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An Inventory Ordering Model for Deteriorating Items with Compounding and Backordering

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Abstract: We consider the optimal order quantity problem for exponentially deteriorating items where the opportunity cost is based on compound interest and backorders are allowed. Our objectives in this research are to develop a model that accurately models deterioration, compound interest and backordering, and determine a near-optimal and intuitive closed-form solution for the proposed model. Deteriorating items include various chemicals, gasoline and petroleum products, fresh produce, bulk and liquid food products, batteries, and some electronic components. These items incur losses over time due to spoilage, evaporation, chemical decomposition, breakdown, or deterioration in general. Exponential deterioration is commonly used to model this phenomenon, which results in a negative exponential inventory level function, which is asymmetric in the sense that the rate of depletion is highest at the beginning of an ordering cycle, and lowest at the end. On the other hand, the rate of deterioration for individual items is the same at both ends of the cycle, which means it is symmetric. Compounding also leads to exponential terms in the opportunity cost function. Both of these factors result in a total cost function that does not have a closed-form optimal solution. We therefore approximate the total cost function using a Taylor series expansion approximation of the exponential function and derive a closed-form solution that is simple and logical, and very close to the exact optimum, which makes it attractive to the practitioners as a quick and accurate calculation. Our closed form solutions for both the basic and the planned backorders models are very close to the exact optimum, as shown by extensive numerical experiments.

Keywords: inventory; deterioration; deteriorating items; exponential decay; compound interest; compounding; Net Present Value; NPV; backordering

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1. Introduction

The basic Economic Order Quantity (EOQ) model has been in widespread use in inventory management for more than a century since its inception due to its simplicity, robustness and its applicability to various cases in practice. The basic EOQ model is based on the tradeoff between two different types of annualized costs relevant in inventory management: the average ordering cost and the average inventory holding cost. The order quantity is the decision variable, and the average annual ordering cost is inversely related to the order quantity, whereas the average annual inventory holding cost increases linearly in the order quantity.

Even though the basic EOQ model is simple and in widespread use, it has some very restrictive assumptions. One of its restrictive assumptions is that the item is durable and can be kept in the inventory for any length of time. There are many products that do not satisfy this assumption: fresh produce, bulk and liquid food products, batteries, electronic parts, etc., deteriorate or break down over time. Exponential deterioration is a commonly used model for this phenomenon, in which the time to deterioration for an individual item follows an exponential distribution. It is also called constant deterioration because the instantaneous deterioration rate (or hazard rate) is constant for this distribution, which is symmetric in the sense that it is the same at both ends of an inventory ordering cycle.

Another assumption of the basic EOQ model is that the items in the inventory have constant and deterministic demand, and they incur an opportunity cost that is calculated as simple interest on the linearly decreasing inventory value over time, during each ordering cycle. In practice, compound interest is more widespread, and simple interest is rarely used in business. Yet another assumption of the basic EOQ model is that the demand for the item during an inventory stockout period is lost and cannot be recovered. Planned backorder models allow for the possibility of demand accumulating during a stockout period, which is immediately satisfied at the beginning of the next ordering cycle, when the new order is received.

Net Present Value (NPV)-based models have been proposed to model the time value of money, by using continuous discounting of all future cash flows. Closed-form solutions are not possible for these models due to exponential terms in the total cost function, and their approximations invariably result in the classical EOQ equation as a closed-form optimal solution. Furthermore, NPV based models are myopic; when discounted, cash flows after the first several years become negligible and have little effect on the solution. In reality, prices go up in the long run and the EOQ has to be recalculated periodically. It is also implicitly assumed that the funds needed for all future cash outflows are available at present in the form of cash, though at discounted values. In reality, most firms have limited working capital, thus the needed cash only becomes available gradually. For these reasons, calculating an average total cost that incorporates compound interest for the first year and then optimizing this total cost seems to be a more accurate way to model compounding.

In this research, our objectives are as follows: (i) to model the inventory for deteriorating items with exponential deterioration, (ii) derive the opportunity cost based on compound interest, (iii) allow for planned backorders, (iv) develop an exact total cost function, and (v) derive a closed-form near-optimal solution that is intuitive and easy to understand by the practitioners in the industry. The exact total cost function that we derive in this paper does not have a closed-form solution, so we approximate it using a Taylor series approximation of the exponential function, and derive a closed form solution that is remarkably close to the exact optimum, even for interest rates as high as 25%. Our closed-form solution has the same form as the basic EOQ equation; with two inventory holding cost rate adjustments: (i) the Annual Percentage Yield (APY) of compound interest, (ii) the rate of deterioration losses.

We can summarize our contributions in this paper as follows: (i) the modeling approach for compound interest that is significantly different and more realistic and accurate compared to the existing Net Present Value (NPV) based models, (ii) easy-to-follow and simple derivations that make this paper accessible to practitioners, (iii) the novel total cost function that is simple enough to be optimized using a spreadsheet optimizer, and (iv) the simple and logical closed-form solution that is remarkably close to the optimum. In addition to not having the shortcomings of the NPV-based models described earlier, our model has another advantage. All NPV based models assume continuous compounding, and there is no straightforward way to adjust for a specific compounding period. Our closed-form optimal solution can be easily modified to any compounding period of choice: daily, weekly, monthly, quarterly, etc.

2. Literature Review

Harris [1] is the founder of inventory management, with their development of the basic EOQ model and its famous square-root formula as its optimal solution. Ghare and Schrader [2] were the first to extend the EOQ model to exponentially deteriorating items and derive a recursive closed-form solution that requires several iterations to converge. Dave and Patel [3], Bahari-Kashani [4], Chung and Ting [5], Chung and Ting [6], and Kim [7] later extended this model to linearly increasing demand and derived heuristic optimal solutions. Dave and Patel [3] proposed only a numerical solution approach, whereas Sachan [8] modified their proposed method and corrected parts of it. Çalışkan [9] modified and improved the model in Ghare and Schrader [2] and derived a truly closed-form solution without using differential calculus, while Çalışkan [10] derived a different, simpler closed-form solution for the model of Chung and Ting [6].

Widyadana et al. [11] approximated the total cost function in a different, but much less accurate way, as was later shown by Çalışkan [12]. Wee [13] extended the exponentially deteriorating items model of Ghare and Schrader [2] to where demand is exponentially declining over time in a given cycle, whereas Hariga and Benkherouf [14] considered exponentially changing demand in general. Benkherouf [15] assumed linearly increasing demand, whereas Benkherouf and Mahmoud [16] considered linearly increasing demand with planed backorders. Aggarwal and Jaggi [17] and Chu et al. [18] extended the basic exponentially deteriorating items model to trade credits, where the supplier offers an extended time to pay for each order.

More recently, Khakzad and Gholamian [19] considered the case of the deteriorating items such as fresh produce, in which items adjacent to the already deteriorated items are removed, and this reduces the average rate of deterioration. Tiwari et al. [20] considered a model in which the item is exponentially deteriorating, and the supplier offers a partial delay in payment. Khan et al. [21] considered a deteriorating item for which the supplier offers an all-units discount policy and the demand depends on the price, and they developed a model to maximize the total profit and determine the optimal values of the selling price and the order interval, which in turn determines the optimal order quantity. Rezagholifam et al. [22] considered non-instantaneous deteriorating items for which the demand is price- and stock-dependent and the storage space is limited, and they developed a model to maximize the total profit and determined the optimal price and order quantity. All of these papers assume simple interest.

Hadley [23] developed the first NPV based model, but proposes no closed-form solution. Porteus [24] proposed a Discounted Average Value (DAV) approach and concluded that the basic EOQ formula is an accurate approximate closed-form solution for it. Teunter et al. [25], van der Laan and Teunter [26], Teunter and Van der Laan [27], van der Laan [28], Çorbacıoğlu and van der Laan [29] also concluded that the basic EOQ formula is a good approximate solution to the NPV model. Sun and Queyranne [30] similarly concluded that the EOQ solution is a good approximation for the optimal solution to the NPV-based model. Haneveld and Teunter [31] considered random demand and a NPV-based total cost function; and their closed-form solution for uniform demand is applicable to the basic EOQ problem with compound interest. Bartmann and Beckmann [32] ([Ch.2]) also proposed a closed-form solution that is applicable to the basic EQO model with compound interest.

Departing from the NPV-based models to incorporate compounding in the basic EOQ model, Çalışkan [33] derived an approximate closed-form solution that (i) is not the classical EOQ equation, (ii) is closer to the optimum compared to the classical EOQ equation, and (iii) is simple, logical and intuitive. Çalışkan [33] also showed that the closed-form solution of Haneveld and Teunter [31] is almost equal to the basic EOQ equation for all reasonable values of the parameters and therefore not very accurate; and the closed-form solution of Bartmann and Beckmann [32] was completely off, except when the demand is approximately equal to one unit. Çalışkan [34] simplified the derivation of the optimal solution for the model in Çalışkan [33].

Zipkin [35] ([pp. 63–64, 71]) also developed a NPV-based model and approximated the total cost function and ended up with the basic EOQ formula as the closed-form optimal solution. Lin et al. [36] re-derived the same result based on Hadley [23]. Grubbström [37] applied Laplace transform to develop a NPV-based model for the Economic Production Quantity (EPQ) problem, but he found that the basic EPQ equation is an approximate closed-form solution for the NPV model. Mahajan and Diatha [38] considered compound interest and used the approximation in Porteus [24], but they also arrived at the basic EOQ equation as the approximate optimal solution. Mahajan and Diatha [38] proposed a model for the deteriorating items extension as well, but they did not propose a closed-form solution; they only demonstrated it by numerically solving an example problem. Rachamadugu [39] considered the EOQ problem with a delay in payment from the vendor (trade credit) and developed a NPV-based model. He analyzed the relationship between the length of the trade credit and the order quantity, and interestingly, pointed out that using the basic EOQ equation as an approximation to the NPV-based model is problematic, as Çalışkan [33] also proved, though without trade credits. van Delft and Vial [40] also studied the basic EOQ model with discounted cash flows and derived a total cost function similar to that of Hadley [23] and Zipkin [35] ([pp 63–64, 71]), Lin et al. [36], Mahajan and Diatha [38], and others. Not surprisingly, they also concluded that the basic EOQ formula is a good approximation for the NPV-based total cost function.

Regardless of the shortcomings of the NPV method in modeling the compound interest based opportunity cost in inventory optimization models, it has been in widespread use. Recently, Alım and Beullens [41] studied an inventory and distribution optimization problem in which an online seller offers their customers a flexible delivery option in which delivery times can be postponed at a discount to the customer and the delivery can be done by the company's own vehicles or third party logistics companies. The model maximizes the profit function that is represented as the NPV of the future cash flows. Another recent study that incorporates the NPV approach in modeling is by Sarkar, Sumon et al. [42], who studied a vendor–buyer model with partial backordering, stochastic lead time, and they minimized the NPV of the expected total cost for the integrated system. Sundararajan et al. [43] studied the case of an inventory item for which the demand is time- and price-dependent, backorders are allowed, inflation is assumed, and the ending inventory is nonzero, and they developed an algorithm to find the optimal selling price, optimal stockout period, optimal replenishment cycle time, and the optimal ending inventory level.

3. Materials and Methods

The notation that we use in this research is in Table 1. The items deteriorate according to an exponential distribution, with an expected time to deterioration (or useful life) of $\frac{1}{\delta}$. Deteriorated items are immediately detected and removed from the inventory. The quantity ordered is received all at once, the demand per unit is deterministic and constant and occurs uniformly over time. The inventory holding cost consists of a financial and a warehousing cost portion. The financial portion is the opportunity cost of capital invested in the inventory. This is represented by the interest rate r, and continuous compounding is used in calculating the total opportunity cost in any period. The warehousing cost portion is calculated using the warehousing cost rate i and it is calculated the usual way for any period. The objective is to determine the order (T) and the fulfillment (T_I) intervals that minimize the total cost of deterioration, warehousing, opportunity, and ordering costs per unit time. The optimal order (Q) and backorder (B) quantities can then be calculated based on their relationship to T and T_I, which we derive later in the paper.

Table 1. The parameters (top) and variables (bottom) of the model.

D	the demand of the item in number of units per year
S	the cost of ordering per order
δ	the rate of deterioration for one unit of the item per year
i	the cost of warehousing per year as a fraction of the unit price of the item
r	the annual interest rate (or opportunity cost) as a fraction
c	the unit cost of the item
T	the length of an ordering cycle (order interval)
T_I	the length of time the demand is fulfilled in a cycle (fulfillment interval)
Q	the number of units to order in each ordering cycle
B	the number of units to backorder in each ordering cycle
t	the elapsed time since the beginning of an ordering cycle
$I(t)$	the number of units in the inventory at time t

3.1. Model Development

At the beginning of an ordering cycle, the received order is used to immediately satisfy the accumulated backorders in the previous period, so the maximum inventory level is $Q - B$. This amount is depleted by both demand and deterioration. The deterioration rate for one unit of the item is δ, so the deterioration rate for the entire inventory at time t is $\delta I(t)$. This leads to the following differential equation:

$$\frac{dI(t)}{dt} = -D - \delta I(t), \qquad 0 \le t \le T_I \tag{1}$$

The boundary conditions are $I(0) = Q - B$ and $I(T_I) = 0$, because the inventory completely depletes at time T_I. The solution to this differential equation is as follows:

$$I(t) = (Q - B)e^{-\delta t} - \frac{D}{\delta}(1 - e^{-\delta t}), \qquad 0 \le t \le T_I \tag{2}$$

Equation (2) exhibits asymmetry in the sense that the depletion rate (slope) at the beginning of the cycle is highest, whereas it is lowest at the end of the cycle. For an individual item, instantaneous deterioration rate is symmetric because it is the same both at the beginning and end of the cycle. We can determine Q and B in terms of T and T_I as follows:

$$Q = \frac{D}{\delta}(e^{\delta T_I} - 1) + D(T - T_I) \tag{3}$$

$$B = D(T - T_I) \tag{4}$$

3.1.1. Deterioration (Waste) Cost

The initial inventory on hand in a given cycle is $Q - B$. From this amount, in T_I time units, DT_I units fulfill the demand, the rest is deterioration, which results in the deterioration cost per cycle as follows:

$$W_c = c(Q - B - DT_I) \tag{5}$$

We can also arrive at the same result as follows:

$$\begin{aligned}
W_c &= \int_0^{T_I} c\delta I(t)dt = c\delta \int_0^{T_I} \left[(Q - B)e^{-\delta t} - \frac{D}{\delta}(1 - e^{-\delta t})\right]dt \\
W_c &= c\delta\left[\int_0^{T_I}(Q - B)e^{-\delta t} - \int_0^{T_I}\frac{D}{\delta}dt + \int_0^{T_I}\frac{D}{\delta}e^{-\delta t}dt\right] \\
W_c &= c\delta\left[-\frac{(Q - B)}{\delta}e^{-\delta t}\Big|_0^{T_I} - \frac{D}{\delta}t\Big|_0^{T_I} - \frac{D}{\delta^2}e^{-\delta t}\Big|_0^{T_I}\right] \\
W_c &= c\delta\left[-\frac{(Q - B)}{\delta}e^{-\delta T_I} + \frac{(Q - B)}{\delta} - \frac{DT_I}{\delta} - \frac{D}{\delta^2}e^{-\delta T_I} + \frac{D}{\delta^2}\right] \\
W_c &= c\delta\left[\left(-\frac{(Q - B)}{\delta} - \frac{D}{\delta^2}e^{-\delta T_I}\right) + \frac{(Q - B)}{\delta} - \frac{DT_I}{\delta} + \frac{D}{\delta^2}\right] \\
W_c &= c\delta\left[-\frac{1}{\delta}\left((Q - B) + \frac{D}{\delta}\right)\left(\frac{D}{D + \delta(Q - B)}\right) + \frac{(Q - B)}{\delta} - \frac{DT_I}{\delta} + \frac{D}{\delta^2}\right] \\
W_c &= c\delta\left[-\frac{D}{\delta^2} + \frac{(Q - B)}{\delta} - \frac{DT_I}{\delta} + \frac{D}{\delta^2}\right] = c(Q - B - DT_I)
\end{aligned} \tag{6}$$

3.1.2. Warehousing Cost

We can similarly calculate the cost of warehousing per cycle as follows:

$$H_c = \int_0^{T_I} icI(t)dt = ic\int_0^{T_I}\left[(Q - B)e^{-\delta t} - \frac{D}{\delta}(1 - e^{-\delta t})\right]dt \tag{7}$$

Based on the equivalence of Equations (5) and (6), we obtain the following:

$$H_c = \frac{ic}{\delta}(Q - B - DT_I) \tag{8}$$

3.1.3. Opportunity Cost

The amount of opportunity cost until the time inventory is fully depleted (T_I) can be calculated as follows:

$$I_I = \int_0^{T_I} cI(t)re^{r(T_I - t)}dt = rc\int_0^{T_I}\left[(Q-B)e^{-\delta t} - \frac{D}{\delta}(1-e^{-\delta t})\right]e^{r(T_I-t)}dt$$
$$I_I = rc(Q-B)\int_0^{T_I} e^{r(T_I-t)-\delta t}dt - \frac{rcD}{\delta}\int_0^{T_I} e^{r(T_I-t)}dt + \frac{rcD}{\delta}\int_0^{T_I} e^{r(T_I-t)-\delta t}dt$$
$$I_I = -\frac{rc(Q-B)}{r+\delta}e^{r(T_I-t)-\delta t}\Big|_0^{T_I} + \frac{cD}{\delta}e^{r(T_I-t)}\Big|_0^{T_I} - \frac{rcD}{\delta(r+\delta)}e^{r(T_I-t)-\delta t}\Big|_0^{T_I}$$
$$I_I = -\frac{rc(Q-B)}{r+\delta}[e^{-\delta T_I} - e^{rT_I}] + \frac{cD}{\delta} - \frac{cD}{\delta}e^{rT_I} - \frac{rcD}{\delta(r+\delta)}[e^{-\delta T_I} - e^{rT_I}]$$
$$I_I = rc\left[\frac{Q-B}{r+\delta} + \frac{D}{\delta(r+\delta)}\right](e^{rT_I} - e^{-\delta T_I}) + \frac{cD}{\delta}(1 - e^{rT_I}) \tag{9}$$

This amount will accumulate further opportunity cost until the end of the cycle, or until time T:

$$I_c = I_I e^{r(T-T_I)} = rc\left[\frac{Q-B}{r+\delta} + \frac{D}{\delta(r+\delta)}\right](e^{rT} - e^{rT-(r+\delta)T_I}) + \frac{cD}{\delta}(e^{r(T-T_I)} - e^{rT})$$
$$I_c = rce^{rT}\left[\frac{Q-B}{r+\delta} + \frac{D}{\delta(r+\delta)}\right](1 - e^{-(r+\delta)T_I}) + \frac{cDe^{rT}}{\delta}(e^{-rT_I} - 1) \tag{10}$$

The opportunity cost per year (I_a) on average is the sum of the opportunity cost for all cycles until the end of the year plus the average compound interest for the fractional cycles, if there is any (I_f), and the number of cycles per year is $\frac{1}{T}$. This results in an annual opportunity cost as follows:

$$I_a = \sum_{i=1}^{\lfloor \frac{1}{T} \rfloor} I_c e^{r(1-iT)} + I_f$$
$$I_a = I_c e^r\left[e^{-rT} + \left(e^{-rT}\right)^2 + \left(e^{-rT}\right)^3 + \ldots + \left(e^{-rT}\right)^{\lfloor \frac{1}{T} \rfloor}\right] + I_f$$
$$I_a = I_c e^r\left[\frac{1 - \left(e^{-rT}\right)^{\lfloor \frac{1}{T} \rfloor + 1}}{1 - e^{-rT}} - 1\right] + I_f \tag{11}$$

In Çalışkan [33], it has been shown that in Equation (11), using the summation equation with $\frac{1}{T}$ instead of $\left\lfloor \frac{1}{T} \right\rfloor$ and removing the second term is a close approximation. Applying this, we obtain the annual opportunity cost as follows:

$$I_a = I_c e^r\left[\frac{1 - \left(e^{-rT}\right)^{\frac{1}{T}+1}}{1 - e^{-rT}} - 1\right] = \frac{I_c e^r(1 - e^{-r})e^{-rT}}{1 - e^{-rT}} = (e^r - 1)\frac{e^{-rT}}{1 - e^{-rT}}I_c$$
$$I_a = (e^r - 1)\frac{e^{-rT}}{1 - e^{-rT}}\left[rce^{rT}\left[\frac{Q-B}{r+\delta} + \frac{D}{\delta(r+\delta)}\right](1 - e^{-(r+\delta)T_I}) + \frac{cDe^{rT}}{\delta}(e^{-rT_I} - 1)\right]$$
$$I_a = \frac{c(e^r - 1)}{1 - e^{-rT}}\left[r\left(\frac{Q-B}{r+\delta} + \frac{D}{\delta(r+\delta)}\right)(1 - e^{-(r+\delta)T_I}) + \frac{D}{\delta}(e^{-rT_I} - 1)\right] \tag{12}$$

Substituting Equation (3) in place of $(Q-B)$ in Equation (12), we obtain I_a as a function of T and T_I:

$$I_a = \frac{c(e^r-1)}{1-e^{-rT}}\left[r\left(\frac{D(e^{\delta T_I}-1)}{\delta(r+\delta)}+\frac{D}{\delta(r+\delta)}\right)(1-e^{-(r+\delta)T_I})+\frac{D}{\delta}(e^{-rT_I}-1)\right]$$
$$I_a = \frac{c(e^r-1)}{1-e^{-rT}}\left[\frac{rDe^{\delta T_I}}{\delta(r+\delta)}(1-e^{-(r+\delta)T_I})+\frac{D}{\delta}(e^{-rT_I}-1)\right]$$
$$I_a = \frac{c(e^r-1)}{1-e^{-rT}}\left[\frac{rD}{\delta(r+\delta)}e^{\delta T_I}-\frac{rD}{\delta(r+\delta)}e^{-rT_I}+\frac{D}{\delta}e^{-rT_I}-\frac{D}{\delta}\right]$$
$$I_a = \frac{c(e^r-1)D}{\delta(1-e^{-rT})}\left[\frac{r}{r+\delta}e^{\delta T_I}+\left(1-\frac{r}{r+\delta}\right)e^{-rT_I}-1\right]$$
$$I_a = \frac{c(e^r-1)D}{\delta(1-e^{-rT})}\left[\frac{r}{r+\delta}e^{\delta T_I}+\frac{\delta}{r+\delta}e^{-rT_I}-1\right] \tag{13}$$

3.1.4. Annual Total Cost

Backordering cost in a given cycle can be calculated as follows:

$$B_c = \int_{T_I}^{T} bD(t-T_I)dt = bD\left(\frac{t^2}{2}-T_I t\right)\Big|_{T_I}^{T}$$
$$B_c = bD\left(\frac{T^2}{2}-T_I T-\frac{T_I^2}{2}+T_I^2\right) = \frac{bD(T-T_I)^2}{2} \tag{14}$$

We can then express the annual average total cost function as follows:

$$TC(Q,B,T,T_I) = \frac{S}{T}+\frac{c}{T}(Q-B-DT_I)+I_a+\frac{ic}{\delta T}(Q-B-DT_I)+\frac{bD(T-T_I)^2}{2T} \tag{15}$$

Substituting Equations (3) and (13) in Equation (15), we obtain:

$$TC(T,T_I) = \frac{S}{T}+\frac{c\delta+ic}{\delta T}\left[\frac{D}{\delta}(e^{\delta T_I}-1)-DT_I\right]$$
$$+\frac{c(e^r-1)D}{\delta(1-e^{-rT})}\left[\frac{r}{r+\delta}e^{\delta T_I}+\frac{\delta}{r+\delta}e^{-rT_I}-1\right]+\frac{bD(T-T_I)^2}{2T} \tag{16}$$

The exponential terms in Equation (16) make it impossible to obtain a closed-form optimal solution. In the next section, we approximate the exponential terms in Equation (16) with polynomials and develop an approximate model that lends itself to a closed-form solution.

3.1.5. The Basic Model

For the basic deteriorating items compound interest model in which backorders are not allowed, $T = T_I$ and $B = 0$. Therefore, the total cost function in terms of T and T_I reduces to:

$$TC(T) = \frac{S}{T}+\frac{c\delta+ic}{\delta T}\left[\frac{D}{\delta}(e^{\delta T}-1)-DT\right]$$
$$+\frac{c(e^r-1)D}{\delta(1-e^{-rT})}\left[\frac{r}{r+\delta}e^{\delta T}+\frac{\delta}{r+\delta}e^{-rT}-1\right] \tag{17}$$

3.2. The Approximate Model

Taylor series expansion of the exponential function is $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$. For small x, terms higher than the second order can be neglected. To calculate the approximation of I_a, we will substitute the first three terms of their Taylor series expansions in place of e^{-rT} and e^{-rT_I} in Equation (13). This is warranted because in practice, r is usually small, less than 0.1 in almost all cases. In addition, both T and T_I are very small, much less than 1 in practice. This results in the following:

$$I_a(T,T_I) = \frac{c(e^r-1)D}{\delta(rT-\frac{r^2T^2}{2})}\left[\frac{r}{r+\delta}\left(1+\delta T_I+\frac{\delta^2T_I^2}{2}\right)+\frac{\delta}{r+\delta}\left(1-rT_I+\frac{r^2T_I^2}{2}\right)-1\right]$$
$$I_a(T,T_I) = \frac{c(e^r-1)D}{\delta(rT-\frac{r^2T^2}{2})}\left[\frac{r}{r+\delta}+\frac{r\delta T_I}{r+\delta}+\frac{r\delta^2T_I^2}{2(r+\delta)}+\frac{\delta}{r+\delta}-\frac{r\delta T_I}{r+\delta}+\frac{\delta r^2T_I^2}{2(r+\delta)}-1\right]$$
$$I_a(T,T_I) = \frac{c(e^r-1)D}{\delta rT\frac{(2-rT)}{2}}\left[\frac{r\delta^2T_I^2+\delta r^2T_I^2}{2(r+\delta)}\right] = \frac{c(e^r-1)D}{\delta rT\frac{(2-rT)}{2}}\left[\frac{\delta rT_I^2(\delta+r)}{2(r+\delta)}\right] = \frac{c(e^r-1)DT_I^2}{T(2-rT)}$$
$$I_a(T,T_I) = \frac{c(e^r-1)DT_I^2}{2T-rT^2} \approx \frac{c(e^r-1)DT_I^2}{2T} \tag{18}$$

Again, rT^2 is very small in practice as we discussed earlier, and we therefore ignore it in the last simplified version of I_a in Equation (18). Applying the same and substituting Equation (18) in Equation (16), we obtain the average annual total cost function as follows:

$$TC(T,T_I) = \frac{S}{T}+\frac{[c\delta+ic+c(e^r-1)]DT_I^2}{2T}+\frac{b(T-T_I)^2}{2T} \tag{19}$$

For the basic model, $T = T_I$, therefore Equation (19) reduces to:

$$TC(T) = \frac{S}{T}+\frac{[c\delta+ic+c(e^r-1)]DT}{2} \tag{20}$$

Let $h' = c\delta+ic+c(e^r-1)$. Then, the second partial derivatives of Equation (19) can be determined as follows:

$$TC_{TT} = \frac{2S}{T^3}+(h'+b)\frac{DT_I^2}{T^3} \tag{21}$$
$$TC_{T_IT_I} = \frac{(h'+b)D}{T} \tag{22}$$
$$TC_{TT_I} = TC_{T_IT} = -\frac{(h'+b)DT_I}{T^2} \tag{23}$$

Convexity of Equation (19) requires $TC_{TT} \geq 0$, $TC_{T_IT_I} \geq 0$ and $TC_{TT}TC_{T_IT_I} - TC_{TT_I}^2 \geq 0$. The first two conditions are satisfied as strict inequalities, and the last condition reduces to:

$$TC_{TT}TC_{T_IT_I} - TC_{TT_I}^2 = \frac{2(h'+b)SD}{T^4} \tag{24}$$

The last condition is also satisfied as a strict inequality. Therefore, Equation (19) is strictly convex. We will now use the first order conditions to determine the optimal solution. Setting the partial derivative of Equation (19) with respect to T_I equal to zero, we obtain:

$$\frac{\partial TC(T,T_I)}{\partial T_I} = \frac{h'DT_I}{T}-\frac{bD(T-T_I)}{T} = 0$$
$$T_I^* = \left(\frac{b}{h'+b}\right)T^* \;\Rightarrow\; T_I^* = \left(\frac{b}{c\delta+ic+c(e^r-1)+b}\right)T^* \tag{25}$$

Substituting Equation (25) in Equation (19), we obtain the following:

$$TC(T) = \frac{S}{T} + \frac{h'D\left(\frac{b}{h'+b}\right)^2 T^2}{2T} + \frac{bD\left[T - \left(\frac{b}{h'+b}\right)T\right]^2}{2T}$$

$$TC(T) = \frac{S}{T} + \frac{DT}{2}\left[\frac{h'b^2}{(h'+b)^2} + \frac{b(h')^2}{(h'+b)^2}\right] \tag{26}$$

$$TC(T) = \frac{S}{T} + \left[\frac{h'b(b+h')}{(h'+b)^2}\right]\frac{DT}{2} = \frac{S}{T} + \left(\frac{b}{h'+b}\right)h'\frac{DT}{2} \tag{27}$$

Equation (27) is the same as the total cost function of the basic EOQ model except that h is replaced by $\left(\frac{b}{h'+b}\right)h'$. Therefore, the optimal order interval can be determined as follows:

$$T^* = \sqrt{\frac{2S}{\left(\frac{b}{h'+b}\right)h'D}} = \sqrt{\frac{2S}{[c\delta + ic + c(e^r - 1)]D}}\sqrt{\frac{c\delta + ic + c(e^r - 1) + b}{b}} \tag{28}$$

For the basic model, Equation (20) is the same as the total cost function of the basic EOQ model with h' instead of $h = i(c + r)$. Therefore, the optimal order interval can be determined as follows:

$$T^* = \sqrt{\frac{2S}{h'D}} = \sqrt{\frac{2S}{[c\delta + ic + c(e^r - 1)]D}} \tag{29}$$

The optimal solution to the approximate model is the same as the EOI (Economic Order Interval) equation of the classical EOQ model, except that the inventory holding cost rate is $h' = c[i + (e^r - 1) + \delta]$ instead of $h = c(i + r)$. The adjustments to the inventory holding cost rate of the classical EOQ model to account for both compound interest and deterioration are meaningful, logical, and easy to explain to the practitioners and decision makers in the industry. Deterioration can be thought of a type of inventory holding cost; $c\delta$ is the cost of waste per unit per year for the items in the inventory, much like the traditional definition of the inventory holding cost rate. Annual (simple) interest rate is r and the Annual Percentage Yield (APY) with continuous compounding is $e^r - 1$. The opportunity cost component in h' can simply be adjusted to allow for specific discrete compounding periods. For instance, e^r can be replaced by $(1 + \frac{r}{12})^{12}$ for monthly compounding; by $(1 + \frac{r}{4})^4$ for quarterly compounding; and by $(1 + \frac{r}{n})^n$ for compounding n times per year in general.

4. Results

In this section, we test the accuracy of our closed form solution by comparing it to the exact optimal solution for both versions of the model. The following are the sets of parameters in our experimental cases: $D = \{10{,}000, 500\}$; $\delta = \{5, 3.75, 2.5, 1.25, 0.1\}$; and $r = \{0.05, 0.1, 0.15, 0.20, 0.25\}$. We use $i = 0$ in all cases because higher i values lessen the impact of compounding, as well as lessening the percentage error between the approximate and exact models. In all cases, we use $c = 10$, $S = 50$ and $b = 20$. The effect of different values of S has been similar in terms of percentage error in our experimentation, so we chose not to vary S for brevity. The effective inventory holding cost rate, calculated as $h' = c[i + (e^r - 1) + \delta]$, varies between 1.51 and 52.84 in all cases; and $b = 20$ is chosen to be somewhere in the middle between the smallest and the largest h'. If b is too large, there is very little backordering in the optimal solution and the backordering model approaches the basic model. If it is too small, then the optimal backorder quantity becomes unreasonably large as compared to the order quantity and the order quantity approaches zero, which is obviously not acceptable in practice.

We calculate the exact optimal solution by minimizing Equations (16) and (17), for the backordering and basic versions of the model, respectively. For the approximate backordering model, we first calculate T^* and T_I^* by using Equations (25) and (28); then calculate Q^* using Equation (3). For both the approximate and the exact backordering models, we calculate the optimal total cost using Equation (16). For the approximate basic model, we first calculate T^* by using Equation (29); then calculate Q^* using Equation (3). For both the approximate and the exact basic models, we calculate the optimal total cost using Equation (17).

As can be seen in Table 2, our approximate model is very close to the optimum in all of the cases. For $D = 10{,}000$, the approximate solution's total cost is between 0.0001% and 0.0044% from the exact optimal total cost. The range is [0.0018%, 0.0871%] for $D = 500$. We can confidently say that our planned backorders model is very accurate and therefore very useful to practitioners, considering that it is very simple, logical and intuitive as well.

Table 2. Optimal order and backorder quantities and optimal total cost for the exact and approximate planned backorders models.

		$D = 10{,}000$							$D = 500$						
		Q^*		B^*		$TC(Q^*, B^*)$			Q^*		B^*		$TC(Q^*, B^*)$		
δ	r	**Exact**	**App.**	**Exact**	**App.**	**Exact**	**App.**	**% Diff.**	**Exact**	**App.**	**Exact**	**App.**	**Exact**	**App.**	**% Diff.**
5.00	0.05	264.65	265.61	189.59	189.26	3791.75	3791.91	0.0042	59.51	60.56	42.63	42.32	852.69	853.38	0.0808
5.00	0.10	264.25	265.19	189.87	189.54	3797.34	3797.49	0.0041	59.42	60.44	42.69	42.38	853.88	854.55	0.0790
5.00	0.15	263.83	264.76	190.15	189.83	3803.12	3803.27	0.0040	59.32	60.33	42.75	42.45	855.11	855.77	0.0773
5.00	0.20	263.40	264.31	190.45	190.14	3809.10	3809.25	0.0039	59.21	60.20	42.82	42.52	856.38	857.03	0.0756
5.00	0.25	262.95	263.85	190.76	190.45	3815.28	3815.42	0.0038	59.10	60.08	42.88	42.59	857.71	858.34	0.0740
3.75	0.05	276.80	277.96	181.37	181.00	3627.42	3627.58	0.0044	62.30	63.56	40.83	40.47	816.51	817.21	0.0860
3.75	0.10	276.11	277.25	181.80	181.44	3636.05	3636.21	0.0043	62.14	63.37	40.92	40.57	818.36	819.04	0.0837
3.75	0.15	275.42	276.53	182.24	181.89	3644.95	3645.10	0.0042	61.97	63.18	41.01	40.67	820.26	820.93	0.0814
3.75	0.20	274.70	275.79	182.70	182.36	3654.11	3654.25	0.0041	61.80	62.98	41.11	40.78	822.23	822.88	0.0793
3.75	0.25	273.97	275.03	183.17	182.83	3663.52	3663.67	0.0040	61.62	62.77	41.21	40.88	824.27	824.90	0.0773
2.50	0.05	299.37	300.83	167.81	167.42	3356.29	3356.44	0.0044	67.47	69.04	37.82	37.44	756.41	757.07	0.0871
2.50	0.10	297.98	299.40	168.57	168.18	3371.46	3371.60	0.0043	67.14	68.67	37.98	37.61	759.68	760.32	0.0840
2.50	0.15	296.57	297.96	169.34	168.97	3386.96	3387.10	0.0041	66.80	68.28	38.15	37.78	763.04	763.66	0.0811
2.50	0.20	295.15	296.49	170.13	169.76	3402.81	3402.94	0.0040	66.46	67.90	38.32	37.96	766.48	767.08	0.0784
2.50	0.25	293.71	295.02	170.94	170.58	3418.97	3419.11	0.0039	66.11	67.51	38.49	38.14	770.01	770.59	0.0759
1.25	0.05	357.12	359.09	140.77	140.39	2815.36	2815.47	0.0037	80.58	82.67	31.76	31.39	635.28	635.73	0.0720
1.25	0.10	352.73	354.61	142.48	142.11	2849.62	2849.72	0.0035	79.55	81.53	32.13	31.78	642.80	643.23	0.0682
1.25	0.15	348.43	350.22	144.19	143.84	2884.09	2884.18	0.0033	78.53	80.42	32.51	32.16	650.38	650.80	0.0648
1.25	0.20	344.20	345.92	145.91	145.58	2918.71	2918.80	0.0031	77.53	79.34	32.88	32.55	658.03	658.44	0.0620
1.25	0.25	340.06	341.72	147.64	147.32	2953.47	2953.56	0.0030	76.54	78.28	33.26	32.94	665.73	666.12	0.0595
0.01	0.05	1296.04	1297.75	38.53	38.55	771.55	771.55	0.0001	289.08	290.80	8.60	8.62	172.92	172.93	0.0018
0.01	0.10	956.98	958.68	52.12	52.18	1044.50	1044.50	0.0002	212.97	214.69	11.61	11.67	234.29	234.29	0.0033
0.01	0.15	793.48	795.22	62.81	62.90	1259.32	1259.33	0.0002	176.29	178.03	13.97	14.06	282.67	282.68	0.0049
0.01	0.20	692.80	694.56	71.89	72.01	1442.01	1442.02	0.0003	153.71	155.46	15.98	16.10	323.87	323.89	0.0066
0.01	0.25	622.94	624.73	79.91	80.05	1603.39	1603.39	0.0004	138.04	139.81	17.76	17.90	360.31	360.34	0.0084

As can be seen in Table 3, our approximate model is also quite accurate, similar to the planned backorders model. For $D = 10{,}000$, the approximate model is between 0.0001% and 0.0273% from the exact optimal total cost. The range is [0.0017%, 0.5394%] for $D = 500$. We can confidently say that our basic model is also very accurate, in addition to being very simple, logical and intuitive.

Table 3. Optimal order quantity and total cost for the exact and approximate basic models.

Parameters		$D = 10{,}000$					$D = 500$				
		Q^*		$TC(Q^*)$			Q^*		$TC(Q^*)$		
δ	r	Exact	App.	Exact	App	% Diff.	Exact	App.	Exact	App	% Diff.
5.00	0.05	142.34	145.77	7190.09	7192.05	0.0273	33.07	36.97	1670.54	1679.55	0.5394
5.00	0.10	141.58	144.97	7227.92	7229.88	0.0271	32.89	36.75	1679.02	1687.98	0.5339
5.00	0.15	140.79	144.14	7267.49	7269.43	0.0268	32.70	36.51	1687.89	1696.81	0.5283
5.00	0.20	139.97	143.29	7308.86	7310.80	0.0265	32.50	36.27	1697.18	1706.05	0.5227
5.00	0.25	139.13	142.41	7352.11	7354.03	0.0262	32.30	36.02	1706.89	1715.72	0.5171
3.75	0.05	163.83	167.23	6227.64	6228.91	0.0205	37.88	41.68	1439.82	1445.63	0.4040
3.75	0.10	162.67	166.02	6271.22	6272.48	0.0202	37.60	41.35	1449.58	1455.36	0.3986
3.75	0.15	161.47	164.78	6316.71	6317.97	0.0199	37.31	41.00	1459.79	1465.53	0.3932
3.75	0.20	160.24	163.50	6364.19	6365.44	0.0196	37.02	40.65	1470.44	1476.15	0.3878
3.75	0.25	158.98	162.19	6413.74	6414.98	0.0194	36.72	40.29	1481.57	1487.24	0.3825
2.50	0.05	199.61	202.96	5092.52	5093.21	0.0136	45.87	49.55	1170.39	1173.53	0.2682
2.50	0.10	197.51	200.80	5145.62	5146.30	0.0133	45.38	48.98	1182.29	1185.40	0.2630
2.50	0.15	195.38	198.60	5200.87	5201.55	0.0130	44.87	48.39	1194.68	1197.76	0.2580
2.50	0.20	193.21	196.36	5258.34	5259.02	0.0128	44.36	47.80	1207.58	1210.64	0.2531
2.50	0.25	191.00	194.09	5318.10	5318.77	0.0126	43.83	47.20	1221.01	1224.04	0.2483
1.25	0.05	278.81	282.07	3628.10	3628.35	0.0067	63.57	67.05	827.21	828.30	0.1321
1.25	0.10	273.17	276.31	3702.10	3702.34	0.0064	62.25	65.60	843.80	844.87	0.1275
1.25	0.15	267.59	270.62	3778.36	3778.59	0.0062	60.95	64.17	860.91	861.98	0.1233
1.25	0.20	262.07	264.99	3856.91	3857.15	0.0060	59.66	62.76	878.56	879.62	0.1196
1.25	0.25	256.62	259.44	3937.83	3938.06	0.0059	58.38	61.37	896.76	897.81	0.1162
0.01	0.05	1276.67	1278.35	783.27	783.27	0.0001	284.81	286.48	175.54	175.55	0.0017
0.01	0.10	930.64	932.25	1074.10	1074.11	0.0001	207.20	208.79	240.89	240.90	0.0029
0.01	0.15	761.59	763.15	1312.20	1312.20	0.0002	169.33	170.87	294.45	294.46	0.0041
0.01	0.20	656.08	657.59	1522.95	1522.95	0.0003	145.72	147.21	341.89	341.91	0.0052
0.01	0.25	581.88	583.36	1716.89	1716.90	0.0003	129.13	130.57	385.58	385.60	0.0063

5. Discussion

In this research, we develop exact total cost functions for both the basic as well as the backordering versions of the deteriorating items EOQ problem under compound interest. The total cost functions are reasonably simple to be solved even using spreadsheet solvers. This greatly increases the likelihood of adoption by the practitioners in the industry, to optimize their inventories where the EOQ model with exponential deterioration is appropriate, and backorders are potentially allowed. Our treatment of the compound interest is novel for this problem, which has been developed by Çalışkan [33]. Even though the exact total cost functions can be used on their own, along with a nonlinear optimizer, we also develop approximation models using the Taylor series expansion of the exponential function. The resulting total cost functions are the same as the total cost functions of the basic and the planned backorders versions of the EOQ model, with the inventory holding cost rate $h = c(i + r)$ of the EOQ model being replaced by $h' = c[i + \delta + (e^r - 1)]$. This is very intuitive as $(e^r - 1)$ is the effective annual interest rate with continuous compounding, and δ is the cost of deterioration per unit of the inventory per year, which is defined the same way as the inventory holding cost rate h of the EOQ model. Our approximate closed-form solutions, with the adjusted or effective inventory holding cost rate h', results in optimal solutions that are remarkably close to the exact optima. This makes the closed-form solutions highly likely to be adopted in practice.

6. Conclusions

We consider the optimal order quantity problem for deteriorating items where the opportunity cost is calculated based on compound interest, and planned backorders are allowed. Deteriorating items include various chemicals, gasoline and petroleum products, fresh produce, batteries and some electronic components. For these types of items, a portion of the inventory deteriorates and becomes unusable, throughout their time in storage, depleting the inventory in addition to the demand. Exponential deterioration is a reasonable model for this phenomenon and it is commonly used in the literature. In exponential deterioration, each unit of the item has an exponential lifetime or time to deterioration distribution. The resulting inventory level with exponential deterioration is an exponentially decreasing function over time, and the resulting total cost functions in these models do not have closed-form solutions. Compounding effects have invariably been modeled in the literature by total cost functions that consist of the Net Present Value (NPV) of future cash outflows. Because of the exponential terms, these models do not have exact closed-form optimal solutions, either. NPV-based models invariably result in the classical EOQ equation as a closed-form solution, when they are approximated. Our approach is a departure from the NPV-based models, in the sense that we do not discount all cash outflows; rather, we calculate the opportunity cost part of the inventory holding cost using compound interest as opposed to the simple interest in the classical EOQ model. Because of the exponential terms in the total cost function arising from both exponential deterioration and compounding, our exact model does not have a closed-form solution, either. However, we approximate our exact model by using the Taylor expansion of the exponential function, and obtain a closed-form solution that is simple, elegant, logical and intuitive, as well as remarkably close to the optimal solution. We demonstrate the closeness of our intuitive closed-form solution by extensive experimental results. Our exact model in this paper is novel and solvable by all nonlinear solvers, including spreadsheet solvers. Both our exact model and its approximate closed-form solution have great likelihood of being adopted in practice, because of their simplicity and accuracy.

7. Limitations and Future Research

We assume exponential deterioration in this research, which also means that the lifetime distribution for each individual item is exponential, which has the memoryless property, so the probability of deterioration does not depend on the part of the life that has already elapsed. This makes the instantaneous deterioration rate constant, hence this model is also called constant deterioration model. In practice, for some items, this may not be the case, and the probability of deterioration may be higher as the elapsed life time gets longer. Future research may use other deterioration models that allow for increased probability of deterioration over time. We also assume that the unit price of the item, as well as the demand are constant and the demand is independent of the price. In many cases in practice, demand may be price-dependent. Future research may consider a demand function that is price-dependent. Finally, we assume that the purchase and ordering costs are immediately paid at the time of order arrival. Future research may extend our model to trade credits in which these costs can be paid after some predetermined trade credit period for each order.

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