \left(\frac{17u_x^4}{16u^4} + \frac{40u_x^2}{u} + \frac{u_{xx}^2}{u^2} + \frac{4u^2}{9} - \frac{2u_x^2 u_{xx}}{u^3} \right). \end{aligned} \quad (21)$$

2. Reduction Integrability Properties

Now, we proceed to the following reduction of the new Equation (12) putting $u \rightarrow \mu v$ as $\mu \rightarrow 0$:

$$u_t = uu_x + u_{xxx} - \frac{3}{4} \left(\frac{u_x^2}{u} \right)_x \Rightarrow v_t = v_{xxx} - \frac{3}{4} \left(\frac{v_x^2}{v} \right)_x, \quad (22)$$

called here KN-3/4 and *a priori* integrable and possessing an infinite hierarchy of conservation laws, which can be easily written down from the hierarchy (13) via the limiting procedure

$$\tilde{H}_j := \lim_{\mu \rightarrow 0} \mu^{-1} H_j|_{u=\mu v}, \quad j \in \mathbb{N}.$$

Remark 2. Here, we would like to remark that Equation (22) above is very similar to the well known Krichever-Novikov (KN-3/2) equation

$$v_t = v_{xxx} - \frac{3}{2} \left(\frac{v_x^2}{v} \right)_x, \quad (23)$$

which differs from (22) only by the coefficient $3/2$ instead of the rational number $3/4$ and was studied before by V. Sokolov [6] by means of the well known Mikhailov-Shabat recursion symmetry analysis technique and by G. Wilson [7], using differential-algebraic Galois group solvability reasonings.

We have reanalyzed these Novikov-Krichever type Equations (22) and (23), having performed the following manipulation:

$$\begin{aligned} v_t &= v_{xxx} - \frac{3}{2} \left(\frac{v_x^2}{v} \right)_x = v_{xxx} - \frac{3}{2} \left(\frac{2v_x v_{xx}}{v} - \frac{v_x^3}{v^2} \right)_x = \\ &= v_{xxx} - 3 \frac{v_x v_{xx}}{v} + \frac{3}{2} \frac{v_x^3}{v^2} \Rightarrow v_{xxx} + c_1 \frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2}, \end{aligned} \quad (24)$$

where $c_1, c_2 \in \mathbb{R}$ are now arbitrary coefficients, and checked the latter equation for the existence of an infinite hierarchy of suitably ordered conservation laws. The corresponding calculations immediately revealed that the coefficients $c_1, c_2 \in \mathbb{R}$ should satisfy two related algebraic relationships:

$$(c_1(c_1 - 3) - 9c_2) = 0 \quad \text{and} \quad (c_1 + 3)(c_1(c_1 - 3) - 9c_2) = 0, \quad (25)$$

whose solutions are the following two cases:

$$\begin{aligned} (i) \quad c_1 &= -3k, c_2 = k(k+1) \text{ for arbitrary } k \in \mathbb{R} \setminus \{1\}, \\ (ii) \quad c_1 &= -3, c_2 \in \mathbb{R} \text{ is arbitrary.} \end{aligned} \quad (26)$$

The first case (i) gives rise to the new integrable bi-Hamiltonian system on the functional manifold M in the form

$$v_t = v_{xxx} - 3k \frac{v_x v_{xx}}{v} + k(k+1) \frac{v_x^3}{v^2}, \quad (27)$$

where $k \in \mathbb{R}$ is an arbitrary parameter, yet $k \neq 1$. For the case (ii) when $k = 1$, Equation (24) reduces to the modified integrable Krichever-Novikov type system

$$v_t = v_{xxx} - 3 \frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2}, \quad (28)$$

possessing an infinite hierarchy of conservation laws and giving rise to the well known Krichever-Novikov bi-Hamiltonian system (23) at $c_2 = 1/2$:

$$v_t = v_{xxx} - \frac{3}{2} \left(\frac{v_x^2}{v} \right)_x. \quad (29)$$

Remark 3. We would like to remark here that originally Krichever and Novikov [8] and Sokolov [6,9] analyzed integrability of the generalized equation

$$w_t = w_{xxx} - \frac{3}{2} \frac{w_{xx}^2}{w_x} + \frac{h(w)}{w_x}, \quad (30)$$

where $h(w) = \sum_{j=1}^3 c_j w^j$ is a third order polynomial with constant coefficients, and reducing to the KN-3/2 (29) upon the change of dependent variable $v := w_x$ and $h(w) = 0$. We have reanalyzed the Equation (30) within the gradient-holonomic integrability scheme [10] for the case of an arbitrary N -s order polynomial $h(w) = \sum_{j=1}^N c_j w^j$ and proved that it conserves the integrability for the polynomial order $N = 4$, that slightly generalizes the former result presented in Reference [6,8,9].

The derived above modified Krichever-Novikov type Equation (28) is also an integrable bi-Hamiltonian flow on the functional manifold M for arbitrary $c_2 \in \mathbb{R}$:

$$v_t = v_{xxx} - 3 \frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2} = -\theta \text{grad} \tilde{H}^{(c_2)}[v], \quad (31)$$

where the Poisson operator

$$\theta = v D_x^{-1} v \quad (32)$$

forms a compatible pair to the operator

$$\eta = D_x v D_x^{-1} v D_x, \quad (33)$$

and the corresponding Hamiltonian function equals

$$\tilde{H}_2^{(c_2)} = \int dx \left(\frac{v_{xxx}}{v} - \frac{2v_x v_{xx}}{v^2} - \frac{3v_{xx}^2}{2v^2} + c_2 \frac{v_x^2 v_{xx}}{v^3} + \frac{2v_x^2 v_{xx}}{v^3} - \frac{3c_2 v_x^4}{4v^4} \right). \quad (34)$$

Moreover, one can also easily check that the next slightly modified Krichever-Novikov-type equation

$$v_t = v_{xxx} - 3 \frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2} - \frac{k_0 v_x}{v^2} \quad (35)$$

for arbitrary $c_2, k_0 \in \mathbb{R}$ is also an integrable bi-Hamiltonian flow, possesses an infinite hierarchy of functionally independent conservation laws, which can be generated recursively:

$$H_1^{(k_0, c_2)} = \int dx \left(\frac{v_x^2}{v^2} + \frac{2k_0}{v^2} \right) \rightarrow \tilde{H}_2^{(k_0, c_2)} \rightarrow \dots \rightarrow \tilde{H}_n^{(k_0, c_2)} \rightarrow \dots \quad (36)$$

via the Magri gradient recursion scheme:

$$\eta \text{grad} \tilde{H}_n^{(k_0, c_2)} = \theta \text{grad} \tilde{H}_{n+1}^{(k_0, c_2)}, \quad (37)$$

for arbitrary $n \in \mathbb{Z}$, using the above mentioned compatible Poisson θ - η pair (32) and (33). The same can be stated about the new integrable Krichever-Novikov type equation

$$v_t = v_{xxx} - \frac{3}{4} \left(\frac{v_x^2}{v} \right)_x, \quad (38)$$

which is also a bi-Hamiltonian flow with respect to a compatible pair of the Poisson operators

$$\theta = v D_x^{-1} v, \quad \eta = D_x v D_x^{-1} v D_x. \quad (39)$$

Remark 4. It appears interesting to observe that the generalized Krichever-Novikov type Equation (24)

$$v_t = v_{xxx} + c_1 \frac{v_x v_{xx}}{v} + c_2 \frac{v_x^3}{v^2} \quad (40)$$

transforms via the change of variables $w := v_x / v$ to the following modified Korteweg de Vries-type equation:

$$w_t = w_{3x} + \frac{(c_1 + 3)}{2} (w^2)_x + (c_1 + c_2 + 1) (w^3)_x, \quad (41)$$

which is, obviously, also integrable for two cases (26), mentioned before:

- (i) $c_1 = -3k$, $c_2 = k(k+1)$ for arbitrary $k \in \mathbb{R} \setminus \{1\}$,
- (ii) $c_1 = -3$, $c_2 \in \mathbb{R}$ is arbitrary.

The case (i) reduces to the well known integrable modified Korteweg de Vries equation

$$w_t = w_{3x} - 3(k-1) w w_x + 3(k-1)^2 w^2 w_x, \quad (42)$$

Respectively, at $k = 1/2$, or equivalently at $c_1 = -3/2$, $c_2 = 3/4$, the modified Korteweg-de Vries Equation (42) reduces to the classical modified Korteweg de Vries equation

$$w_t = w_{3x} + \frac{3}{2} w w_x + \frac{3}{4} w^2 w_x, \quad (43)$$

which, evidently, is also integrable and bi-Hamiltonian on the functional manifold M . The second case (ii) of Equation (41) also reduces to the classical integrable modified Korteweg-de Vries equation

$$w_t = w_{3x} + 3(c_2 - 2) w^2 w_x. \quad (44)$$

The special case $k = 1$, which is equivalent to the choice $c_1 = -3$, corresponds at $c_2 = 2$ exactly to the strictly linear equation, whose exact integrability is trivial.

3. The Integrability of the Riemann-Type Hydrodynamical Systems via the Gradient-Holonomic Integrability Scheme

In this section, we will dwell on the integrability theory aspects of a new Riemann type hierarchy

$$D_t^{N-1}u = \bar{z}_x^s, \quad D_t\bar{z} = 0, \quad (45)$$

where $s, N \in \mathbb{N}$ are arbitrary natural numbers, in the frame of the gradient-holonomic integrability scheme, devised and applied jointly with Maxim Pavlov and collaborators. This hierarchy was proposed before in Reference [11] as a nontrivial generalization of the infinite hierarchy of the Riemann type flows, suggested recently by M. Pavlov and D. Holm [12,13] in the form of dynamical systems on a 2π -periodic functional manifold $\bar{M}^N \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^N)$

$$D_t^N u = 0, \quad (46)$$

where the vector $(u, D_t u, D_t^2 u, \dots, D_t^{N-1} u, \bar{z})^T \in \bar{M}^N$, and the differentiations

$$D_x := \partial/\partial x, \quad D_t := \partial/\partial t + u\partial/\partial x \quad (47)$$

satisfy the Lie-algebraic commutator relationship

$$[D_x, D_t] = u_x D_x, \quad (48)$$

and $t \in \mathbb{R}$ is an evolution parameter.

The mentioned above dynamical systems

$$D_t^{N-1}u = \bar{z}_x^s, \quad D_t\bar{z} = 0, \quad (49)$$

at $s = 1$, $s = 2$ and $N = 2$, $n = 3$, respectively, were extensively studied by many researchers. They appeared to be related to nontrivial generalizations of the Camassa-Holm and Degasperis-Procesi systems. The case $s = 2$ and $N = 2$ is a generalization of the known Gurevich-Zybin dynamical system in cosmology, whose integrability was analyzed by M. Pavlov in Reference [12] and later in the works [4,10,14] within the gradient-holonomic scheme. There was shown that this system, namely:

$$D_t u = \bar{z}_x^2, \quad D_t \bar{z} = 0, \quad (50)$$

is a smooth integrable bi-Hamiltonian flow on the 2π -periodic functional manifold \bar{M}_2 . This flow has the following Lax type representation

$$D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/\bar{z}_x) & -\bar{z}_{xx}/\bar{z}_x \end{pmatrix} f, \quad D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda\bar{z}_x & u_x \end{pmatrix} f, \quad (51)$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$.

Dynamical system (49) for the case $s = 2$ and $N = 3$ is equivalent to the following evolution flow on a 2π -periodic functional manifold $\bar{M}_3 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$ for a point $(u, v, \bar{z})^T \in \bar{M}_3$:

$$D_t u = v, \quad D_t v = \bar{z}_x^2, \quad D_t \bar{z} = 0. \quad (52)$$

3.1. Poissonian Structure on \bar{M}_3

Let us rewrite the dynamical system (52) in the following component-wise form

$$\frac{d}{dt} \begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix}, \quad (53)$$

where $\bar{K} : \bar{M}_3 \rightarrow (\bar{M}_3)$ is the corresponding vector field on \bar{M}_3 , and construct the Poissonian structures on \bar{M}_3 . To do that, we need to obtain additional solutions to the basic Noether-Lax gradient equation on the functional manifold \bar{M}_3

$$D_t \bar{\psi} + \bar{K}'^*[u, v, \bar{z}] \bar{\psi} = \text{grad } \bar{\mathcal{L}}. \quad (54)$$

Here, the matrix operator

$$\bar{K}'^*[u, v, \bar{z}] := \begin{pmatrix} 0 & -v_x & -\bar{z}_x \\ 1 & u_x & 0 \\ 0 & -2\partial \bar{z}_x & u_x \end{pmatrix} \quad (55)$$

is an endomorphism of the cotangent space $T^*(\bar{M}_3)$, adjoint to the corresponding Fréchet derivative $\bar{K}'[u, v, \bar{z}] : T(\bar{M}_3) \rightarrow T(\bar{M}_3)$ at $(u, v, \bar{z}) \in \bar{M}_3$ with respect to the bi-linear form $(\cdot | \cdot) : T^*(\bar{M}_3) \times T(\bar{M}_3) \rightarrow \mathbb{R}$. Equation (55) in the component-wise form can be rewritten as

$$\begin{aligned} D_t \bar{\psi}^{(1)} &= v_x \bar{\psi}^{(2)} + \bar{z}_x \bar{\psi}^{(3)} + \delta \bar{\mathcal{L}} / \delta u, \\ D_t \bar{\psi}^{(2)} &= -\bar{\psi}^{(1)} - u_x \bar{\psi}^{(2)} + \delta \bar{\mathcal{L}} / \delta v, \\ D_t \bar{\psi}^{(3)} &= 2(\bar{z}_x \bar{\psi}^{(2)})_x - u_x \bar{\psi}^{(3)} + \delta \bar{\mathcal{L}} / \delta \bar{z}, \end{aligned} \quad (56)$$

where $\bar{\psi} := (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^\top \in T^*(\bar{M}_3)$. The following system of linear differential relationships follows from (56)

$$\begin{aligned} D_t^3 \bar{\psi}^{(2)} &= -2\bar{z}_x^2 \bar{\psi}_x^{(2)} + D_t^2 \partial^{-1} (\delta \bar{\mathcal{L}} / \delta v) - \\ &\quad - \partial^{-1} \langle \text{grad } \bar{\mathcal{L}} | (u_x, v_x, \bar{z}_x)^\top \rangle, \\ D_t \bar{\psi}^{(2)} &= -\bar{\psi}^{(1)} + \partial^{-1} (\delta \bar{\mathcal{L}} / \delta v), \\ D_t \bar{\psi}^{(3)} &= 2\bar{z}_x \bar{\psi}_x^{(2)} + \partial^{-1} (\delta \bar{\mathcal{L}} / \delta \bar{z}), \end{aligned} \quad (57)$$

where $(\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^\top := (\tilde{\psi}_x^{(1)}, \tilde{\psi}_x^{(2)}, \tilde{\psi}_x^{(3)})^\top$. Here, the next operator relation

$$[D_t, (\alpha D_x)^j] = 0, \quad (58)$$

holds for any $j \in \mathbb{N}$ and the function $\alpha := 1/\bar{z}_x$, for which $D_t \bar{z} = 0$. The relationship (58) follows from the commutator relationship $[D_x, D_t] = u_x D_x$.

Let us now construct a differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$ which is invariant with respect to two differentiations $D_x := \partial/\partial x$, $D_t := \partial/\partial t + u\partial/\partial x$ and generated by a fixed functional variable $u \in \mathbb{R}\{\{x, t\}\}$. These differentiations satisfy the Lie-algebraic commutator relationship (48) together with the constraint (50). Taking into account that, for any $m \in \mathbb{N}$, any additive set $I_m := \{\sum_{j=1, \overline{m}} a_j \alpha^j : a_j \in \mathcal{K}\{u\}, j = \overline{1, m}\} \subset \mathcal{K}\{u\}$ is an ideal in the functional ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$, we can solve the first equation of the linear system (57) above and next recursively solve the remaining two equations. We can obtain that the three vector elements

$$\begin{aligned} \tilde{\psi}_0 &= (-v, u, -2\bar{z}_x)^\top, \quad \bar{\mathcal{L}}_0 = 0; \\ \tilde{\psi}_\theta &= (-u_x/\bar{z}_x, 1/\bar{z}_x, (u_x^2 - 2v_x)/(2\bar{z}_x^2))^\top, \quad \bar{\mathcal{L}}_\theta = 0; \\ \tilde{\psi}_\eta &= (u/2, -x/2, \partial^{-1}[(2v_x - u_x^2)/(2\bar{z}_x)])^\top, \quad \bar{\mathcal{L}}_\eta = (D_x \tilde{\psi}_\eta, \bar{K}) - H_\theta, \end{aligned} \quad (59)$$

are solutions to the linear system (57). The first two elements of (59) lead to the trivial conservation laws $(\bar{\psi}_0, \bar{K}) = 0 = (\bar{\psi}_\theta, \bar{K})$. For the $\bar{\psi}_\eta$ we obtain the Volterra asymmetric vector $\bar{\psi}_\eta := D_x \bar{\psi}_\eta : \bar{\psi}'_\eta \neq \bar{\psi}'_\eta{}^*$, which give rise to the following co-symplectic expression:

$$\bar{\eta}^{-1} := \bar{\psi}'_\eta - \bar{\psi}'_\eta{}^* = \begin{pmatrix} \partial & 0 & -\partial \frac{u_x}{\bar{z}_x} \\ 0 & 0 & \partial \frac{1}{\bar{z}_x} \\ -\frac{u_x}{\bar{z}_x} \partial & \frac{1}{\bar{z}_x} \partial & \frac{u_x}{2\bar{z}_x} \partial \frac{u_x}{\bar{z}_x} - \frac{v_x}{\bar{z}_x} \partial \frac{1}{\bar{z}_x} - \frac{1}{\bar{z}_x} \partial \frac{v_x}{\bar{z}_x} \end{pmatrix}. \quad (60)$$

Then, the Poisson operator $\bar{\eta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ can be obtained as

$$\bar{\eta} = \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} \bar{z}_x \\ 0 & \bar{z}_x \partial^{-1} & 0 \end{pmatrix}, \quad (61)$$

and the Hamiltonian representation with respect to the Poisson operator (61) is

$$\bar{K}[u, v, \bar{z}] = -\bar{\eta} \text{grad} \bar{H}_\eta, \quad (62)$$

where the Hamiltonian function $\bar{H}_\eta : \bar{M}_3 \rightarrow \mathbb{R}$ is given by the polynomial functional

$$\bar{H}_\eta = \frac{1}{2} \int_0^{2\pi} dx (2u\bar{z}_x^2 - v^2 - u^2 v_x). \quad (63)$$

The same way makes it possible to derive the second Poisson operator $\bar{\vartheta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ on the manifold \bar{M}_3 , which is compatible with the Poisson operator $\bar{\eta} : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ (61):

$$\bar{\vartheta} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2D^{-1} \end{pmatrix}. \quad (64)$$

3.2. Lax-Type Integrability Analysis

Next, we return to the Lax type integrability analysis of the dynamical system (53) on the functional manifold \bar{M}_3

$$\frac{d}{dt} \begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u, v, \bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix}. \quad (65)$$

As the Poissonian operators (64) and (61) on the manifold \bar{M}_3 are compatible [4,10,15,16], then for arbitrary $\lambda \in \mathbb{R}$ the operator pencil $(\bar{\vartheta} + \lambda \bar{\eta}) : T^*(\bar{M}_3) \rightarrow T(\bar{M}_3)$ is also Poissonian, and then all operators of the form

$$\bar{\vartheta}_n := \bar{\vartheta}(\bar{\vartheta}^{-1} \bar{\eta})^n \quad (66)$$

for arbitrary $n \in \mathbb{Z}$ are Poissonian too on the functional manifold \bar{M}_3 . Now, it is easy to reconstruct the related infinite hierarchy of the mutually commuting conservation laws

$$\begin{aligned} \bar{\gamma}_j &= \int_0^1 d\mu (\text{grad} \bar{\gamma}_j[\mu u, \mu v, \mu \bar{z}], (u, v, \bar{z})^\top), \\ \text{grad} \bar{\gamma}_j[u, v, \bar{z}] &:= \bar{\Lambda}^j \text{grad} \bar{H}_\eta, \end{aligned} \quad (67)$$

for the dynamical system (65), using the recursion property of the Poissonian pair (61), (64) and the homotopy formula [4,10,17]. Here, $j \in \mathbb{Z}$ and the corresponding recursion operator $\bar{\Lambda} := \bar{\vartheta}^{-1}\bar{\eta} : T^*(\bar{M}_3) \rightarrow T^*(\bar{M}_3)$ satisfies the associated Lax-type commutator relationship

$$\frac{d\bar{\Lambda}}{dt} = [\bar{\Lambda}, \bar{K}'^*]. \quad (68)$$

Remark 5. It is evident that the trace-functionals

$$\bar{\gamma}_n := \int_0^{2\pi} \text{Tr}(\bar{\Lambda}^n) dx, \quad (69)$$

where $\text{Tr} := \text{res}_{D-1} \text{tr} : \text{End}(T^*(\bar{M}_3) \rightarrow C^\infty(\bar{M}_3; C^\infty([0, 2\pi]; \mathbb{R})))$ is the usual Adler type trace operation on the algebra of periodic pseudo-differential operators $\text{PDO}(\mathbb{R}/\{2\pi\mathbb{Z}\})$, are conservation laws for our dynamical system (65) for any $n \in \mathbb{Z}$. In particular, this property was put into the background of the Shabat-Mikhailov integrability classification scheme.

The following result is based on the analysis and observations above.

Proposition 2. The Riemann type hydrodynamic system (65) on the functional manifold \bar{M}_3 is a bi-Hamiltonian dynamical system with respect to the compatible Poissonian structures $\bar{\vartheta}, \bar{\eta} : T^*(\bar{M}_3) \rightarrow T^*(\bar{M}_3)$ (64) and (61) and possesses an infinite hierarchy of mutually commuting conservation laws (67).

4. Differential-Algebraic Integrability Analysis: $N = 3$

Let us consider the previously introduced differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$, which is generated by a fixed functional variable $u \in \mathbb{R}\{\{x, t\}\}$ and is invariant with respect to differentiations $D_x := \partial/\partial x$, $D_t := \partial/\partial t + u\partial/\partial x$. These differentiations satisfy the Lie-algebraic commutator relationship (48) together with the constraint (50), which is expressed in the following differential-algebraic functional form

$$D_t^3 u = -2D_t^2 u D_x u.$$

Now, we take into account the fact that the Lax-type representation for (65) can be interpreted [10,11] as the existence of a finite-dimensional invariant ideal $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ realizing the finite-dimensional representation of relationship (48). The ideal can be constructed as

$$\mathcal{I}\{u\} := \{\lambda^2 u f_1 + \lambda v f_2 + \bar{z}_x f_3 \in \mathcal{K}\{u\} : f_j \in \mathcal{K}, 1 \leq j \leq 3, \lambda \in \mathbb{R}\}, \quad (70)$$

where $v = D_t u$, $\bar{z}_x^2 = D_t^2 u$, $D_t \bar{z} = 0$. The finite-dimensional representations of the D_x - and D_t -differentiations can be constructed if we find [11] the D_t -invariant kernel $\ker D_t \subset \mathcal{I}\{u\}$ and then check its invariance with respect to the D_x -differentiation. The kernel can be found as follows:

$$\ker D_t = \{f \in \mathcal{K}^3\{u\} : D_t f = q(\lambda) f, \quad \lambda \in \mathbb{R}\}, \quad (71)$$

where $q(\lambda) := q[u, v, \bar{z}; \lambda] \in \text{End } \mathcal{K}\{u\}^3$ is given as

$$q(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -2\lambda \bar{z}_x \bar{z}_{xx} & u_x \end{pmatrix}. \quad (72)$$

The D_x -differentiation representation in the space $\mathcal{K}\{u\}^3$ can be constructed if we find a matrix $l(\lambda) := l[u, v, \bar{z}; \lambda] \in \text{End}\mathcal{K}\{u\}^3$, which satisfies the linear relationship for $f \in \mathcal{K}\{u\}^3$

$$D_x f = l(\lambda) f \quad (73)$$

when the corresponding ideal

$$\mathcal{R}\{u\} := \{\langle g|f \rangle_{\mathcal{K}^3} : f \in \ker D_t \subset \mathcal{K}^3\{u\}, g \in \mathcal{K}^3\} \quad (74)$$

is D_x -invariant with respect to the matrix differentiation representation (73). The matrix

$$l(\lambda) = \begin{pmatrix} \lambda^2 u \bar{z}_x & \lambda v \bar{z}_x & \bar{z}_x^2 \\ -t \lambda^3 u \bar{z}_x & -t \lambda^2 v \bar{z}_x & -t \lambda \bar{z}_x^2 \\ \lambda^4 (tuv - u^2) - \lambda^2 u_x \bar{z}_x^{-1} & -\lambda v_x \bar{z}_x^{-1} + \lambda^3 (tv^2 - uv) & \lambda^2 \bar{z}_x (u - tv) - \bar{z}_{xx} \bar{z}_x^{-1} \end{pmatrix} \quad (75)$$

can be found by straightforward calculations. The following proposition is stated.

Proposition 3. *The generalized Riemann-type dynamical system (65) for $N = 3$ is a bi-Hamiltonian integrable flow with a non-autonomous Lax-type representation for $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$*

$$D_t f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -2\lambda \bar{z}_x \bar{z}_{xx} & u_x \end{pmatrix} f, \quad (76)$$

$$D_x f = \begin{pmatrix} \lambda^2 u \bar{z}_x & \lambda v \bar{z}_x & \bar{z}_x^2 \\ -t \lambda^3 u \bar{z}_x & -t \lambda^2 v \bar{z}_x & -t \lambda \bar{z}_x^2 \\ \lambda^4 (tuv - u^2) - \lambda^2 u_x \bar{z}_x^{-1} & -\lambda v_x \bar{z}_x^{-1} + \lambda^3 (tv^2 - uv) & \lambda^2 \bar{z}_x (u - tv) - \bar{z}_{xx} \bar{z}_x^{-1} \end{pmatrix} f,$$

with the arbitrary spectral parameter $\lambda \in \mathbb{R}$.

Equation (76) depends explicitly on the temporal evolution parameter $t \in \mathbb{R}$, which is not usual. Nonetheless, the matrices (72) and (75) satisfy the Zakharov–Shabat compatibility condition

$$D_t l(\lambda) = [q(\lambda), l(\lambda)] + D_x l(\lambda) - u_x l(\lambda), \quad (77)$$

for all $\lambda \in \mathbb{R}$. This follows from the linear Lax type relationships (71), (73) and the commutator condition (48). We can assume that the dynamical system (65) possesses a usual autonomous Lax type representation, taking into account that it has a compatible Poissonian pair (61) and (64), which depends only on the variables $(u, v, \bar{z})^\top \in \bar{M}_3$. This representation can be found by means of a corresponding gauge transformation of the linear relationships (71) and (73).

5. Concluding Remarks

We reanalyzed Dubrovin's integrability-based classification scheme for perturbed KdV equations through the lens of the gradient-holonomic method. This approach gave us possibility to extract additional information such a bi-Hamiltonian structures and conservation laws. The gradient-holonomic approach also allowed us to introduce and perform an in-depth integrability analysis of a special case of a novel hierarchy of Riemann-type hydrodynamical systems, including conservation laws, Lax representations, and bi-Hamiltonian structure.

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